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Equilibrium fluctuations for a chain of anharmonic oscillators in the Euler scaling limit

Stefano OLLA Lu XU

Abstract

We study the macroscopic behavior of the fluctuations in equilibrium for the conserved quantities of an anharmonic chain of oscillators under hyperbolic scaling of space and time. Under a stochastic perturbation of the dynamics conservative of such quantities, we prove that these fluctuations evolve macroscopically following the linearized Euler system of equations.

1 Introduction

The deduction of Euler equations for a compressible gas from the microscopic dynamics under a space-time scaling limit is one of the main problems in statistical mechanics [13]. With a generic assumption of *local equilibrium*, Euler equations can be formally obtained in the limit, but a mathematical proof starting from deterministic Hamiltonian dynamics is still an open problem. The eventual appearance of shock waves complicates further the problem, and in this case, it is expected the convergence to weak entropic solutions of Euler equations.

Some mathematical results have been obtained by perturbing the Hamiltonian dynamics by random terms that conserve energy and momentum, in such way that the dynamics has enough ergodicity to generate some form of local equilibrium (cf. [16, 1]). These results are obtained by relative entropy techniques and restricted to the smooth regime of the Euler equations. The noise introduced in these works are essentially random collisions between close particles and it is acting only on the velocities. Under such random perturbation, the only *macroscopic* conserved quantities are those that evolve macroscopically with the Euler equations. Actually, random dynamics and local equilibrium are only a tool in order to obtain the separation of scales between microscopic and macroscopic modes necessary in order to close the Euler equations. In the deterministic dynamics of harmonic oscillators with random masses (not ergodic), Anderson localization provides such separation of scales [8].

In this article we study the evolution of the fluctuations of the conserved quantities. When the system is in equilibrium at certain averaged values of the conserved quantities, these have Gaussian macroscopic fluctuations. The aim is to prove that these fluctuations, in the macroscopic space-time scaling limit, evolve deterministically following the linearized Euler equations. It turns out that this is more difficult than proving the hydrodynamic limit, as it requires the control of the space-time variance of the currents of the conserved quantities. More precisely it demands to prove that the currents are *equivalent* (in the norm introduced by the space-time variance) to linear functions of the conserved quantities. This step is usually called *Boltzmann-Gibbs principle* (cf. [2, 10]). This is the main part of the proof, and it forces us to consider *elliptic* type of stochastic perturbations, i.e., noise terms that acts also on the positions, not only on the velocities, still maintaining the same conserved quantities.

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The system we consider is the one-dimensional chain of coupled anharmonic oscillators, similar to the one considered in [1]. It can be described as a lattice system on \mathbb{Z} , where at every site $x \in \mathbb{Z}$ there is an oscillator numbered with x . The momenta (or velocity, since we set the masses equal to 1) of the particle x is denote by $p_x \in \mathbb{R}$, while $r_x \in \mathbb{R}$ denote the distance between particle $x - 1$ and x . The anharmonic interaction between $x - 1$ and x is given by $V(r_x)$, satisfying certain conditions. The dynamics is such that there are three locally conserved quantities: p_x , r_x , and the energy $e_x = p_x^2/2 + V(r_x)$, and the stochastic perturbations assure that these are the only ones. With $w_x = (p_x, r_x, e_x)$, the hydrodynamic limit is given by the convergence, for any continuous function G on \mathbb{R} with compact support,

$$\frac{1}{N} \sum_{x \in \mathbb{Z}} w_x(Nt) G\left(\frac{x}{N}\right) \xrightarrow{N \rightarrow \infty} \int_{\mathbb{R}} G(y) \mathfrak{w}(t, y) dy, \quad (1.1)$$

where $\mathfrak{w}(t, y)$ is the solution of the *compressible Euler equations*

$$\partial_t \mathfrak{w}(t, y) = \partial_y F(\mathfrak{w}(t, y)). \quad (1.2)$$

Here $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the vector of the macroscopic fluxes, computed from the local equilibrium (defined by (2.3)). In the smooth regime of (1.2), this can be proven by relative entropy techniques (at least for finite macroscopic volume with boundary conditions, cf. [1]).

We consider here the system in equilibrium starting with the Gibbs measure

$$\prod_{x \in \mathbb{Z}} \exp\{\lambda \cdot w_x - \mathcal{G}(\lambda)\} dp_x dr_x, \quad (1.3)$$

for a given $\lambda = (\beta\bar{p}, \beta\tau, -\beta) \in \mathbb{R}^2 \times \mathbb{R}_-$, where \mathcal{G} is the Gibbs potential given by

$$\mathcal{G}(\lambda) = \ln \left[\int_{\mathbb{R}} \exp\{-\beta(V(r) - \tau r)\} dr \right] + \frac{\beta\bar{p}^2}{2} + \frac{1}{2} \ln \frac{2\pi}{\beta} < \infty. \quad (1.4)$$

Denote by E_λ the expectation with respect to the measure defined in (1.3). Correspondingly there are equilibrium values $\bar{w}(\lambda) = E_\lambda[w_x]$ for the conserved quantities. The empirical distribution of the vibration of the conserved quantities is given by

$$\frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} [w_x(Nt) - \bar{w}(\lambda)] \delta_{x/N}(dy). \quad (1.5)$$

This is expected to converge to the solution of the linearized system

$$\partial_t \tilde{\mathfrak{w}}(t, y) = F'(\bar{w}(\lambda)) \partial_y \tilde{\mathfrak{w}}(t, y), \quad (1.6)$$

with a Gaussian initial distribution stationary for (1.6). Here F' denotes the Jacobian matrix associated to F .

While the non-equilibrium hydrodynamic limit can be proven by adding a simple exchange of p_x with p_{x+1} at random independent times (cf. [1]), in order to prove (1.6) we need to add, for each bond $(x, x + 1)$, a perturbation that exchanges $(p_x, p_{x+1}, r_x, r_{x+1})$ in such way that $r_x + r_{x+1}$, $p_x + p_{x+1}$, $e_x + e_{x+1}$ are conserved. The corresponding micro canonical surface is a one-dimensional circle, where we add a diffusive Wiener process. Then the main part of the article is the proof of a bound on the spectral gap for this stochastic dynamic process for K particles, that is independent of the values of the conserved quantities.

We believe that such macroscopic behavior of the equilibrium fluctuations should be valid also for the deterministic (non-linear) dynamics, but even the case with a stochastic perturbation acting only on the velocities remains an open problem. In the diffusive scaling with stochastic perturbations acting only on velocities but without conserving neither momentum nor energy, equilibrium fluctuations convergence has been proven in [15].

Finally, we comment briefly on longer time scales. The hyperbolic scale describes the time for the system to reach its mechanical equilibrium. Beyond that, it takes more time to reach the thermal equilibrium. Generally speaking, superdiffusions of energy are conjectured for one-dimensional Hamiltonian system conserving momentum, see [17], and proven rigorously for harmonic chains with conservative noise in [9]. Moreover, heat equations are obtained as diffusive space-time scaling limit in energy models, where noises are strong enough to destroy the momentum conservation law, see [7, 14].

2 Model and main result

We assume that the interaction potential V is a smooth function on \mathbb{R} satisfying the following condition:

$$(A1) \quad V \in C^2(\mathbb{R}), \quad \lim_{|r| \rightarrow \infty} |V| = \infty, \quad \inf_{\mathbb{R}} V'' > 0 \quad \text{and} \quad \sup_{\mathbb{R}} V'' < \infty.$$

Under (A1), there exists some unique point $r_0 \in \mathbb{R}$, such that $V(r) > V(r_0)$ for all $r \neq r_0$. By replacing V with a new potential $V_*(r) = V(r + r_0) - V(r_0)$, we can assume without loss of generality that $V \geq 0$ and $V(0) = 0$.

The configuration evolves in time as a Markov process in Ω with generator

$$\mathcal{L}_\gamma = \mathcal{A} + \gamma \mathcal{S},$$

where

$$\mathcal{A} = \sum_{x \in \mathbb{Z}} p_x (\partial_{r_x} - \partial_{r_{x+1}}) + (V'(r_{x+1}) - V'(r_x)) \partial_{p_x}, \quad \mathcal{S} = \frac{1}{2} \sum_{x \in \mathbb{Z}} \mathcal{X}_{x,x+1}^2,$$

$$\mathcal{X}_{x,x+1} = (p_{x+1} - p_x) (\partial_{r_{x+1}} - \partial_{r_x}) - (V'(r_{x+1}) - V'(r_x)) (\partial_{p_{x+1}} - \partial_{p_x}).$$

Here $\gamma > 0$ is the intensity of noise, \mathcal{A} is the Liouville operator associated to the Hamiltonian system, and \mathcal{S} generates the stochastic perturbation. The noise in the dynamics is conceived in such way that the three balanced (conserved) quantities p_x , r_x and e_x of \mathcal{A} are still preserved by \mathcal{S} .

Let us denote by π_λ the equilibrium Gibbs measure given by (1.3). Let \mathcal{H}_λ be Hilbert space of functions f on Ω such that $E_\lambda[f^2] < \infty$, and $\langle \cdot, \cdot \rangle_\lambda$ be the scalar product on it. Observe that for all smooth cylinder (i.e., local) functions f, g on Ω ,

$$\langle \mathcal{A}f, g \rangle_\lambda = \langle -f, \mathcal{A}g \rangle_\lambda, \quad \langle \mathcal{S}f, g \rangle_\lambda = \langle f, \mathcal{S}g \rangle_\lambda.$$

In particular, \mathcal{A} is anti-symmetric on \mathcal{H}_λ and \mathcal{S} is symmetric. Moreover,

$$\langle -\mathcal{L}_\gamma f, f \rangle_\lambda = \gamma \langle -\mathcal{S}f, f \rangle_\lambda = \frac{\gamma}{2} \sum_{x \in \mathbb{Z}} E_\lambda [(\mathcal{X}_{x,x+1} f)^2]$$

gives the Dirichlet form corresponding to \mathcal{L}_γ . We assume here that smooth cylinder functions form a core for \mathcal{L} in \mathcal{H}_λ , and the existence of the equilibrium dynamics generated by \mathcal{L} with initial distribution π_λ as well. We believe that proofs of such existence can be performed with standard techniques, similar as done in [15], see also [6], but it goes beyond the purpose of this article.

Denote by $\{\eta_t; t \geq 0\}$ the Markov process generated by $N\mathcal{L}_\gamma$, where the factor $N \in \mathbb{N}_+$ is the scaling parameter. The diffusion generated by $N\mathcal{L}_\gamma$ can also be formally expressed by the

following infinite system of stochastic differential equations:

$$\left\{ \begin{array}{l} dp_x(t) = N[V'(r_{x+1}) - V'(r_x)]dt + \frac{\gamma N}{2}(p_{x+1} - p_x)[V''(r_{x+1}) + V''(r_x)]dt \\ \quad - \frac{\gamma N}{2}(p_x - p_{x-1})[V''(r_x) + V''(r_{x-1})]dt \\ \quad - \sqrt{\gamma N}[V'(r_x) - V'(r_{x-1})]dB_t^{x-1} + \sqrt{\gamma N}[V'(r_{x+1}) - V'(r_x)]dB_t^x, \\ dr_x(t) = N(p_x - p_{x-1})dt + \gamma N[V'(r_{x+1}) - 2V'(r_x) + V'(r_{x-1})]dt \\ \quad + \sqrt{\gamma N}(p_x - p_{x-1})dB_t^{x-1} - \sqrt{\gamma N}(p_{x+1} - p_x)dB_t^x, \end{array} \right.$$

where $\{(B_t^x)_{t \geq 0}; x \in \mathbb{Z}\}$ is an infinite system of independent Brownian motions.

The random variables $\{w_x, x \in \mathbb{Z}\}$ are independent identical distributed under π_λ , and its average is $\bar{w}(\lambda) = E_\lambda[w_0] = \nabla \mathcal{G}(\lambda)$. In particular, the average velocity, inter-particle distance and energy can be expressed by the functions of λ as

$$E_\lambda[p_x] = \bar{p}, \quad \bar{r}(\lambda) \triangleq E_\lambda[r_x] = \beta^{-1} \partial_\tau \mathcal{G},$$

$$\bar{e}(\lambda) \triangleq E_\lambda[e_x] = -\partial_\beta \mathcal{G} + \bar{p}^2 + \tau \bar{r} = -\partial_\beta \mathcal{G}(0, \beta\tau, -\beta) + \frac{\bar{p}^2}{2} + \tau \bar{r}.$$

Noting that $\bar{e} \geq \bar{p}^2/2 + V(\bar{r})$, we define the internal energy in equilibrium by

$$U(\lambda) \triangleq \bar{e} - \frac{\bar{p}^2}{2} = -\partial_\beta \mathcal{G}(0, \beta\tau, -\beta) + \tau \bar{r}.$$

Moreover, the covariance matrix of w_x is

$$\sigma^2(\lambda) \triangleq E_\lambda [(w_x - \bar{w}(\lambda)) \otimes (w_x - \bar{w}(\lambda))^T] = \text{Hess } \mathcal{G}(\lambda),$$

where $\text{Hess } \mathcal{G}$ denotes the Hessian matrix of \mathcal{G} .

Hence the inverse temperature and tension can be expressed by the formulas

$$\beta(r, U) = \partial_U \mathcal{S}(r, U), \quad \tau(r, U) = -\beta^{-1} \partial_r \mathcal{S}(r, U), \quad (2.1)$$

where \mathcal{S} is the *thermodynamic entropy* given by

$$\mathcal{S}(r, U) = \inf_{\beta > 0, \tau \in \mathbb{R}} \{ \beta(U - \tau r) + \mathcal{G}(0, \beta\tau, -\beta) \}, \quad \forall r \in \mathbb{R}, U > 0. \quad (2.2)$$

With $\tau(\bar{w}) = \tau(\bar{r}, \bar{e} - \bar{p}^2/2)$, the *macroscopic flux vector function* is given by

$$F(\bar{w}) = (\tau(\bar{w}), \bar{p}, \bar{p}\tau(\bar{w})). \quad (2.3)$$

In the following contents, $S(\mathbb{R})$ stands for the Schwartz space over \mathbb{R} , consisting of all smooth functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\|f\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}} \left\{ \left| x^\alpha \frac{d^\beta}{dx^\beta} f(x) \right| \right\} < \infty, \quad \forall \alpha, \beta \in \mathbb{N}_+.$$

A sequence of functions f_m converges in $S(\mathbb{R})$ to a function f , if $\|f_m - f\|_{\alpha, \beta} \rightarrow 0$ as $m \rightarrow \infty$ for all $\alpha, \beta \in \mathbb{N}_+$. The dense subset of $S(\mathbb{R})$ consisting of all smooth functions f with compact support is denoted by $C_0^\infty(\mathbb{R})$. The dual space of $S(\mathbb{R})$ is denoted by $S'(\mathbb{R})$.

Denote by $\mathbb{P}_{\lambda, N}$ the probability measure on the path space $C([0, \infty); \Omega)$ induced by $\{\eta_t; t \geq 0\}$ with initial condition π_λ , and let $\mathbb{E}_{\lambda, N}$ be the corresponding expectation. Define the *empirical conserved quantities fluctuation field* on $G \in S(\mathbb{R})$ by

$$\mathcal{Y}_N(t, G) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} G\left(\frac{x}{N}\right) (w_x(\eta_t) - \bar{w}(\lambda)). \quad (2.4)$$

Let $\|\cdot\|_{L^2}$ be the norm of $L^2(\mathbb{R}, dx)$, and notice that

$$\mathbb{E}_{\lambda, N} \left[|\mathcal{Y}_N(t, G)|^2 \right] = \frac{1}{N} \sum_{x \in \mathbb{Z}} G \left(\frac{x}{N} \right)^2 \text{Tr} [\sigma^2(\lambda)] \leq C_\lambda \|G\|_{L^2}^2,$$

so the summation on the right-hand side of (2.4) is well-defined in $L^2(\Omega, \pi_\lambda)$. Classical central limit theorem yields that for fixed $t \geq 0$ we have the following convergence in law to a three-dimensional Gaussian distribution

$$\mathcal{Y}_N(t, G) \Rightarrow \mathcal{N} \left(0, \sigma^2(\lambda) \int_{\mathbb{R}} G^2(y) dy \right), \quad \forall G \in S(\mathbb{R}).$$

To describe the limit process, consider the $S'(\mathbb{R})^3$ -valued stochastic process

$$\{\tilde{\mathfrak{w}}(t, \cdot) = (\tilde{\mathfrak{p}}(t, \cdot), \tilde{\mathfrak{r}}(t, \cdot), \tilde{\mathfrak{e}}(t, \cdot)); t \geq 0\},$$

which is determined by the solution of the deterministic evolution equation

$$\partial_t \tilde{\mathfrak{w}}(t, y) = F'(\bar{w}(\lambda)) \partial_y \tilde{\mathfrak{w}}(t, y), \quad \forall t > 0, \forall y \in \mathbb{R}, \quad (2.5)$$

and starting from the initial measure

$$\tilde{\mathfrak{w}}(0, y) = \sigma(\lambda) \dot{B}_y, \quad (2.6)$$

where F' denotes the Jacobian matrix of F , and $\{\dot{B}_y\}_{y \in \mathbb{R}}$ is a three-dimensional standard white noise on \mathbb{R} .

For $T > 0$, denote by \mathbb{Q}_N the distribution of $\mathcal{Y}_N(t)$, and by \mathbb{Q} that of $\tilde{\mathfrak{w}}(t, \cdot)$, both on the trajectory space $C([0, T]; (S'(\mathbb{R}))^3)$. Our main result is stated as below.

Theorem 2.1. *Assume that V satisfies the condition (A1). Then for every $T > 0$, the sequence of probability measures $\{\mathbb{Q}_N; N \geq 1\}$ converges weakly to \mathbb{Q} as $N \rightarrow \infty$.*

3 Proof of the main theorem

For given $G \in C_0^\infty(\mathbb{R})$, the following decomposition holds $\mathbb{P}_{\lambda, N}$ almost surely:

$$\mathcal{Y}_N(t, G) - \mathcal{Y}_N(0, G) = \mathcal{I}_{N,1}(t, G) + \frac{\gamma}{2} \mathcal{I}_{N,2}(t, G) + \sqrt{\gamma} \mathcal{M}_N(t, G). \quad (3.1)$$

Here $\mathcal{I}_{N,1}$, $\mathcal{I}_{N,2}$ and \mathcal{M}_N are defined respectively as

$$\begin{cases} \mathcal{I}_{N,1}(t, G) = \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} \nabla_N G(x) J_{x, x+1}(\eta_s) ds, \\ \mathcal{I}_{N,2}(t, G) = \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} \nabla_N G(x) j_{x, x+1}(\eta_s) ds, \\ \mathcal{M}_N(t, G) = \frac{1}{N} \int_0^t \sum_{x \in \mathbb{Z}} \nabla_N G(x) m_{x, x+1}(\eta_s) dB_s^x, \end{cases}$$

where ∇_N is the discrete derivative operator defined by

$$\nabla_N G(x) = N \left[G \left(\frac{x+1}{N} \right) - G \left(\frac{x}{N} \right) \right],$$

$J_{x,x+1}, j_{x,x+1}$ are the instantaneous microscopic currents between $x, x+1$:

$$\begin{aligned} \mathcal{A} \begin{pmatrix} p_x \\ r_x \\ e_x \end{pmatrix} &= J_{x-1,x} - J_{x,x+1}, & J_{x,x+1} &= \begin{pmatrix} -V'(r_{x+1}) \\ -p_x \\ -p_x V'(r_{x+1}) \end{pmatrix}, \\ \mathcal{S} \begin{pmatrix} p_x \\ r_x \\ e_x \end{pmatrix} &= j_{x-1,x} - j_{x,x+1}, & j_{x,x+1} &= -\frac{1}{2} \mathcal{X}_{x,x+1}^2 \begin{pmatrix} p_x \\ r_x \\ e_x \end{pmatrix}, \end{aligned}$$

and $m_{x,x+1}$ is given by

$$m_{x,x+1} = -\mathcal{X}_{x,x+1} \begin{pmatrix} p_x \\ r_x \\ e_x \end{pmatrix} = \begin{pmatrix} V'(r_x) - V'(r_{x+1}) \\ p_{x+1} - p_x \\ p_{x+1} V'(r_x) - V'(r_{x+1}) p_x \end{pmatrix}.$$

Note that $F(\bar{w}(\lambda)) = -E_\lambda[J_{x,x+1}]$. To get the linearized Euler system in (2.5), we introduce a linear approximation for the fluctuation of $J_{x,x+1}$. Define a random field Φ by

$$\Phi(\omega) = J_{0,1}(\omega) + F(\bar{w}(\lambda)) + F'(\bar{w}(\lambda))(w_0(\omega) - \bar{w}(\lambda)), \quad \forall \omega \in \Omega. \quad (3.2)$$

In order to prove Theorem 2.1, we need the following lemmas.

Lemma 3.1. *For every $G \in S(\mathbb{R})$ and $T > 0$,*

$$\mathbb{E}_{\lambda,N} \left[\sup_{t \in [0,T]} \lambda |\mathcal{I}_{N,2}(t, G)|^2 + \sup_{t \in [0,T]} |\mathcal{M}_N(t, G)|^2 \right] \leq \frac{CT \|G'\|_{L^2}^2}{N}.$$

Lemma 3.2. *For every $G \in S(\mathbb{R})$ and $T > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\lambda,N} \left[\sup_{t \in [0,T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} G' \left(\frac{x}{N} \right) \tau_x \Phi(\eta_s) ds \right|^2 \right] = 0,$$

where τ_x is the translation on Ω : $\tau_x \omega = \omega(\cdot + x)$, and $\tau_x \Phi = \Phi \circ \tau_x$.

Lemma 3.1 is proved at the end of this section. Lemma 3.2 is the main point. We postpone its proof to Section 4. Now we prove Theorem 2.1 as consequence of them.

Proof. For test function $G \in S(\mathbb{R})$, denote by $\mathbb{Q}_{N,G}$ the distribution of $\mathcal{Y}_N(\cdot, G)$ on the trajectory space $C([0, T]; \mathbb{R}^3)$. It suffices to show that

- (i) $\{\mathbb{Q}_{N,G}; N \geq 1\}$ is tight, and
- (ii) each limit point of $\{\mathbb{Q}_N\}$ satisfies the integrate form of (2.5) and (2.6).

Indeed, by (i) and [12, Theorem 3.1], $\{\mathbb{Q}_N; N \geq 1\}$ is tight on $C([0, T]; (S'(\mathbb{R}))^3)$. Since the solution to (2.5) and (2.6) is unique, (ii) implies the weak convergence.

For (i), it suffices to consider only $\mathcal{I}_{N,1}(\cdot, G)$, thanks to Lemma 3.1. For all $0 \leq s < t \leq T$, Cauchy-Schwarz inequality yields that

$$\mathbb{E}_{\lambda,N} \left[|\mathcal{I}_{N,1}(t, G) - \mathcal{I}_{N,1}(s, G)|^2 \right] \leq \frac{|t-s|^2}{N} E_\lambda \left[\left| \sum \nabla_N G(x) \tau_x \bar{J}_{0,1} \right|^2 \right],$$

where $\bar{J}_{0,1} = J_{0,1} - E_\lambda[J_{0,1}]$. Since $\langle \bar{J}_{0,1} \cdot \tau_x \bar{J}_{0,1} \rangle_\lambda = 0$ for $x \neq 0$,

$$E_\lambda \left[\left| \sum \nabla_N G(x) \tau_x \bar{J}_{0,1} \right|^2 \right] \leq C E_\lambda \left[|\bar{J}_{0,1}|^2 \right] \sum |\nabla_N G(x)|^2$$

with some universal constant C . Therefore, if $G \in C_0^\infty(\mathbb{R})$,

$$\mathbb{E}_{\lambda,N} \left[\left| \mathcal{I}_{N,1}(t, G) - \mathcal{I}_{N,1}(s, G) \right|^2 \right] \leq C' |t - s|^2 \|G'\|_{L^2}^2.$$

Since the bound above depends only on the L^2 -norm of G' , it can be extended to any $G \in S(\mathbb{R})$ and the tightness follows from the standard argument.

For (ii), pick an arbitrary weak limit point \mathbb{Q} of $\{\mathbb{Q}_N\}$, and let \mathcal{Y}_t be a process subject to \mathbb{Q} . Write $\Delta(x, N) = \nabla_N G(x) - G'(x/N)$. By similar argument as above,

$$\mathbb{E}_{\lambda,N} \left[\sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} \Delta(x, N) \bar{J}_{x, x+1}(\eta_s) ds \right|^2 \right] \leq \frac{T^2}{N} E_\lambda \left[|\bar{J}_{0,1}|^2 \right] \sum_{x \in \mathbb{Z}} |\Delta(x, N)|^2,$$

and vanishes as $N \rightarrow \infty$. By this, Lemma 3.1 and 3.2, \mathcal{Y}_t satisfies that

$$\mathcal{Y}_t(G) - \mathcal{Y}_0(G) = -F'(\bar{w}(\lambda)) \int_0^t \mathcal{Y}_s(G') ds, \quad \forall G \in S(\mathbb{R}).$$

The evolution equation (2.5) then follows. Since the initial condition (2.6) follows directly from the classical central limit theorem, the proof is completed. \square

Now we proceed to prove Lemma 3.1. We make use of the following estimate of the space-time variance of $\{\eta_t\}$. For any function $f \in \mathcal{H}_\lambda$, we have

$$\mathbb{E}_{\lambda,N} \left[\sup_{t \in [0, T]} \left| \int_0^t f(\eta_s) ds \right|^2 \right] \leq 14T \cdot \sup_h \left\{ 2\langle f, h \rangle_\lambda - \gamma N \langle h, -Sh \rangle_\lambda \right\}, \quad (3.3)$$

where the superior in the right-hand side is taken over all bounded smooth cylinder functions. A proof of (3.3) can be found in [11, Sec. 2.5].

Proof of Lemma 3.1. First pick $G \in C_0^\infty$. By applying (3.3) on $\mathcal{I}_{N,2}$, one obtains that

$$\mathbb{E}_{\lambda,N} \left[\sup_{t \in [0, T]} \left| \mathcal{I}_{N,2}(t, G) \right|^2 \right] \leq \frac{14T}{N} \cdot \sup_h \left\{ 2 \sum_{x \in \mathbb{Z}} \nabla_N G(x) \langle \tau_x j_{0,1}, h \rangle_\lambda - \gamma N \langle h, -Sh \rangle_\lambda \right\}.$$

Recalling that $j_{0,1} = \frac{1}{2} \mathcal{X}_{0,1} m_{0,1}$, by Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \sum \nabla_N G(x) \langle \tau_x j_{0,1}, h \rangle_\lambda \right|^2 &= \frac{1}{4} \left| \sum \nabla_N G(x) \langle m_{x, x+1}, \mathcal{X}_{x, x+1} h \rangle_\lambda \right|^2 \\ &\leq \frac{1}{4} \sum_{x \in \mathbb{Z}} |\nabla_N G(x)|^2 E_\lambda [m_{0,1}^2] \sum_{x \in \mathbb{Z}} E_\lambda [(\mathcal{X}_{x, x+1} h)^2] \\ &\leq CN \|G'\|_{L^2}^2 E_\lambda [m_{0,1}^2] \langle h, -Sh \rangle_\lambda, \end{aligned}$$

with some constant C . Thus, by optimizing h ,

$$\mathbb{E}_{\lambda,N} \left[\sup_{t \in [0, T]} \left| \mathcal{I}_{N,2}(t, G) \right|^2 \right] \leq \frac{C'T \|G'\|_{L^2}^2}{\gamma N}.$$

On the other hand, by Doob's maximal inequality,

$$\mathbb{E}_{\lambda,N} \left[\sup_{t \in [0, T]} \left| \mathcal{M}_N(t, G) \right|^2 \right] \leq 4 \mathbb{E}_{\lambda,N} \left[\left| \mathcal{M}_N(T, G) \right|^2 \right].$$

Since π_λ is stationary and $m_{0,1}$ is a local function on Ω ,

$$\mathbb{E}_{\lambda,N} \left[\left| \mathcal{M}_N(T, G) \right|^2 \right] = \frac{T}{N^2} E_\lambda \left[\left| \sum \nabla_N G(x) \tau_x m_{0,1} \right|^2 \right] \leq \frac{CT \|G'\|_{L^2}^2}{N}.$$

Since all these bounds depends only on $\|G'\|_{L^2}$, they can be extended to any $G \in S(\mathbb{R})$, and Lemma then 3.1 follows. \square

4 Boltzmann-Gibbs principle

In this section we state the proof of Lemma 3.2. It is called Boltzmann-Gibbs principle, firstly established for zero range jump process, see [2]. It aims at determining the space-time fluctuation of a local function by linear function on the conserved quantities. Here we prove it along the arguments in [10, Sec. 11.1]. To this end, we make use of a spectral gap bound of \mathcal{S} , as well as the equivalence of grand and micro canonical ensembles. These results are established in Section 5 and Section 6, respectively.

Proof of Lemma 3.2. Recall that Φ defined in (3.2) is a local function depending on p_0 , r_0 and r_1 . The first step is to consider a microscopic block $\{0, 1, \dots, K\}$. Note that $1 < K < N$ and K increases to ∞ after N . Define

$$\Phi_K = \frac{1}{K} \sum_{x=0}^{K-1} \Phi_x, \quad \langle \Phi_K \rangle = E_\lambda \left[\Phi_K \mid \frac{w_0 + w_1 + \dots + w_K}{K+1} \right].$$

Denoting by $\varphi_K = \Phi - \Phi_K$ and $\psi_K = \Phi_K - \langle \Phi_K \rangle$, then

$$\Phi = \varphi_K + \psi_K + \langle \Phi_K \rangle.$$

We prove the estimate for each term respectively. Firstly, by Cauchy-Schwarz inequality,

$$\begin{aligned} & \mathbb{E}_{\lambda, N} \left[\sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} G' \left(\frac{x}{N} \right) \tau_x \varphi_K(\eta(s)) ds \right|^2 \right] \\ & \leq \frac{T^2}{K^2 N} E_\lambda \left[\left| \sum_{y=1}^K \sum_{x \in \mathbb{Z}} \left[G' \left(\frac{x}{N} \right) - G' \left(\frac{x-y}{N} \right) \right] \tau_x \Phi \right|^2 \right] \\ & \leq \frac{T^2}{KN} \sum_{y=1}^K E_\lambda \left[\left| \sum_{x \in \mathbb{Z}} \left[G' \left(\frac{x}{N} \right) - G' \left(\frac{x-y}{N} \right) \right] \tau_x \Phi \right|^2 \right]. \end{aligned}$$

Since Φ and $\tau_x \Phi$ are orthogonal for $|x| > 1$, this is bounded from above by $C_G K^2 N^{-2}$ with some $C_G < \infty$, thus vanishes as $N \rightarrow \infty$ for every fixed K .

For the fluctuation of ψ_K , (3.3) yields that

$$\begin{aligned} & \mathbb{E}_{\lambda, N} \left[\sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} G' \left(\frac{x}{N} \right) \tau_x \psi_K(\eta(s)) ds \right|^2 \right] \\ & \leq \frac{14T}{N} \cdot \sup_h \left\{ 2 \sum_{x \in \mathbb{Z}} G' \left(\frac{x}{N} \right) \langle \tau_x \psi_K, h \rangle_\lambda - \gamma N \langle h, -\mathcal{S}h \rangle_\lambda \right\}, \end{aligned} \tag{4.1}$$

where the superior in the second line is taken over all bounded smooth cylinder functions h on Ω . Noting that ψ_K is measurable with respect to the σ -field \mathcal{F}_K generated by $\{(p_x, r_x); 0 \leq x \leq K\}$, consider an operator $\mathcal{S}_{[0, K]}$ given by

$$\mathcal{S}_{[0, K]} = \frac{1}{2} \sum_{x=0}^{K-1} \mathcal{X}_{x, x+1}^2.$$

In view of Proposition 5.1, $\mathcal{S}_{[0, K]}$ has a spectral gap bound, and the equation

$$-\mathcal{S}_{[0, K]} F_K = \psi_K$$

can be solved by some \mathcal{F}_K -measurable function F_K such that for some constant C_K ,

$$\langle F_K, -\mathcal{S}_{[0, K]} F_K \rangle_\lambda \leq C_K.$$

Applying Cauchy-Schwarz inequality and the estimate above,

$$\begin{aligned}
& \left| \sum_x G' \left(\frac{x}{N} \right) \langle \tau_x \psi_K, h \rangle_\lambda \right|^2 = \left| \sum_x G' \left(\frac{x}{N} \right) \sum_{y=0}^K \langle \mathcal{X}_{y,y+1} F_K, \mathcal{X}_{y,y+1} \tau_{-x} h \rangle_\lambda \right|^2 \\
& \leq \left(\sum_x \left[G' \left(\frac{x}{N} \right) \right]^2 \sum_{y=0}^K \langle (\mathcal{X}_{y,y+1} F_K)^2 \rangle_\lambda \right) \left(\sum_x \sum_{y=0}^K \langle (\mathcal{X}_{y,y+1} \tau_{-x} h)^2 \rangle_\lambda \right) \\
& \leq C_K \sum_x \left[G' \left(\frac{x}{N} \right) \right]^2 K \langle h, -Sh \rangle_\lambda \leq KC_K N \|G'\|_{L^2}^2 \langle h, -Sh \rangle_\lambda.
\end{aligned}$$

Thus, the second line in (4.1) is $O(N^{-1})$ for every fixed K .

For the fluctuation of $\langle \Phi_K \rangle$, noting that $\langle \Phi_K \rangle$ and $\tau_x \langle \Phi_K \rangle$ are orthogonal in \mathcal{H}_λ for $|x| > K$, direct calculation shows that

$$\mathbb{E}_{\lambda, N} \left[\sup_{t \in [0, T]} \left| \frac{1}{\sqrt{N}} \int_0^t \sum_{x \in \mathbb{Z}} G' \left(\frac{x}{N} \right) \tau_x \langle \Phi_K \rangle (\eta(s)) ds \right|^2 \right] \leq KC_G T^2 E_\lambda [\langle \Phi_K \rangle^2],$$

Applying Proposition 6.3 (the equivalence of ensembles) to the \mathcal{F}_K -measurable function $\langle \Phi_K \rangle$, we know that $E_\lambda[\langle \Phi_K \rangle^2] = O(K^{-2})$. Therefore, the expression above vanishes as $K \rightarrow \infty$ uniformly in N . This completes the proof. \square

5 Spectral gap for \mathcal{S}

To begin with, we introduce the definition of micro canonical manifold. For each $K \geq 2$, let $\pi_{K, \lambda}$ be the marginal distribution of π_λ on $\{(p_k, r_k), k = 1, \dots, K\}$. Given $w \in \mathbb{R}^3$, the *micro canonical manifold* $\Omega_{K, w}$ is defined in the following:

$$\Omega_{K, w} = \left\{ (p_k, r_k), 1 \leq k \leq K \mid \frac{1}{K} \sum_{k=1}^K w_k = w \right\}. \quad (5.1)$$

In view of the conditions on V , $\Omega_{K, w}$ is a compact and connected manifold if $w = (p, r, e)$ and $e \geq p^2/2 + V(r)$. Define the conditioned measure

$$\pi_{K, w} \triangleq \pi_{K, \lambda} [\cdot \mid \Omega_{K, w}]. \quad (5.2)$$

The measure $\pi_{K, w}$ is called the *micro canonical measure* on $\Omega_{K, w}$. Notice that the definition of $\pi_{K, w}$ does not depend on the choice of λ and $\pi_{K, w}$ is the uniform measure on $\Omega_{K, w}$. Hereafter we use $E_{K, w}$ to denote the expectation with respect to $\pi_{K, w}$.

The main result of this section is stated as follows.

Proposition 5.1. *Suppose that the potential V satisfies (A1), then for any $K \geq 2$ there exists a finite constant C_K depending only on V , such that the following estimate*

$$E_{K, w} [(f - E_{K, w}[f])^2] \leq C_K \sum_{k=1}^{K-1} E_{K, w} [(\mathcal{X}_{k, k+1} f)^2] \quad (5.3)$$

holds for all w and bounded smooth function f . Moreover,

$$C_K \leq CK^2 \left(\frac{\delta_+}{\delta_-} \right)^{3K},$$

where C is some universal constant, $\delta_- = \inf_{\mathbb{R}} V''(r)$ and $\delta_+ = \inf_{\mathbb{R}} V''(r)$.

The proof of Proposition 5.1 is divided into Lemma 5.2, 5.3 and 5.4 below.

Lemma 5.2. *There exists a finite constant C such that*

$$E_{2,w} [(f - E_{2,w}[f])^2] \leq CE_{2,w} [(\mathcal{X}_{1,2}f)^2]$$

for all w and bounded smooth function f on (p_1, r_1, p_2, r_2) .

Proof. For $(p_1, r_1, p_2, r_2) \in \mathbb{R}^4$, define

$$p = p(p_1, p_2) = \frac{p_1 + p_2}{2}, \quad r = r(r_1, r_2) = \frac{r_1 + r_2}{2},$$

and $\mathfrak{E} = \mathfrak{E}(p_1, r_1, p_2, r_2) \geq 0$ given by

$$\mathfrak{E} = \frac{e_1 + e_2}{2} - \frac{p^2}{2} - V(r) = \frac{(p_1 - p_2)^2}{8} + \frac{V(r_1) + V(r_2)}{2} - V\left(\frac{r_1 + r_2}{2}\right).$$

Furthermore, let $\theta \in [0, 2\pi)$ satisfy that $\sqrt{\mathfrak{E}} \cos \theta = \sqrt{2}(p_1 - p_2)/4$ and

$$\sqrt{\mathfrak{E}} \sin \theta = \operatorname{sgn}(r_1 - r_2) \sqrt{\frac{V(r_1) + V(r_2)}{2} - V\left(\frac{r_1 + r_2}{2}\right)}.$$

The Jacobian determinant of the bijection $(p_1, r_1, p_2, r_2) \rightarrow (p, r, \mathfrak{E}, \theta)$ is

$$\mathfrak{J}(p, r, \mathfrak{E}, \theta) = \sqrt{2} \cdot \frac{\sqrt{V(r_1) + V(r_2) - 2V(r)}}{|V'(r_1) - V'(r_2)|}.$$

Recall that $0 < \delta_- \leq V''(r) \leq \delta_+ < \infty$, we have

$$0 < \frac{\sqrt{\delta_-}}{\sqrt{2}\delta_+} \leq \mathfrak{J}(p, r, \mathfrak{E}, \theta) \leq \frac{\sqrt{\delta_+}}{\sqrt{2}\delta_-}. \quad (5.4)$$

For a bounded smooth function $f = f(p_1, r_1, p_2, r_2)$, define $f_*(p, r, \mathfrak{E}, \theta) = f(p_1, r_1, p_2, r_2)$, and let $\langle f_* \rangle = \int_0^{2\pi} f_*(p, r, \mathfrak{E}, \theta) d\theta$. By simple calculations,

$$E_{2,w} [(f - \langle f_* \rangle)^2] = \frac{\int_0^{2\pi} [f_*(p, r, \mathfrak{E}, \theta) - \langle f_* \rangle]^2 \mathfrak{J}(p, r, \mathfrak{E}, \theta) d\theta}{\int_0^{2\pi} \mathfrak{J}(p, r, \mathfrak{E}, \theta) d\theta}.$$

On the other hand, since $\mathcal{X}_{1,2}f = \mathfrak{J}^{-1} \partial_\theta f_*$, we have

$$E_{2,w} [(\mathcal{X}_{1,2}f)^2] = \frac{\int_0^{2\pi} [\partial_\theta f_*(p, r, \mathfrak{E}, \theta)]^2 \mathfrak{J}^{-1}(p, r, \mathfrak{E}, \theta) d\theta}{\int_0^{2\pi} \mathfrak{J}(p, r, \mathfrak{E}, \theta) d\theta}.$$

Applying Poincaré inequality and (5.4), we obtain that

$$E_{2,w} [(f - E_{2,w}[f])^2] \leq E_{2,w} [(f - \langle f_* \rangle)^2] \leq \frac{C\delta_+}{2\delta_-^2} E_{2,w} [(\mathcal{X}_{1,2}f)^2]$$

holds with some universal constant $C < \infty$. □

To state the next lemma, for each pair of $i < j$, define a σ -algebra $\mathcal{F}_{i,j}$ by

$$\mathcal{F}_{i,j} = \sigma(\{(p_k, r_k); 1 \leq k \leq K, k \neq i, j\}).$$

Lemma 5.3. *There exists a finite constant C such that*

$$\sum_{1 \leq i < j \leq K} E_{K,w} [(f - E_{K,w}[f|\mathcal{F}_{i,j}])^2] \leq CK^3 \sum_{k=1}^{K-1} E_{K,w} [(f - E_{K,w}[f|\mathcal{F}_{k,k+1}])^2]$$

for all $K \geq 3$, w and bounded smooth function f .

Proof. The lemma is proved along the ideas used in [14, Lemma 12.4]. We first introduce some notations. For $1 \leq k \leq K$, write $x_k = (p_k, r_k)$ and $\mathbf{x} = (x_1, \dots, x_K)$. Recall the bijection defined in the proof of the Lemma 5.2. To simplify the notations, write

$$(p_{i,j}, r_{i,j}, \mathfrak{E}_{i,j}, \theta_{i,j}) = (p, r, \mathfrak{E}, \theta)(x_i, x_j), \quad \forall i < j.$$

For $\theta \in [0, 2\pi)$, denote the Jacobian determinant by

$$\mathfrak{J}_{\mathbf{x},i,j}(\theta) = \mathfrak{J}(p_{i,j}, r_{i,j}, \mathfrak{E}_{i,j}, \theta).$$

For $i < j$, $\theta \in [0, 2\pi]$ and $\mathbf{x} = (x_1, \dots, x_K)$, define a vector $\rho_{i,j}^\theta \mathbf{x}$ by

$$(\rho_{i,j}^\theta \mathbf{x})_k = \begin{cases} g_1(p_{i,j}, r_{i,j}, \mathfrak{E}_{i,j}, \theta), & k = i; \\ g_2(p_{i,j}, r_{i,j}, \mathfrak{E}_{i,j}, \theta), & k = j; \\ x_k, & k \neq i, j, \end{cases}$$

where (g_1, g_2) denotes the inverse map of $(x_1, x_2) \rightarrow (p, r, \mathfrak{E}, \theta)$. Observe that $\rho_{i,j}^\theta \mathbf{x} = \mathbf{x}$ when $\theta = \theta_{i,j}$, and for every smooth function f ,

$$E_{K,w}[f|\mathcal{F}_{i,j}] = \frac{1}{\mathcal{J}_{x_i+x_j}} \int_0^{2\pi} f(\rho_{i,j}^\theta \mathbf{x}) \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta,$$

where $\mathcal{J}_{x_i+x_j} = \int_0^{2\pi} \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta$. On the other hand, let $\tau_{i,j} \mathbf{x}$ be the vector given by

$$(\tau_{i,j} \mathbf{x})_i = x_j, \quad (\tau_{i,j} \mathbf{x})_j = x_i, \quad (\tau_{i,j} \mathbf{x})_k = x_k, \quad \forall k \neq i, j.$$

Moreover for $1 \leq i < j \leq K$, we inductively define that

$$\sigma_{i,i} = \tilde{\sigma}_{i,i} = id, \quad \sigma_{i,j} = \tau_{j-1,j} \circ \sigma_{i,j-1}, \quad \tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,j-1} \circ \tau_{j-1,j}.$$

Observe that for any $i < j$ and $\theta \in [0, 2\pi)$, $\rho_{i,j}^\theta \equiv \tilde{\sigma}_{i,j-1} \circ \rho_{j-1,j}^\theta \circ \sigma_{i,j-1}$.

For a smooth function f , by Cauchy-Schwarz inequality,

$$(f - E_{K,w}[f|\mathcal{F}_{i,j}])^2 \leq \frac{1}{\mathcal{J}_{x_i+x_j}} \int_0^{2\pi} [f(\rho_{i,j}^\theta \mathbf{x}) - f(\mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta.$$

The right-hand side is bounded from above by $3(f_1 + f_2 + f_3)$, where

$$\begin{aligned} f_1 &= \frac{1}{\mathcal{J}_{x_i+x_j}} \int_0^{2\pi} [f(\sigma_{i,j-1} \mathbf{x}) - f(\mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta, \\ f_2 &= \frac{1}{\mathcal{J}_{x_i+x_j}} \int_0^{2\pi} [f(\rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x}) - f(\sigma_{i,j-1} \mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta, \\ f_3 &= \frac{1}{\mathcal{J}_{x_i+x_j}} \int_0^{2\pi} [f(\tilde{\sigma}_{i,j-1} \circ \rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x}) - f(\rho_{j-1,j}^\theta \circ \sigma_{i,j-1} \mathbf{x})]^2 \mathfrak{J}_{\mathbf{x},i,j}(\theta) d\theta. \end{aligned}$$

For f_1 , noticing that $f_1 = (f(\sigma_{i,j-1}\mathbf{x}) - f(\mathbf{x}))^2$, hence

$$\begin{aligned} E_{K,w}[f_1] &\leq K \sum_{k=i}^{j-2} E_{K,w} [(f \circ \sigma_{i,k+1} - f \circ \sigma_{i,k})^2] \\ &= K \sum_{k=i}^{j-2} E_{K,w} [(f \circ \tau_{k,k+1} - f)^2]. \end{aligned}$$

Since $E_{K,w}[f \circ \tau_{k,k+1} | \mathcal{F}_{k,k+1}] = E_{K,w}[f | \mathcal{F}_{k,k+1}]$, by the estimate above and Cauchy-Schwarz inequality, we obtain that

$$E_{K,w}[f_1] \leq 4K \sum_{k=i}^{j-2} E_{K,w} [(f - E_{K,w}[f | \mathcal{F}_{k,k+1}])^2].$$

For f_2 , by applying the change of variable $\mathbf{y} = \sigma_{i,j-1}\mathbf{x}$, we obtain that

$$E_{K,w}[f_2] = E_{K,w} \left[\frac{1}{\mathcal{J}_{y_{j-1}+y_j}} \int_0^{2\pi} [f(\rho_{j-1,j}^\theta \mathbf{y}) - f(\mathbf{y})]^2 \mathfrak{J}_{\mathbf{y},i,j}(\theta) d\theta \right].$$

Therefore, we can calculate this term as

$$\begin{aligned} E_{K,w}[f_2] &= 2E_{K,w}[f^2] - 2E_{K,w}[f E_{K,w}[f | \mathcal{F}_{j-1,j}]] \\ &= 2E_{K,w} [(f - E_{K,w}[f | \mathcal{F}_{j-1,j}])^2]. \end{aligned}$$

For f_3 , the same change of variable yields that

$$E_{K,w}[f_3] = E_{K,w} [E_{K,w} [(f \circ \tilde{\sigma}_{i,j-1} - f)^2 | \mathcal{F}_{j-1,j}]] = E_{K,w} [(f \circ \tilde{\sigma}_{i,j-1} - f)^2].$$

Since $\tilde{\sigma}_{k,j-1} = \tau_{k,k+1} \circ \tilde{\sigma}_{k+1,j-1}$, by repeating the calculation in f_1 ,

$$E_{K,w}[f_3] \leq 4K \sum_{k=i}^{j-2} E_{K,w} [(f - E_{K,w}[f | \mathcal{F}_{k,k+1}])^2].$$

Hence, with some universal constant $C < \infty$ we have

$$E_{K,w} [(f - E_{K,w}[f | \mathcal{F}_{i,j}])^2] \leq CK \sum_{k=i}^{j-1} E_{K,w} [(f \circ \tau_{k,k+1} - f)^2].$$

Lemma 5.3 follows by summing up this estimate with i and j . \square

For each $k = 1, \dots, K-1$, in view of Lemma 5.2,

$$E_{K,w} [(f - E_{K,w}[f | \mathcal{F}_{k,k+1}])^2 | \mathcal{F}_{i,i+1}] \leq CE_{K,w} [(\mathcal{X}_{i,i+1} f)^2 | \mathcal{F}_{i,i+1}].$$

Substituting this to (5.3) and applying Lemma 5.3, one observes that the proof of Proposition 5.1 can be completed by the next lemma.

Lemma 5.4. *There exists a finite constant C such that*

$$E_{K,w} [(f - E_{K,w}[f])^2] \leq \frac{C}{K} \left(\frac{\delta_+}{\delta_-} \right)^{3K-3} \sum_{1 \leq i < j \leq n} E_{K,w} [(f - E_{K,w}[f | \mathcal{F}_{i,j}])^2] \quad (5.5)$$

for all $K \geq 3$, w and bounded smooth function f .

To show this lemma, we make use of the spectral gap bound of Kac walk. For $a \in \mathbb{R}^2$ and $R > \|a\|_{\mathbb{R}^2}^2$, consider the $(2K - 3)$ -dimensional sphere

$$S_K(a, R) = \left\{ x_1, \dots, x_K \in \mathbb{R}^2 \mid \frac{1}{K} \sum_{k=1}^K x_k = a, \frac{1}{K} \sum_{k=1}^K |x_k|^2 = R \right\}.$$

Denote by $\mu_K(a, R)$ the uniform measure on $S_K(a, R)$. Recall that $\mathcal{F}_{i,j} = \sigma\{x_k; k \neq i, j\}$ for $1 \leq i < j \leq n$. We have the following result.

Lemma 5.5. *There exists a constant C such that*

$$E_{\mu_K(a,R)} [(f - E_{\mu_K(a,R)}[f])^2] \leq \frac{C}{K} \sum_{1 \leq i < j \leq n} E_{\mu_K(a,R)} [(f - E_{\mu_K(a,R)}[f|\mathcal{F}_{i,j}])^2]$$

for all $K \geq 3$, a , R and bounded smooth function f .

Lemma 5.5 can be proved by the arguments in [4] and [5]. Here we give the proof of Lemma 5.4 based on Lemma 5.5.

Proof. To begin with, we know from (A1) that for $r \neq r'$ and $K \geq 1$,

$$\frac{\sqrt{2(K+1)}}{\sqrt{K}} c_- \leq \frac{|V'(r) - V'(r')|}{\sqrt{V(r) + KV(r') - (K+1)V(\frac{r+Kr'}{K+1})}} \leq \frac{\sqrt{2(K+1)}}{\sqrt{K}} c_+, \quad (5.6)$$

where $c_- = \delta_- / \sqrt{\delta_+}$ and $c_+ = \delta_+ / \sqrt{\delta_-}$. For each $K \geq 3$, we construct a bijection $\iota_K : \mathbb{R}^{2K} \rightarrow \mathbb{R}^{2K}$, satisfying the following two conditions.

- (i) For $w = (p, r, e)$, $a = (p, r)$ and $R = 2e - 2V(r) + r^2$, $\iota_K(S_K(a, R)) = \Omega_{K,w}$;
- (ii) The Jacobian matrix ι'_K of ι_K satisfies that $c_-^{K-1} \leq |\det(\iota'_K)| \leq c_+^{K-1}$.

Indeed, given a bounded, measurable, positive function g on $\Omega_{K,w}$, by (i) we know that $\iota_K g := g \circ \iota_K$ defines a function on $S_K(a, R)$, and (ii) yields that

$$c^{-(K-1)} E_{\mu_K(a,R)}[\iota_K g] \leq E_{K,w}[g] \leq c^{K-1} E_{\mu_K(a,R)}[\iota_K g],$$

where $c = c_+/c_- = (\delta_+/\delta_-)^{3/2}$. For a bounded smooth function f , apply the estimate above to $g = (f - E_{\mu_K(a,R)}[\iota_K f])^2$ and we obtain

$$E_{K,w} [(f - E_{K,w}[f])^2] \leq E_{K,w}[g] \leq c^{K-1} E_{\mu_K(a,R)}[\iota_K g].$$

On the other hand, take $h_{i,j} = (f - E_{K,w}[f|\mathcal{F}_{i,j}])^2$ and similarly,

$$E_{\mu_K(a,R)} [(\iota_K f - E_{\mu_K(a,R)}[\iota_K f|\mathcal{F}_{i,j}])^2] \leq E_{\mu_K(a,R)}[\iota_K h_{i,j}] \leq c^{K-1} E_{K,w}[h_{i,j}].$$

Substituting $\iota_K f$ for f in Lemma 5.5, we get

$$\begin{aligned} E_{K,w} [(f - E_{K,w}[f])^2] &\leq c^{K-1} E_{\mu_K(a,R)} [(\iota_K f - E_{\mu_K(a,R)}[\iota_K f])^2] \\ &\leq \frac{C c^{K-1}}{K} \sum_{i < j} E_{\mu_K(a,R)} [(\iota_K f - E_{\mu_K(a,R)}[\iota_K f|\mathcal{F}_{i,j}])^2] \\ &\leq \frac{C c^{2(K-1)}}{K} \sum_{i < j} E_{K,w} [(f - E_{K,w}[f|\mathcal{F}_{i,j}])^2]. \end{aligned}$$

Lemma 5.4 then follows.

Now fix $K \geq 3$ and we construct the map ι_K . Recall that $x_k = (p_k, r_k)$ and define

$$\alpha_k = \frac{1}{k} \sum_{i=1}^k r_i, \quad \forall 1 \leq k \leq K.$$

Consider two maps $\zeta, \zeta_* : \mathbb{R}^K \rightarrow \mathbb{R}^K$. The first map ζ is given by

$$\zeta(r_1, \dots, r_K) = (r'_1, \dots, r'_K),$$

such that $r'_K = \alpha_K$, and

$$(r'_k)^2 = \frac{2k}{k+1} [V(r_{k+1}) + kV(\alpha_k) - (k+1)V(\alpha_{k+1})], \quad \forall 1 \leq k \leq K-1,$$

where the sign of r'_k is chosen in accordance with $r_k - \alpha_K$. Meanwhile, ζ_* is given by

$$\zeta_*(r'_1, \dots, r'_K) = (r''_1, \dots, r''_K),$$

such that

$$r''_k = \begin{cases} r'_K - \sum_{i=1}^{K-1} \frac{r'_i}{i}, & \text{for } k = 1, \\ r'_K + r'_{k-1} - \sum_{i=k}^{K-1} \frac{r'_i}{i}, & \text{for } 2 \leq k \leq K-1, \\ r'_K + r'_{K-1}, & \text{for } k = K. \end{cases}$$

Denote by J and J_* the Jacobian matrixes of ζ and ζ_* , respectively. To compute J , noticing that $\partial_{r_i} r'_k = \partial_{r_k} r'_k$ for all $i \leq k$, and $\partial_{r_i} r'_k = 0$ for all $i > k$, we have

$$J = \begin{bmatrix} \frac{\partial r'_1}{\partial r_1} & \frac{\partial r'_1}{\partial r_2} & 0 & \dots & 0 \\ \frac{\partial r'_2}{\partial r_2} & \frac{\partial r'_2}{\partial r_2} & \frac{\partial r'_2}{\partial r_3} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \frac{\partial r'_{K-1}}{\partial r_{K-1}} & \frac{\partial r'_{K-1}}{\partial r_K} & \dots & \frac{\partial r'_{K-1}}{\partial r_K} \\ \frac{\partial r'_K}{\partial r_K} & \frac{\partial r'_K}{\partial r_K} & \frac{\partial r'_K}{\partial r_K} & \dots & \frac{\partial r'_K}{\partial r_K} \end{bmatrix}.$$

Hence, its determinant reads

$$|\det(J)| = \left| \frac{\partial r'_K}{\partial r_K} \right| \cdot \prod_{k=1}^{K-1} \left| \frac{\partial r'_k}{\partial r_k} - \frac{\partial r'_k}{\partial r_{k+1}} \right|.$$

In view of the definition of ζ , $\partial_{r_K} r'_K = 1/K$ and for $k = 1, \dots, K-1$, and

$$\frac{\partial r'_k}{\partial r_i} = \begin{cases} \frac{k}{(k+1)r'_k} [V'(\alpha_k) - V'(\alpha_{k+1})], & \text{if } 1 \leq i \leq k, \\ \frac{k}{(k+1)r'_k} [V'(r_{k+1}) - V'(\alpha_{k+1})], & \text{if } i = k+1. \end{cases}$$

Therefore, we can write

$$|\det(J)| = \frac{1}{K} \prod_{k=1}^{K-1} \frac{\sqrt{k}}{\sqrt{2(k+1)}} \frac{|V'(r_{k+1}) - V'(\alpha_k)|}{\sqrt{V(r_{k+1}) + kV(\alpha_k) - (k+1)V(\alpha_{k+1})}}.$$

Applying the estimate in (5.6), we obtain that

$$\frac{c_-^{K-1}}{K} \leq |\det(J)| \leq \frac{c_+^{K-1}}{K}.$$

Meanwhile it is easy to calculate that $|\det(J_*)| = K$. Therefore, define

$$\iota_K : (p_1, \dots, p_K, r_1, \dots, r_K) \mapsto (p_1, \dots, p_K, (\zeta_* \circ \zeta)^{-1}(r_1, \dots, r_K)),$$

then ι_K satisfies (i). On the other hand, by the definition of ζ ,

$$\sum_{k=1}^K r_k = Kr'_K, \quad \sum_{k=1}^K V(r_k) = KV(r'_K) + \sum_{k=1}^{K-1} \frac{k+1}{k} \frac{(r'_k)^2}{2}.$$

Similarly, by the definition of ζ_* ,

$$\sum_{k=1}^K r''_k = Kr'_n, \quad \sum_{k=1}^K \frac{(r''_k)^2}{2} = \frac{K(r'_K)^2}{2} + \sum_{k=1}^{K-1} \frac{k+1}{k} \frac{(r'_k)^2}{2},$$

hence (ii) is also satisfied. \square

6 Equivalence of ensembles

This section is devoted to the equivalence of ensembles under a multi-dimensional framework, which has been used in the proof of Lemma 3.2. In the main result, Proposition 6.3, we prove a Lebowitz–Percus–Verlet type formula (cf. [3]).

In this section, Ω denotes a general topological space equipped with a positive measure ν . Let $\mathbf{f} = (f_1, \dots, f_d)$ be a random vector on Ω , and we assume that it has compact level sets. For a d -dimensional coefficient λ , define

$$Z(\lambda) = \log \left[\int_{\Omega} \exp \{ \lambda \cdot \mathbf{f}(\omega) \} \nu(d\omega) \right].$$

Assume a nonempty domain $D \subseteq \mathbb{R}^d$ such that $Z(\lambda) < \infty$ for $\lambda \in D$, and

(a) $Z \in C_b^4(D)$, Hess $Z(\lambda)$ is positive-definite for $\lambda \in D$.

For simplicity we denote $\mathbf{u}_\lambda = \nabla Z(\lambda)$ and $\Sigma_\lambda = \text{Hess } Z(\lambda)$. For $\lambda \in D$, let ν_λ be the *tilted probability measure* on Ω , given by the formula

$$\nu_\lambda(d\omega) \triangleq \exp \{ \lambda \cdot \mathbf{f}(\omega) - Z(\lambda) \} \nu(d\omega).$$

Observe that under ν_λ , the average of \mathbf{f} is \mathbf{u}_λ and the covariance matrix is Σ_λ . Let Φ_λ denote the characteristic function of $\mathbf{f} - \mathbf{u}_\lambda$:

$$\Phi_\lambda(\mathbf{h}) = \int_{\Omega} \exp \{ i\mathbf{h} \cdot (\mathbf{f}(\omega) - \mathbf{u}_\lambda) \} \nu_\lambda(d\omega), \quad \forall \mathbf{h} \in \mathbb{R}^d.$$

We also assume that

(b) there exists some $\epsilon_0 > 0$ such that $|\Phi_\lambda(\mathbf{h})| \leq |\mathbf{h}|^{-\epsilon_0}$ if $|\mathbf{h}|$ is large enough.

Remark 6.1. In the model introduced in Section 1, $\Omega = \mathbb{R}^2$ and ν is the Lebesgue measure on Ω . By taking $d = 3$ and $\lambda = (\beta\bar{p}, \beta\tau, -\beta)$, $Z(\lambda)$ is the Gibbs potential in (1.4), and ν_λ is the marginal distribution of the corresponding Gibbs measure.

Notice that Z is convex, we consider its *Fenchel-Legendre transform*

$$Z^*(\mathbf{u}) = \sup_{\lambda \in D} \{ \lambda \cdot \mathbf{u} - Z(\lambda) \}.$$

Let $D^* = \{\mathbf{u} \in \mathbb{R}^d : Z^*(\mathbf{u}) < \infty\}$. The superior is reached at a unique $\lambda(\mathbf{u}) \in D$, satisfying that $\mathbf{u} = \nabla Z|_{\lambda=\lambda(\mathbf{u})}$ and $\lambda(\mathbf{u}) = \nabla Z^*(\mathbf{u})$. Notice that the map $\mathbf{u} \mapsto \lambda(\mathbf{u})$ is a one-to-one correspondence from D^* to D , and its inverse is given by $\lambda \mapsto \mathbf{u}_\lambda$. The Hessian matrices of Z and Z^* then satisfy the relation

$$\text{Hess } Z^*|_{\mathbf{u}=\mathbf{u}_\lambda} = [\text{Hess } Z(\lambda)]^{-1} = \Sigma_\lambda^{-1}.$$

We define the rate function $I_\lambda(\mathbf{u})$ by

$$I_\lambda(\mathbf{u}) = Z^*(\mathbf{u}) - Z^*(\mathbf{u}_\lambda) - \nabla Z^*(\mathbf{u}_\lambda) \cdot (\mathbf{u} - \mathbf{u}_\lambda). \quad (6.1)$$

Denote by M_λ the largest eigenvalue of Σ_λ . By the arguments above it is not hard to conclude that for any constant $M > M_\lambda$, we have

$$I_\lambda(\mathbf{u}) \geq (2M)^{-1} |\mathbf{u} - \mathbf{u}_\lambda|^2 \quad (6.2)$$

holds if $|\mathbf{u} - \mathbf{u}_\lambda|$ is small enough.

For $n \geq 1$, equip the product space Ω^n with measure $\nu_\lambda^n = \otimes_j d\nu_\lambda(\omega_j)$, and define

$$\mathbf{f}_{(n)}(\omega_1, \dots, \omega_n) = \frac{1}{n} \sum_{j=1}^n \mathbf{f}(\omega_j), \quad \forall (\omega_1, \dots, \omega_n) \in \Omega^n.$$

In view of 6.2 we can obtain the following large deviation property.

Lemma 6.2. *For any $M > M_\lambda$, there exists some δ_M such that*

$$\nu_\lambda^n \{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta\} \leq 2^d \exp\left(-\frac{nM\delta^2}{d}\right),$$

holds for all $n \geq 1$ when $|\delta| < \delta_M$.

Proof. Let $\Gamma \subseteq \mathbb{R}^d$ be the collection of vectors whose coordinates are all ± 1 . Notice that the following inequality holds for all $\mathbf{x} \in \mathbb{R}^d$:

$$e^{|\mathbf{x}|} \leq \prod_{j=1}^d e^{|x_j|} \leq \prod_{j=1}^d (e^{-x_j} + e^{x_j}) = \sum_{\gamma \in \Gamma} e^{\gamma \cdot \mathbf{x}}.$$

By exponential Chebyshev's inequality and the above estimate, for $\theta > 0$,

$$\begin{aligned} \nu_\lambda^n \{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta\} &\leq \sum_{\gamma \in \Gamma} e^{-n\theta\delta} \int_{|\mathbf{f}_{(n)} - \mathbf{u}_\lambda| \geq \delta} \exp\{n\theta\gamma \cdot (\mathbf{f}_{(n)} - \mathbf{u}_\lambda)\} d\nu_\lambda^n \\ &\leq \sum_{\gamma \in \Gamma} \exp\{-n\theta u' + nZ(\lambda + \theta\gamma) - nZ(\lambda)\}, \end{aligned}$$

where $u' = \gamma \cdot \mathbf{u}_\lambda + \delta$. To optimize this estimate, define

$$I_{\lambda, \gamma}(u') = \sup_{\theta > 0} \{\theta u' - Z(\lambda + \theta\gamma) + Z(\lambda)\} = \sup_{\theta \in \mathbb{R}} \{\theta u - Z(\lambda + \theta\gamma) + Z(\lambda)\}.$$

The last equality is due to the fact that $u' - \partial_\theta Z(\lambda + \theta\gamma)|_{\theta=0} = \delta > 0$. Notice that $I_{\lambda, \gamma}$ is the rate function defined in (6.1) corresponding to the measure ν_λ and the function $\gamma \cdot \mathbf{f}$. By the arguments which has been used to derive (6.2), one obtains that $I_{\lambda, \gamma}(u') \geq M_\lambda |\gamma|^{-2} \delta^2$. The estimate in Lemma 6.2 then follows directly. \square

For a nice measurable function G on Ω^n , we consider the canonical expectation $\varphi_{n,G}$ and the micro canonical expectation $\psi_{n,G}$, respectively given by

$$\varphi_{n,G}(\lambda) \triangleq E_{\nu_\lambda^n}[G], \quad \psi_{n,G}(\mathbf{u}) \triangleq E_{\nu_\lambda^n}[G | \mathbf{f}_{(n)} = \mathbf{u}].$$

By equivalence of ensembles we mean that $|\varphi_{n,G}(\lambda) - \psi_{n,G}(\mathbf{u}_\lambda)|$ vanishes when volume grows to infinity. To state the result, let $\|G\|_{\lambda,n}$ be the L^2 norm of G with respect to ν_λ^n , and assume that there is a constant $C_\lambda < \infty$ such that

(c.1) $\|G\|_{\lambda,n} \leq C_\lambda, |\psi_{n,G}(\mathbf{u}_\lambda)| \leq C_\lambda \|G\|_{\lambda,n};$

(c.2) $\psi_{n,G}$ is three times differentiable on D , and for all $\gamma \in \mathbb{R}^d$,

$$|\nabla \psi_{n,G}|_{\mathbf{u}=\mathbf{u}_\lambda}| \leq C_\lambda \sqrt{n} \|G\|_{2,\lambda,n}, \quad |[\text{Hess } \psi_{n,G}|_{\mathbf{u}=\mathbf{u}_\lambda}] \gamma| \leq C_\lambda n \|G\|_{\lambda,n} \cdot |\gamma|;$$

(c.3) with a constant $b_\lambda > \sqrt{8dM_\lambda}$ and a sequence $c_{\lambda,n} \rightarrow 0$ as $n \rightarrow \infty$,

$$\sup_{|\mathbf{u}-\mathbf{u}_\lambda| < \delta_n} \frac{1}{3!} \sum_{|\alpha|=3} \partial_\alpha \psi_{n,G}(\mathbf{u}) \gamma^\alpha \leq B_{\lambda,n} \|G\|_{\lambda,n} |\gamma|^3, \quad \forall \gamma \in \mathbb{R}^d,$$

where δ_n and $B_{\lambda,n}$ are constants given by

$$\delta_n = b_\lambda \sqrt{\frac{\log n}{n}}, \quad B_{\lambda,n} = \frac{c_{\lambda,n}}{\log \log n} \sqrt{\frac{n}{\log n}}.$$

The equivalence of ensembles is stated below.

Proposition 6.3. *Assume \mathbf{a} , \mathbf{b} and **c.1-c.3** above. Then*

$$\left| \varphi_{n,G}(\lambda) - \psi_{n,G}(\mathbf{u}_\lambda) - \frac{1}{2n} \text{Tr} [\nabla(\Sigma_\lambda^{-1} \nabla \varphi_{n,G}(\lambda))] \right| \leq \frac{K_n}{n} \|G\|_{\lambda,n},$$

where the sequence $\{K_n\}$ satisfies that $\lim_{n \rightarrow \infty} K_n = 0$.

Proof. The proof goes exactly the same as the one-dimensional case showed in [3], based on a local central limit theorem with an expansion of error, as well as an estimate on the large deviation probability of $\mathbf{f}_{(n)}$ in Lemma 6.2. \square

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