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Backbone colouring and algorithms for TDMA scheduling

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Abstract

We investigate graph colouring models for the purpose of optimizing TDMA link scheduling in Wireless Networks. Inspired by the BPRN-colouring model recently introduced by Rocha and Sasaki, we introduce a new colouring model, namely the BMRN-colouring model, which can be used to model link scheduling problems where particular types of collisions must be avoided during the node transmissions.

In this paper, we initiate the study of the BMRN-colouring model by providing several bounds
on the minimum number of colours needed to BMRN-colour digraphs, as well as several complexity results establishing the hardness of finding optimal colourings. We also give a special
focus on these considerations for planar digraph topologies, for which we provide refined results.
Some of these results extend to the BPRN-colouring model as well.

12 **1** Introduction

13 1.1 Motivation

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A Radio Network consists in a set of nodes distributed in space that communicate via broadcast radio 14 waves, with all messages sent from a node transmitted to all nodes in its range. The range of a node is 15 therefore the region in space within which it can communicate to others. Node a can transmit to node 16 b if b is within the range of a, and we say there is a directional *link ab*. Typically, communication is 17 done in a *multi-hop* fashion, with intermediate nodes forwarding data from a source to a destination 18 that is distant. Since the channels are shared, transmissions are subject to collisions, that result in 19 undesired effects such as loss of data and network efficiency. To avoid such collisions, medium 20 access control techniques have been designed. 21

In the Time-Division Multiple Access (TDMA) method [10], time is divided into *frames* of fixed 22 duration. Each frame is itself divided into a number of fixed duration time slots. A link schedule 23 is then an assignment of time slots to the network transmissions. When the link ab is scheduled, 24 it is required that b receives its message from a free of collision, although we do not require the 25 same for other nodes receiving a message from a. The TDMA method is being used in standards 26 such as IEEE 802.16 [17] and IEEE 802.11s [18], providing guaranteed Quality-of-Service (QoS). In 27 particular, TDMA MAC protocols are being used for the increasingly popular Wireless Sensor and 28 Mesh Networks [5]. Although there are particularities in the distinct network scenarios, an efficient 29 use of TDMA methods is related to an increase in network throughput, and to a reduction of delays 30 and packet losses. 31 Given a network and an interference model, the challenge is to find a link schedule that avoids 32

³² conflicts and minimizes the number of used time slots [6, 9]. Minimizing the number of time slots is

¹ important, since it affects network throughput and multi-hop transmission delays. That is, if there are ² N_f different time slots in a frame, then, since a node or link is only active during its associated time ³ slot, it becomes active only during a fraction $\frac{1}{N_f}$ of time. For this reason, finding a broadcast or link ⁴ scheduling that minimizes the number of slots needed in a frame can lead to an improvement in the ⁵ network efficiency. ⁶ In a typical network, only a subset of the existing links become active. This is because the

existence of a link *ab* does not necessarily imply that *a* ever communicates with *b*. As an example,
consider a network formed by a Wi-Fi router that is connected to the internet by cable, and that is
surrounded by a number of devices such as computers and smartphones, connected to the internet
through it. The nearby devices may be in each other's ranges, but their transmissions are not intended
to each other, so that their mutual links do not need to be scheduled. In the rest of this paper we refer
to the subnetwork whose links are the active ones as the network *backbone*.

13 1.2 Modelling

Such problems on radio networks can be modelled as graph colouring problems, in the following 14 way. A sensor network can be represented as a digraph D with vertex set V(D) and arc set A(D). 15 The vertices of D correspond to the nodes, and there is an arc from a vertex u to a vertex v if the 16 node corresponding to v is in the transmission range of the node corresponding to u. The backbone is 17 represented by a spanning subdigraph B of D, whose arcs correspond exactly to the backbone links. 18 The arcs in $A(D) \setminus A(B)$ correspond to the networks links that are not used by the network. The pair 19 (D, B) is called a *backboned digraph* throughout. Now, we consider the time slots as colours to be 20 assigned to the arcs of B. Hence, a link schedule of k time slots is a k-arc-colouring of B that must 21 satisfy some constraints, depicting the conflicts which must be avoided during transmissions. 22 In practice, transmission collisions may occur for various reasons (physical constraints, device 23 constraints, etc.), such as the following four ones, which shall be considered throughout this paper. 24

- **Type-1 constraint**: During a time slot, a node cannot both transmit and receive messages along the backbone.
- \Rightarrow If uv and vw are two arcs of B, then they cannot be assigned the same colour.
- **Type-2 constraint**: During a time slot, a node receiving a message along the backbone must not receive a message from another transmitting node.
- \Rightarrow If u_1v_1 is an arc of B, and e is either an arc u_2v_1 of B, or an arc u_2v_2 of B such that $u_2v_1 \in A(D) \setminus A(B)$, then uv and e cannot be assigned the same colour.
- **Type-3 constraint**: During a time slot, a node transmits at most one message along the backbone.
- \Rightarrow If uv_1 and uv_2 are two arcs of B, then they cannot be assigned the same colour.
- **Type-3*** **constraint**: A node transmits all its messages along the backbone during one time slot only.
- \Rightarrow If uv_1 and uv_2 are two arcs of B, then they must be assigned the same colour.

1.3 Colouring variants over the four types of constraints

39 1.3.1 BPRN-colouring

⁴⁰ Arc-colourings of backboned digraphs fulfilling combinations of the constraints above were already ⁴¹ considered in literature. In particular, an arc-colouring of a backboned digraph (D, B) verifying

type-1, type-2 and type-3 constraints is called a Backbone-Packet-Radio-Network-colouring (BPRN-1 colouring for short). Note that when B is an out-branching of D, i.e., a spanning oriented tree all 2 arcs of which are oriented away from the root, BPRN-colourings model link schedules where, in the 3 radio network, the root node wants to send a private message to all other nodes. The BPRN-chromatic 4 index of (D, B), denoted by BPRN(D, B), is the minimum number of colours in a BPRN-colouring 5 of (D, B). 6 The BPRN-colouring model is a generalization of the *PRN-colouring* model, first proposed by 7 Arika [4], that appeared in other works about link scheduling [9, 13, 15]. It was introduced by 8 Rocha and Sasaki in [14], who, motivated by applications in Wireless Sensor Networks, studied 9 the case where the backbone is an in-branching, i.e., an oriented tree where all arcs are oriented 10 towards the root. Among several results on BPRN-colourings, they exhibited bounds on the BPRN-11 chromatic index of backboned digraphs (D, B) where D is complete or a cycle. They also proved 12

that, in general, determining the BPRN-chromatic index is NP-hard, even when restricted to bipartite

14 backboned digraphs.

15 1.3.2 BMRN-colouring

Let (D, B) be a backboned digraph. An arc-colouring of (D, B) verifying type-1 and type-2 con-16 straints only is called a *Backbone-Multicast-Radio-Network-colouring* (BMRN-colouring for short). 17 The BMRN-chromatic index of (D, B), denoted by BMRN(D, B), is the minimum number of colours 18 in a BMRN-colouring of (D, B). To the best of our knowledge, BMRN-colourings were not stud-19 20 ied in literature. According to the definitions, BMRN-colourings are subject to less constraints than BPRN-colourings, and can thus be regarded as a simpler colouring notion. In particular, results 21 on BPRN-colourings hold for BMRN-colourings. Moreover, the particular case when B is an out-22 branching models the case when a root wants to send the same message to all other vertices via a 23 multicast tree. 24

25 1.3.3 BMRN*-colouring

For a backboned digraph (D, B), an arc-colouring verifying type-1, type-2, and type-3^{*} constraints is 26 called a *Star-Backbone-Multicast-Radio-Network-colouring* or simply *BMRN*-colouring* of (D, B). 27 The BMRN*-chromatic index of (D, B), denoted by BMRN*(D, B), is the minimum number of 28 colours in a BMRN*-colouring of (D, B). A first main motivation to consider BMRN*-colourings is 29 to derive results on the BMRN-colourings, since BMRN*-colourings are BMRN-colourings. How-30 ever, it should be pointed that BMRN*-colourings are also a model to broadcast scheduling, in which 31 each node of the network is scheduled to a time slot, during which all of its neighbours must re-32 ceive its transmission free of collisions. Other works such as [12, 13] study broadcast scheduling by 33 considering that the network links are symmetric, the network being modelled by a graph and the 34 scheduling problem corresponding to finding a distance-2 colouring that uses the smallest number of 35 colours. This approach can be seen as a simplification of our problem, since a typical network may 36 have devices with different transmission ranges, resulting in a directed graph. 37

The BMRN*-colouring model can therefore be seen as a more realistic model for broadcast scheduling. This motivates the investigation of more results on this model and its corresponding algorithms. Other works such as [8] already point that distance-2 colouring of graphs is not the most accurate model to broadcast scheduling in real wireless networks, and consider the directed version. But to the best of our knowledge, we are the first to model the problem as an arc-colouring problem,

⁴³ and to relate it to link scheduling.

1 1.4 Our results

It should be intuitive to the readers that the three colouring variants presented in the previous section are related. From the definitions, BPRN-colourings and BMRN*-colourings are also BMRNcolourings. Section 3 is devoted to understanding some of these relations further.

⁵ In Section 4, we exhibit upper bounds on the BMRN-chromatic, BPRN-chromatic and BMRN*-

⁶ chromatic indices of a spanned digraph, that is a backboned digraph (D,T) where T is an out-

⁷ branching in D. Firstly, in Subsection 4.1, we establish general bounds in terms of the maximum

⁸ degree and maximum out-degree of the digraph and show that these bounds are tight up to a small con-

stant factor. For example, we show that for every spanned digraph (D,T) we have BMRN $(D,T) \leq D$

 $2\Delta^+(D) + 1$ (where $\Delta^+(D)$ denotes the maximum out-degree of a vertex of D) and BMRN $(D, T) \le 2\Delta^+(D) + 1$ (where $\Delta^+(D)$) is a set of D by the set of D and D by the set of D by the set of D and D by the set of D by the set of D and D by the set of D

¹¹ $2\Delta(D) - 1$ (where $\Delta(D)$ denotes the maximum degree of the underlying graph of D) while there are ¹² spanned digraph (D,T) for which BMRN $(D,T) \ge 2\Delta^+(D) - 1$ and BMRN $(D,T) \ge 2\Delta(D) - 3$.

¹³ Secondly, in Subsection 4.2, we give more refined upper bounds in the realistic case where a planar

topology is considered. We show that $BMRN(D,T) \leq BMRN^*(D,T) \leq 8$ for every planar spanned

to poly is considered we show the Dirich $(D, T) \subseteq D$ and $(D, T) \subseteq 0$ for every outerplanar spanned digraph to (D, T), and that BMRN $(D, T) \leq$ BMRN $^*(D, T) \leq 5$ for every outerplanar spanned digraph (D, T).

In Section 5, we show that determining any of the BMRN-, BPRN- and BMRN*-chromatic indices of a spanned digraph is \mathcal{NP} -hard in general, and that this remains true if one is allowed to choose the out-branching of the input spanned digraph. We also show that given a planar spanned digraph (D, T), deciding whether BMRN $(D, T) \leq 3$ is \mathcal{NP} -complete.

21 **2** Definitions, notations and preliminaries

The square G^2 of an undirected graph G is the graph with same vertex set as G and in which two vertices are adjacent if and only if they are at distance at most 2 in G.

The underlying simple graph U(D) of a digraph D is the graph defined by V(U(D)) = V(D)and $E(\tilde{U}(D)) = \{uv \mid uv \in A(D) \text{ or } vu \in A(D)\}$. We call D symmetric if $vu \in A(D)$ whenever $uv \in A(D)$. Observe that if D is symmetric, then $\tilde{U}(D)$ is obtained by replacing each directed 2-cycle by an edge. The digraph obtained from D by contraction of the arc a = xy in A(D) is the digraph D/a obtained from $D - \{x, y\}$ by adding a new vertex v_a and the arc $v_a w$ (resp. wv_a) for every vertex u in $V(D) \setminus \{x, y\}$ such that $xw \in A(D)$ or $yw \in A(D)$ (resp. $wx \in A(D)$ or $wy \in A(D)$).

A digraph D is subcubic if $\Delta(D) \leq 3$.

The notion of minor of a digraph that we use in this paper corresponds to the notion of minor in the underlying graph. That is, a digraph D' is a *minor* of D if it can be obtained from D by a succession of vertex-deletions, arc-deletions, and arc-contractions. A family \mathcal{D} of digraphs is *minor-closed* if for every $D \in \mathcal{D}$, all minors of D are also in \mathcal{D} .

The union of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$. If D_1 and D_2 are disjoint (i.e., $V(D_1) \neq V(D_2)$), then it is the *disjoint union* of D_1 and D_2 . A *matching* in a digraph D is a disjoint union of arcs. A matching M is *perfect* in D if every vertex of D is either the tail or the head of an arc of M. A *matched digraph* is a pair (D, M) where D is a digraph and M is a perfect matching of D.

In a digraph, we say that a vertex u dominates a vertex v if uv is an arc. An out-star is a digraph consisting in a vertex, called the *centre*, dominating all the others, called *spades*. A *galaxy* is a disjoint union of out-stars. For a digraph D, an out-generator of D is a vertex r such that D admits an out-branching rooted at r. In other words, there is a directed path from r to every other vertex of D.

If A is a directed path or cycle, and x, y are two vertices of A, then we denote by A[x, y] the 1 directed subpath of A with origin x and terminus y. 2

Drawing conventions: In every figure depicting a backboned digraph (D, B), the bold arcs are the 3 arcs of B, the dashed arcs are the arcs not in B, and the edges are pairs of opposite arcs. Moreover, if 4 B is an out-branching, then the white vertex is its root. 5

Relations between BPRN-, BMRN- and BMRN*-indices 3 6

According to the definitions, for every backboned digraph (D, B) we have the following:

 $BMRN(D, B) \leq BPRN(D, B)$ and $BMRN(D, B) \leq BMRN^*(D, B).$

- Conversely, one can easily see that BPRN(D, B) cannot be bounded by a function of BMRN(D, B). 7
- To see this, consider, for example, the directed out-star with k spades \vec{S}_k ; we clearly have BMRN $(\vec{S}_k, \vec{S}_k) =$ 8 1 and BPRN $(\vec{S}_k, \vec{S}_k) = k$.
- 9
- On the other hand, we show that BMRN $^*(D, T)$ is bounded above by a function of BMRN(D, T). 10

The upper bound function we exhibit is actually best possible. 11

Theorem 1. For every backboned digraph (D, B), we have $BMRN^*(D, B) < 2^{BMRN(D,B)} - 1$. 12

Proof. Let ϕ be a BMRN-colouring of (D, B). For every vertex $v \in V(D)$, let

$$f(v) = \{\phi(vw) \mid w \in N^+(v)\}.$$

Now, let q be the arc-colouring of B defined by q(xy) = f(x) for every arc $xy \in A(B)$. We claim 13 that g is a BMRN*-colouring of (D, B): 14

• For every two arcs uv and vw of B, we have $\phi(uv) \notin f(v)$ because ϕ is a BMRN-colouring of 15 (D, B). Hence $f(u) \neq f(v)$, and $q(uv) \neq q(vw)$. Thus type-1 constraints are satisfied. 16

• For every two arcs u_1v_1 and u_2v_2 of B such that $u_1v_2 \in A(D)$, we have $\phi(u_2v_2) \notin f(u_1)$ 17 because ϕ is a BMRN-colouring of (D, B). Hence $f(u_1) \neq f(u_2)$, and $q(u_1v_1) \neq q(u_2v_2)$. 18 Thus type-2 constraints are satisfied. 19

• By definition, any two arcs of B with the same tail are assigned the same colour by q. Hence 20 type-3* constraints are satisfied. 21

The conclusion is now obtained by noting that if ϕ takes values in a set S of k colours, then g 22 takes values in a set of at most $2^k - 1$ colours, namely the non-empty subsets of S. 23

We now prove that the upper bound in Theorem 1 is best possible. 24

Proposition 2. For every positive integer k, there exist a digraph D and a spanning galaxy B of D 25 such that BMRN(D, B) = k and $BMRN^*(D, B) = 2^k - 1$. 26

Proof. Let X be a set of k vertices. Let us construct a galaxy B as follows: for each non-empty 27 subset U of X, we create an out-star $S^+(U)$ with centre x_U and spades $y_U(u)$ for all $u \in U$. Let now 28 D be the digraph obtained from B by adding arcs as follows. For every two elements $u \neq t$ of X, and 29 every subset T such that $t \in T$, we add the arc $x_{\{u\}}y_T(t)$. This ensures that, in any BMRN*-colouring 30 of (D, B), the arcs $x_{\{u\}}y_{\{u\}}(u)$ and $x_Ty_T(t)$ get different colours. Then, for every pair $\{U, T\}$ of 31 distinct subsets of X with $|U| \leq |T|$, choose a vertex t in $T \setminus U$, and add the arc $x_U y_T(t)$ (if not 32

already present). Observe that this arc implies that, in any BMRN*-colouring of (D, B), the arcs of

- ² $S^+(U)$ and the arcs of $S^+(T)$ are assigned different colours. Consequently, BMRN^{*} $(D, B) \ge 2^k 1$.
- Now consider the arc-colouring ϕ of B where every arc $x_U y_U(u)$ is assigned colour u. One easily
- ⁴ checks that ϕ is a k-BMRN-colouring of (D, B). Indeed, the arcs of $A(D) \setminus A(B)$ are of the form
- 5 $x_U y_T(t)$ with $t \in T \setminus U$. According to how ϕ was defined, this means that the arcs with tail x_U are
- 6 assigned, by ϕ , a colour in U, while the arc $x_T y_T(t)$ is assigned colour t. Hence all type-2 constraints
- ⁷ are satisfied, and, consequently, BMRN $(D, B) \le k$. Furthermore, we actually have equality, as every
- 8 two arcs $x_{\{u\}}y_{\{u\}}(u)$, $x_{\{t\}}y_{\{t\}}(t)$ for $u \neq t$ must receive distinct colours, due to the arc $x_{\{u\}}y_{\{t\}}(t)$.
- So all k arcs of the form $x_{\{u\}}y_{\{u\}}(u)$ ($u \in U$) must be coloured differently.
- From Theorem 1, we now get that $BMRN^*(D, B) = 2^k 1$.
- The proof of Proposition 2 can be easily modified to hold for spanned digraphs (D, B).

Proposition 3. For every positive integer k, there exists a spanned digraph (D,T) such that BMRN(D,T) = k and $BMRN^*(D,T) = 2^k - 1$.

Proof. We start from the pair (D, B) constructed in the proof of Proposition 2. We construct a spanned digraph (D, T) as follows. We take an out-branching B^+ with vertex set $\{z_U \mid U \subseteq X, |U| \ge 1\}$ and root z_X . For every arc $a_U a_T$ of B^+ , we add a directed path of length 3 with initial vertex a spade of $S^+(U)$ (i.e., $y_U(u)$ for some $u \in U$), terminal vertex x_T , and new (private) internal vertices. This forms our out-branching T. One then easily checks that BMRN(D, T) =BMRN(D, B) = k and BMRN $^*(D, T) =$ BMRN $^*(D, B) = 2^k - 1$.

²⁰ **4** Bounds on BPRN(D, B), BMRN(D, B) and BMRN $^*(D, B)$

²¹ BPRN-, BMRN- and BMRN*-colourings of backboned digraphs can be viewed as particular cases of

the classical colouring of graphs, where one aims at assigning colours to the vertices so that a proper

23 *colouring* is attained, i.e., a colouring where no two adjacent vertices have the same colour. Let us

²⁴ now explain this in detail, for a backboned digraph (D, B).



Figure 1: A spanned digraph (D, T) (left), and the BMRN-constraint graph $C_{\text{BMRN}}(D, T)$ (right).

The *BMRN-constraint graph* of (D, B) (see Figure 1 for an example) is the undirected graph $C_{BMRN}(D, B)$ defined as follows:

• $V(C_{BMRN}(D,B)) = A(B);$

• There is an edge in $C_{\text{BMRN}}(D, B)$ between two vertices corresponding to arcs, say, u_1v_1 and u_2v_2 if either $u_1 = v_2$ or $u_2 = v_1$ (type-1 constraint), or $u_1v_2 \in A(D)$ or $u_2v_1 \in A(D)$ (type-2 constraint).

⁴ By construction, there is a trivial one-to-one correspondence between the BMRN-colourings of (D, B)⁵ and the proper colourings of $C_{\text{BMRN}}(D, B)$. In particular, we have $\text{BMRN}(D, B) = \chi(C_{\text{BMRN}}(D, B))$, ⁶ where χ is the usual chromatic number of undirected graphs.

The *BPRN-constraint graph* of (D, B) is the graph $C_{\text{BPRN}}(D, B)$ defined as follows:

•
$$V(C_{\text{BPRN}}(D,B)) = A(B);$$

• There is an edge in $C_{\text{BPRN}}(D, B)$ between two vertices corresponding to arcs, say, u_1v_1 and u_2v_2 if either $u_1 = v_2$ or $u_2 = v_1$ (type-1 constraint), or $u_1v_2 \in A(D)$ or $u_2v_1 \in A(D)$ (type-2 constraint), or $u_1 = u_2$ (type-3 constraint).

By construction, there is a trivial one-to-one correspondence between the BPRN-colourings of (D, B)and the proper colourings of $C_{\text{BPRN}}(D, B)$, and $\text{BPRN}(D, B) = \chi(C_{\text{BPRN}}(D, B))$.

The *BMRN*^{*}-constraint graph of (D, B), denoted by $C_{\text{BMRN}^*}(D, B)$, can be obtained from $C_{\text{BMRN}}(D, B)$ by identifying all vertices corresponding to arcs of *B* with same tail. It can also be defined as follows:

16 •
$$V(C_{\text{BMRN}^*}(D,B)) = V(D)$$

• u_1u_2 is an edge in $C_{\text{BMRN}*}(D, B)$ if there are two arcs u_1v_1 and u_2v_2 of B such that either $u_1 = v_2$ or $u_2 = v_1$ (type-1 constraint), or $u_1v_2 \in A(D)$ or $u_2v_1 \in A(D)$ (type-2 constraint).

Again, by construction, there is a one-to-one correspondence between the BMRN*-colourings of (D, B) and the proper colourings of $C_{\text{BMRN}*}(D, B)$. So BMRN* $(D, B) = \chi(C_{\text{BMRN}*}(D, B))$.

In this section, we give general bounds on BPRN(D, B), BMRN(D, B) and BMRN $^*(D, B)$, mainly by studying properties of $C_{\text{BPRN}}(D, B)$, $C_{\text{BMRN}}(D, B)$ and $C_{\text{BMRN}^*}(D, B)$.

23 4.1 General upper bounds

24 4.1.1 BMRN*-colouring

Observe that if u_1u_2 is an edge of $C_{\text{BMRN}^*}(D, B)$, then u_1 and u_2 are at distance at most 2 in D. This means that $C_{\text{BMRN}^*}(D, B)$ is a subgraph of $\tilde{U}(D)^2$. Hence

$$\mathbf{BMRN}^*(D,B) \le \chi(\tilde{U}(D)^2) \le \Delta(\tilde{U}(D)^2) + 1 \le \Delta(\tilde{U}(D))^2 + 1 \le \Delta(D)^2 + 1,$$

which provides a first upper bound on BMRN^{*}(D, B) (and, hence, on BMRN(D, B)) in terms of $\Delta(D)$.

We now prove a better upper bound on BMRN^{*}(D, T) for spanned digraphs (D, T). It is obtained

from a particular orientation of $C_{\text{BMRN}^*}(D,T)$ and the following well-known lemma, whose short proof is given for sake of completeness.

Lemma 4. For every digraph D, we have $\chi(D) \leq 2\Delta^+(D) + 1$.

Proof. For every subdigraph H of D, we have

$$\sum_{v \in V(H)} d_H(v) = 2 \cdot |A(H)| = 2 \sum_{v \in V(H)} d_H^+(v) \le 2 \cdot |V(H)| \cdot \Delta^+(H) \le 2 \cdot |V(H)| \cdot \Delta^+(D).$$

Hence every subgraph H of D has a vertex of degree at most $2\Delta^+(D)$. In other words, D is $2\Delta^+(D)$ -

degenerate, and thus $\chi(D) \leq 2\Delta^+(D) + 1$.

Theorem 5. For every spanned digraph (D,T), we have

$$BMRN^*(D,T) \le 2\Delta^+(D) + 3$$
 and $BMRN^*(D,T) \le 2\Delta(D) + 1$.

¹ *Proof.* Let \vec{C} be the orientation of $C_{\text{BMRN}^*}(D,T)$ defined by

•
$$V(\vec{C}) = V(D)$$
, and

• u_1 dominates u_2 in \vec{C} if either

4 –
$$u_2u_1 \in A(T)$$
 (type-1 constraint), or

- there exists v_1 such that $u_2v_1 \in A(T)$ and $u_1v_1 \in A(D)$ (type-2 constraint).

For every $u \in V(D)$, we have $d^+_{\vec{C}}(u) \le d^-_T(u) + \sum_{v \in N^+_D(u)} d^-_T(v)$. Now since T is an out-branching, we have $d^-_T(x) \le 1$ for every vertex x, so

$$d^+_{\vec{C}}(u) \le d^-_T(u) + d^+_D(u) \le d^+_D(u) + 1 \le \Delta^+(D) + 1,$$

so $\Delta^+(\vec{C}) \leq \Delta^+(D) + 1$. Thus, by Lemma 4, we get

BMRN^{*}
$$(D,T) = \chi(\vec{C}) \le 2\Delta^+(\vec{C}) + 1 \le 2\Delta^+(D) + 3.$$

6 Similarly $d^+_{\vec{C}}(u) \leq d^-_T(u) + d^+_D(u) - 1 \leq \Delta(D) - 1$, and so we get $\text{BMRN}^*(D,T) \leq 2\Delta(D) - 1$. \Box

7 4.1.2 BPRN-colouring

⁸ As in the previous section, using an orientation of the BPRN-constraint graph $C_{\text{BPRN}}(D,T)$ (for some

spanned digraph (D,T)), we now establish upper bounds on BPRN(D,T) in terms of $\Delta^+(D)$ and $\Delta(D)$.

Theorem 6. For every spanned digraph (D, T), we have

$$BPRN(D,T) \le 2\Delta^+(D) + 1$$
 and $BPRN(D,T) \le 2\Delta(D) - 1$.

¹¹ *Proof.* Let \vec{C} be the orientation of $C_{\text{BPRN}}(D,T)$ defined by

•
$$V(\vec{C}) = A(T)$$
, and

- u_1v_1 dominates u_2v_2 in \vec{C} if either
- $v_2 = u_1$ (type-1 constraint),
- 15 $u_1v_2 \in A(D) \setminus A(T)$ (type-2 constraint), or
- 16 $u_1 = u_2$ that is $u_1v_2 \in A(T)$ (type-3 constraint).

For every arc $uv \in A(T)$, we have $d^+_{\vec{C}}(uv) \leq d^-_T(u) + \sum_{w \in N^+_D(u) \setminus \{v\}} d^-_T(w)$. Since T is an outbranching, we have

$$d^+_{\vec{C}}(uv) \le d^-_T(u) + d^+_D(u) - 1 \le d^+_D(u) \le \Delta^+(D),$$

so $\Delta^+(\vec{C}) \leq \Delta^+(D)$. Thus, by Lemma 4, we get

$$BPRN(D,T) = \chi(\vec{C}) \le 2\Delta^{+}(\vec{C}) + 1 \le 2\Delta^{+}(D) + 1.$$

17 Similarly $d^+_{\vec{C}}(uv) \leq d^-_T(u) + d^+_D(u) - 1 \leq \Delta(D) - 1$, and so we get $\text{BPRN}(D,T) \leq 2\Delta(D) - 1$. \Box

1 4.1.3 BMRN-colouring

- 2 Since BMRN $(D, B) \leq$ BPRN(D, B) holds for every backboned digraph (D, B), we directly get the
- ³ following from Theorem 6.

Corollary 7. For every spanned digraph (D,T), we have

 $BMRN(D,T) \le 2\Delta^+(D) + 1$ and $BMRN(D,T) \le 2\Delta(D) - 1$.

4 4.1.4 Tightness of the bounds

The bounds of Theorems 5 and 6 and Corollary 7 are tight up to a small additive factor, as shown in
 the following proposition.

Proposition 8. For every $k \ge 2$, there exists a spanned digraph (D_k, T_k) with $\Delta^+(D_k) = k$, $\Delta(D_k) = k + 1$, and $BMRN(D_k, T_k) = 2k - 1$.

Proof. Let (D_k, T_k) be the spanned digraph defined as follows (see Figure 2):

•
$$V(D_k) = V(T_k) = \bigcup_{i=1}^{2k-1} \{t_i, u_i, v_i\},$$

• $A(T_k) = \bigcup_{i=1}^{2k-2} \{t_i t_{i+1}\} \cup \bigcup_{i=1}^{2k-1} \{t_i u_i, u_i v_i\}, \text{ and}$
• $A(D_k) = A(T_k) \cup \bigcup_{i=1}^{2k-1} \{u_i v_{i+j} \mid 1 \le j \le k-1\}$ (indices are modulo $2k - 1$).



Figure 2: The spanned digraph (D_3, T_3) described in the proof of Proposition 8.

⁹ Clearly $\Delta^+(D_k) = k$, and $\Delta(D_k) = k + 1$. Now, in every BMRN-colouring of (D_k, T_k) , ¹⁰ every two of the $u_i v_i$'s must receive distinct colours because they are involved in a type-2 constraint. ¹¹ Consequently, BMRN $(D_k, T_k) \ge 2k - 1$. It is easy to check that (2k - 1)-BMRN-colourings of ¹² (D_k, T_k) actually exist (by, e.g., generalizing the colouring scheme of (D_3, T_3) depicted in Figure 2), ¹³ so BMRN $(D_k, T_k) = 2k - 1$.

Remark 9. Note that in the construction described in the proof of Proposition 8, we have $\Delta^+(T_k) =$ 2. Therefore, Theorems 5 and 6 and Corollary 7 are tight in the sense that there is no pair (ϵ, f) where ϵ is a positive real and f a function such that $\text{BMRN}(D,T) \leq (2-\varepsilon)\Delta^+(D) + f(\Delta^+(T))$ for any spanned digraph (D,T).

The bounds of Theorems 5 and 6 and Corollary 7 are almost tight (as shown in Proposition 8), 1 but not tight. If $\Delta(D) = 1$, then D and T have two vertices and one arc, so BMRN(D,T) =2 BMRN^{*}(D,T) = BPRN(D,T) = 1. If $\Delta(D)^+ = 1$, then D is a directed path or cycle and T is a 3 directed path, in which case $BMRN(D,T) = BMRN^*(D,T) = BPRN(D,T) = 2$ (if $A(T) \ge 2$). 4 If $\Delta(D) = 2$, then D is an oriented path or cycle and T is an oriented path, and one easily sees 5 that BMRN(D,T), BMRN $^{*}(D,T)$, BPRN $(D,T) \leq 3$ Moreover the upper bound 3 is attained by 6 the spanned digraph (D,T) with $V(D) = V(T) = \{v_1, v_2, v_3, v_4\}, A(T) = \{v_1v_2, v_2v_3, v_3v_4\}$ and 7 $A(D) = A(T) \cup \{v_1 v_4\}.$ 8 For larger values of $\Delta(D)$ and $\Delta^+(D)$, one could wonder whether the bounds of Theorems 5 and 9

¹⁰ 6 and Corollary 7 can be improved. We thus address the following questions:

11 **Question 10.**

• What is the maximum value $M_{\Delta^+}(k)$ (resp. $M_{\Delta}(k)$) of BMRN(D,T) over all spanned digraphs (D,T) with $\Delta^+(D) \le k$ (resp. $\Delta(D) \le k$)?

• What is the maximum value $M^*_{\Delta^+}(k)$ (resp. $M^*_{\Delta}(k)$) of BMRN^{*}(D,T) over all spanned digraphs (D,T) with $\Delta^+(D) \le k$ (resp. $\Delta(D) \le k$)?

• What is the maximum value $P_{\Delta^+}(k)$ (resp. $P_{\Delta}(k)$) of BPRN(D, T) over all spanned digraphs (D, T) with $\Delta^+(D) \le k$ (resp. $\Delta(D) \le k$)?

In the rest of this section, we make a first step towards these questions by studying $M_{\Delta}(3)$ and $M^*_{\Delta}(3)$.

19 Corollary 7 and Proposition 8 yield $3 \le M_{\Delta}(3) \le 5$. We show that $M_{\Delta}(3) = M^*_{\Delta}(3) = 4$.

Figure 3 shows a spanned digraph (D,T) such that $\Delta(D) = 3$ and BMRN(D,T) = 4 (one easily checks that its BMRN-constraint graph is K_4 , the complete graph on four vertices). Hence $M_{\Delta}^*(3) \ge M_{\Delta}(3) \ge 4$.



Figure 3: A spanned digraph (D, T) with $\Delta(D) = 3$ and BMRN(D, T) = 4.

We now prove that $M_{\Delta}(3) \leq M_{\Delta}^*(3) \leq 4$.

Theorem 11. For every subcubic spanned digraph (D, T), we have BMRN^{*} $(D, T) \le 4$.

Proof. Let D be a subcubic digraph and T an out-branching of D. Let $D' = D \setminus A(T)$. We partition

²⁶ V(D) = V(T) into four sets according to their in- and out-degrees in T. Recall that the root r is

the unique vertex such that $d_T^-(r) = 0$, and that the *leaves* are the vertices with out-degree 0 in T. A vertex v is *flat* if $d_T^+(v) = d_T^-(v) = 1$ and it is *branching* if $d_T^-(v) = 1$ and $d_T^+(v) = 2$. If u is a flat vertex, then we denote by u^+ its out-neighbour in T.

To prove that BMRN^{*} $(D,T) \le 4$, we shall prove that $C^* = C_{\text{BMRN}^*}(D,B)$ is 3-degenerate, and thus 4-colourable.

Suppose for a contradiction that C^* is not 3-degenerate. Then it has a subgraph H such that $\delta(H) \ge 4$. The graph H contains no leaves of T because they are isolated vertices in C^* .

Consider a flat vertex u in V(H). The only possibility for it to have degree 4 in C^* is that there exist three distinct vertices v, u_1 , u_2 in V(H) such that there exists w such that $vw \in A(T)$ and 1 $uw \in A(D'), u_1u^+ \in A(D')$, and either $u_2u^+ \in A(D')$ or $u_2 = u^+$. In the latter case, note that u_2 2 has degree at most 3 in C^* ; so let us suppose the first situation occurs. Note that u_1 and u_2 are either 3 flat vertices or the root. Therefore if u is a flat vertex in H, then u^+ is a leaf which is the tail of two 4 arcs u_1u^+ and u_2u^+ of A(D') (with u_1, u_2 either flat vertices or the root).

Let U be the set of flat vertices in H and let U^+ be the set of leaves dominated by a vertex of U in T. We have $|U| = |U^+|$. Moreover, in D' each vertex of U^+ has two in-neighbours in $U \cup \{r\}$, while every vertex of U has at most one out-neighbour of U^+ and r has at most two out-neighbours in U^+ . Hence $2|U| = 2|U^+| \le |U| + 2$, so $|U| \le 2$. Furthermore, if |U| = 2, say $U = \{u_1, u_2\}$, then $u_1u_2^+, u_2u_1^+ \in A(D')$ and so u_1 (and u_2 as well) has degree 3 in C^* , a contradiction; and $U = \{u\}$ is not possible because u^+ must have an in-neighbour in $U \setminus \{u\}$. Therefore $U = \emptyset$.

Now V(H) contains only branching vertices and possibly the root r. Thus all arcs in D' originate from r. Let v be a vertex in V(H) such that its out-neighbours in T are not in H; recall that a such vertex exists since all leaves of T are not in H. Then v can only be adjacent in H to r, and its in-neighbour in T, a contradiction to the fact that $\delta(H) \ge 4$.

We believe that a similar result holds for BPRN-colourings of spanned digraphs with maximum degree 3. Note in particular that the spanned digraph (D, T) in Figure 3 verifies BPRN $(D, T) \le 4$.

17 Question 12. Does every subcubic spanned digraph (D,T) satisfy BPRN $(D,T) \le 4$?

4.2 Upper bounds for some families of digraphs

19 4.2.1 Minor-closed families of digraphs

²⁰ We start off by pointing out the following obvious result for BMRN*-colouring.

Lemma 13. Let D be a digraph and B_1 and B_2 be two subdigraphs of D. Then

$$BMRN^{*}(D, B_{1} \cup B_{2}) \leq BMRN^{*}(D, B_{1}) + BMRN^{*}(D, B_{2}).$$

In view of Lemma 13, to get upper bounds on BMRN^{*}(D, T) for spanned digraphs (D, T), it might be interesting to get upper bounds on BMRN^{*}(D, S) when S is a galaxy. We now use this approach for minor-closed families of backboned digraphs. For any class \mathcal{F} of digraphs, we define $\chi(\mathcal{F}) = \max{\chi(D) \mid D \in \mathcal{F}}$.

Theorem 14. Let \mathcal{F} be a minor-closed family of digraphs. If $D \in \mathcal{F}$ and S is a galaxy in D, then BMRN^{*} $(D, S) \leq \chi(\mathcal{F})$.

Proof. Consider the BMRN*-constraint graph $C_{\text{BMRN}*}(D, S)$. One easily sees that it is the minor of *D* obtained by contracting the arcs of *S*. Hence BMRN* $(D, S) = \chi(C_{\text{BMRN}*}(D, S)) \le \chi(\mathcal{F})$. \Box

²⁹ Theorem 14 and Lemma 13 immediately imply the following.

Corollary 15. Let \mathcal{F} be a minor-closed family of digraphs. If (D, B) is a backboned digraph with $D \in \mathcal{F}$, then $BMRN^*(D, B) \leq \chi(B)\chi(\mathcal{F})$. In particular, for every spanned digraph (D, T) where $D \in \mathcal{F}$, we have $BMRN^*(D, T) \leq 2\chi(\mathcal{F})$.

4.3 Upper bounds for planar spanned digraphs

³⁴ Corollary 15 and the Four-Colour Theorem [1, 2, 3] yield the following.

Corollary 16. For every planar spanned digraph (D,T), we have

$$BMRN(D,T) \leq BMRN^*(D,T) \leq 8.$$

There exist planar spanned digraphs (D,T) verifying BMRN(D,T) = 7. One such example is given in Figure 4. One easily checks that the BMRN-constraint graph of this example is K_7 , the complete graph on seven vertices.



Figure 4: A planar spanned digraph (D, T) with BMRN(D, T) = 7.

However, we still do not know whether all planar spanned digraphs have BMRN- or BMRN* chromatic index at most 7.

6 Question 17. Is it true that, for every planar spanned digraph (D, T), we have:

7 (a) $BMRN(D,T) \le 7?$

8 (b) BMRN* $(D, T) \le 7?$

⁹ In the next section, we manage to answer such questions for outerplanar spanned digraphs.

10 4.4 Upper bounds for outerplanar spanned digraphs

Since outerplanar graphs have chromatic number at most 3, Corollary 15 yields that BMRN* $(D, T) \le$ 6 for every outerplanar spanned digraph (D, T). The aim of this section is to improve this bound and show the following theorem.

Theorem 18. For every outerplanar spanned digraph (D,T), we have

$$BMRN(D,T) \leq BMRN^*(D,T) \leq 5.$$

The bound 5 of Theorem 18 is best possible as shown by the example depicted in Figure 5. One easily sees that, for this spanned digraph (D,T), we have BMRN $(D,T) \ge 5$ because its BMRNconstraint graph contains a K_5 .



Figure 5: An outerplanar spanned digraph (D, T) with BMRN $(D, T) \ge 5$.

¹⁷ A digraph *D* is *outerplanar-maximal* if it is outerplanar and adding any new arc to *D* results in ¹⁸ a non-outerplanar digraph. An outerplanar-maximal digraph is symmetric and its underlying simple ¹⁹ graph $G = \tilde{U}(D)$ of *D* is 2-connected and *inner triangulated* (i.e., all faces except the outer one are ²⁰ 3-faces).

Let (D, T) be an outerplanar spanned digraph. A *chord* is an arc of D which is not incident to the outer face, and a *T*-chord is a chord in A(T). The proof of Theorem 18 is by induction on the number 1 of T-chords. We first prove the following which corresponds to the basis of the induction, that is the 2 case when there is no T-chord.

Lemma 19. Let (D,T) be a 2-connected outerplanar spanned digraph such that all arcs of T are on the outer face of D. Then BMRN^{*} $(D,T) \le 5$.

Proof. By considering a minimum counterexample (i.e., with the minimum number of vertices). Free 5 to add arcs, we may assume that D is outerplanar-maximal. Since D is a symmetric digraph, in what 6 follows we sometimes regard it as an undirected graph G, in which every pair of arcs $\{uv, vu\}$ is 7 replaced becomes an edge uv. 8 Let us number the vertices of D by v_1, \ldots, v_n so that $v_i v_{i+1}$ or $v_{i+1} v_i$ is an arc of T for all 9 $1 \leq i \leq n-1$. For $1 \leq i \leq n-1$, let a_i be the arc of T between v_i and v_{i+1} . In other words, 10 $\{a_i\} = \{v_i v_{i+1}, v_{i+1} v_i\} \cap A(T).$ 11 The span of an arc $v_i v_j$ of D or an edge $v_i v_j$ of G is |j - i|. In particular, all arcs of T have span 12 1. We denote by T[i, j] the subdipath of T induced by $\{v_i, \ldots, v_j\}$. Let i_0 be the index of the root r 13 of T (i.e., $r = v_{i_0}$). 14

The *edge-tree* of (D,T) is the out-tree whose vertices are the edges of G and such that every edge v_iv_j (with i < j) of span at least 2 dominates the two edges v_iv_k and v_kv_j such that i < k < j and $v_iv_kv_jv_i$ is a 3-cycle in G. Observe that the root of the edge-tree of (D,T) is v_1v_n , and its leaves are the edges of span 1. Furthermore, the span of every edge that is not a leaf is the sum of the spans of the two edges it dominates.

If $v_i v_j$ is an edge of G and T[i, j] is a directed path, then the $v_i v_j$ -reduced spanned digraph is the spanned digraph $(D_{i,j}, T_{i,j})$ defined as follows:

• $D_{i,j}$ is obtained from $D - \{v_{i+1}, \dots, v_{j-1}\}$ by adding a vertex $x_{i,j}$ and the arcs of the two directed cycles $(v_i, x_{i,j}, v_i)$ and $(v_j, x_{i,j}, v_j)$.

• If T[i, j] is a directed path from v_i to v_j (resp. from v_j to v_i), then $T_{i,j}$ is obtained from $T - \{v_{i+1}, \ldots, v_{j-1}\}$ by adding a new vertex $x_{i,j}$ and the arcs $v_i x_{i,j}$ and $x_{i,j} v_j$ (resp. $v_j x_{i,j}$ and $x_{i,j} v_i$).

In this case, we denote by $b = b_{i,j}$ (resp. $b' = b'_{i,j}$) the arc of $T_{i,j}$ between v_i (resp. v_j) and $x_{i,j}$.

If $v_i v_j$ is an edge of G and T[i, j] is not a directed path, then the $v_i v_j$ -reduced spanned digraph is the spanned digraph $(D_{i,j}, T_{i,j})$ defined as follows:

• $D_{i,j}$ is obtained from $D - \{v_{i+1}, \dots, v_{j-1}\}$ by adding two vertices $x_{i,j}$ and $y_{i,j}$ and the arcs of the three directed cycles $(v_i, x_{i,j}, v_i), (x_{i,j}, y_{i,j}, x_{i,j})$, and $(v_j, y_{i,j}, v_j)$.

• $T_{i,j}$ is obtained from $T - \{v_{i+1}, \dots, v_{j-1}\}$ by adding the two vertices $x_{i,j}$ and $y_{i,j}$ and the arcs $x_{i,j}v_i, x_{i,j}y_{i,j}$, and $y_{i,j}v_j$.

In that case, we denote by $b = b_{i,j}$ (resp. $b' = b'_{i,j}$) the arc $x_{i,j}v_i$ (resp. $y_{i,j}v_j$).

Observe moreover that if $v_i v_j$ has span at least 3 or span 2 and T[i, j] is a directed path, then the $v_i v_j$ -reduced spanned digraph has smaller order than (D, T). Therefore by minimality, it admits a 5-BMRN*-colouring ϕ . Moreover, the colours assigned to a_{i-1} , a_j , b and b' are all distinct, except possibly $\phi(a_{i-1}) = \phi(a_j)$ when $a_{i-1} = v_i v_{i-1}$ and $a_j = v_j v_{j+1}$.

The general idea of the proof is to show that there is an edge $v_i v_j$ such that any 5-BMRN^{*}colouring of the $v_i v_j$ -reduced spanned digraph can be modified to get a 5-BMRN^{*}-colouring of (D,T), which is a contradiction.

42 **Claim 19.1.** If $v_i v_{i+3}$ is an edge of span 3, then $i < i_0 < i + 3$.

1 Subproof. Assume for a contradiction that G has an edge $v_i v_{i+3}$ of span 3 with $i_0 \le i$ or $i \ge i+3$. 2 By symmetry, we may assume that $i_0 \le i$. Hence T[i, i+3] is a directed path from v_i to v_{i+3} .

By minimality of (D, T), the $v_i v_{i+3}$ -reduced spanned digraph $(D_{i,i+3}, T_{i,i+3})$ has a 5-BMRN^{*}colouring ϕ . Without loss of generality, $\phi(b) = 2$, $\phi(b') = 3$, $\phi(a_{i-1}) \in \{1, 2\}$ (if a_{i-1} exists), and $\phi(a_{i+3}) \in \{1, 4\}$ (if a_{i+3} exists). Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 5$, and $\phi(a_{i+2}) = 3$. One easily checks that this yields a 5-BMRN^{*}-colouring of (D, T), a contradiction.

⁷ Claim 19.2. If $v_i v_{i+4}$ is an edge of span 4 and $v_i v_{i+2}$ and $v_{i+2} v_{i+4}$ are edges, then $i_0 < i$ or ⁸ $i_0 > i + 4$.

9 Subproof. Assume for a contradiction that v_iv_{i+4} , v_iv_{i+2} and $v_{i+2}v_{i+4}$ are edges and $i \le i_0 \le i+4$. 10 By symmetry, we may assume $i_0 \in \{i, i+1, i+2\}$. By minimality of (D, T), the v_iv_{i+4} -reduced 11 spanned digraph has a 5-BMRN*-colouring ϕ . Without loss of generality, $\phi(a_1) = 1$ (if a_{i-1} exists), 12 $\phi(b) = 2$ if $i_0 \ne i$ and $\phi(b) = 1$ if $i_0 = i$, $\phi(b') = 3$, and $\phi(a_{i+3}) \in \{1, 4\}$ (if a_{i+3} exists).

13 Set $\phi(a_i) = 1$ if $i_0 = i$ and $\phi(a_i) = 2$ if $i_0 \in \{i + 1, i + 2\}$, $\phi(a_{i+1}) = 2$ if $i_0 \in \{i, i + 1\}$ 14 and $\phi(a_{i+1}) = 5$ if $i_0 = i + 2$, $\phi(a_{i+2}) = 5$ and $\phi(a_{i+3}) = 3$. This gives a 5-BMRN*-colouring of 15 (D, T), a contradiction.

Consider a *deepest* edge e of span 2 in the edge-tree of (D,T), that is an edge with longest distance from v_1v_n in the edge-tree. Set $e_1 = v_iv_{i+2}$. Let e_2 be the edge that dominates e_1 in the edge-tree, and let e'_1 be the second edge that is dominated by e_2 . Since e_1 is a deepest edge of span 2, necessarily e'_1 is an edge of span 1 or 2, for otherwise the branch of the edge-tree spanned at e'_1 would contain an edge of span 2 deeper than e_1 . Therefore the span of e_2 , which is the sum of the spans of e_1 and e'_1 , is either 3 or 4. Without loss of generality, either $e_2 = v_iv_{i+3}$ or $e_2 = v_iv_{i+4}$ and $v_{i+2}v_{i+4}$ is an edge.

23 <u>Case 1:</u> $e_2 = v_i v_{i+3}$.

Then $i_0 \in \{i + 1, i + 2\}$ by Claim 19.1. Let $e_3 = v_k v_l$ be the edge that dominates e_2 in the 24 edge-tree, and let e'_2 be the second edge dominated by e_3 . Since e_1 was the deepest edge of span 25 2, e'_2 has span at most 4. If e'_2 has span 4, then swapping the names of e_2 and e'_2 we are in Case 2. 26 If e'_2 has span 3, then e'_2 contradicts Claim 19.1. Hence e'_2 has span 1 or 2. Henceforth, we must 27 be in one of the cases below. For each of them, we take a 5-BMRN*-colouring ϕ of the e_3 -reduced 28 spanned digraph, which exists by minimality of (D, T). Without loss of generality, we may assume 29 that $\phi(a_{k-1}) = 1$ (if a_{k-1} exists), $\phi(b) = 2$, $\phi(b') = 3$, and $\phi(a_l) \in \{1, 4\}$ (if a_l exists). We now 30 show for each case how to derive a 5-BMRN*-colouring of (D, T), which is a contradiction. 31

•
$$e_3 = v_i v_{i+4}$$
.
Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 2$ if $i_0 = i+1$ and $\phi(a_{i+1}) = 5$ if $i_0 = i+2$, $\phi(a_{i+2}) = 5$, and
 $\phi(a_{i+3}) = 3$.

35 • $e_3 = v_{i-1}v_{i+3}$.

Set $\phi(a_{i-1}) = 2$, $\phi(a_i) = 5$, $\phi(a_{i+1}) = 5$ if $i_0 = i+1$ and $\phi(a_{i+1}) = 3$ if $i_0 = i+2$, and $\phi(a_{i+2}) = 3$.

• $e_3 = v_i v_{i+5}$. Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 2$ if $i_0 = i + 1$ and $\phi(a_{i+1}) = 5$ if $i_0 = i + 2$, $\phi(a_{i+2}) = 5$, $\phi(a_{i+3})$ 40 is a colour of $\{1, 4\} \setminus \{\phi(a_{i+5})\}$, and $\phi(a_{i+4}) = 3$.

41 • $e_3 = v_{i-2}v_{i+4}$.

42 Set
$$\phi(a_{i-2}) = 2$$
, $\phi(a_{i-1}) = 4$, $\phi(a_i) = 5$, $\phi(a_{i+1}) = 5$ if $i_0 = i+1$ and $\phi(a_{i+2}) = 3$ if

43 $i_0 = i + 2$, and $\phi(a_{i+3}) = 3$.

1 Case 2: $e_2 = v_i v_{i+4}$ and $v_{i+2} v_{i+4}$ is an edge.

By Claim 19.2, and by symmetry, we may assume that $i_0 < i$. Let $e_3 = v_k v_l$ be the edge that dominates e_2 in the edge-tree, and let e'_2 be the second edge dominated by e_3 . Since e_1 was the deepest edge of span 2, e'_2 has span at most 4. Furthermore if e'_2 has span 4, then it dominates two edges of span 2.

For each of them, we take a 5-BMRN*-colouring ϕ of the e_3 -reduced spanned digraph, which exists by minimality of (D, T). Without loss of generality, we may assume that $\phi(a_{k-1}) \in \{1, 2\}$ (if a_{k-1} exists), $\phi(b) = 2$, $\phi(b') = 3$, and $\phi(a_l) \in \{1, 4\}$ (if a_l exists). Moreover $\phi(a_{k-1}) = 2$ only if $a_0 = k$ and $\phi(a_l)$ only if $k \le i_0 \le l$.

We now show for each case how to derive a 5-BMRN*-colouring of (D, T), which is a contradiction. We first consider the cases when k = i. In those cases, $\phi(a_l) = 4$ because $i_0 < i$.

(a) $e_3 = v_i v_{i+5}$. 12 Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 4$, $\phi(a_{i+2}) = 5$, $\phi(a_{i+3}) = 1$, and $\phi(a_{i+4}) = 3$. 13 (b) $e_3 = v_i v_{i+6}$. 14 Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 3$, $\phi(a_{i+2}) = 4$, $\phi(a_{i+3}) = 1$, $\phi(a_{i+4}) = 5$, and $\phi(a_{i+5}) = 3$. 15 (c) $e_3 = v_i v_{i+7}$. 16 In this case, $e'_2 = v_{i+4}v_{i+7}$ and contradicts Claim 19.1. 17 (d) $e_3 = v_i v_{i+8}$. 18 In this case, $v_{i+4}v_{i+6}$, $v_{i+6}v_{i+8}$, and $v_{i+4}v_{i+8}$ are edges. Then, set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 3$, 19 $\phi(a_{i+2}) = 5, \phi(a_{i+3}) = 1, \phi(a_{i+4}) = 4, \phi(a_{i+5}) = 5, \phi(a_{i+6}) = 2 \text{ and } \phi(a_{i+7}) = 3.$ 20 We now consider the cases when l = i + 4. 21 (e) $e_3 = v_{i-1}v_{i+4}$. 22 Set $\phi(a_{i-1}) = 2$, $\phi(a_i) = 4$, $\phi(a_{i+1}) = 5$, $\phi(a_{i+2}) = 1$, and $\phi(a_{i+3}) = 3$. 23 (f) $e_3 = v_{i-2}v_{i+4}$. 24 On the one hand, if $i_0 \leq i - 2$, then set $\phi(a_{i-2}) = 2$, $\phi(a_{i-1}) = 5$, $\phi(a_i) = \phi(a_{i+4})$, 25 $\{\phi(a_{i+1})\} = \{1,4\} \setminus \{\phi(a_i)\}, \ \phi(a_{i+2}) = 2, \ \text{and} \ \phi(a_{i+3}) = 3.$ On the other hand, if $i_0 =$ 26 i-1, then set $\phi(a_{i-2}) = 2$, $\phi(a_{i-1}) = 2$, $\phi(a_i) = \phi(a_{i+4})$, $\{\phi(a_{i+1})\} = \{1,4\} \setminus \{\phi(a_i)\}$, 27 $\phi(a_{i+2}) = 5$, and $\phi(a_{i+3}) = 3$. 28 (g) $e_3 = v_{i-3}v_{i+4}$. 29 In that case, by Claim 19.1, $i_0 \in \{i-2, i-1\}$. Then, set $\phi(a_{i-3}) = 2$, $\phi(a_{i-2}) = 2$ if 30 $i_0 = i - 2$ and $\phi(a_{i-2}) = 3$ if $i_0 = i - 1$, $\phi(a_{i-1}) = 3$, $\phi(a_i) = 1$, $\phi(a_{i+1}) = 2$, $\phi(a_{i+2}) = 5$, 31 and $\phi(a_{i+3}) = 3$. 32 (h) $e_3 = v_i v_{i+8}$. 33 We are in Case (d), with e_2 and e'_2 swapped. 34 *Proof of Theorem 18.* By induction on the number of T-chords and next on the order of D. Without 35 loss of generality, we may assume that D is outerplanar-maximal. Let C be the oriented cycle around 36

37 the outer face.

If there is no *T*-chord, then we have the result by Lemma 19. Assume now that there is a *T*-chord uv. This chord divides *D* into two outerplanar digraphs, D_1 with outer face $C_1 = C[u, v] \cup \{uv\}$ and D_2 with outer face $C_2 = C[v, u] \cup \{uv\}$. Without loss of generality, we may assume that the root *r* of *T* is in D_1 . For i = 1, 2, let D_i^* be the symmetric outerplanar graph obtained from D_i by adding the arc vu (if necessary), and let $T_i = T \cap D_i$. Observe that T_i is an out-branching of D_i^* and that the number of T_i -chords in (D_i^*, T_i) is less than the number of *T*-chords in (D, T).

- ¹ We distinguish several cases.
- ² <u>Case 1</u>: There is an arc vw_1 in T_1 .
- \mathbf{B} We distinguish two subcases depending on whether u is the root of T or not.
- 4 Subcase 1.1: u is the root of T.

By the induction hypothesis, there is a 5-BMRN*-colouring ϕ_i of each (D_i^*, T_i) . Free to permute the colours, we may assume that $\phi_1(uv) = \phi_2(uv)$ and that the arcs of T_2 with tail v(if some exist) are coloured (by ϕ_2) with $\phi_1(vw_1)$. One can easily check that the colouring ϕ of A(T), defined by $\phi(a) = \phi_1(a)$ if $a \in A(T_1)$ and $\phi(a) = \phi_2(a)$ if $a \in A(T_2)$, is a 5-BMRN*-colouring of (D, T).

10 Subcase 1.2: u is not the root of T.

In that case, u has a unique in-neighbour t in T, which must be in T_1 . Let D_2^+ be the subdigraph 11 of D induced by $V(D_2) \cup \{t\}$, and T_2^+ be the out-branching of D_2^+ obtained from T by adding 12 t and the arc tu. Observe that the number of T_2^+ -chords in (D_2^+, T_2^+) is not greater than the 13 number of T-chords in (D,T) and $|V(D_2^+)| < |V(D)|$. By the induction hypothesis, there are 14 5-BMRN*-colourings ϕ_1 of (D_1, T_1) and ϕ_2^+ of (D_2, T_2) . Free to permute the colours, we may 15 assume that $\phi_1(uv) = \phi_2^+(uv)$, $\phi_1(tu) = \phi_2^+(tu)$ and that the arcs of T_2 with tail v (if some 16 exist) are coloured (by ϕ_2^+) with $\phi_1(vw_1)$. Note that this is possible because, in both colourings, 17 tu and the arcs with tail v receive distinct colours because vu is an arc. One can easily check 18 that the colouring ϕ of A(T), defined by $\phi(a) = \phi_1(a)$ if $a \in A(T_1)$ and $\phi(a) = \phi_2^+(a)$ if 19 $a \in A(T_2^+)$, is a 5-BMRN^{*}-colouring of (D, T). 20

²¹ <u>Case 2</u>: There is no arc with tail v in T_1 .

Since uv is a T-chord, $|V(D_1)| \ge 3$ and so u has a unique in-neighbour t in T_1 , which is also its unique in-neighbour in T.

- 24 Subcase 2.1: $|V(D_1)| \ge 4$.
- We get a 5-BMRN*-colouring of (D, T) exactly as in Subcase 1.2.
- 26 Subcase 2.2: $|V(D_1)| \ge 3$.

We may assume that uv is the unique T-chord, for otherwise Case 1 or Subcase 2.1 would 27 apply, and we would be done. Let v' be the neighbour of u in C distinct from t. Let D' be 28 the digraph obtained from D by replacing the arcs tv and vt by the arcs tv' and v't. Observe 29 that (D,T) and (D',T) have the same BMRN*-chromatic index because those four arcs do 30 not create any new constraint. If $uv' \notin A(T)$, then, by Lemma 19, (D', T) has a 5-BMRN^{*}-31 colouring which is also a 5-BMRN*-colouring of (D,T). Henceforth, we may assume that 32 $uv' \in A(T)$. In particular, T - t is the union of two directed paths, P with first arc uv and P' 33 with first arc uv'. 34

Assume u has at most five neighbours in D_2^* . Recall that D is symmetric, so every neighbour is both an in- and an out-neighbour. Two of these neighbours are v and v'. By Lemma 19, (D_2^*, T_2) admits a 5-BMRN*-colouring ϕ_2 . Now at most four colours are forbidden for tu, namely $\phi_2(uv) = \phi_2(uv')$ and the colours assigned to arcs with tail a neighbour of u in D_2 distinct from v and v'. Hence one can extend ϕ_2 to (D, T) by assigning to tu a non-forbidden colour. Henceforth, we may assume that u has at least five neighbours in D_2^* .

Free to consider (D', T) instead of (D, T), we may assume that u has at least three neighbours in P. Let z be the last neighbour of u along P. Let z^- be the in-neighbour of z in P and let z^+ be the out-neighbour of z in P if it has one and $z^+ = z$ otherwise, and let z^* be the terminal vertex of P.

- Let D_3 be the subdigraph induced by the vertices of $V(P[u, z^+]) \cup \{t\}$. By Lemma 19, (D_3, T_3)
- admits a 5-BMRN*-colouring ϕ_3 . Observe moreover that the colours assigned to tu, uv, z^-z

1 and zz^+ (if $z \neq z^+$) are all distinct.

Let D_4 be the digraph obtained from the subdigraph induced by the vertices of $V(P[z^-, z^*]) \cup$ 2 V(P') by adding the 2-cycle (u, z^-, u) . Let $T_4 = \{tu\} \cup P' \cup \{uz^-\} \cup P[z^-, z^*]$. The number 3 of T_4 -chords in (D_4, T_4) is equal to the number of T-chords in (D, T) and $|V(D_4)| < |V(D)|$, 4 because there are at least three neighbours of u in P[v, z]. Thus, by minimality of (D, T), 5 (D_4, T_4) admits a 5-BMRN*-colouring. In addition, the colours assigned to tu, uv, z^-z and 6 zz^+ (if $z \neq z^+$) are all distinct. Hence, free to permute the colours, we may assume that ϕ_3 7 and ϕ_4 agree on those four arcs. Now one easily checks that the colouring ϕ of A(T), defined 8 by $\phi(a) = \phi_3(a)$ if $a \in A(T_3)$ and $\phi(a) = \phi_4(a)$ if $a \in A(T_4)$, is a 5-BMRN*-colouring of 9 (D,T).10

11 5 Complexity

12 5.1 Determining the exact value of an index

- In this section, we prove several results showing that the problems of determining BMRN(D,T),
- ¹⁴ BPRN(D,T) and BMRN $^*(D,T)$ are \mathcal{NP} -hard, even when restricted to particular spanned digraphs
- 15 (D,T). We define these decision problems in the usual way:
- 16 k-BMRN-COLOURING
- ¹⁷ Input: A spanned digraph (D, T).
- $\underbrace{\text{Question:}}_{18} \quad \underbrace{\text{Question:}}_{0} \text{ Do we have } \text{BMRN}(D,T) \leq k?$
- 20 *k*-BPRN-COLOURING
- ²¹ Input: A spanned digraph (D,T).
- <u>Question:</u> Do we have BPRN $(D,T) \le k$?
- 24 k-BMRN*-Colouring
- ²⁵ Input: A spanned digraph (D, T).
- $\underbrace{\text{Question:}}_{27} \quad \underbrace{\text{Question:}}_{27} \text{ Do we have BMRN}^*(D,T) \leq k?$

Recall that finding a k-BMRN-colouring (resp., k-BPRN-colouring, k-BMRN*-colouring) of (D,T) is equivalent to finding a k-colouring of $C_{\text{BMRN}}(D,T)$ (resp., $C_{\text{BPRN}}(D,T)$, $C_{\text{BMRN}*}(D,T)$).

Furthermore, the usual k-COLOURING problem is well-known to be polynomial-time solvable when

- k = 2 and \mathcal{NP} -complete for all $k \ge 3$. Since the constraint graphs of (D, T) can clearly be con-
- ³² structed in polynomial time, we directly get that 2-BMRN-COLOURING, 2-BPRN-COLOURING,

³³ and 2-BMRN*-COLOURING are polynomial-time solvable.

Still from the previous colouring equivalence, we now establish the hardness of the three problems above for all $k \geq 3$, using the following construction. Let \vec{G} be an oriented graph. The *matched* digraph associated to \vec{G} is the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ defined by:

•
$$V(D_{\vec{G}}) = V(M_{\vec{G}}) = \bigcup_{x \in V(\vec{G})} \{u_x, v_x\},$$

• $A(M_{\vec{G}}) = \{u_x v_x \mid x \in V(\vec{G})\}, \text{ and}$
• $A(D_{\vec{G}}) = A(M_{\vec{G}}) \cup \{u_x v_y \mid xy \in A(\vec{G})\}.$

One can easily check that \vec{G} is nothing but the BMRN-constraint graph $C_{\text{BMRN}}(D_{\vec{G}}, M_{\vec{G}})$. Observe

moreover that $\Delta^+(D_{\vec{G}}) = \Delta^+(\vec{G}) + 1.$

Theorem 20. For every $k \ge 3$, k-BMRN-COLOURING, k-BPRN-COLOURING and k-BMRN^{*}-

² COLOURING are \mathcal{NP} -complete, even when restricted to spanned digraphs (D,T) where T is a di-

з *rected path*.

4 Proof. Fix $k \ge 3$. The problems are clearly in \mathcal{NP} . Let us now prove that they are \mathcal{NP} -hard. 5 Observe that when T is a directed path, the type-3 and type-3^{*} constraints are vacuously fulfilled by 6 any arc-colouring. Therefore, in that case, any BMRN-colouring of (D, T) is also a BPRN-colouring 7 and a BMRN^{*}-colouring. It thus suffices to prove the \mathcal{NP} -completeness of k-BMRN-COLOURING 8 for such restricted instances. 9 The reduction is from h COLOURING. Let C he a graph with vertex set $\{r_{1}, \dots, r_{n}\}$. Consider

The reduction is from k-COLOURING. Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$. Consider 9 any orientation \vec{G} of G. Let (D,T) be the spanned digraph obtained from the associated matched 10 digraph $(D_{\vec{G}}, M_{\vec{G}})$ by adding the arc $v_{x_i}u_{x_{i+1}}$ for all $1 \leq i \leq n-1$, i.e., $D = D_{\vec{G}} \cup \{v_{x_i}u_{x_{i+1}} \mid 1 \leq i \leq n-1\}$ 11 $i \leq n-1$ and $T = M_{\vec{G}} \cup \{v_{x_i}u_{x_{i+1}} \mid 1 \leq i \leq n-1\}$. By construction, T is a directed path. One 12 easily sees that each $v_{x_i}u_{x_{i+1}}$ is subject to at most two colouring constraints, or, in other words, that 13 the associated vertex has degree 2 in the BMRN-constraint graph $C_{\text{BMRN}}(D, T)$. Hence, since $k \ge 3$, 14 the graph $C_{\text{BMRN}}(D,T)$ is k-colourable if and only if G is k-colourable. In other words, (D,T) is 15 k-BMRN-colourable if and only if G is k-colourable. 16

We proved, in Theorem 11, that subcubic spanned digraphs have BMRN-chromatic index at most 4. As already pointed out, by looking at the BMRN-constraint graph, it can be decided in polynomial time whether the BMRN-chromatic index of a given subcubic spanned digraph is at most 2. It is not clear, however, whether it can easily be decided whether this index is 3 or 4; so we leave the following question open:

21 following question open:

Question 21. Is 3-BMRN-COLOURING \mathcal{NP} -complete when restricted to subcubic spanned digraphs?

24 5.2 Finding a particular out-branching

²⁵ We now consider problems where, for a given digraph D, one aims at finding an out-branching T

(rooted at some given vertex r or not), such that BMRN(D, T), (resp. BMRN(D, T) and $BPRN^*(D, T)$)

is small. More precisely, we consider the following decision problems (which we define for BMRN-

colouring only, but they can be derived for BPRN-colouring and BMRN*-colouring in an obvious
way):

30 *k*-BMRN-ROOT

³¹ Input: A digraph D and an out-generator r of D.

<u>32</u> Question: Is there an out-branching T of D spanned at r such that $BMRN(D,T) \le k$?

- 34 *k*-BMRN-BRANCHING
- $_{35}$ Input: A digraph D.
- $\frac{36}{37}$ Question: Is there an out-branching T of D such that BMRN $(D,T) \le k$?

We show in Theorem 22 that all arising problems are \mathcal{NP} -complete for all $k \geq 3$. We then

³⁹ prove the *NP*-completeness of 2-BMRN-ROOT, 2-BMRN-BRANCHING, 2-BMRN*-ROOT and 2-

40 BMRN*-BRANCHING in Theorems 23 and 24. The same result for 2-BMRN-ROOT and 2-BMRN-

41 BRANCHING is then proved in Theorem 25.

⁴² **Theorem 22.** For every $k \ge 3$, k-BMRN-ROOT, k-BMRN-BRANCHING, k-BMRN*-ROOT, k-⁴³ BMRN*-BRANCHING, k-BPRN-ROOT and k-BPRN-BRANCHING are \mathcal{NP} -complete.

⁴⁴ *Proof.* The problems are clearly in \mathcal{NP} . We now prove that they are \mathcal{NP} -hard, by focusing, again,

on BMRN-colourings (for the same reasons as in the proof of Theorem 20). Fix $k \ge 3$. The reduction

is from k-COLOURING. Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$. Let \vec{G} be the orientation of G 1 such that if $x_i x_j$ is an arc, then i > j. Again consider, as in the proof of Theorem 20, the associated 2 matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ of \vec{G} . Let D be the digraph obtained from $D_{\vec{G}}$ by adding a vertex r and 3 the arcs ru_{x_1} and $v_{x_i}u_{x_{i+1}}$ for all $1 \le i \le n-1$. One easily sees that the unique out-branching of D 4 is the directed path $T = (r, u_{x_1}, v_{x_1}, u_{x_2}, v_{x_2}, \dots, u_{x_n}, v_{x_n})$. Moreover, all $u_{x_i}v_{x_{i+1}}$'s and ru_{x_1} have 5 degree at most 2 in the BMRN-constraint graph $C_{\text{BMRN}}(D,T)$ of (D,T). Hence, since $k \geq 3$, we 6

have that (D, T) is k-BMRN-colourable if and only if G is k-colourable. 7

Theorem 23. 2-BMRN-ROOT and 2-BMRN-BRANCHING are *NP*-complete. 8

Proof. The proof of the \mathcal{NP} -hardness is by reduction from MONOTONE NOT-ALL-EQUAL 3-SAT. In 9

this variant, the boolean formula \mathcal{F} consists of clauses all of whose literals are non-negated variables. 10

We want to decide whether \mathcal{F} admits an *NAE-assignment*, that is a truth assignment such that every 11

clause has a true literal and a false literal. MONOTONE NOT-ALL-EQUAL 3-SAT was shown \mathcal{NP} -12

complete by Schaefer [16]. 13



Figure 6: The two gadgets used in the proof of Theorem 23: the $\{r_i, r_{i+1}\}$ -connector gadget (left), and the clause gadget for a clause $C_{\ell} = x_i \lor x_j \lor x_k$ (right).

Let \mathcal{F} be an instance of MONOTONE NOT-ALL-EQUAL 3-SAT with variables x_1, \ldots, x_n and 14 clauses C_1, \ldots, C_m . Let D be the digraph constructed as follows. For $1 \le i \le n$, add the vertices 15 r_i, v_i and the arc $r_i v_i$. For every $1 \le i \le n-1$, we add an $\{r_i, r_{i+1}\}$ -connector gadget (see the left 16 of Figure 6), which is a digraph with vertex set $\{r_i, p_i, p'_i, q_i, r_{i+1}\}$ (where r_i and r_{i+1} are existing 17 vertices) and arc set $\{r_iq_i, q_ir_{i+1}, r_ip_i, p_ip'_i, p'_ir_{i+1}\}$. For every clause $C_\ell = x_i \vee x_j \vee x_j$, we add a 18 *clause gadget* (see the right of Figure 6) with vertex set $\{v_i, v_j, v_k, s_\ell, s_\ell', u_\ell, t_\ell, t_\ell'\}$ (where v_i, v_j, v_k 19 already exist) and arc set $\{v_i s_\ell, v_j s_\ell, s_\ell t_\ell, s_\ell u_\ell, v_k s'_\ell, s'_\ell t'_\ell, s'_\ell u_\ell\}$. 20

Observe that r_1 is the unique out-generator of D. Furthermore, every out-branching T of D must 21 contain every arc $r_i v_i$, $1 \le i \le n$, because each v_i has in-degree 1. Let us show that D has an out-22 branching T (with root r_1) such that BMRN $(D,T) \leq 2$ if and only if \mathcal{F} admits an NAE-assignment. 23 Assume first that there is an out-branching T of D such that $BMRN(D,T) \leq 2$. Let ϕ be a 24 2-BMRN-colouring of (D,T). Let ψ be the truth assignment defined by $\psi(x_i) = true$ if and only 25 if $\phi(r_i v_i) = 1$. This is well-defined because all $r_i v_i$'s are in A(T). We claim that ψ is an NAE-26 assignment of \mathcal{F} . Suppose for a contradiction that there is a clause $C_{\ell} = x_i \vee x_j \vee x_k$ such that all 27 literals have the same value. Without loss of generality, $\psi(x_i) = \psi(x_j) = \psi(x_k) = true$. Then, 28 by definition, $\phi(r_i v_i) = \phi(r_j v_j) = \phi(r_k v_k) = 1$. Now $v_k s'_\ell$ and $s'_\ell t'_\ell$ must be in T (because s'_ℓ and 29 t'_{ℓ} have in-degree 1), and we have $\phi(v_k s'_{\ell}) = 2$ and $\phi(s'_{\ell} t'_{\ell}) = 1$. Furthermore, T contains an arc 30 $a \in \{v_i s_\ell, v_j s_\ell\}$ and $s_\ell t_\ell$, and $\phi(a) = 2$ and $\phi(s_\ell t_\ell) = 1$. Finally, one arc a' of $s_\ell u_\ell$ and $s'_\ell u_\ell$ is 31 in A(T). Because of type-1 constraints, we must have $\phi(a') = 1$, which creates a type-2 constraint 32 since $\phi(s_{\ell}t_{\ell}) = \phi(s'_{\ell}t'_{\ell}) = 1$. This is a contradiction. 33

Reciprocally, assume that there exists an NAE-assignment ψ of \mathcal{F} . We shall construct an out-34 branching T of D and a 2-BMRN-colouring ϕ of (D,T). For $1 \leq i \leq n$, add $r_i v_i$ to A(T) and 35 set $\phi(r_i v_i) = 1$ if $\psi(x_i) = true$, and $\phi(r_i v_i) = 2$ if $\psi(x_i) = false$. For every clause $C_{\ell} =$ 36

 $x_i \vee x_j \vee x_k$, either $\psi(x_i) \neq \psi(x_k)$ or $\psi(x_j) \neq \psi(x_k)$ because ψ is an NAE-assignment. Let a be $v_i s_\ell$ if $\psi(x_i) \neq \psi(x_k)$ and let a be $v_j s_\ell$ otherwise. Add $a, v_k s'_\ell, s_\ell t_\ell, s'_\ell t'_\ell$, and $s_\ell u_\ell$ to A(T), and let $\phi(a) = \phi(s'_\ell t'_\ell) = \phi(r_k v_k)$ and $\phi(s_\ell t_\ell) = \phi(s_\ell u_\ell) = \phi(v_k s'_\ell) = 3 - \phi(a)$. For each $1 \leq i \leq n - 1$, 4 do the following:

• If $\phi(r_i v_i) = \phi(r_{i+1} v_{i+1})$, then add $r_i q_i$, $r_i p_i$, $p_i p'_i$ and $q_i r_{i+1}$ to A(T) and set $\phi(r_i q_i) = \phi(r_i p_i) = \phi(r_i v_i)$ and $\phi(p_i p'_i) = \phi(q_i r_{i+1}) = 3 - \phi(r_i v_i)$;

• If $\phi(r_i v_i) \neq \phi(r_{i+1} v_{i+1})$, then add $r_i q_i$, $r_i p_i$, $p_i p'_i$ and $p'_i r_{i+1}$ to A(T) and set $\phi(r_i q_i) = \phi(r_i p_i) = \phi(p'_i r_{i+1}) = \phi(r_i v_i)$ and $\phi(p_i p'_i) = 3 - \phi(r_i v_i)$.

⁹ Now, it is simple matter to check that T is an out-branching of D and ϕ a 2-BMRN-colouring of ¹⁰ (D,T).

¹¹ The exact same proof yields the following.

12 **Theorem 24.** 2-BMRN*-ROOT and 2-BMRN*-BRANCHING are *NP*-complete.

¹³ In the next result, we prove the similar result for 2-BPRN-ROOT and 2-BPRN-BRANCHING:

14 **Theorem 25.** 2-BPRN-ROOT and 2-BPRN-BRANCHING are \mathcal{NP} -complete.

¹⁵ *Proof.* The proof of the \mathcal{NP} -hardness of the two problems is by reduction from a restriction of ¹⁶ the DIRECTED HAMILTONIAN CYCLE problem, which asks whether a given digraph has a directed ¹⁷ Hamiltonian cycle. In [11], Plesńik proved that DIRECTED HAMILTONIAN CYCLE remains \mathcal{NP} -¹⁸ complete when restricted to *small-degree digraphs*, that are digraphs in which the in-degree and ¹⁹ out-degree of each vertex are either 1 or 2.

A *track* is a vertex with in-degree 1 and out-degree 1, an *out-switch* is a vertex with in-degree 1 and out-degree 2, and an *in-switch* is a vertex with in-degree 2 and out-degree 1. A digraph is *nice* if every vertex is either a track, an out-switch or an in-switch.

²³ Claim 25.1. DIRECTED HAMILTONIAN CYCLE remains *NP*-complete for nice digraphs.

Subproof. Reduction from DIRECTED HAMILTONIAN CYCLE restricted to small-degree digraphs. Consider a small-degree digraph D. Let D' be the digraph obtained by "exploding" each vertex v to an arc v^-v^+ . Formally,

•
$$V(D') = \bigcup_{v \in V(D)} \{v^-, v^+\}$$
, and
• $A(D') = \{v^-v^+ \mid v \in V(D)\} \cup \{u^+v^- \mid uv \in A(D)\}$

 \Diamond

²⁴ Clearly D' is nice and it has a directed Hamiltonian cycle if and only if D has one.

We now give a reduction from DIRECTED HAMILTONIAN CYCLE restricted to nice digraphs to 2-BPRN-ROOT and 2-BPRN-BRANCHING. Let D be a nice digraph. Choose a vertex v of D with in-degree 1. Let u be its in-neighbour. Observe that every directed Hamiltonian cycle of D must contain the arc uv. Let D' be the digraph obtained from D - uv by adding two vertices r and s and the arcs rv and us. Note that a directed Hamiltonian path in D' necessarily starts in r and ends in s, so D' has a directed Hamiltonian path if and only if D has a directed Hamiltonian cycle.

We now construct a digraph H such that H admits an out-branching T such that BPRN(H, T) = 2if and only if D' has a directed Hamiltonian path, and so if and only if D has a directed Hamiltonian cycle. To that end, we first associate a gadget in H to each vertex of D', in the following way:



Figure 7: The out-gadget O_v . A directed Hamiltonian path P of O_v is displayed with bold arcs. The black and gray bold arcs form a 2-BPRN-colouring of (O_v, P) .

• For each of r and s, we add to H two gadgets that are actually exactly the vertices r and s. For the vertex gadget corresponding to r (resp. s), we call r (resp. s) its *exit* (resp. *entry*).

1

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- Consider a track v of D'. In H, we associate a *track gadget* T_v which is a directed path of length 3. The origin v^- of T_v is its *entry*, and the terminus v^+ is its *exit*.
- Consider an out-switch v of D'. In H, we associate an *out-gadget* O_v depicted in Figure 7.
 This gadget has one *entry* v⁻ and two *exits* v⁺₁, v⁺₂.
- Consider an in-switch v of D'. In H, we associate an *in-gadget* I_v , which is just an arc v^-v^+ . • We call v^- the *entry* of I_v , and v^+ its *exit*.

Note that all gadgets have at most one entry. Furthermore, only out-gadgets have two exits. To finish the construction of H, we now connect the gadgets as follows. Consider every arc uv of D'; then:

• If u is an out-switch, then we choose a degree-2 exit of the gadget associated to u in H, and we add an arc from that exit to the unique entry of the gadget associated to v. In other words, for the gadgets with two exits, each exit is added exactly one out-going arc.

• Otherwise, we add an arc from the unique exit of the gadget associated to *u* in *H* to the unique entry of the gadget associated to *v*.

Note that an out-gadget has only two directed Hamiltonian paths: they both start in its entry and they end in different exits. Moreover once a directed Hamiltonian path of H enters a gadget (track, out-gadget or in-gadget), it has to go through all the vertices of the gadget at once, due to the number of entries and exits. Therefore the digraph H has a directed Hamiltonian path if and only if D' has one. In particular, recall that all directed Hamiltonian paths start in r and end in s.

We claim that we have the desired equivalence between D' and H. Assume first that there is an out-branching T of H such that (H, T) has a 2-BPRN-colouring. Because of type-1 and type-3 constraints, note that T must be a directed Hamiltonian path of H.

Reciprocally, assume H has a directed Hamiltonian path of H. Reciprocally, assume H has a directed Hamiltonian path P. We claim that colouring its arcs alternately with colours 1 and 2 (starting, say, with 1) yields a 2-BPRN-colouring of (H, P). Indeed, type-1 constraints are satisfied by definition, and, since P is a path, type-3 constraints are trivially satisfied. To be convinced that no type-2 constraint can arise, consider the following arguments. First of all, all arcs of H not in P either belong to an out-gadget (which raise no type-2 constraint, see Figure 7), or go from an exit of an out-gadget to the entry of an in-gadget. Also, it can be checked that all arcs of P entering and exiting any gadget necessarily have colour 1. Under this assumption,

 $_{32}$ for every arc uv from an exit of an out-gadget to the entry of an in-gadget, the only arc of P out-going

from u has colour 2, while the only arc of P in-coming to v has colour 1. Hence no type-2 constraint arises.

Hence there is an out-branching T of H such that (H, T) has a 2-BPRN-colouring if and only if

⁴ *H* has a directed Hamiltonian path, so if and only if D' has a directed Hamiltonian path.

5 5.3 Restriction to planar spanned digraphs

6 In this section, we consider PLANAR k-BMRN-COLOURING which is the restriction of k-BMRN-

⁷ COLOURING to planar spanned digraphs. For k = 2, the problem is polynomial-time solvable

⁸ because k-BMRN-COLOURING is polynomial-time solvable. For $k \ge 8$, PLANAR k-BMRN-

9 COLOURING is trivial as the answer is always 'Yes' by Corollary 16.

10 **Theorem 26.** PLANAR 3-BMRN-COLOURING is \mathcal{NP} -complete.

¹¹ *Proof.* The proof is by reduction from PLANAR 3-COLOURING which consists in deciding whether ¹² a given planar graph is 3-colourable. This problem was proved \mathcal{NP} -complete in [7].

From a planar graph G, we shall construct, in polynomial time, a spanned planar digraph (D, T)such that $\chi(G) \leq 3$ if and only if BMRN $(D, T) \leq 3$.

Since G is planar, it admits a planar straight-line embedding \tilde{G} in the plane. Moreover, free to move slightly some vertices, we may assume that no two vertices of \tilde{G} lie on a same horizontal or vertical line. Let \vec{G} be the orientation of G obtained by orienting every edge towards its higher vertex in \tilde{G} : hence if $uv \in A(\vec{G})$, then vertex v lies above u.

¹⁹ Let us consider the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ associated to \vec{G} . As observed earlier, G is actually ²⁰ the BMRN-constraint graph $C_{\text{BMRN}}(D_{\vec{G}}, M_{\vec{G}})$. So G is 3-colourable if and only if $(D_{\vec{G}}, M_{\vec{G}})$ is 3-²¹ BMRN-colourable.

Our goal is now to extend $(D_{\vec{G}}, M_{\vec{G}})$ into a planar spanned digraph (D, T), with preserving the 22 colouring equivalence with G. From \tilde{G} , one can easily derive a plane straight-line embedding of $D_{\vec{G}}$ 23 in which all arcs $u_x v_x$, for $x \in V(\vec{G})$, are drawn horizontally and from left to right, of tiny length 24 ϵ and whose middle is x. Towards getting (D,T), we first add, to $(D_{\vec{G}}, M_{\vec{G}})$, a vertex r below all 25 other vertices (and thus on the outer face), which shall be the root of T. For every inner face F of \hat{G} , 26 let m(F) be the lowest vertex of F (for the particular case of the outer face F, let m(F) = r). For 27 every face F of $(D_{\vec{G}}, M_{\vec{G}})$, let M(F) be the set of vertices x such that u_x is inside F. Observe that 28 the vertices of M(F) are incident to F, and that $m(F) \notin M(F)$. For each face F, we add a directed 29 path P_x (both in D and T) of length 3 from $u_{m(F)}$ to u_x for each vertex $x \in M(F)$. Of course, we 30 do this in such a way that the added paths do not cross. This results in the planar spanned digraph 31 (D,T) where $D = D_{\vec{G}} \cup \bigcup_{x \in V(G)} P_x$ and $T = M_{\vec{G}} \cup \bigcup_{x \in V(G)} P_x$. 32

Let us now check that $\text{BMRN}(D_{\vec{G}}, M_{\vec{G}}) \leq 3$ if and only if $\text{BMRN}(D, T) \leq 3$. Since $(D_{\vec{G}}, M_{\vec{G}})$ is the restriction of (D, T) to $V(D_{\vec{G}})$, every 3-BMRN-colouring of (D, T) induces a 3-BMRNcolouring of $(D_{\vec{G}}, M_{\vec{G}})$. Conversely, every 3-BMRN-colouring ϕ of $(D_{\vec{G}}, M_{\vec{G}})$ can be easily extended in a 3-BMRN-colouring of (D, T), for the following reasons. First, the added paths P_x do not create any new constraint between the arcs of $M_{\vec{G}}$. Moreover, for every face F and every $x \in M(F)$, we can assign the colour $\phi(u_{m(F)}, v_{m(F)})$ to the first arc of P_x , and then extend the colouring to the two other arcs, which are subject to at most two constraints in (D, T).

Hence (D,T) is a planar spanned digraph which is 3-BMRN-colourable if and only if G is 3colourable.

PLANAR *k*-COLOURING is trivial for all $k \ge 4$, as the answer is always 'Yes' according to the Four-Colour Theorem. Henceforth, the above proof cannot be generalized to show that PLANAR *k*-BMRN-COLOURING is \mathcal{NP} -complete for *k* larger than 3. 1 Question 27. For every $k \in \{4, 5, 6, 7\}$, what is the complexity of PLANAR k-BMRN-COLOURING?

² Similarly, one can compute the chromatic number of an outerplanar graph in polynomial time. So

3 we cannot establish the hardness of the restriction of k-BMRN-COLOURING to outerplanar digraphs

⁴ via a proof similar to that of Theorem 26. We thus leave the following question open:

⁵ Question 28. For any outerplanar spanned digraph (D,T), can BMRN(D,T) and BMRN $^*(D,T)$

⁶ be determined in polynomial time?

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