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# Backbone colouring and algorithms for TDMA scheduling

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## Abstract

We investigate graph colouring models for the purpose of optimizing TDMA link scheduling in Wireless Networks. Inspired by the BPRN-colouring model recently introduced by Rocha and Sasaki, we introduce a new colouring model, namely the BMRN-colouring model, which can be used to model link scheduling problems where particular types of collisions must be avoided during the node transmissions.

In this paper, we initiate the study of the BMRN-colouring model by providing several bounds on the minimum number of colours needed to BMRN-colour digraphs, as well as several complexity results establishing the hardness of finding optimal colourings. We also give a special focus on these considerations for planar digraph topologies, for which we provide refined results. Some of these results extend to the BPRN-colouring model as well.

## 1 Introduction

### 1.1 Motivation

A *Radio Network* consists in a set of *nodes* distributed in space that communicate via broadcast radio waves, with all messages sent from a node transmitted to all nodes in its range. The *range* of a node is therefore the region in space within which it can communicate to others. Node  $a$  can transmit to node  $b$  if  $b$  is within the range of  $a$ , and we say there is a directional *link*  $ab$ . Typically, communication is done in a *multi-hop* fashion, with intermediate nodes forwarding data from a source to a destination that is distant. Since the channels are shared, transmissions are subject to collisions, that result in undesired effects such as loss of data and network efficiency. To avoid such collisions, medium access control techniques have been designed.

In the Time-Division Multiple Access (TDMA) method [10], time is divided into *frames* of fixed duration. Each frame is itself divided into a number of fixed duration *time slots*. A *link schedule* is then an assignment of time slots to the network transmissions. When the link  $ab$  is scheduled, it is required that  $b$  receives its message from  $a$  free of collision, although we do not require the same for other nodes receiving a message from  $a$ . The TDMA method is being used in standards such as IEEE 802.16 [17] and IEEE 802.11s [18], providing guaranteed Quality-of-Service (QoS). In particular, TDMA MAC protocols are being used for the increasingly popular Wireless Sensor and Mesh Networks [5]. Although there are particularities in the distinct network scenarios, an efficient use of TDMA methods is related to an increase in network throughput, and to a reduction of delays and packet losses.

Given a network and an interference model, the challenge is to find a link schedule that avoids conflicts and minimizes the number of used time slots [6, 9]. Minimizing the number of time slots is

important, since it affects network throughput and multi-hop transmission delays. That is, if there are  $N_f$  different time slots in a frame, then, since a node or link is only active during its associated time slot, it becomes active only during a fraction  $\frac{1}{N_f}$  of time. For this reason, finding a broadcast or link scheduling that minimizes the number of slots needed in a frame can lead to an improvement in the network efficiency.

In a typical network, only a subset of the existing links become active. This is because the existence of a link  $ab$  does not necessarily imply that  $a$  ever communicates with  $b$ . As an example, consider a network formed by a Wi-Fi router that is connected to the internet by cable, and that is surrounded by a number of devices such as computers and smartphones, connected to the internet through it. The nearby devices may be in each other's ranges, but their transmissions are not intended to each other, so that their mutual links do not need to be scheduled. In the rest of this paper we refer to the subnetwork whose links are the active ones as the network *backbone*.

## 1.2 Modelling

Such problems on radio networks can be modelled as graph colouring problems, in the following way. A sensor network can be represented as a digraph  $D$  with vertex set  $V(D)$  and arc set  $A(D)$ . The vertices of  $D$  correspond to the nodes, and there is an arc from a vertex  $u$  to a vertex  $v$  if the node corresponding to  $v$  is in the transmission range of the node corresponding to  $u$ . The backbone is represented by a spanning subdigraph  $B$  of  $D$ , whose arcs correspond exactly to the backbone links. The arcs in  $A(D) \setminus A(B)$  correspond to the networks links that are not used by the network. The pair  $(D, B)$  is called a *backboned digraph* throughout. Now, we consider the time slots as colours to be assigned to the arcs of  $B$ . Hence, a link schedule of  $k$  time slots is a  $k$ -arc-colouring of  $B$  that must satisfy some constraints, depicting the conflicts which must be avoided during transmissions.

In practice, transmission collisions may occur for various reasons (physical constraints, device constraints, etc.), such as the following four ones, which shall be considered throughout this paper.

- **Type-1 constraint:** *During a time slot, a node cannot both transmit and receive messages along the backbone.*  
 $\Rightarrow$  If  $uv$  and  $vw$  are two arcs of  $B$ , then they cannot be assigned the same colour.
- **Type-2 constraint:** *During a time slot, a node receiving a message along the backbone must not receive a message from another transmitting node.*  
 $\Rightarrow$  If  $u_1v_1$  is an arc of  $B$ , and  $e$  is either an arc  $u_2v_1$  of  $B$ , or an arc  $u_2v_2$  of  $B$  such that  $u_2v_1 \in A(D) \setminus A(B)$ , then  $uv$  and  $e$  cannot be assigned the same colour.
- **Type-3 constraint:** *During a time slot, a node transmits at most one message along the backbone.*  
 $\Rightarrow$  If  $uw_1$  and  $uw_2$  are two arcs of  $B$ , then they cannot be assigned the same colour.
- **Type-3\* constraint:** *A node transmits all its messages along the backbone during one time slot only.*  
 $\Rightarrow$  If  $uw_1$  and  $uw_2$  are two arcs of  $B$ , then they must be assigned the same colour.

## 1.3 Colouring variants over the four types of constraints

### 1.3.1 BPRN-colouring

Arc-colourings of backboned digraphs fulfilling combinations of the constraints above were already considered in literature. In particular, an arc-colouring of a backboned digraph  $(D, B)$  verifying

1 type-1, type-2 and type-3 constraints is called a *Backbone-Packet-Radio-Network-colouring* (*BPRN-*  
2 *colouring* for short). Note that when  $B$  is an *out-branching* of  $D$ , i.e., a spanning oriented tree all  
3 arcs of which are oriented away from the root, BPRN-colourings model link schedules where, in the  
4 radio network, the root node wants to send a private message to all other nodes. The *BPRN-chromatic*  
5 *index* of  $(D, B)$ , denoted by  $\text{BPRN}(D, B)$ , is the minimum number of colours in a BPRN-colouring  
6 of  $(D, B)$ .

7 The BPRN-colouring model is a generalization of the *PRN-colouring* model, first proposed by  
8 Arika [4], that appeared in other works about link scheduling [9, 13, 15]. It was introduced by  
9 Rocha and Sasaki in [14], who, motivated by applications in Wireless Sensor Networks, studied  
10 the case where the backbone is an in-branching, i.e., an oriented tree where all arcs are oriented  
11 towards the root. Among several results on BPRN-colourings, they exhibited bounds on the BPRN-  
12 chromatic index of backboned digraphs  $(D, B)$  where  $D$  is complete or a cycle. They also proved  
13 that, in general, determining the BPRN-chromatic index is  $\mathcal{NP}$ -hard, even when restricted to bipartite  
14 backboned digraphs.

### 15 **1.3.2 BMRN-colouring**

16 Let  $(D, B)$  be a backboned digraph. An arc-colouring of  $(D, B)$  verifying type-1 and type-2 con-  
17 straints only is called a *Backbone-Multicast-Radio-Network-colouring* (*BMRN-colouring* for short).  
18 The *BMRN-chromatic index* of  $(D, B)$ , denoted by  $\text{BMRN}(D, B)$ , is the minimum number of colours  
19 in a BMRN-colouring of  $(D, B)$ . To the best of our knowledge, BMRN-colourings were not stud-  
20 ied in literature. According to the definitions, BMRN-colourings are subject to less constraints than  
21 BPRN-colourings, and can thus be regarded as a simpler colouring notion. In particular, results  
22 on BPRN-colourings hold for BMRN-colourings. Moreover, the particular case when  $B$  is an out-  
23 branching models the case when a root wants to send the same message to all other vertices via a  
24 multicast tree.

### 25 **1.3.3 BMRN\*-colouring**

26 For a backboned digraph  $(D, B)$ , an arc-colouring verifying type-1, type-2, and type-3\* constraints is  
27 called a *Star-Backbone-Multicast-Radio-Network-colouring* or simply *BMRN\*-colouring* of  $(D, B)$ .  
28 The *BMRN\*-chromatic index* of  $(D, B)$ , denoted by  $\text{BMRN}^*(D, B)$ , is the minimum number of  
29 colours in a BMRN\*-colouring of  $(D, B)$ . A first main motivation to consider BMRN\*-colourings is  
30 to derive results on the BMRN-colourings, since BMRN\*-colourings are BMRN-colourings. How-  
31 ever, it should be pointed that BMRN\*-colourings are also a model to *broadcast scheduling*, in which  
32 each node of the network is scheduled to a time slot, during which all of its neighbours must re-  
33 ceive its transmission free of collisions. Other works such as [12, 13] study broadcast scheduling by  
34 considering that the network links are symmetric, the network being modelled by a graph and the  
35 scheduling problem corresponding to finding a distance-2 colouring that uses the smallest number of  
36 colours. This approach can be seen as a simplification of our problem, since a typical network may  
37 have devices with different transmission ranges, resulting in a directed graph.

38 The BMRN\*-colouring model can therefore be seen as a more realistic model for broadcast  
39 scheduling. This motivates the investigation of more results on this model and its corresponding  
40 algorithms. Other works such as [8] already point that distance-2 colouring of graphs is not the most  
41 accurate model to broadcast scheduling in real wireless networks, and consider the directed version.  
42 But to the best of our knowledge, we are the first to model the problem as an arc-colouring problem,  
43 and to relate it to link scheduling.

## 1.4 Our results

It should be intuitive to the readers that the three colouring variants presented in the previous section are related. From the definitions, BPRN-colourings and BMRN\*-colourings are also BMRN-colourings. Section 3 is devoted to understanding some of these relations further.

In Section 4, we exhibit upper bounds on the BMRN-chromatic, BPRN-chromatic and BMRN\*-chromatic indices of a *spanned digraph*, that is a backboned digraph  $(D, T)$  where  $T$  is an out-branching in  $D$ . Firstly, in Subsection 4.1, we establish general bounds in terms of the maximum degree and maximum out-degree of the digraph and show that these bounds are tight up to a small constant factor. For example, we show that for every spanned digraph  $(D, T)$  we have  $\text{BMRN}(D, T) \leq 2\Delta^+(D) + 1$  (where  $\Delta^+(D)$  denotes the maximum out-degree of a vertex of  $D$ ) and  $\text{BMRN}(D, T) \leq 2\Delta(D) - 1$  (where  $\Delta(D)$  denotes the maximum degree of the underlying graph of  $D$ ) while there are spanned digraph  $(D, T)$  for which  $\text{BMRN}(D, T) \geq 2\Delta^+(D) - 1$  and  $\text{BMRN}(D, T) \geq 2\Delta(D) - 3$ . Secondly, in Subsection 4.2, we give more refined upper bounds in the realistic case where a planar topology is considered. We show that  $\text{BMRN}(D, T) \leq \text{BMRN}^*(D, T) \leq 8$  for every planar spanned digraph  $(D, T)$ , and that  $\text{BMRN}(D, T) \leq \text{BMRN}^*(D, T) \leq 5$  for every outerplanar spanned digraph  $(D, T)$ .

In Section 5, we show that determining any of the BMRN-, BPRN- and BMRN\*-chromatic indices of a spanned digraph is  $\mathcal{NP}$ -hard in general, and that this remains true if one is allowed to choose the out-branching of the input spanned digraph. We also show that given a planar spanned digraph  $(D, T)$ , deciding whether  $\text{BMRN}(D, T) \leq 3$  is  $\mathcal{NP}$ -complete.

## 2 Definitions, notations and preliminaries

The *square*  $G^2$  of an undirected graph  $G$  is the graph with same vertex set as  $G$  and in which two vertices are adjacent if and only if they are at distance at most 2 in  $G$ .

The *underlying simple graph*  $\tilde{U}(D)$  of a digraph  $D$  is the graph defined by  $V(\tilde{U}(D)) = V(D)$  and  $E(\tilde{U}(D)) = \{uv \mid uv \in A(D) \text{ or } vu \in A(D)\}$ . We call  $D$  *symmetric* if  $vu \in A(D)$  whenever  $uv \in A(D)$ . Observe that if  $D$  is symmetric, then  $\tilde{U}(D)$  is obtained by replacing each directed 2-cycle by an edge. The digraph obtained from  $D$  by *contraction* of the arc  $a = xy$  in  $A(D)$  is the digraph  $D/a$  obtained from  $D - \{x, y\}$  by adding a new vertex  $v_a$  and the arc  $v_a w$  (resp.  $w v_a$ ) for every vertex  $u$  in  $V(D) \setminus \{x, y\}$  such that  $xw \in A(D)$  or  $yw \in A(D)$  (resp.  $wx \in A(D)$  or  $wy \in A(D)$ ).

A digraph  $D$  is *subcubic* if  $\Delta(D) \leq 3$ .

The notion of minor of a digraph that we use in this paper corresponds to the notion of minor in the underlying graph. That is, a digraph  $D'$  is a *minor* of  $D$  if it can be obtained from  $D$  by a succession of vertex-deletions, arc-deletions, and arc-contractions. A family  $\mathcal{D}$  of digraphs is *minor-closed* if for every  $D \in \mathcal{D}$ , all minors of  $D$  are also in  $\mathcal{D}$ .

The *union* of two digraphs  $D_1$  and  $D_2$  is the digraph  $D_1 \cup D_2$  with vertex set  $V(D_1) \cup V(D_2)$  and arc set  $A(D_1) \cup A(D_2)$ . If  $D_1$  and  $D_2$  are disjoint (i.e.,  $V(D_1) \cap V(D_2) = \emptyset$ ), then it is the *disjoint union* of  $D_1$  and  $D_2$ . A *matching* in a digraph  $D$  is a disjoint union of arcs. A matching  $M$  is *perfect* in  $D$  if every vertex of  $D$  is either the tail or the head of an arc of  $M$ . A *matched digraph* is a pair  $(D, M)$  where  $D$  is a digraph and  $M$  is a perfect matching of  $D$ .

In a digraph, we say that a vertex  $u$  *dominates* a vertex  $v$  if  $uv$  is an arc. An *out-star* is a digraph consisting in a vertex, called the *centre*, dominating all the others, called *spades*. A *galaxy* is a disjoint union of out-stars. For a digraph  $D$ , an *out-generator* of  $D$  is a vertex  $r$  such that  $D$  admits an out-branching rooted at  $r$ . In other words, there is a directed path from  $r$  to every other vertex of  $D$ .

1 If  $A$  is a directed path or cycle, and  $x, y$  are two vertices of  $A$ , then we denote by  $A[x, y]$  the  
2 directed subpath of  $A$  with origin  $x$  and terminus  $y$ .

3 **Drawing conventions:** In every figure depicting a backboned digraph  $(D, B)$ , the bold arcs are the  
4 arcs of  $B$ , the dashed arcs are the arcs not in  $B$ , and the edges are pairs of opposite arcs. Moreover, if  
5  $B$  is an out-branching, then the white vertex is its root.

### 6 3 Relations between BPRN-, BMRN- and BMRN\*-indices

According to the definitions, for every backboned digraph  $(D, B)$  we have the following:

$$\text{BMRN}(D, B) \leq \text{BPRN}(D, B) \quad \text{and} \quad \text{BMRN}(D, B) \leq \text{BMRN}^*(D, B).$$

7 Conversely, one can easily see that  $\text{BPRN}(D, B)$  cannot be bounded by a function of  $\text{BMRN}(D, B)$ .  
8 To see this, consider, for example, the directed out-star with  $k$  spades  $\vec{S}_k$ ; we clearly have  $\text{BMRN}(\vec{S}_k, \vec{S}_k) =$   
9 1 and  $\text{BPRN}(\vec{S}_k, \vec{S}_k) = k$ .

10 On the other hand, we show that  $\text{BMRN}^*(D, T)$  is bounded above by a function of  $\text{BMRN}(D, T)$ .  
11 The upper bound function we exhibit is actually best possible.

12 **Theorem 1.** *For every backboned digraph  $(D, B)$ , we have  $\text{BMRN}^*(D, B) \leq 2^{\text{BMRN}(D, B)} - 1$ .*

*Proof.* Let  $\phi$  be a BMRN-colouring of  $(D, B)$ . For every vertex  $v \in V(D)$ , let

$$f(v) = \{\phi(vw) \mid w \in N^+(v)\}.$$

13 Now, let  $g$  be the arc-colouring of  $B$  defined by  $g(xy) = f(x)$  for every arc  $xy \in A(B)$ . We claim  
14 that  $g$  is a BMRN\*-colouring of  $(D, B)$ :

- 15 • For every two arcs  $uv$  and  $vw$  of  $B$ , we have  $\phi(uv) \notin f(v)$  because  $\phi$  is a BMRN-colouring of  
16  $(D, B)$ . Hence  $f(u) \neq f(v)$ , and  $g(uv) \neq g(vw)$ . Thus type-1 constraints are satisfied.
- 17 • For every two arcs  $u_1v_1$  and  $u_2v_2$  of  $B$  such that  $u_1v_2 \in A(D)$ , we have  $\phi(u_2v_2) \notin f(u_1)$   
18 because  $\phi$  is a BMRN-colouring of  $(D, B)$ . Hence  $f(u_1) \neq f(u_2)$ , and  $g(u_1v_1) \neq g(u_2v_2)$ .  
19 Thus type-2 constraints are satisfied.
- 20 • By definition, any two arcs of  $B$  with the same tail are assigned the same colour by  $g$ . Hence  
21 type-3\* constraints are satisfied.

22 The conclusion is now obtained by noting that if  $\phi$  takes values in a set  $S$  of  $k$  colours, then  $g$   
23 takes values in a set of at most  $2^k - 1$  colours, namely the non-empty subsets of  $S$ .  $\square$

24 We now prove that the upper bound in Theorem 1 is best possible.

25 **Proposition 2.** *For every positive integer  $k$ , there exist a digraph  $D$  and a spanning galaxy  $B$  of  $D$   
26 such that  $\text{BMRN}(D, B) = k$  and  $\text{BMRN}^*(D, B) = 2^k - 1$ .*

27 *Proof.* Let  $X$  be a set of  $k$  vertices. Let us construct a galaxy  $B$  as follows: for each non-empty  
28 subset  $U$  of  $X$ , we create an out-star  $S^+(U)$  with centre  $x_U$  and spades  $y_U(u)$  for all  $u \in U$ . Let now  
29  $D$  be the digraph obtained from  $B$  by adding arcs as follows. For every two elements  $u \neq t$  of  $X$ , and  
30 every subset  $T$  such that  $t \in T$ , we add the arc  $x_{\{u\}}y_T(t)$ . This ensures that, in any BMRN\*-colouring  
31 of  $(D, B)$ , the arcs  $x_{\{u\}}y_{\{u\}}(u)$  and  $x_Ty_T(t)$  get different colours. Then, for every pair  $\{U, T\}$  of  
32 distinct subsets of  $X$  with  $|U| \leq |T|$ , choose a vertex  $t$  in  $T \setminus U$ , and add the arc  $x_Uy_T(t)$  (if not

1 already present). Observe that this arc implies that, in any BMRN\*-colouring of  $(D, B)$ , the arcs of  $S^+(U)$  and the arcs of  $S^+(T)$  are assigned different colours. Consequently,  $\text{BMRN}^*(D, B) \geq 2^k - 1$ .

2  
3 Now consider the arc-colouring  $\phi$  of  $B$  where every arc  $x_U y_U(u)$  is assigned colour  $u$ . One easily  
4 checks that  $\phi$  is a  $k$ -BMRN-colouring of  $(D, B)$ . Indeed, the arcs of  $A(D) \setminus A(B)$  are of the form  
5  $x_U y_T(t)$  with  $t \in T \setminus U$ . According to how  $\phi$  was defined, this means that the arcs with tail  $x_U$  are  
6 assigned, by  $\phi$ , a colour in  $U$ , while the arc  $x_T y_T(t)$  is assigned colour  $t$ . Hence all type-2 constraints  
7 are satisfied, and, consequently,  $\text{BMRN}(D, B) \leq k$ . Furthermore, we actually have equality, as every  
8 two arcs  $x_{\{u\}} y_{\{u\}}(u)$ ,  $x_{\{t\}} y_{\{t\}}(t)$  for  $u \neq t$  must receive distinct colours, due to the arc  $x_{\{u\}} y_{\{t\}}(t)$ .  
9 So all  $k$  arcs of the form  $x_{\{u\}} y_{\{u\}}(u)$  ( $u \in U$ ) must be coloured differently.

10 From Theorem 1, we now get that  $\text{BMRN}^*(D, B) = 2^k - 1$ . □

11 The proof of Proposition 2 can be easily modified to hold for spanned digraphs  $(D, B)$ .

12 **Proposition 3.** *For every positive integer  $k$ , there exists a spanned digraph  $(D, T)$  such that  $\text{BMRN}(D, T) =$   
13  $k$  and  $\text{BMRN}^*(D, T) = 2^k - 1$ .*

14 *Proof.* We start from the pair  $(D, B)$  constructed in the proof of Proposition 2. We construct a  
15 spanned digraph  $(D, T)$  as follows. We take an out-branching  $B^+$  with vertex set  $\{z_U \mid U \subseteq$   
16  $X, |U| \geq 1\}$  and root  $z_X$ . For every arc  $a_U a_T$  of  $B^+$ , we add a directed path of length 3 with  
17 initial vertex a spade of  $S^+(U)$  (i.e.,  $y_U(u)$  for some  $u \in U$ ), terminal vertex  $x_T$ , and new (private)  
18 internal vertices. This forms our out-branching  $T$ . One then easily checks that  $\text{BMRN}(D, T) =$   
19  $\text{BMRN}(D, B) = k$  and  $\text{BMRN}^*(D, T) = \text{BMRN}^*(D, B) = 2^k - 1$ . □

## 20 4 Bounds on BPRN( $D, B$ ), BMRN( $D, B$ ) and BMRN\*( $D, B$ )

21 BPRN-, BMRN- and BMRN\*-colourings of backbone digraphs can be viewed as particular cases of  
22 the classical colouring of graphs, where one aims at assigning colours to the vertices so that a *proper*  
23 *colouring* is attained, i.e., a colouring where no two adjacent vertices have the same colour. Let us  
24 now explain this in detail, for a backbone digraph  $(D, B)$ .

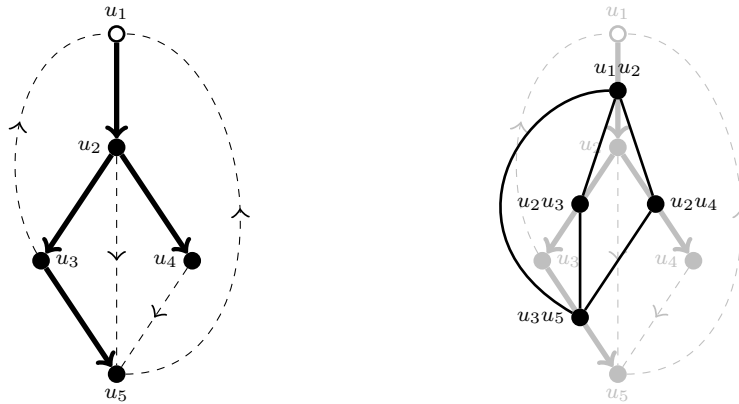


Figure 1: A spanned digraph  $(D, T)$  (left), and the BMRN-constraint graph  $C_{\text{BMRN}}(D, T)$  (right).

25 The *BMRN-constraint graph* of  $(D, B)$  (see Figure 1 for an example) is the undirected graph  
26  $C_{\text{BMRN}}(D, B)$  defined as follows:

- 27 •  $V(C_{\text{BMRN}}(D, B)) = A(B)$ ;

- There is an edge in  $C_{\text{BMRN}}(D, B)$  between two vertices corresponding to arcs, say,  $u_1v_1$  and  $u_2v_2$  if either  $u_1 = v_2$  or  $u_2 = v_1$  (type-1 constraint), or  $u_1v_2 \in A(D)$  or  $u_2v_1 \in A(D)$  (type-2 constraint).

By construction, there is a trivial one-to-one correspondence between the BMRN-colourings of  $(D, B)$  and the proper colourings of  $C_{\text{BMRN}}(D, B)$ . In particular, we have  $\text{BMRN}(D, B) = \chi(C_{\text{BMRN}}(D, B))$ , where  $\chi$  is the usual chromatic number of undirected graphs.

The *BPRN-constraint graph* of  $(D, B)$  is the graph  $C_{\text{BPRN}}(D, B)$  defined as follows:

- $V(C_{\text{BPRN}}(D, B)) = A(B)$ ;
- There is an edge in  $C_{\text{BPRN}}(D, B)$  between two vertices corresponding to arcs, say,  $u_1v_1$  and  $u_2v_2$  if either  $u_1 = v_2$  or  $u_2 = v_1$  (type-1 constraint), or  $u_1v_2 \in A(D)$  or  $u_2v_1 \in A(D)$  (type-2 constraint), or  $u_1 = u_2$  (type-3 constraint).

By construction, there is a trivial one-to-one correspondence between the BPRN-colourings of  $(D, B)$  and the proper colourings of  $C_{\text{BPRN}}(D, B)$ , and  $\text{BPRN}(D, B) = \chi(C_{\text{BPRN}}(D, B))$ .

The *BMRN\*-constraint graph* of  $(D, B)$ , denoted by  $C_{\text{BMRN}^*}(D, B)$ , can be obtained from  $C_{\text{BMRN}}(D, B)$  by identifying all vertices corresponding to arcs of  $B$  with same tail. It can also be defined as follows:

- $V(C_{\text{BMRN}^*}(D, B)) = V(D)$ ;
- $u_1u_2$  is an edge in  $C_{\text{BMRN}^*}(D, B)$  if there are two arcs  $u_1v_1$  and  $u_2v_2$  of  $B$  such that either  $u_1 = v_2$  or  $u_2 = v_1$  (type-1 constraint), or  $u_1v_2 \in A(D)$  or  $u_2v_1 \in A(D)$  (type-2 constraint).

Again, by construction, there is a one-to-one correspondence between the BMRN\*-colourings of  $(D, B)$  and the proper colourings of  $C_{\text{BMRN}^*}(D, B)$ . So  $\text{BMRN}^*(D, B) = \chi(C_{\text{BMRN}^*}(D, B))$ .

In this section, we give general bounds on  $\text{BPRN}(D, B)$ ,  $\text{BMRN}(D, B)$  and  $\text{BMRN}^*(D, B)$ , mainly by studying properties of  $C_{\text{BPRN}}(D, B)$ ,  $C_{\text{BMRN}}(D, B)$  and  $C_{\text{BMRN}^*}(D, B)$ .

## 4.1 General upper bounds

### 4.1.1 BMRN\*-colouring

Observe that if  $u_1u_2$  is an edge of  $C_{\text{BMRN}^*}(D, B)$ , then  $u_1$  and  $u_2$  are at distance at most 2 in  $D$ . This means that  $C_{\text{BMRN}^*}(D, B)$  is a subgraph of  $\tilde{U}(D)^2$ . Hence

$$\text{BMRN}^*(D, B) \leq \chi(\tilde{U}(D)^2) \leq \Delta(\tilde{U}(D)^2) + 1 \leq \Delta(\tilde{U}(D))^2 + 1 \leq \Delta(D)^2 + 1,$$

which provides a first upper bound on  $\text{BMRN}^*(D, B)$  (and, hence, on  $\text{BMRN}(D, B)$ ) in terms of  $\Delta(D)$ .

We now prove a better upper bound on  $\text{BMRN}^*(D, T)$  for spanned digraphs  $(D, T)$ . It is obtained from a particular orientation of  $C_{\text{BMRN}^*}(D, T)$  and the following well-known lemma, whose short proof is given for sake of completeness.

**Lemma 4.** *For every digraph  $D$ , we have  $\chi(D) \leq 2\Delta^+(D) + 1$ .*

*Proof.* For every subdigraph  $H$  of  $D$ , we have

$$\sum_{v \in V(H)} d_H(v) = 2 \cdot |A(H)| = 2 \sum_{v \in V(H)} d_H^+(v) \leq 2 \cdot |V(H)| \cdot \Delta^+(H) \leq 2 \cdot |V(H)| \cdot \Delta^+(D).$$

Hence every subgraph  $H$  of  $D$  has a vertex of degree at most  $2\Delta^+(D)$ . In other words,  $D$  is  $2\Delta^+(D)$ -degenerate, and thus  $\chi(D) \leq 2\Delta^+(D) + 1$ .  $\square$



**Theorem 5.** For every spanned digraph  $(D, T)$ , we have

$$BMRN^*(D, T) \leq 2\Delta^+(D) + 3 \quad \text{and} \quad BMRN^*(D, T) \leq 2\Delta(D) + 1.$$

1 *Proof.* Let  $\vec{C}$  be the orientation of  $C_{BMRN^*}(D, T)$  defined by

- 2 •  $V(\vec{C}) = V(D)$ , and
- 3 •  $u_1$  dominates  $u_2$  in  $\vec{C}$  if either
  - 4 –  $u_2u_1 \in A(T)$  (type-1 constraint), or
  - 5 – there exists  $v_1$  such that  $u_2v_1 \in A(T)$  and  $u_1v_1 \in A(D)$  (type-2 constraint).

For every  $u \in V(D)$ , we have  $d_{\vec{C}}^+(u) \leq d_T^-(u) + \sum_{v \in N_D^+(u)} d_T^-(v)$ . Now since  $T$  is an out-branching, we have  $d_T^-(x) \leq 1$  for every vertex  $x$ , so

$$d_{\vec{C}}^+(u) \leq d_T^-(u) + d_D^+(u) \leq d_D^+(u) + 1 \leq \Delta^+(D) + 1,$$

so  $\Delta^+(\vec{C}) \leq \Delta^+(D) + 1$ . Thus, by Lemma 4, we get

$$BMRN^*(D, T) = \chi(\vec{C}) \leq 2\Delta^+(\vec{C}) + 1 \leq 2\Delta^+(D) + 3.$$

6 Similarly  $d_{\vec{C}}^+(u) \leq d_T^-(u) + d_D^+(u) - 1 \leq \Delta(D) - 1$ , and so we get  $BMRN^*(D, T) \leq 2\Delta(D) - 1$ .  $\square$

#### 7 4.1.2 BPRN-colouring

8 As in the previous section, using an orientation of the BPRN-constraint graph  $C_{BPRN}(D, T)$  (for some  
9 spanned digraph  $(D, T)$ ), we now establish upper bounds on  $BPRN(D, T)$  in terms of  $\Delta^+(D)$  and  
10  $\Delta(D)$ .

**Theorem 6.** For every spanned digraph  $(D, T)$ , we have

$$BPRN(D, T) \leq 2\Delta^+(D) + 1 \quad \text{and} \quad BPRN(D, T) \leq 2\Delta(D) - 1.$$

11 *Proof.* Let  $\vec{C}$  be the orientation of  $C_{BPRN}(D, T)$  defined by

- 12 •  $V(\vec{C}) = A(T)$ , and
- 13 •  $u_1v_1$  dominates  $u_2v_2$  in  $\vec{C}$  if either
  - 14 –  $v_2 = u_1$  (type-1 constraint),
  - 15 –  $u_1v_2 \in A(D) \setminus A(T)$  (type-2 constraint), or
  - 16 –  $u_1 = u_2$  that is  $u_1v_2 \in A(T)$  (type-3 constraint).

For every arc  $uv \in A(T)$ , we have  $d_{\vec{C}}^+(uv) \leq d_T^-(u) + \sum_{w \in N_D^+(u) \setminus \{v\}} d_T^-(w)$ . Since  $T$  is an out-branching, we have

$$d_{\vec{C}}^+(uv) \leq d_T^-(u) + d_D^+(u) - 1 \leq d_D^+(u) \leq \Delta^+(D),$$

so  $\Delta^+(\vec{C}) \leq \Delta^+(D)$ . Thus, by Lemma 4, we get

$$BPRN(D, T) = \chi(\vec{C}) \leq 2\Delta^+(\vec{C}) + 1 \leq 2\Delta^+(D) + 1.$$

17 Similarly  $d_{\vec{C}}^+(uv) \leq d_T^-(u) + d_D^+(u) - 1 \leq \Delta(D) - 1$ , and so we get  $BPRN(D, T) \leq 2\Delta(D) - 1$ .  $\square$

### 4.1.3 BMRN-colouring

Since  $\text{BMRN}(D, B) \leq \text{BPRN}(D, B)$  holds for every backboned digraph  $(D, B)$ , we directly get the following from Theorem 6.

**Corollary 7.** *For every spanned digraph  $(D, T)$ , we have*

$$\text{BMRN}(D, T) \leq 2\Delta^+(D) + 1 \quad \text{and} \quad \text{BMRN}(D, T) \leq 2\Delta(D) - 1.$$

### 4.1.4 Tightness of the bounds

The bounds of Theorems 5 and 6 and Corollary 7 are tight up to a small additive factor, as shown in the following proposition.

**Proposition 8.** *For every  $k \geq 2$ , there exists a spanned digraph  $(D_k, T_k)$  with  $\Delta^+(D_k) = k$ ,  $\Delta(D_k) = k + 1$ , and  $\text{BMRN}(D_k, T_k) = 2k - 1$ .*

*Proof.* Let  $(D_k, T_k)$  be the spanned digraph defined as follows (see Figure 2):

- $V(D_k) = V(T_k) = \bigcup_{i=1}^{2k-1} \{t_i, u_i, v_i\}$ ,
- $A(T_k) = \bigcup_{i=1}^{2k-2} \{t_i t_{i+1}\} \cup \bigcup_{i=1}^{2k-1} \{t_i u_i, u_i v_i\}$ , and
- $A(D_k) = A(T_k) \cup \bigcup_{i=1}^{2k-1} \{u_i v_{i+j} \mid 1 \leq j \leq k-1\}$  (indices are modulo  $2k-1$ ).

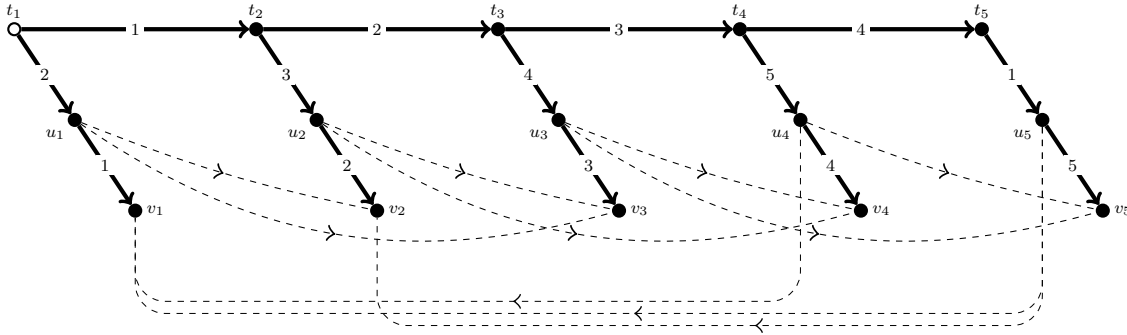


Figure 2: The spanned digraph  $(D_3, T_3)$  described in the proof of Proposition 8.

Clearly  $\Delta^+(D_k) = k$ , and  $\Delta(D_k) = k + 1$ . Now, in every BMRN-colouring of  $(D_k, T_k)$ , every two of the  $u_i v_i$ 's must receive distinct colours because they are involved in a type-2 constraint. Consequently,  $\text{BMRN}(D_k, T_k) \geq 2k - 1$ . It is easy to check that  $(2k - 1)$ -BMRN-colourings of  $(D_k, T_k)$  actually exist (by, e.g., generalizing the colouring scheme of  $(D_3, T_3)$  depicted in Figure 2), so  $\text{BMRN}(D_k, T_k) = 2k - 1$ .  $\square$

**Remark 9.** Note that in the construction described in the proof of Proposition 8, we have  $\Delta^+(T_k) = 2$ . Therefore, Theorems 5 and 6 and Corollary 7 are tight in the sense that there is no pair  $(\epsilon, f)$  where  $\epsilon$  is a positive real and  $f$  a function such that  $\text{BMRN}(D, T) \leq (2 - \epsilon)\Delta^+(D) + f(\Delta^+(T))$  for any spanned digraph  $(D, T)$ .

1 The bounds of Theorems 5 and 6 and Corollary 7 are almost tight (as shown in Proposition 8),  
2 but not tight. If  $\Delta(D) = 1$ , then  $D$  and  $T$  have two vertices and one arc, so  $\text{BMRN}(D, T) =$   
3  $\text{BMRN}^*(D, T) = \text{BPRN}(D, T) = 1$ . If  $\Delta(D)^+ = 1$ , then  $D$  is a directed path or cycle and  $T$  is a  
4 directed path, in which case  $\text{BMRN}(D, T) = \text{BMRN}^*(D, T) = \text{BPRN}(D, T) = 2$  (if  $A(T) \geq 2$ ).  
5 If  $\Delta(D) = 2$ , then  $D$  is an oriented path or cycle and  $T$  is an oriented path, and one easily sees  
6 that  $\text{BMRN}(D, T), \text{BMRN}^*(D, T), \text{BPRN}(D, T) \leq 3$ . Moreover the upper bound 3 is attained by  
7 the spanned digraph  $(D, T)$  with  $V(D) = V(T) = \{v_1, v_2, v_3, v_4\}$ ,  $A(T) = \{v_1v_2, v_2v_3, v_3v_4\}$  and  
8  $A(D) = A(T) \cup \{v_1v_4\}$ .

9 For larger values of  $\Delta(D)$  and  $\Delta^+(D)$ , one could wonder whether the bounds of Theorems 5 and  
10 6 and Corollary 7 can be improved. We thus address the following questions:

11 **Question 10.**

- 12 • What is the maximum value  $M_{\Delta^+}(k)$  (resp.  $M_{\Delta}(k)$ ) of  $\text{BMRN}(D, T)$  over all spanned di-  
13 graphs  $(D, T)$  with  $\Delta^+(D) \leq k$  (resp.  $\Delta(D) \leq k$ )?
- 14 • What is the maximum value  $M_{\Delta^+}^*(k)$  (resp.  $M_{\Delta}^*(k)$ ) of  $\text{BMRN}^*(D, T)$  over all spanned di-  
15 graphs  $(D, T)$  with  $\Delta^+(D) \leq k$  (resp.  $\Delta(D) \leq k$ )?
- 16 • What is the maximum value  $P_{\Delta^+}(k)$  (resp.  $P_{\Delta}(k)$ ) of  $\text{BPRN}(D, T)$  over all spanned digraphs  
17  $(D, T)$  with  $\Delta^+(D) \leq k$  (resp.  $\Delta(D) \leq k$ )?

18 In the rest of this section, we make a first step towards these questions by studying  $M_{\Delta}(3)$  and  $M_{\Delta}^*(3)$ .  
19 Corollary 7 and Proposition 8 yield  $3 \leq M_{\Delta}(3) \leq 5$ . We show that  $M_{\Delta}(3) = M_{\Delta}^*(3) = 4$ .

20 Figure 3 shows a spanned digraph  $(D, T)$  such that  $\Delta(D) = 3$  and  $\text{BMRN}(D, T) = 4$  (one  
21 easily checks that its BMRN-constraint graph is  $K_4$ , the complete graph on four vertices). Hence  
22  $M_{\Delta}^*(3) \geq M_{\Delta}(3) \geq 4$ .

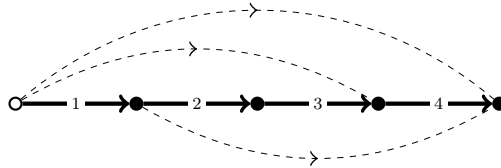


Figure 3: A spanned digraph  $(D, T)$  with  $\Delta(D) = 3$  and  $\text{BMRN}(D, T) = 4$ .

23 We now prove that  $M_{\Delta}(3) \leq M_{\Delta}^*(3) \leq 4$ .

24 **Theorem 11.** *For every subcubic spanned digraph  $(D, T)$ , we have  $\text{BMRN}^*(D, T) \leq 4$ .*

25 *Proof.* Let  $D$  be a subcubic digraph and  $T$  an out-branching of  $D$ . Let  $D' = D \setminus A(T)$ . We partition  
26  $V(D) = V(T)$  into four sets according to their in- and out-degrees in  $T$ . Recall that the *root*  $r$  is  
27 the unique vertex such that  $d_T^-(r) = 0$ , and that the *leaves* are the vertices with out-degree 0 in  $T$ . A  
28 vertex  $v$  is *flat* if  $d_T^+(v) = d_T^-(v) = 1$  and it is *branching* if  $d_T^-(v) = 1$  and  $d_T^+(v) = 2$ . If  $u$  is a flat  
29 vertex, then we denote by  $u^+$  its out-neighbour in  $T$ .

30 To prove that  $\text{BMRN}^*(D, T) \leq 4$ , we shall prove that  $C^* = C_{\text{BMRN}^*}(D, T)$  is 3-degenerate, and  
31 thus 4-colourable.

32 Suppose for a contradiction that  $C^*$  is not 3-degenerate. Then it has a subgraph  $H$  such that  
33  $\delta(H) \geq 4$ . The graph  $H$  contains no leaves of  $T$  because they are isolated vertices in  $C^*$ .

34 Consider a flat vertex  $u$  in  $V(H)$ . The only possibility for it to have degree 4 in  $C^*$  is that there  
35 exist three distinct vertices  $v, u_1, u_2$  in  $V(H)$  such that there exists  $w$  such that  $vw \in A(T)$  and

1  $uw \in A(D')$ ,  $u_1u^+ \in A(D')$ , and either  $u_2u^+ \in A(D')$  or  $u_2 = u^+$ . In the latter case, note that  $u_2$   
2 has degree at most 3 in  $C^*$ ; so let us suppose the first situation occurs. Note that  $u_1$  and  $u_2$  are either  
3 flat vertices or the root. Therefore if  $u$  is a flat vertex in  $H$ , then  $u^+$  is a leaf which is the tail of two  
4 arcs  $u_1u^+$  and  $u_2u^+$  of  $A(D')$  (with  $u_1, u_2$  either flat vertices or the root).

5 Let  $U$  be the set of flat vertices in  $H$  and let  $U^+$  be the set of leaves dominated by a vertex of  $U$   
6 in  $T$ . We have  $|U| = |U^+|$ . Moreover, in  $D'$  each vertex of  $U^+$  has two in-neighbours in  $U \cup \{r\}$ ,  
7 while every vertex of  $U$  has at most one out-neighbour of  $U^+$  and  $r$  has at most two out-neighbours  
8 in  $U^+$ . Hence  $2|U| = 2|U^+| \leq |U| + 2$ , so  $|U| \leq 2$ . Furthermore, if  $|U| = 2$ , say  $U = \{u_1, u_2\}$ , then  
9  $u_1u_2^+, u_2u_1^+ \in A(D')$  and so  $u_1$  (and  $u_2$  as well) has degree 3 in  $C^*$ , a contradiction; and  $U = \{u\}$   
10 is not possible because  $u^+$  must have an in-neighbour in  $U \setminus \{u\}$ . Therefore  $U = \emptyset$ .

11 Now  $V(H)$  contains only branching vertices and possibly the root  $r$ . Thus all arcs in  $D'$  originate  
12 from  $r$ . Let  $v$  be a vertex in  $V(H)$  such that its out-neighbours in  $T$  are not in  $H$ ; recall that a such  
13 vertex exists since all leaves of  $T$  are not in  $H$ . Then  $v$  can only be adjacent in  $H$  to  $r$ , and its  
14 in-neighbour in  $T$ , a contradiction to the fact that  $\delta(H) \geq 4$ .  $\square$

15 We believe that a similar result holds for BPRN-colourings of spanned digraphs with maximum  
16 degree 3. Note in particular that the spanned digraph  $(D, T)$  in Figure 3 verifies  $\text{BPRN}(D, T) \leq 4$ .

17 **Question 12.** Does every subcubic spanned digraph  $(D, T)$  satisfy  $\text{BPRN}(D, T) \leq 4$ ?

## 18 4.2 Upper bounds for some families of digraphs

### 19 4.2.1 Minor-closed families of digraphs

20 We start off by pointing out the following obvious result for  $\text{BMRN}^*$ -colouring.

**Lemma 13.** *Let  $D$  be a digraph and  $B_1$  and  $B_2$  be two subdigraphs of  $D$ . Then*

$$\text{BMRN}^*(D, B_1 \cup B_2) \leq \text{BMRN}^*(D, B_1) + \text{BMRN}^*(D, B_2).$$

21 In view of Lemma 13, to get upper bounds on  $\text{BMRN}^*(D, T)$  for spanned digraphs  $(D, T)$ , it  
22 might be interesting to get upper bounds on  $\text{BMRN}^*(D, S)$  when  $S$  is a galaxy. We now use this  
23 approach for minor-closed families of backboned digraphs. For any class  $\mathcal{F}$  of digraphs, we define  
24  $\chi(\mathcal{F}) = \max\{\chi(D) \mid D \in \mathcal{F}\}$ .

25 **Theorem 14.** *Let  $\mathcal{F}$  be a minor-closed family of digraphs. If  $D \in \mathcal{F}$  and  $S$  is a galaxy in  $D$ , then*  
26  $\text{BMRN}^*(D, S) \leq \chi(\mathcal{F})$ .

27 *Proof.* Consider the  $\text{BMRN}^*$ -constraint graph  $C_{\text{BMRN}^*}(D, S)$ . One easily sees that it is the minor of  
28  $D$  obtained by contracting the arcs of  $S$ . Hence  $\text{BMRN}^*(D, S) = \chi(C_{\text{BMRN}^*}(D, S)) \leq \chi(\mathcal{F})$ .  $\square$

29 Theorem 14 and Lemma 13 immediately imply the following.

30 **Corollary 15.** *Let  $\mathcal{F}$  be a minor-closed family of digraphs. If  $(D, B)$  is a backboned digraph with*  
31  $D \in \mathcal{F}$ , *then  $\text{BMRN}^*(D, B) \leq \chi(B)\chi(\mathcal{F})$ . In particular, for every spanned digraph  $(D, T)$  where*  
32  $D \in \mathcal{F}$ , *we have  $\text{BMRN}^*(D, T) \leq 2\chi(\mathcal{F})$ .*

## 33 4.3 Upper bounds for planar spanned digraphs

34 Corollary 15 and the Four-Colour Theorem [1, 2, 3] yield the following.

**Corollary 16.** *For every planar spanned digraph  $(D, T)$ , we have*

$$\text{BMRN}(D, T) \leq \text{BMRN}^*(D, T) \leq 8.$$

1 There exist planar spanned digraphs  $(D, T)$  verifying  $\text{BMRN}(D, T) = 7$ . One such example  
 2 is given in Figure 4. One easily checks that the BMRN-constraint graph of this example is  $K_7$ , the  
 3 complete graph on seven vertices.

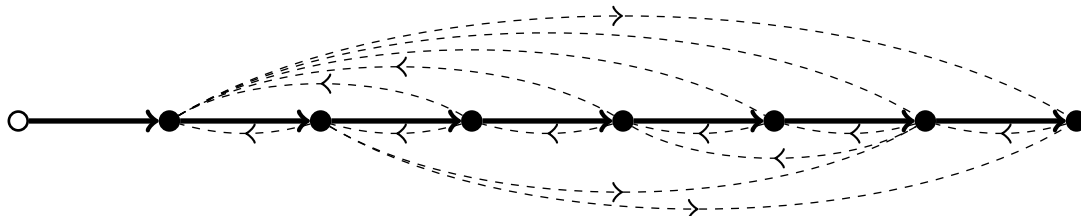


Figure 4: A planar spanned digraph  $(D, T)$  with  $\text{BMRN}(D, T) = 7$ .

4 However, we still do not know whether all planar spanned digraphs have BMRN- or BMRN\*-  
 5 chromatic index at most 7.

6 **Question 17.** Is it true that, for every planar spanned digraph  $(D, T)$ , we have:

7 (a)  $\text{BMRN}(D, T) \leq 7$ ?

8 (b)  $\text{BMRN}^*(D, T) \leq 7$ ?

9 In the next section, we manage to answer such questions for outerplanar spanned digraphs.

#### 10 4.4 Upper bounds for outerplanar spanned digraphs

11 Since outerplanar graphs have chromatic number at most 3, Corollary 15 yields that  $\text{BMRN}^*(D, T) \leq$   
 12 6 for every outerplanar spanned digraph  $(D, T)$ . The aim of this section is to improve this bound and  
 13 show the following theorem.

**Theorem 18.** For every outerplanar spanned digraph  $(D, T)$ , we have

$$\text{BMRN}(D, T) \leq \text{BMRN}^*(D, T) \leq 5.$$

14 The bound 5 of Theorem 18 is best possible as shown by the example depicted in Figure 5. One  
 15 easily sees that, for this spanned digraph  $(D, T)$ , we have  $\text{BMRN}(D, T) \geq 5$  because its BMRN-  
 16 constraint graph contains a  $K_5$ .

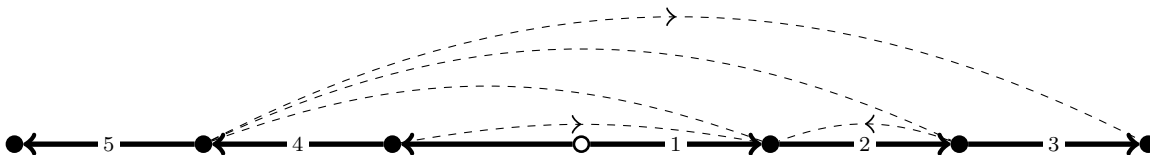


Figure 5: An outerplanar spanned digraph  $(D, T)$  with  $\text{BMRN}(D, T) \geq 5$ .

17 A digraph  $D$  is *outerplanar-maximal* if it is outerplanar and adding any new arc to  $D$  results in  
 18 a non-outerplanar digraph. An outerplanar-maximal digraph is symmetric and its underlying simple  
 19 graph  $G = \tilde{U}(D)$  of  $D$  is 2-connected and *inner triangulated* (i.e., all faces except the outer one are  
 20 3-faces).

21 Let  $(D, T)$  be an outerplanar spanned digraph. A *chord* is an arc of  $D$  which is not incident to the  
 22 outer face, and a *T-chord* is a chord in  $A(T)$ . The proof of Theorem 18 is by induction on the number

1 of  $T$ -chords. We first prove the following which corresponds to the basis of the induction, that is the  
 2 case when there is no  $T$ -chord.

3 **Lemma 19.** *Let  $(D, T)$  be a 2-connected outerplanar spanned digraph such that all arcs of  $T$  are on  
 4 the outer face of  $D$ . Then  $BMRN^*(D, T) \leq 5$ .*

5 *Proof.* By considering a minimum counterexample (i.e., with the minimum number of vertices). Free  
 6 to add arcs, we may assume that  $D$  is outerplanar-maximal. Since  $D$  is a symmetric digraph, in what  
 7 follows we sometimes regard it as an undirected graph  $G$ , in which every pair of arcs  $\{uv, vu\}$  is  
 8 replaced becomes an edge  $uv$ .

9 Let us number the vertices of  $D$  by  $v_1, \dots, v_n$  so that  $v_i v_{i+1}$  or  $v_{i+1} v_i$  is an arc of  $T$  for all  
 10  $1 \leq i \leq n - 1$ . For  $1 \leq i \leq n - 1$ , let  $a_i$  be the arc of  $T$  between  $v_i$  and  $v_{i+1}$ . In other words,  
 11  $\{a_i\} = \{v_i v_{i+1}, v_{i+1} v_i\} \cap A(T)$ .

12 The *span* of an arc  $v_i v_j$  of  $D$  or an edge  $v_i v_j$  of  $G$  is  $|j - i|$ . In particular, all arcs of  $T$  have span  
 13 1. We denote by  $T[i, j]$  the subdipath of  $T$  induced by  $\{v_i, \dots, v_j\}$ . Let  $i_0$  be the index of the root  $r$   
 14 of  $T$  (i.e.,  $r = v_{i_0}$ ).

15 The *edge-tree* of  $(D, T)$  is the out-tree whose vertices are the edges of  $G$  and such that every edge  
 16  $v_i v_j$  (with  $i < j$ ) of span at least 2 dominates the two edges  $v_i v_k$  and  $v_k v_j$  such that  $i < k < j$  and  
 17  $v_i v_k v_j v_i$  is a 3-cycle in  $G$ . Observe that the root of the edge-tree of  $(D, T)$  is  $v_1 v_n$ , and its leaves are  
 18 the edges of span 1. Furthermore, the span of every edge that is not a leaf is the sum of the spans of  
 19 the two edges it dominates.

20 If  $v_i v_j$  is an edge of  $G$  and  $T[i, j]$  is a directed path, then the  $v_i v_j$ -*reduced spanned digraph* is the  
 21 spanned digraph  $(D_{i,j}, T_{i,j})$  defined as follows:

- 22 •  $D_{i,j}$  is obtained from  $D - \{v_{i+1}, \dots, v_{j-1}\}$  by adding a vertex  $x_{i,j}$  and the arcs of the two  
 23 directed cycles  $(v_i, x_{i,j}, v_i)$  and  $(v_j, x_{i,j}, v_j)$ .
- 24 • If  $T[i, j]$  is a directed path from  $v_i$  to  $v_j$  (resp. from  $v_j$  to  $v_i$ ), then  $T_{i,j}$  is obtained from  
 25  $T - \{v_{i+1}, \dots, v_{j-1}\}$  by adding a new vertex  $x_{i,j}$  and the arcs  $v_i x_{i,j}$  and  $x_{i,j} v_j$  (resp.  $v_j x_{i,j}$   
 26 and  $x_{i,j} v_i$ ).

27 In this case, we denote by  $b = b_{i,j}$  (resp.  $b' = b'_{i,j}$ ) the arc of  $T_{i,j}$  between  $v_i$  (resp.  $v_j$ ) and  $x_{i,j}$ .

28 If  $v_i v_j$  is an edge of  $G$  and  $T[i, j]$  is not a directed path, then the  $v_i v_j$ -*reduced spanned digraph*  
 29 is the spanned digraph  $(D_{i,j}, T_{i,j})$  defined as follows:

- 30 •  $D_{i,j}$  is obtained from  $D - \{v_{i+1}, \dots, v_{j-1}\}$  by adding two vertices  $x_{i,j}$  and  $y_{i,j}$  and the arcs of  
 31 the three directed cycles  $(v_i, x_{i,j}, v_i)$ ,  $(x_{i,j}, y_{i,j}, x_{i,j})$ , and  $(v_j, y_{i,j}, v_j)$ .
- 32 •  $T_{i,j}$  is obtained from  $T - \{v_{i+1}, \dots, v_{j-1}\}$  by adding the two vertices  $x_{i,j}$  and  $y_{i,j}$  and the arcs  
 33  $x_{i,j} v_i$ ,  $x_{i,j} y_{i,j}$ , and  $y_{i,j} v_j$ .

34 In that case, we denote by  $b = b_{i,j}$  (resp.  $b' = b'_{i,j}$ ) the arc  $x_{i,j} v_i$  (resp.  $y_{i,j} v_j$ ).

35 Observe moreover that if  $v_i v_j$  has span at least 3 or span 2 and  $T[i, j]$  is a directed path, then the  
 36  $v_i v_j$ -reduced spanned digraph has smaller order than  $(D, T)$ . Therefore by minimality, it admits a  
 37 5-BMRN\*-colouring  $\phi$ . Moreover, the colours assigned to  $a_{i-1}$ ,  $a_j$ ,  $b$  and  $b'$  are all distinct, except  
 38 possibly  $\phi(a_{i-1}) = \phi(a_j)$  when  $a_{i-1} = v_i v_{i-1}$  and  $a_j = v_j v_{j+1}$ .

39 The general idea of the proof is to show that there is an edge  $v_i v_j$  such that any 5-BMRN\*-  
 40 colouring of the  $v_i v_j$ -reduced spanned digraph can be modified to get a 5-BMRN\*-colouring of  
 41  $(D, T)$ , which is a contradiction.

42 **Claim 19.1.** *If  $v_i v_{i+3}$  is an edge of span 3, then  $i < i_0 < i + 3$ .*

1 *Subproof.* Assume for a contradiction that  $G$  has an edge  $v_i v_{i+3}$  of span 3 with  $i_0 \leq i$  or  $i \geq i + 3$ .  
2 By symmetry, we may assume that  $i_0 \leq i$ . Hence  $T[i, i + 3]$  is a directed path from  $v_i$  to  $v_{i+3}$ .  
3 By minimality of  $(D, T)$ , the  $v_i v_{i+3}$ -reduced spanned digraph  $(D_{i,i+3}, T_{i,i+3})$  has a 5-BMRN\*-  
4 colouring  $\phi$ . Without loss of generality,  $\phi(b) = 2$ ,  $\phi(b') = 3$ ,  $\phi(a_{i-1}) \in \{1, 2\}$  (if  $a_{i-1}$  exists), and  
5  $\phi(a_{i+3}) \in \{1, 4\}$  (if  $a_{i+3}$  exists). Set  $\phi(a_i) = 2$ ,  $\phi(a_{i+1}) = 5$ , and  $\phi(a_{i+2}) = 3$ . One easily checks  
6 that this yields a 5-BMRN\*-colouring of  $(D, T)$ , a contradiction.  $\diamond$

7 **Claim 19.2.** *If  $v_i v_{i+4}$  is an edge of span 4 and  $v_i v_{i+2}$  and  $v_{i+2} v_{i+4}$  are edges, then  $i_0 < i$  or*  
8  *$i_0 > i + 4$ .*

9 *Subproof.* Assume for a contradiction that  $v_i v_{i+4}$ ,  $v_i v_{i+2}$  and  $v_{i+2} v_{i+4}$  are edges and  $i \leq i_0 \leq i + 4$ .  
10 By symmetry, we may assume  $i_0 \in \{i, i + 1, i + 2\}$ . By minimality of  $(D, T)$ , the  $v_i v_{i+4}$ -reduced  
11 spanned digraph has a 5-BMRN\*-colouring  $\phi$ . Without loss of generality,  $\phi(a_1) = 1$  (if  $a_{i-1}$  exists),  
12  $\phi(b) = 2$  if  $i_0 \neq i$  and  $\phi(b) = 1$  if  $i_0 = i$ ,  $\phi(b') = 3$ , and  $\phi(a_{i+3}) \in \{1, 4\}$  (if  $a_{i+3}$  exists).  
13 Set  $\phi(a_i) = 1$  if  $i_0 = i$  and  $\phi(a_i) = 2$  if  $i_0 \in \{i + 1, i + 2\}$ ,  $\phi(a_{i+1}) = 2$  if  $i_0 \in \{i, i + 1\}$   
14 and  $\phi(a_{i+1}) = 5$  if  $i_0 = i + 2$ ,  $\phi(a_{i+2}) = 5$  and  $\phi(a_{i+3}) = 3$ . This gives a 5-BMRN\*-colouring of  
15  $(D, T)$ , a contradiction.  $\diamond$

16 Consider a *deepest* edge  $e$  of span 2 in the edge-tree of  $(D, T)$ , that is an edge with longest  
17 distance from  $v_1 v_n$  in the edge-tree. Set  $e_1 = v_i v_{i+2}$ . Let  $e_2$  be the edge that dominates  $e_1$  in the  
18 edge-tree, and let  $e'_1$  be the second edge that is dominated by  $e_2$ . Since  $e_1$  is a deepest edge of span 2,  
19 necessarily  $e'_1$  is an edge of span 1 or 2, for otherwise the branch of the edge-tree spanned at  $e'_1$  would  
20 contain an edge of span 2 deeper than  $e_1$ . Therefore the span of  $e_2$ , which is the sum of the spans of  
21  $e_1$  and  $e'_1$ , is either 3 or 4. Without loss of generality, either  $e_2 = v_i v_{i+3}$  or  $e_2 = v_i v_{i+4}$  and  $v_{i+2} v_{i+4}$   
22 is an edge.

23 Case 1:  $e_2 = v_i v_{i+3}$ .

24 Then  $i_0 \in \{i + 1, i + 2\}$  by Claim 19.1. Let  $e_3 = v_k v_l$  be the edge that dominates  $e_2$  in the  
25 edge-tree, and let  $e'_2$  be the second edge dominated by  $e_3$ . Since  $e_1$  was the deepest edge of span  
26 2,  $e'_2$  has span at most 4. If  $e'_2$  has span 4, then swapping the names of  $e_2$  and  $e'_2$  we are in Case 2.  
27 If  $e'_2$  has span 3, then  $e'_2$  contradicts Claim 19.1. Hence  $e'_2$  has span 1 or 2. Henceforth, we must  
28 be in one of the cases below. For each of them, we take a 5-BMRN\*-colouring  $\phi$  of the  $e_3$ -reduced  
29 spanned digraph, which exists by minimality of  $(D, T)$ . Without loss of generality, we may assume  
30 that  $\phi(a_{k-1}) = 1$  (if  $a_{k-1}$  exists),  $\phi(b) = 2$ ,  $\phi(b') = 3$ , and  $\phi(a_l) \in \{1, 4\}$  (if  $a_l$  exists). We now  
31 show for each case how to derive a 5-BMRN\*-colouring of  $(D, T)$ , which is a contradiction.

- 32 •  $e_3 = v_i v_{i+4}$ .  
33 Set  $\phi(a_i) = 2$ ,  $\phi(a_{i+1}) = 2$  if  $i_0 = i + 1$  and  $\phi(a_{i+1}) = 5$  if  $i_0 = i + 2$ ,  $\phi(a_{i+2}) = 5$ , and  
34  $\phi(a_{i+3}) = 3$ .
- 35 •  $e_3 = v_{i-1} v_{i+3}$ .  
36 Set  $\phi(a_{i-1}) = 2$ ,  $\phi(a_i) = 5$ ,  $\phi(a_{i+1}) = 5$  if  $i_0 = i + 1$  and  $\phi(a_{i+1}) = 3$  if  $i_0 = i + 2$ , and  
37  $\phi(a_{i+2}) = 3$ .
- 38 •  $e_3 = v_i v_{i+5}$ .  
39 Set  $\phi(a_i) = 2$ ,  $\phi(a_{i+1}) = 2$  if  $i_0 = i + 1$  and  $\phi(a_{i+1}) = 5$  if  $i_0 = i + 2$ ,  $\phi(a_{i+2}) = 5$ ,  $\phi(a_{i+3})$   
40 is a colour of  $\{1, 4\} \setminus \{\phi(a_{i+5})\}$ , and  $\phi(a_{i+4}) = 3$ .
- 41 •  $e_3 = v_{i-2} v_{i+4}$ .  
42 Set  $\phi(a_{i-2}) = 2$ ,  $\phi(a_{i-1}) = 4$ ,  $\phi(a_i) = 5$ ,  $\phi(a_{i+1}) = 5$  if  $i_0 = i + 1$  and  $\phi(a_{i+2}) = 3$  if  
43  $i_0 = i + 2$ , and  $\phi(a_{i+3}) = 3$ .

1 Case 2:  $e_2 = v_i v_{i+4}$  and  $v_{i+2} v_{i+4}$  is an edge.

2 By Claim 19.2, and by symmetry, we may assume that  $i_0 < i$ . Let  $e_3 = v_k v_l$  be the edge that  
3 dominates  $e_2$  in the edge-tree, and let  $e'_2$  be the second edge dominated by  $e_3$ . Since  $e_1$  was the  
4 deepest edge of span 2,  $e'_2$  has span at most 4. Furthermore if  $e'_2$  has span 4, then it dominates two  
5 edges of span 2.

6 For each of them, we take a 5-BMRN\*-colouring  $\phi$  of the  $e_3$ -reduced spanned digraph, which  
7 exists by minimality of  $(D, T)$ . Without loss of generality, we may assume that  $\phi(a_{k-1}) \in \{1, 2\}$  (if  
8  $a_{k-1}$  exists),  $\phi(b) = 2$ ,  $\phi(b') = 3$ , and  $\phi(a_l) \in \{1, 4\}$  (if  $a_l$  exists). Moreover  $\phi(a_{k-1}) = 2$  only if  
9  $i_0 = k$  and  $\phi(a_l)$  only if  $k \leq i_0 \leq l$ .

10 We now show for each case how to derive a 5-BMRN\*-colouring of  $(D, T)$ , which is a contradic-  
11 tion. We first consider the cases when  $k = i$ . In those cases,  $\phi(a_l) = 4$  because  $i_0 < i$ .

12 (a)  $e_3 = v_i v_{i+5}$ .

13 Set  $\phi(a_i) = 2$ ,  $\phi(a_{i+1}) = 4$ ,  $\phi(a_{i+2}) = 5$ ,  $\phi(a_{i+3}) = 1$ , and  $\phi(a_{i+4}) = 3$ .

14 (b)  $e_3 = v_i v_{i+6}$ .

15 Set  $\phi(a_i) = 2$ ,  $\phi(a_{i+1}) = 3$ ,  $\phi(a_{i+2}) = 4$ ,  $\phi(a_{i+3}) = 1$ ,  $\phi(a_{i+4}) = 5$ , and  $\phi(a_{i+5}) = 3$ .

16 (c)  $e_3 = v_i v_{i+7}$ .

17 In this case,  $e'_2 = v_{i+4} v_{i+7}$  and contradicts Claim 19.1.

18 (d)  $e_3 = v_i v_{i+8}$ .

19 In this case,  $v_{i+4} v_{i+6}$ ,  $v_{i+6} v_{i+8}$ , and  $v_{i+4} v_{i+8}$  are edges. Then, set  $\phi(a_i) = 2$ ,  $\phi(a_{i+1}) = 3$ ,  
20  $\phi(a_{i+2}) = 5$ ,  $\phi(a_{i+3}) = 1$ ,  $\phi(a_{i+4}) = 4$ ,  $\phi(a_{i+5}) = 5$ ,  $\phi(a_{i+6}) = 2$  and  $\phi(a_{i+7}) = 3$ .

21 We now consider the cases when  $l = i + 4$ .

22 (e)  $e_3 = v_{i-1} v_{i+4}$ .

23 Set  $\phi(a_{i-1}) = 2$ ,  $\phi(a_i) = 4$ ,  $\phi(a_{i+1}) = 5$ ,  $\phi(a_{i+2}) = 1$ , and  $\phi(a_{i+3}) = 3$ .

24 (f)  $e_3 = v_{i-2} v_{i+4}$ .

25 On the one hand, if  $i_0 \leq i - 2$ , then set  $\phi(a_{i-2}) = 2$ ,  $\phi(a_{i-1}) = 5$ ,  $\phi(a_i) = \phi(a_{i+4})$ ,  
26  $\{\phi(a_{i+1})\} = \{1, 4\} \setminus \{\phi(a_i)\}$ ,  $\phi(a_{i+2}) = 2$ , and  $\phi(a_{i+3}) = 3$ . On the other hand, if  $i_0 =$   
27  $i - 1$ , then set  $\phi(a_{i-2}) = 2$ ,  $\phi(a_{i-1}) = 2$ ,  $\phi(a_i) = \phi(a_{i+4})$ ,  $\{\phi(a_{i+1})\} = \{1, 4\} \setminus \{\phi(a_i)\}$ ,  
28  $\phi(a_{i+2}) = 5$ , and  $\phi(a_{i+3}) = 3$ .

29 (g)  $e_3 = v_{i-3} v_{i+4}$ .

30 In that case, by Claim 19.1,  $i_0 \in \{i - 2, i - 1\}$ . Then, set  $\phi(a_{i-3}) = 2$ ,  $\phi(a_{i-2}) = 2$  if  
31  $i_0 = i - 2$  and  $\phi(a_{i-2}) = 3$  if  $i_0 = i - 1$ ,  $\phi(a_{i-1}) = 3$ ,  $\phi(a_i) = 1$ ,  $\phi(a_{i+1}) = 2$ ,  $\phi(a_{i+2}) = 5$ ,  
32 and  $\phi(a_{i+3}) = 3$ .

33 (h)  $e_3 = v_i v_{i+8}$ .

34 We are in Case (d), with  $e_2$  and  $e'_2$  swapped. □

35 *Proof of Theorem 18.* By induction on the number of  $T$ -chords and next on the order of  $D$ . Without  
36 loss of generality, we may assume that  $D$  is outerplanar-maximal. Let  $C$  be the oriented cycle around  
37 the outer face.

38 If there is no  $T$ -chord, then we have the result by Lemma 19. Assume now that there is a  $T$ -chord  
39  $uv$ . This chord divides  $D$  into two outerplanar digraphs,  $D_1$  with outer face  $C_1 = C[u, v] \cup \{uv\}$  and  
40  $D_2$  with outer face  $C_2 = C[v, u] \cup \{uv\}$ . Without loss of generality, we may assume that the root  $r$   
41 of  $T$  is in  $D_1$ . For  $i = 1, 2$ , let  $D_i^*$  be the symmetric outerplanar graph obtained from  $D_i$  by adding  
42 the arc  $vu$  (if necessary), and let  $T_i = T \cap D_i$ . Observe that  $T_i$  is an out-branching of  $D_i^*$  and that the  
43 number of  $T_i$ -chords in  $(D_i^*, T_i)$  is less than the number of  $T$ -chords in  $(D, T)$ .



1 We distinguish several cases.

2 Case 1: There is an arc  $vw_1$  in  $T_1$ .

3 We distinguish two subcases depending on whether  $u$  is the root of  $T$  or not.

4 Subcase 1.1:  $u$  is the root of  $T$ .

5 By the induction hypothesis, there is a 5-BMRN\*-colouring  $\phi_i$  of each  $(D_i^*, T_i)$ . Free to per-  
6 mutate the colours, we may assume that  $\phi_1(uv) = \phi_2(uv)$  and that the arcs of  $T_2$  with tail  $v$   
7 (if some exist) are coloured (by  $\phi_2$ ) with  $\phi_1(vw_1)$ . One can easily check that the colouring  
8  $\phi$  of  $A(T)$ , defined by  $\phi(a) = \phi_1(a)$  if  $a \in A(T_1)$  and  $\phi(a) = \phi_2(a)$  if  $a \in A(T_2)$ , is a  
9 5-BMRN\*-colouring of  $(D, T)$ .

10 Subcase 1.2:  $u$  is not the root of  $T$ .

11 In that case,  $u$  has a unique in-neighbour  $t$  in  $T$ , which must be in  $T_1$ . Let  $D_2^+$  be the subdigraph  
12 of  $D$  induced by  $V(D_2) \cup \{t\}$ , and  $T_2^+$  be the out-branching of  $D_2^+$  obtained from  $T$  by adding  
13  $t$  and the arc  $tu$ . Observe that the number of  $T_2^+$ -chords in  $(D_2^+, T_2^+)$  is not greater than the  
14 number of  $T$ -chords in  $(D, T)$  and  $|V(D_2^+)| < |V(D)|$ . By the induction hypothesis, there are  
15 5-BMRN\*-colourings  $\phi_1$  of  $(D_1, T_1)$  and  $\phi_2^+$  of  $(D_2^+, T_2^+)$ . Free to permute the colours, we may  
16 assume that  $\phi_1(uv) = \phi_2^+(uv)$ ,  $\phi_1(tu) = \phi_2^+(tu)$  and that the arcs of  $T_2$  with tail  $v$  (if some  
17 exist) are coloured (by  $\phi_2^+$ ) with  $\phi_1(vw_1)$ . Note that this is possible because, in both colourings,  
18  $tu$  and the arcs with tail  $v$  receive distinct colours because  $vu$  is an arc. One can easily check  
19 that the colouring  $\phi$  of  $A(T)$ , defined by  $\phi(a) = \phi_1(a)$  if  $a \in A(T_1)$  and  $\phi(a) = \phi_2^+(a)$  if  
20  $a \in A(T_2^+)$ , is a 5-BMRN\*-colouring of  $(D, T)$ .

21 Case 2: There is no arc with tail  $v$  in  $T_1$ .

22 Since  $uv$  is a  $T$ -chord,  $|V(D_1)| \geq 3$  and so  $u$  has a unique in-neighbour  $t$  in  $T_1$ , which is also its  
23 unique in-neighbour in  $T$ .

24 Subcase 2.1:  $|V(D_1)| \geq 4$ .

25 We get a 5-BMRN\*-colouring of  $(D, T)$  exactly as in Subcase 1.2.

26 Subcase 2.2:  $|V(D_1)| \geq 3$ .

27 We may assume that  $uv$  is the unique  $T$ -chord, for otherwise Case 1 or Subcase 2.1 would  
28 apply, and we would be done. Let  $v'$  be the neighbour of  $u$  in  $C$  distinct from  $t$ . Let  $D'$  be  
29 the digraph obtained from  $D$  by replacing the arcs  $tv$  and  $vt$  by the arcs  $tv'$  and  $v't$ . Observe  
30 that  $(D, T)$  and  $(D', T)$  have the same BMRN\*-chromatic index because those four arcs do  
31 not create any new constraint. If  $uv' \notin A(T)$ , then, by Lemma 19,  $(D', T)$  has a 5-BMRN\*-  
32 colouring which is also a 5-BMRN\*-colouring of  $(D, T)$ . Henceforth, we may assume that  
33  $uv' \in A(T)$ . In particular,  $T - t$  is the union of two directed paths,  $P$  with first arc  $uv$  and  $P'$   
34 with first arc  $uv'$ .

35 Assume  $u$  has at most five neighbours in  $D_2^*$ . Recall that  $D$  is symmetric, so every neighbour  
36 is both an in- and an out-neighbour. Two of these neighbours are  $v$  and  $v'$ . By Lemma 19,  
37  $(D_2^*, T_2)$  admits a 5-BMRN\*-colouring  $\phi_2$ . Now at most four colours are forbidden for  $tu$ ,  
38 namely  $\phi_2(uv) = \phi_2(uv')$  and the colours assigned to arcs with tail a neighbour of  $u$  in  $D_2$   
39 distinct from  $v$  and  $v'$ . Hence one can extend  $\phi_2$  to  $(D, T)$  by assigning to  $tu$  a non-forbidden  
40 colour. Henceforth, we may assume that  $u$  has at least five neighbours in  $D_2^*$ .

41 Free to consider  $(D', T)$  instead of  $(D, T)$ , we may assume that  $u$  has at least three neighbours  
42 in  $P$ . Let  $z$  be the last neighbour of  $u$  along  $P$ . Let  $z^-$  be the in-neighbour of  $z$  in  $P$  and let  $z^+$   
43 be the out-neighbour of  $z$  in  $P$  if it has one and  $z^+ = z$  otherwise, and let  $z^*$  be the terminal  
44 vertex of  $P$ .

45 Let  $D_3$  be the subdigraph induced by the vertices of  $V(P[u, z^+]) \cup \{t\}$ . By Lemma 19,  $(D_3, T_3)$   
46 admits a 5-BMRN\*-colouring  $\phi_3$ . Observe moreover that the colours assigned to  $tu, uv, z^-z$

1 and  $zz^+$  (if  $z \neq z^+$ ) are all distinct.

2 Let  $D_4$  be the digraph obtained from the subdigraph induced by the vertices of  $V(P[z^-, z^*]) \cup$   
 3  $V(P')$  by adding the 2-cycle  $(u, z^-, u)$ . Let  $T_4 = \{tu\} \cup P' \cup \{uz^-\} \cup P[z^-, z^*]$ . The number  
 4 of  $T_4$ -chords in  $(D_4, T_4)$  is equal to the number of  $T$ -chords in  $(D, T)$  and  $|V(D_4)| < |V(D)|$ ,  
 5 because there are at least three neighbours of  $u$  in  $P[v, z]$ . Thus, by minimality of  $(D, T)$ ,  
 6  $(D_4, T_4)$  admits a 5-BMRN\*-colouring. In addition, the colours assigned to  $tu, uv, z^-z$  and  
 7  $zz^+$  (if  $z \neq z^+$ ) are all distinct. Hence, free to permute the colours, we may assume that  $\phi_3$   
 8 and  $\phi_4$  agree on those four arcs. Now one easily checks that the colouring  $\phi$  of  $A(T)$ , defined  
 9 by  $\phi(a) = \phi_3(a)$  if  $a \in A(T_3)$  and  $\phi(a) = \phi_4(a)$  if  $a \in A(T_4)$ , is a 5-BMRN\*-colouring of  
 10  $(D, T)$ .  $\square$

## 11 5 Complexity

### 12 5.1 Determining the exact value of an index

13 In this section, we prove several results showing that the problems of determining  $\text{BMRN}(D, T)$ ,  
 14  $\text{BPRN}(D, T)$  and  $\text{BMRN}^*(D, T)$  are  $\mathcal{NP}$ -hard, even when restricted to particular spanned digraphs  
 15  $(D, T)$ . We define these decision problems in the usual way:

16  $k$ -BMRN-COLOURING

17 Input: A spanned digraph  $(D, T)$ .

18 Question: Do we have  $\text{BMRN}(D, T) \leq k$ ?

20  $k$ -BPRN-COLOURING

21 Input: A spanned digraph  $(D, T)$ .

22 Question: Do we have  $\text{BPRN}(D, T) \leq k$  ?

24  $k$ -BMRN\*-COLOURING

25 Input: A spanned digraph  $(D, T)$ .

26 Question: Do we have  $\text{BMRN}^*(D, T) \leq k$ ?

28 Recall that finding a  $k$ -BMRN-colouring (resp.,  $k$ -BPRN-colouring,  $k$ -BMRN\*-colouring) of  
 29  $(D, T)$  is equivalent to finding a  $k$ -colouring of  $C_{\text{BMRN}}(D, T)$  (resp.,  $C_{\text{BPRN}}(D, T)$ ,  $C_{\text{BMRN}^*}(D, T)$ ).  
 30 Furthermore, the usual  $k$ -COLOURING problem is well-known to be polynomial-time solvable when  
 31  $k = 2$  and  $\mathcal{NP}$ -complete for all  $k \geq 3$ . Since the constraint graphs of  $(D, T)$  can clearly be con-  
 32 structed in polynomial time, we directly get that 2-BMRN-COLOURING, 2-BPRN-COLOURING,  
 33 and 2-BMRN\*-COLOURING are polynomial-time solvable.

Still from the previous colouring equivalence, we now establish the hardness of the three problems  
 above for all  $k \geq 3$ , using the following construction. Let  $\vec{G}$  be an oriented graph. The *matched*  
*digraph associated to  $\vec{G}$*  is the matched digraph  $(D_{\vec{G}}, M_{\vec{G}})$  defined by:

$$\begin{aligned} \bullet V(D_{\vec{G}}) &= V(M_{\vec{G}}) = \bigcup_{x \in V(\vec{G})} \{u_x, v_x\}, \\ \bullet A(M_{\vec{G}}) &= \{u_x v_x \mid x \in V(\vec{G})\}, \text{ and} \\ \bullet A(D_{\vec{G}}) &= A(M_{\vec{G}}) \cup \{u_x v_y \mid xy \in A(\vec{G})\}. \end{aligned}$$

34 One can easily check that  $\vec{G}$  is nothing but the BMRN-constraint graph  $C_{\text{BMRN}}(D_{\vec{G}}, M_{\vec{G}})$ . Observe  
 35 moreover that  $\Delta^+(D_{\vec{G}}) = \Delta^+(\vec{G}) + 1$ .

1 **Theorem 20.** For every  $k \geq 3$ ,  $k$ -BMRN-COLOURING,  $k$ -BPRN-COLOURING and  $k$ -BMRN\*-  
2 COLOURING are  $\mathcal{NP}$ -complete, even when restricted to spanned digraphs  $(D, T)$  where  $T$  is a di-  
3 rected path.

4 *Proof.* Fix  $k \geq 3$ . The problems are clearly in  $\mathcal{NP}$ . Let us now prove that they are  $\mathcal{NP}$ -hard.  
5 Observe that when  $T$  is a directed path, the type-3 and type-3\* constraints are vacuously fulfilled by  
6 any arc-colouring. Therefore, in that case, any BMRN-colouring of  $(D, T)$  is also a BPRN-colouring  
7 and a BMRN\*-colouring. It thus suffices to prove the  $\mathcal{NP}$ -completeness of  $k$ -BMRN-COLOURING  
8 for such restricted instances.

9 The reduction is from  $k$ -COLOURING. Let  $G$  be a graph with vertex set  $\{x_1, \dots, x_n\}$ . Consider  
10 any orientation  $\vec{G}$  of  $G$ . Let  $(D, T)$  be the spanned digraph obtained from the associated matched  
11 digraph  $(D_{\vec{G}}, M_{\vec{G}})$  by adding the arc  $v_{x_i}u_{x_{i+1}}$  for all  $1 \leq i \leq n-1$ , i.e.,  $D = D_{\vec{G}} \cup \{v_{x_i}u_{x_{i+1}} \mid 1 \leq$   
12  $i \leq n-1\}$  and  $T = M_{\vec{G}} \cup \{v_{x_i}u_{x_{i+1}} \mid 1 \leq i \leq n-1\}$ . By construction,  $T$  is a directed path. One  
13 easily sees that each  $v_{x_i}u_{x_{i+1}}$  is subject to at most two colouring constraints, or, in other words, that  
14 the associated vertex has degree 2 in the BMRN-constraint graph  $C_{\text{BMRN}}(D, T)$ . Hence, since  $k \geq 3$ ,  
15 the graph  $C_{\text{BMRN}}(D, T)$  is  $k$ -colourable if and only if  $G$  is  $k$ -colourable. In other words,  $(D, T)$  is  
16  $k$ -BMRN-colourable if and only if  $G$  is  $k$ -colourable.  $\square$

17 We proved, in Theorem 11, that subcubic spanned digraphs have BMRN-chromatic index at  
18 most 4. As already pointed out, by looking at the BMRN-constraint graph, it can be decided in poly-  
19 nomial time whether the BMRN-chromatic index of a given subcubic spanned digraph is at most 2.  
20 It is not clear, however, whether it can easily be decided whether this index is 3 or 4; so we leave the  
21 following question open:

22 **Question 21.** Is 3-BMRN-COLOURING  $\mathcal{NP}$ -complete when restricted to subcubic spanned di-  
23 graphs?

## 24 5.2 Finding a particular out-branching

25 We now consider problems where, for a given digraph  $D$ , one aims at finding an out-branching  $T$   
26 (rooted at some given vertex  $r$  or not), such that  $\text{BMRN}(D, T)$ , (resp.  $\text{BMRN}^*(D, T)$  and  $\text{BPRN}^*(D, T)$ )  
27 is small. More precisely, we consider the following decision problems (which we define for BMRN-  
28 colouring only, but they can be derived for BPRN-colouring and BMRN\*-colouring in an obvious  
29 way):

30  $k$ -BMRN-ROOT

31 Input: A digraph  $D$  and an out-generator  $r$  of  $D$ .

32 Question: Is there an out-branching  $T$  of  $D$  spanned at  $r$  such that  $\text{BMRN}(D, T) \leq k$ ?

34  $k$ -BMRN-BRANCHING

35 Input: A digraph  $D$ .

36 Question: Is there an out-branching  $T$  of  $D$  such that  $\text{BMRN}(D, T) \leq k$ ?

38 We show in Theorem 22 that all arising problems are  $\mathcal{NP}$ -complete for all  $k \geq 3$ . We then  
39 prove the  $\mathcal{NP}$ -completeness of 2-BMRN-ROOT, 2-BMRN-BRANCHING, 2-BMRN\*-ROOT and 2-  
40 BMRN\*-BRANCHING in Theorems 23 and 24. The same result for 2-BMRN-ROOT and 2-BMRN-  
41 BRANCHING is then proved in Theorem 25.

42 **Theorem 22.** For every  $k \geq 3$ ,  $k$ -BMRN-ROOT,  $k$ -BMRN-BRANCHING,  $k$ -BMRN\*-ROOT,  $k$ -  
43 BMRN\*-BRANCHING,  $k$ -BPRN-ROOT and  $k$ -BPRN-BRANCHING are  $\mathcal{NP}$ -complete.

44 *Proof.* The problems are clearly in  $\mathcal{NP}$ . We now prove that they are  $\mathcal{NP}$ -hard, by focusing, again,  
45 on BMRN-colourings (for the same reasons as in the proof of Theorem 20). Fix  $k \geq 3$ . The reduction

1 is from  $k$ -COLOURING. Let  $G$  be a graph with vertex set  $\{x_1, \dots, x_n\}$ . Let  $\vec{G}$  be the orientation of  $G$   
2 such that if  $x_i x_j$  is an arc, then  $i > j$ . Again consider, as in the proof of Theorem 20, the associated  
3 matched digraph  $(D_{\vec{G}}, M_{\vec{G}})$  of  $\vec{G}$ . Let  $D$  be the digraph obtained from  $D_{\vec{G}}$  by adding a vertex  $r$  and  
4 the arcs  $ru_{x_1}$  and  $v_{x_i}u_{x_{i+1}}$  for all  $1 \leq i \leq n-1$ . One easily sees that the unique out-branching of  $D$   
5 is the directed path  $T = (r, u_{x_1}, v_{x_1}, u_{x_2}, v_{x_2}, \dots, u_{x_n}, v_{x_n})$ . Moreover, all  $u_{x_i}v_{x_{i+1}}$ 's and  $ru_{x_1}$  have  
6 degree at most 2 in the BMRN-constraint graph  $C_{\text{BMRN}}(D, T)$  of  $(D, T)$ . Hence, since  $k \geq 3$ , we  
7 have that  $(D, T)$  is  $k$ -BMRN-colourable if and only if  $G$  is  $k$ -colourable.  $\square$

8 **Theorem 23.** *2-BMRN-ROOT and 2-BMRN-BRANCHING are  $\mathcal{NP}$ -complete.*

9 *Proof.* The proof of the  $\mathcal{NP}$ -hardness is by reduction from MONOTONE NOT-ALL-EQUAL 3-SAT. In  
10 this variant, the boolean formula  $\mathcal{F}$  consists of clauses all of whose literals are non-negated variables.  
11 We want to decide whether  $\mathcal{F}$  admits an *NAE-assignment*, that is a truth assignment such that every  
12 clause has a true literal and a false literal. MONOTONE NOT-ALL-EQUAL 3-SAT was shown  $\mathcal{NP}$ -  
13 complete by Schaefer [16].

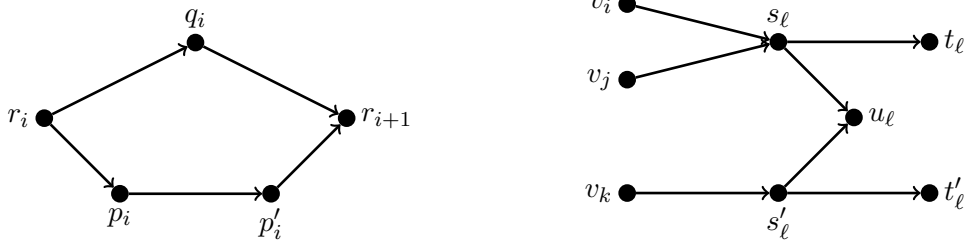


Figure 6: The two gadgets used in the proof of Theorem 23: the  $\{r_i, r_{i+1}\}$ -connector gadget (left), and the clause gadget for a clause  $C_\ell = x_i \vee x_j \vee x_k$  (right).

14 Let  $\mathcal{F}$  be an instance of MONOTONE NOT-ALL-EQUAL 3-SAT with variables  $x_1, \dots, x_n$  and  
15 clauses  $C_1, \dots, C_m$ . Let  $D$  be the digraph constructed as follows. For  $1 \leq i \leq n$ , add the vertices  
16  $r_i, v_i$  and the arc  $r_i v_i$ . For every  $1 \leq i \leq n-1$ , we add an  $\{r_i, r_{i+1}\}$ -connector gadget (see the left  
17 of Figure 6), which is a digraph with vertex set  $\{r_i, p_i, p'_i, q_i, r_{i+1}\}$  (where  $r_i$  and  $r_{i+1}$  are existing  
18 vertices) and arc set  $\{r_i q_i, q_i r_{i+1}, r_i p_i, p_i p'_i, p'_i r_{i+1}\}$ . For every clause  $C_\ell = x_i \vee x_j \vee x_k$ , we add a  
19 clause gadget (see the right of Figure 6) with vertex set  $\{v_i, v_j, v_k, s_\ell, s'_\ell, u_\ell, t_\ell, t'_\ell\}$  (where  $v_i, v_j, v_k$   
20 already exist) and arc set  $\{v_i s_\ell, v_j s_\ell, s_\ell t_\ell, s_\ell u_\ell, v_k s'_\ell, s'_\ell t'_\ell, s'_\ell u_\ell\}$ .

21 Observe that  $r_1$  is the unique out-generator of  $D$ . Furthermore, every out-branching  $T$  of  $D$  must  
22 contain every arc  $r_i v_i$ ,  $1 \leq i \leq n$ , because each  $v_i$  has in-degree 1. Let us show that  $D$  has an out-  
23 branching  $T$  (with root  $r_1$ ) such that  $\text{BMRN}(D, T) \leq 2$  if and only if  $\mathcal{F}$  admits an NAE-assignment.

24 Assume first that there is an out-branching  $T$  of  $D$  such that  $\text{BMRN}(D, T) \leq 2$ . Let  $\phi$  be a  
25 2-BMRN-colouring of  $(D, T)$ . Let  $\psi$  be the truth assignment defined by  $\psi(x_i) = \text{true}$  if and only  
26 if  $\phi(r_i v_i) = 1$ . This is well-defined because all  $r_i v_i$ 's are in  $A(T)$ . We claim that  $\psi$  is an NAE-  
27 assignment of  $\mathcal{F}$ . Suppose for a contradiction that there is a clause  $C_\ell = x_i \vee x_j \vee x_k$  such that all  
28 literals have the same value. Without loss of generality,  $\psi(x_i) = \psi(x_j) = \psi(x_k) = \text{true}$ . Then,  
29 by definition,  $\phi(r_i v_i) = \phi(r_j v_j) = \phi(r_k v_k) = 1$ . Now  $v_k s'_\ell$  and  $s'_\ell t'_\ell$  must be in  $T$  (because  $s'_\ell$  and  
30  $t'_\ell$  have in-degree 1), and we have  $\phi(v_k s'_\ell) = 2$  and  $\phi(s'_\ell t'_\ell) = 1$ . Furthermore,  $T$  contains an arc  
31  $a \in \{v_i s_\ell, v_j s_\ell\}$  and  $s_\ell t_\ell$ , and  $\phi(a) = 2$  and  $\phi(s_\ell t_\ell) = 1$ . Finally, one arc  $a'$  of  $s_\ell u_\ell$  and  $s'_\ell u_\ell$  is  
32 in  $A(T)$ . Because of type-1 constraints, we must have  $\phi(a') = 1$ , which creates a type-2 constraint  
33 since  $\phi(s_\ell t_\ell) = \phi(s'_\ell t'_\ell) = 1$ . This is a contradiction.

34 Reciprocally, assume that there exists an NAE-assignment  $\psi$  of  $\mathcal{F}$ . We shall construct an out-  
35 branching  $T$  of  $D$  and a 2-BMRN-colouring  $\phi$  of  $(D, T)$ . For  $1 \leq i \leq n$ , add  $r_i v_i$  to  $A(T)$  and  
36 set  $\phi(r_i v_i) = 1$  if  $\psi(x_i) = \text{true}$ , and  $\phi(r_i v_i) = 2$  if  $\psi(x_i) = \text{false}$ . For every clause  $C_\ell =$

1  $x_i \vee x_j \vee x_k$ , either  $\psi(x_i) \neq \psi(x_k)$  or  $\psi(x_j) \neq \psi(x_k)$  because  $\psi$  is an NAE-assignment. Let  $a$  be  
2  $v_i s_\ell$  if  $\psi(x_i) \neq \psi(x_k)$  and let  $a$  be  $v_j s_\ell$  otherwise. Add  $a$ ,  $v_k s'_\ell$ ,  $s_\ell t_\ell$ ,  $s'_\ell t'_\ell$ , and  $s_\ell u_\ell$  to  $A(T)$ , and let  
3  $\phi(a) = \phi(s'_\ell t'_\ell) = \phi(r_k v_k)$  and  $\phi(s_\ell t_\ell) = \phi(s_\ell u_\ell) = \phi(v_k s'_\ell) = 3 - \phi(a)$ . For each  $1 \leq i \leq n - 1$ ,  
4 do the following:

- 5 • If  $\phi(r_i v_i) = \phi(r_{i+1} v_{i+1})$ , then add  $r_i q_i$ ,  $r_i p_i$ ,  $p_i p'_i$  and  $q_i r_{i+1}$  to  $A(T)$  and set  $\phi(r_i q_i) =$   
6  $\phi(r_i p_i) = \phi(r_i v_i)$  and  $\phi(p_i p'_i) = \phi(q_i r_{i+1}) = 3 - \phi(r_i v_i)$ ;
- 7 • If  $\phi(r_i v_i) \neq \phi(r_{i+1} v_{i+1})$ , then add  $r_i q_i$ ,  $r_i p_i$ ,  $p_i p'_i$  and  $p'_i r_{i+1}$  to  $A(T)$  and set  $\phi(r_i q_i) =$   
8  $\phi(r_i p_i) = \phi(p'_i r_{i+1}) = \phi(r_i v_i)$  and  $\phi(p_i p'_i) = 3 - \phi(r_i v_i)$ .

9 Now, it is simple matter to check that  $T$  is an out-branching of  $D$  and  $\phi$  a 2-BMRN-colouring of  
10  $(D, T)$ . □

11 The exact same proof yields the following.

12 **Theorem 24.** 2-BMRN\*-ROOT and 2-BMRN\*-BRANCHING are  $\mathcal{NP}$ -complete.

13 In the next result, we prove the similar result for 2-BPRN-ROOT and 2-BPRN-BRANCHING:

14 **Theorem 25.** 2-BPRN-ROOT and 2-BPRN-BRANCHING are  $\mathcal{NP}$ -complete.

15 *Proof.* The proof of the  $\mathcal{NP}$ -hardness of the two problems is by reduction from a restriction of  
16 the DIRECTED HAMILTONIAN CYCLE problem, which asks whether a given digraph has a directed  
17 Hamiltonian cycle. In [11], Plesník proved that DIRECTED HAMILTONIAN CYCLE remains  $\mathcal{NP}$ -  
18 complete when restricted to *small-degree digraphs*, that are digraphs in which the in-degree and  
19 out-degree of each vertex are either 1 or 2.

20 A *track* is a vertex with in-degree 1 and out-degree 1, an *out-switch* is a vertex with in-degree 1  
21 and out-degree 2, and an *in-switch* is a vertex with in-degree 2 and out-degree 1. A digraph is *nice* if  
22 every vertex is either a track, an out-switch or an in-switch.

23 **Claim 25.1.** DIRECTED HAMILTONIAN CYCLE remains  $\mathcal{NP}$ -complete for nice digraphs.

*Subproof.* Reduction from DIRECTED HAMILTONIAN CYCLE restricted to small-degree digraphs.  
Consider a small-degree digraph  $D$ . Let  $D'$  be the digraph obtained by “exploding” each vertex  $v$  to  
an arc  $v^- v^+$ . Formally,

$$\begin{aligned} \bullet V(D') &= \bigcup_{v \in V(D)} \{v^-, v^+\}, \text{ and} \\ \bullet A(D') &= \{v^- v^+ \mid v \in V(D)\} \cup \{u^+ v^- \mid uv \in A(D)\}. \end{aligned}$$

24 Clearly  $D'$  is nice and it has a directed Hamiltonian cycle if and only if  $D$  has one. ◇

25 We now give a reduction from DIRECTED HAMILTONIAN CYCLE restricted to nice digraphs to  
26 2-BPRN-ROOT and 2-BPRN-BRANCHING. Let  $D$  be a nice digraph. Choose a vertex  $v$  of  $D$  with  
27 in-degree 1. Let  $u$  be its in-neighbour. Observe that every directed Hamiltonian cycle of  $D$  must  
28 contain the arc  $uv$ . Let  $D'$  be the digraph obtained from  $D - uv$  by adding two vertices  $r$  and  $s$   
29 and the arcs  $rv$  and  $us$ . Note that a directed Hamiltonian path in  $D'$  necessarily starts in  $r$  and ends in  $s$ ,  
30 so  $D'$  has a directed Hamiltonian path if and only if  $D$  has a directed Hamiltonian cycle.

31 We now construct a digraph  $H$  such that  $H$  admits an out-branching  $T$  such that  $\text{BPRN}(H, T) = 2$   
32 if and only if  $D'$  has a directed Hamiltonian path, and so if and only if  $D$  has a directed Hamiltonian  
33 cycle. To that end, we first associate a gadget in  $H$  to each vertex of  $D'$ , in the following way:

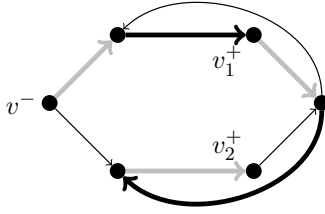


Figure 7: The out-gadget  $O_v$ . A directed Hamiltonian path  $P$  of  $O_v$  is displayed with bold arcs. The black and gray bold arcs form a 2-BPRN-colouring of  $(O_v, P)$ .

- 1     • For each of  $r$  and  $s$ , we add to  $H$  two gadgets that are actually exactly the vertices  $r$  and  $s$ . For
- 2     the vertex gadget corresponding to  $r$  (resp.  $s$ ), we call  $r$  (resp.  $s$ ) its *exit* (resp. *entry*).
- 3     • Consider a track  $v$  of  $D'$ . In  $H$ , we associate a *track gadget*  $T_v$  which is a directed path of
- 4     length 3. The origin  $v^-$  of  $T_v$  is its *entry*, and the terminus  $v^+$  is its *exit*.
- 5     • Consider an out-switch  $v$  of  $D'$ . In  $H$ , we associate an *out-gadget*  $O_v$  depicted in Figure 7.
- 6     This gadget has one *entry*  $v^-$  and two *exits*  $v_1^+, v_2^+$ .
- 7     • Consider an in-switch  $v$  of  $D'$ . In  $H$ , we associate an *in-gadget*  $I_v$ , which is just an arc  $v^-v^+$ .
- 8     We call  $v^-$  the *entry* of  $I_v$ , and  $v^+$  its *exit*.

9     Note that all gadgets have at most one entry. Furthermore, only out-gadgets have two exits. To  
10    finish the construction of  $H$ , we now connect the gadgets as follows. Consider every arc  $uv$  of  $D'$ ;  
11    then:

- 12    • If  $u$  is an out-switch, then we choose a degree-2 exit of the gadget associated to  $u$  in  $H$ , and we
- 13    add an arc from that exit to the unique entry of the gadget associated to  $v$ . In other words, for
- 14    the gadgets with two exits, each exit is added exactly one out-going arc.
- 15    • Otherwise, we add an arc from the unique exit of the gadget associated to  $u$  in  $H$  to the unique
- 16    entry of the gadget associated to  $v$ .

17    Note that an out-gadget has only two directed Hamiltonian paths: they both start in its entry and  
18    they end in different exits. Moreover once a directed Hamiltonian path of  $H$  enters a gadget (track,  
19    out-gadget or in-gadget), it has to go through all the vertices of the gadget at once, due to the number  
20    of entries and exits. Therefore the digraph  $H$  has a directed Hamiltonian path if and only if  $D'$  has  
21    one. In particular, recall that all directed Hamiltonian paths start in  $r$  and end in  $s$ .

22    We claim that we have the desired equivalence between  $D'$  and  $H$ . Assume first that there is  
23    an out-branching  $T$  of  $H$  such that  $(H, T)$  has a 2-BPRN-colouring. Because of type-1 and type-3  
24    constraints, note that  $T$  must be a directed Hamiltonian path of  $H$ .

25    Reciprocally, assume  $H$  has a directed Hamiltonian path  $P$ . We claim that colouring its arcs alter-  
26    nately with colours 1 and 2 (starting, say, with 1) yields a 2-BPRN-colouring of  $(H, P)$ . Indeed,  
27    type-1 constraints are satisfied by definition, and, since  $P$  is a path, type-3 constraints are trivially  
28    satisfied. To be convinced that no type-2 constraint can arise, consider the following arguments. First  
29    of all, all arcs of  $H$  not in  $P$  either belong to an out-gadget (which raise no type-2 constraint, see  
30    Figure 7), or go from an exit of an out-gadget to the entry of an in-gadget. Also, it can be checked  
31    that all arcs of  $P$  entering and exiting any gadget necessarily have colour 1. Under this assumption,  
32    for every arc  $uv$  from an exit of an out-gadget to the entry of an in-gadget, the only arc of  $P$  out-going

1 from  $u$  has colour 2, while the only arc of  $P$  in-coming to  $v$  has colour 1. Hence no type-2 constraint  
 2 arises.

3 Hence there is an out-branching  $T$  of  $H$  such that  $(H, T)$  has a 2-BPRN-colouring if and only if  
 4  $H$  has a directed Hamiltonian path, so if and only if  $D'$  has a directed Hamiltonian path.  $\square$

### 5 5.3 Restriction to planar spanned digraphs

6 In this section, we consider PLANAR  $k$ -BMRN-COLOURING which is the restriction of  $k$ -BMRN-  
 7 COLOURING to planar spanned digraphs. For  $k = 2$ , the problem is polynomial-time solvable  
 8 because  $k$ -BMRN-COLOURING is polynomial-time solvable. For  $k \geq 8$ , PLANAR  $k$ -BMRN-  
 9 COLOURING is trivial as the answer is always ‘Yes’ by Corollary 16.

10 **Theorem 26.** PLANAR 3-BMRN-COLOURING is  $\mathcal{NP}$ -complete.

11 *Proof.* The proof is by reduction from PLANAR 3-COLOURING which consists in deciding whether  
 12 a given planar graph is 3-colourable. This problem was proved  $\mathcal{NP}$ -complete in [7].

13 From a planar graph  $G$ , we shall construct, in polynomial time, a spanned planar digraph  $(D, T)$   
 14 such that  $\chi(G) \leq 3$  if and only if  $\text{BMRN}(D, T) \leq 3$ .

15 Since  $G$  is planar, it admits a planar straight-line embedding  $\tilde{G}$  in the plane. Moreover, free to  
 16 move slightly some vertices, we may assume that no two vertices of  $\tilde{G}$  lie on a same horizontal or  
 17 vertical line. Let  $\vec{G}$  be the orientation of  $G$  obtained by orienting every edge towards its higher vertex  
 18 in  $\tilde{G}$ : hence if  $uv \in A(\vec{G})$ , then vertex  $v$  lies above  $u$ .

19 Let us consider the matched digraph  $(D_{\vec{G}}, M_{\vec{G}})$  associated to  $\vec{G}$ . As observed earlier,  $G$  is actually  
 20 the BMRN-constraint graph  $C_{\text{BMRN}}(D_{\vec{G}}, M_{\vec{G}})$ . So  $G$  is 3-colourable if and only if  $(D_{\vec{G}}, M_{\vec{G}})$  is 3-  
 21 BMRN-colourable.

22 Our goal is now to extend  $(D_{\vec{G}}, M_{\vec{G}})$  into a planar spanned digraph  $(D, T)$ , with preserving the  
 23 colouring equivalence with  $G$ . From  $\tilde{G}$ , one can easily derive a plane straight-line embedding of  $D_{\vec{G}}$   
 24 in which all arcs  $u_x v_x$ , for  $x \in V(\vec{G})$ , are drawn horizontally and from left to right, of tiny length  
 25  $\epsilon$  and whose middle is  $x$ . Towards getting  $(D, T)$ , we first add, to  $(D_{\vec{G}}, M_{\vec{G}})$ , a vertex  $r$  below all  
 26 other vertices (and thus on the outer face), which shall be the root of  $T$ . For every inner face  $F$  of  $\tilde{G}$ ,  
 27 let  $m(F)$  be the lowest vertex of  $F$  (for the particular case of the outer face  $F$ , let  $m(F) = r$ ). For  
 28 every face  $F$  of  $(D_{\vec{G}}, M_{\vec{G}})$ , let  $M(F)$  be the set of vertices  $x$  such that  $u_x$  is inside  $F$ . Observe that  
 29 the vertices of  $M(F)$  are incident to  $F$ , and that  $m(F) \notin M(F)$ . For each face  $F$ , we add a directed  
 30 path  $P_x$  (both in  $D$  and  $T$ ) of length 3 from  $u_{m(F)}$  to  $u_x$  for each vertex  $x \in M(F)$ . Of course, we  
 31 do this in such a way that the added paths do not cross. This results in the planar spanned digraph  
 32  $(D, T)$  where  $D = D_{\vec{G}} \cup \bigcup_{x \in V(G)} P_x$  and  $T = M_{\vec{G}} \cup \bigcup_{x \in V(G)} P_x$ .

33 Let us now check that  $\text{BMRN}(D_{\vec{G}}, M_{\vec{G}}) \leq 3$  if and only if  $\text{BMRN}(D, T) \leq 3$ . Since  $(D_{\vec{G}}, M_{\vec{G}})$   
 34 is the restriction of  $(D, T)$  to  $V(D_{\vec{G}})$ , every 3-BMRN-colouring of  $(D, T)$  induces a 3-BMRN-  
 35 colouring of  $(D_{\vec{G}}, M_{\vec{G}})$ . Conversely, every 3-BMRN-colouring  $\phi$  of  $(D_{\vec{G}}, M_{\vec{G}})$  can be easily ex-  
 36 tended in a 3-BMRN-colouring of  $(D, T)$ , for the following reasons. First, the added paths  $P_x$  do not  
 37 create any new constraint between the arcs of  $M_{\vec{G}}$ . Moreover, for every face  $F$  and every  $x \in M(F)$ ,  
 38 we can assign the colour  $\phi(u_{m(F)}, v_{m(F)})$  to the first arc of  $P_x$ , and then extend the colouring to the  
 39 two other arcs, which are subject to at most two constraints in  $(D, T)$ .

40 Hence  $(D, T)$  is a planar spanned digraph which is 3-BMRN-colourable if and only if  $G$  is 3-  
 41 colourable.  $\square$

42 PLANAR  $k$ -COLOURING is trivial for all  $k \geq 4$ , as the answer is always ‘Yes’ according to the  
 43 Four-Colour Theorem. Henceforth, the above proof cannot be generalized to show that PLANAR  
 44  $k$ -BMRN-COLOURING is  $\mathcal{NP}$ -complete for  $k$  larger than 3.

1 **Question 27.** For every  $k \in \{4, 5, 6, 7\}$ , what is the complexity of PLANAR  $k$ -BMRN-COLOURING?

2 Similarly, one can compute the chromatic number of an outerplanar graph in polynomial time. So  
3 we cannot establish the hardness of the restriction of  $k$ -BMRN-COLOURING to outerplanar digraphs  
4 via a proof similar to that of Theorem 26. We thus leave the following question open:

5 **Question 28.** For any outerplanar spanned digraph  $(D, T)$ , can  $\text{BMRN}(D, T)$  and  $\text{BMRN}^*(D, T)$   
6 be determined in polynomial time?

## 7 References

- 8 [1] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*  
9 21:429–490, 1977.
- 10 [2] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illi-*  
11 *nois J.Math.* 21:491–567, 1977.
- 12 [3] K. Appel and W. Haken. Every Planar Map is Four Colorable. *Contemp. Math.* 98. American  
13 Mathematical Society, Providence, RI, 1989.
- 14 [4] E. Arikian. Some complexity results about packet radio networks (corresp.). *IEEE Transactions*  
15 *on Information Theory*, 30(4):681–685, 1984.
- 16 [5] P. Djukic. *Scheduling Algorithms for TDMA Wireless Multihop Networks*. PhD thesis, 2007.
- 17 [6] S. Even, O Goldreich, S. Moran, and P. Tong. On the NP-completeness of certain network  
18 testing problems. *Networks*, 14(1):1–24, 1984.
- 19 [7] M.R. Garey, D.S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems.  
20 *Theoretical Computer Science* 1(3):237–267, 1976.
- 21 [8] M. L. Huson. *A New Model for Scheduling Radio Networks*. PhD thesis, 1995.
- 22 [9] E. L. Lloyd and S. Ramanathan. On the complexity of link scheduling in multi-hop radio net-  
23 works. In *Proceedings of the 26th Conference Information Sciences and Systems*, 1992.
- 24 [10] R. Nelson and L. Kleinrock. Spatial TDMA: A collision-free multihop channel access protocol.  
25 *IEEE Transactions on Communications*, 33(9):934–944, 1985.
- 26 [11] J. Plesník. The NP-completeness of the Hamiltonian cycle problem in planar digraphs with  
27 degree bound two. *Information Processing Letters*, 8(4):199–201, 1979.
- 28 [12] S. Ramanathan. *Scheduling Algorithms for Multi-hop Radio Networks*. PhD thesis, 1992.
- 29 [13] S. Ramanathan and E. L. Lloyd. Scheduling algorithms for multihop radio networks. *IEEE/ACM*  
30 *Transactions on Networking*, 1(2):166–177, 1993.
- 31 [14] L. Rocha and D. Sasaki. The Backbone PRN coloring for TDMA link scheduling in wireless  
32 multihop networks. *Networks*, to appear.
- 33 [15] A. Sen and M. L. Huson. A new model for scheduling packet radio networks. *Wireless Networks*,  
34 3(1):71–82, 1997.
- 35 [16] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th Annual*  
36 *ACM Symposium on Theory of Computing (STOC 10)*, pages 216–226, New York, 1978. ACM.



- 1 [17] IEEE standard for local and metropolitan area networks part 16: Air interface for fixed broad-  
2 band wireless access systems, 2004.
- 3 [18] IEEE p802.11s/d1.01, draft standard for information technology - telecommunications and in-  
4 formation exchange between systems - local and metropolitan area networks specific require-  
5 ments part 11: Wireless lan medium access control (mac) and physical layer (phy) specifications  
6 amendment: ESS mesh networking, 2006.