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# Majorana neutrino masses in the renormalization group equations for lepton flavor violation

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We suppose that the observed neutrino masses can be parametrized by a lepton number violating dimension-five operator, and calculate the mixing of double insertions of this operator into lepton flavor changing dimension-six operators of the standard model effective theory. This allows to predict the log-enhanced, but  $m_\nu^2$ -suppressed lepton flavor violation that is generic to high-scale Majorana neutrino mass models. We also consider the two Higgs doublet model, where the second Higgs allows the construction of three additional dimension-five operators, and evaluate the corresponding anomalous dimensions. The sensitivity of current searches for lepton flavor violation to these additional Wilson coefficients is then examined.

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## I. INTRODUCTION

Neutrinos are elusive and enigmatic particles: uncolored, uncharged, and very light. Nonetheless, their observed masses and mixing angles [1] imply that lepton flavor violation (LFV) must occur, where we define LFV as flavor-changing contact interactions of charged leptons (for a review, see, e.g., [2]). Since these do not occur in the standard model (SM), LFV is considered to be “new physics”, and searched for in a wide variety of experiments [1,3–12]. Neutrinos could also induce another kind of new physics: if their small masses are “Majorana,” they are lepton number violating (LNV), and could for instance mediate neutrinoless double- $\beta$ -decay [13]. Below the weak scale, such masses appear as renormalizable terms in the Lagrangian, but in the full SU(2) gauge invariant standard model, they arise as a nonrenormalizable, dimension-five operator.

In this paper, we assume that neutrino masses are Majorana, and generated by New Physics in the lepton sector at a scale  $\Lambda \gg m_W$ . We focus on the theory above

$m_W$  but below  $\Lambda$ , where it can be described in the framework of the standard model effective field theory<sup>1</sup> (SMEFT). The neutrino masses are parametrized by operators of dimension five, and LFV is parametrized by operators of dimension-six. Our aim is to obtain the log-enhanced loop contributions of two LNV operators to LFV processes, which arise in the renormalization group equations (RGEs). In particular, we calculate the anomalous dimensions that mix two dimension-five operators into a dimension-six operator. The renormalization group running of the dimension-five operators has been extensively studied in the literature [16–18], and the mixing of the dimension-six operators among themselves have been evaluated at one-loop [19] in the “Warsaw”-basis [20] of SMEFT operators. The mixing of two dimension-five operators into dimension-six operators was calculated in [21], using the Buchmuller-Wyler [22] basis at dimension-six. We perform this calculation using the “Warsaw”-basis, and our results appear to disagree with [21].

The mixing of neutrino masses into LFV amplitudes is  $\mathcal{O}(m_\nu/m_W)^2 \ln(\Lambda/m_W)$ , so negligibly small, but completes the anomalous dimensions required to perform a one-loop renormalization-group analysis of the SMEFT at dimension-six. Indeed, this mixing does not involve any SM couplings, so in a coupling expansion, would be the leading contribution to the one-loop RGEs of the SMEFT; it is only small because the dimension five coefficients are small. In addition, we explore an extension of the SMEFT with two

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<sup>1</sup>For an introduction to EFT, see e.g., [14,15].

Higgs doublets [23], where the second Higgs doublet lives at a scale  $m_{22}$  between  $m_W$  and significantly below the lepton number/flavor-changing scale  $\Lambda$ , and we impose that LFV at the weak scale is still described by the dimension-six operators of the SMEFT. In this scenario, there are four LNV dimension-five operators above  $m_{22}$ , but only one combination of coefficients contributes to neutrino masses. We calculate the mixing of these LNV operators into the LFV operators of the SMEFT, and estimate the sensitivity of current LFV experiments to their coefficients.

The paper is organized as follows. In Sec. II we introduce the notation of our standard model and two-Higgs doublet model calculation. The main results are presented in Sec. III, where we discuss the general structure of our calculation and give the relevant counterterms, anomalous dimensions and renormalization group equations. Section IV discusses the phenomenological implications of both results before we conclude. We provide the relevant Feynman rules, further details of the calculation (including a careful treatment of the flavor structures), and the renormalization group in the Appendices A–C. The LFV operators of the SMEFT are recalled in Appendices D and E gives the current experimental constraints on some LFV coefficients of the SMEFT at the weak scale. Appendix F provides a comparison with the previous calculation of [21] and Appendix G presents the lepton conserving contributions to the anomalous dimensions.

## II. NOTATION AND REVIEW

The SM Lagrangian for leptons can be written as

$$\mathcal{L}_{\text{lep}} = i\bar{\ell}_\alpha \gamma^\mu D_\mu \ell_\alpha + i\bar{e}_\alpha \gamma^\mu D_\mu e_\alpha - ([Y_e]_{\alpha\beta} \bar{\ell}_\alpha H e_\beta + \text{H.c.}) \quad (2.1)$$

where Greek letters represent lepton generation indices in the charged-lepton mass eigenstate basis,  $[Y_e]$  is the diagonal charged-lepton Yukawa matrix with eigenvalues  $y_\alpha$ ,  $\ell$  is a doublet of left-handed leptons, and  $e$  is a right-handed charged-lepton singlet. The explicit form of the lepton and Higgs doublets is

$$\ell = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad H = \begin{pmatrix} H^+ \\ H_0 \end{pmatrix}, \quad (2.2)$$

which have hypercharge  $Y(\ell) = -1/2$  and  $Y(H) = 1/2$  respectively. The covariant derivative for a lepton doublet is

$$(D_\mu \ell)_\alpha^i = \left( \delta_{ij} \partial_\mu + i\frac{g}{2} \tau_{ij}^a W_\mu^a + i\delta_{ij} g' Y(\ell) B_\mu \right) \ell_\alpha^j, \quad (2.3)$$

where  $\tau^a$  are the Pauli matrices. This sign convention for the covariant derivative agrees with [19].

Heavy new physics can be parametrized by adding nonrenormalizable operators to the SM Lagrangian that respect the SM gauge symmetries [22]. There is only a

single operator at dimension-five in the SM, which is the lepton number violating ‘‘Weinberg’’ operator [24] which is responsible for Majorana masses of left-handed neutrinos. The resulting effective Lagrangian at dimension-five is

$$\delta\mathcal{L}_5 = \frac{C_5^{\alpha\beta}}{2\Lambda} (\bar{\ell}_\alpha \varepsilon H^*) (\ell_\beta^c \varepsilon H^*) + \frac{C_5^{\alpha\beta*}}{2\Lambda} (\bar{\ell}_\beta^c \varepsilon H) (\ell_\alpha \varepsilon H), \quad (2.4)$$

where  $\varepsilon$  is the totally antisymmetric rank-2 Levi-Civita symbol with  $\varepsilon_{12} = +1$ , all implicit SU(2) indices inside brackets are contracted, and the charge conjugation acts on the SU(2) component  $\ell^i$  of the lepton doublet as  $(\ell^i)^c = C\bar{\ell}^{iT}$ . The charge conjugation matrix  $C$  fulfils the properties of the charge-conjugation matrix used in [25].<sup>2</sup> The coefficient  $C_5^{\alpha\beta}$  is symmetric under the interchange of the generation indices  $\alpha, \beta$ , the new physics scale  $\Lambda$  is assumed  $\gg m_W$ , and the second term is the Hermitian conjugate of the first.

In the broken theory, with  $H_0 = v + (h/\sqrt{2})$ ,  $v \simeq m_t$ , this gives a Majorana neutrino mass matrix

$$\delta\mathcal{L} = -\frac{1}{2} [m_\nu]_{\alpha\beta} \bar{\nu}_\alpha \nu_\beta^c + \text{H.c.} \quad [m_\nu]_{\alpha\beta} = -\frac{v^2}{\Lambda} C_5^{\alpha\beta} \quad (2.5)$$

In the charged lepton mass eigenstate basis, this mass matrix is diagonalized by the Pontecorvo-Maki-Nakagawa-Sakata matrix  $[m_\nu]_{\alpha\beta} = U_{\alpha i} m_{\nu i} U_{\beta i}$ .

At dimension-six, we will be interested in SM-gauge invariant operators that violate lepton flavor; a complete list is given in Appendix D. Following the conventions of [19,20], they are added to the Lagrangian as:

$$\delta\mathcal{L}_6 = \sum_{X,\zeta} \frac{C_X^\zeta}{\Lambda^2} \mathcal{O}_X^\zeta + \text{H.c.} \quad (2.6)$$

where  $X$  is an operator label and  $\zeta$  represents all required generation indices which are summed over all generations. Of particular interest are the operators that can be generated at one-loop with two insertions of dimension-five operators, as illustrated in Fig. 1. With SM particle content, these operators involve two Higgs doublets and two lepton doublets, four lepton doublets, or three Higgs doublets and leptons of both chiralities. In the ‘‘Warsaw’’ basis [20], the possibilities at dimension-six are

$$\begin{aligned} \mathcal{O}_{H\ell(1)}^{\alpha\beta} &= \frac{i}{2} (H^\dagger \overleftrightarrow{D}_\mu H) (\bar{\ell}_\alpha \gamma^\mu \ell_\beta) \\ \mathcal{O}_{H\ell(3)}^{\alpha\beta} &= \frac{i}{2} (H^\dagger \overleftrightarrow{D}_\mu^a H) (\bar{\ell}_\alpha \gamma^\mu \tau^a \ell_\beta) \\ \mathcal{O}_{eH}^{\alpha\beta} &= (H^\dagger H) \bar{\ell}_\alpha H e_\beta & \mathcal{O}_{\ell\ell}^{\alpha\beta\gamma\delta} &= \frac{1}{2} (\bar{\ell}_\alpha \gamma_\mu \ell_\beta) (\bar{\ell}_\gamma \gamma^\mu \ell_\delta) \end{aligned} \quad (2.7)$$

<sup>2</sup>Note that this definition of the dimension-five operator is the hermitian conjugate of the one used in [20] where  $C = i\gamma^2\gamma^0$  in the Dirac representation, since in this representation  $C^{-1} = -C$ .

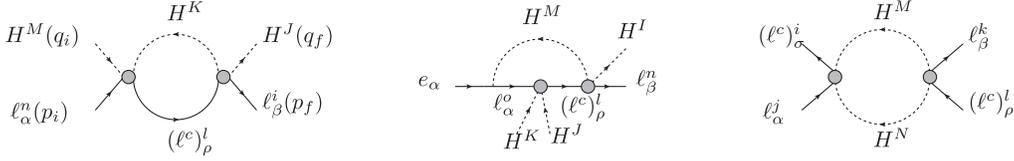


FIG. 1. Diagrams involving two insertions of dimension-five operators, that can contribute to dimension-six lepton-flavor-violating operators. SU(2) indices run from  $I, \dots, O$  and  $i, \dots, o$ , lepton flavor indices are  $\alpha, \beta, \rho, \sigma$ .

where we normalize the ‘‘Hermitian’’ operators with a factor of  $1/2$  (see Appendix D for a discussion) in order to agree with [19,20], and

$$\begin{aligned} i(H^\dagger \overleftrightarrow{D}_\mu H) &\equiv i(H^\dagger D_\mu H) - i(D_\mu H)^\dagger H \\ &= H^\dagger (i\partial_\mu H) - i(\partial_\mu H)^\dagger H - gH^\dagger \tau^a W_\mu^a H \\ &\quad - 2Y(H)g'H^\dagger B_\mu H, \\ i(H^\dagger \overleftrightarrow{D}_\mu^a H) &\equiv i(H^\dagger \tau^a D_\mu H) - i(D_\mu H)^\dagger \tau^a H. \end{aligned} \quad (2.8)$$

The choice of operator basis implies a choice of operators that vanish by the equations of motion (EOMs). For example  $i\not{D}\ell_\alpha - [Y_e]^{a\sigma} H e_\sigma = 0$  implies that the following operators

$$\begin{aligned} \mathcal{O}_{v(1)}^{\alpha\beta} &= i(H^\dagger H)(\overline{\ell}_\alpha \overleftrightarrow{D}\ell_\beta) \\ &\quad - (H^\dagger H)(\overline{\ell}_\alpha H e_\sigma [Y_e^T]_{\sigma\beta} + [Y_e^*]_{\alpha\sigma} \overline{\ell}_\sigma H^\dagger \ell_\beta), \\ \mathcal{O}_{v(3)}^{\alpha\beta} &= i(H^\dagger \tau^a H)(\overline{\ell}_\alpha \overleftrightarrow{D}^a \ell_\beta) \\ &\quad - (H^\dagger H)(\overline{\ell}_\alpha H e_\sigma [Y_e^T]_{\sigma\beta} + [Y_e^*]_{\alpha\sigma} \overline{\ell}_\sigma H^\dagger \ell_\beta), \end{aligned} \quad (2.9)$$

are EOM-vanishing operators. The role of these operators becomes clear by noting that in intermediate steps of our off-shell calculations, additional structures appear that can conveniently be matched onto combinations of EOM-vanishing operators and operators of the Warsaw basis. For example the structures involving two Higgs fields and a covariant derivative of a lepton doublet are expressed in terms of the above operators as:

$$\begin{aligned} \mathcal{S}_{HD\ell(1)}^{\alpha\beta} &= i(H^\dagger H)(\overline{\ell}_\alpha \overleftrightarrow{D}\ell_\beta) \\ &= \mathcal{O}_{v(1)}^{\alpha\beta} + \mathcal{O}_{eH}^{\alpha\sigma} [Y_e^T]_{\sigma\beta} + [Y_e^*]_{\alpha\sigma} \mathcal{O}_{eH}^{\dagger\sigma\beta}, \\ \mathcal{S}_{HD\ell(3)}^{\alpha\beta} &= i(H^\dagger \tau^a H)(\overline{\ell}_\alpha \overleftrightarrow{D}^a \ell_\beta) \\ &= \mathcal{O}_{v(3)}^{\alpha\beta} + \mathcal{O}_{eH}^{\alpha\sigma} [Y_e^T]_{\sigma\beta} + [Y_e^*]_{\alpha\sigma} \mathcal{O}_{eH}^{\dagger\sigma\beta}. \end{aligned} \quad (2.10)$$

In practice, if the coefficients  $C_{HD\ell(1)}^{\beta\alpha}$  and  $C_{HD\ell(3)}^{\beta\alpha}$  of these structures are present, they are equivalent to  $C_{eH}^{\beta\alpha} = C_{HD\ell(1)}^{\beta\alpha} [Y_e]^{a\sigma} + C_{HD\ell(3)}^{\beta\alpha} [Y_e]^{a\sigma}$  (and the Hermitian conjugate relation).

### A. In the case of the 2HDM

In this section, we consider the addition of a second Higgs doublet  $H_2$  to the SM, of the same hypercharge as the SM Higgs (which we relabel  $H_1$ ). The LFV induced by double-insertions of dimension-five operators could be more significant in this model, because there are several dimension-five operators, so neutrino masses cannot constrain them all. However, a complete analysis of LFV in the 2HDM would require extending the operator basis at dimension-six and calculating the additional terms in the RGEs, which is beyond the scope of this work. So for simplicity, we make three restrictions:

- (1) First, we consider only the dimension-six LFV operators of the SMEFT. This is the appropriate set of dimension-six operators just above  $m_W$ , provided that  $H_2$  has no vev, and that the mass  $m_{22}$  of the additional Higgses is sufficiently high:  $m_W^2 \ll m_{22}^2 \ll \Lambda^2$ . In our phenomenological analysis we extend this range to the scenario  $m_W^2 \lesssim m_{22}^2 \ll \Lambda^2$ , by considering a Higgs potential where the additional Higgses are not directly observable at the LHC, and where the Yukawa couplings of  $H_2$  are vanishing. Such a scenario would e.g., be realized in the inert two Higgs doublet model [26–29] and setting the scale  $m_{22}$  close to the electroweak scale will not require the consideration of additional renormalization group effects in the SMEFT.
- (2) Second, we suppose that at the high scale  $\Lambda$  no dimension-six LFV operators are generated. This is unrealistic, but allows us to focus on the LFV generated by double-insertions of the dimension-five operators.
- (3) Third, we suppose there is no LFV in the renormalizable couplings of the 2HDM (in particular, in the lepton Yukawas), so that when matching the 2HDM + dimension-five operators onto the SMEFT at the intermediate scale  $m_{22}$ , no additional LFV operators are generated.

Consider first the renormalizable Lagrangian. The Yukawa couplings can be written [30]:

$$\begin{aligned} \delta\mathcal{L}_{2\text{HDM}} &= -(\overline{\nu}, \overline{e}_L)[Y^{(1)}] \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix} e - \overline{e}[Y^{(1)}]^\dagger H_1^\dagger \ell \\ &\quad - (\overline{\nu}, \overline{e}_L)[Y^{(2)}] \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix} e - \overline{e}[Y^{(2)}]^\dagger H_2^\dagger \ell, \end{aligned} \quad (2.11)$$

where the flavor indices are implicit, and the basis in  $(H_1, H_2)$  space is taken to be the ‘‘Higgs basis’’ where  $\langle H_2 \rangle = 0$ . We suppose that  $[Y^{(1)}]$  and  $[Y^{(2)}]$  are simultaneously diagonalizable on their lepton flavor indices.

The second Yukawa coupling changes the equations of motion for the leptons, so the 2HDM version of the equation-of-motion vanishing operators [given in Eq. (2.9) for the single Higgs model] should be modified. As a result, the operators  $\mathcal{O}_{HD\ell(1)}$  and  $\mathcal{O}_{HD\ell(3)}$  should not be replaced only by the SMEFT operator  $\mathcal{O}_{eH}$ , as given in Eq. (2.10), but also by an operator with an external  $H_2$  leg. However, since we neglect dimension-six operators with external  $H_2$ , we use the relations (2.9) and (2.10) also in the 2HDM case.

In this ‘‘Higgs’’ basis, the most general Higgs potential is

$$\begin{aligned}
V = & m_{11}^2 H_1^\dagger H_1 + m_{22}^2 H_2^\dagger H_2 - [m_{12}^2 H_1^\dagger H_2 + \text{H.c.}] \\
& + \frac{1}{2} \lambda_1 (H_1^\dagger H_1)^2 + \frac{1}{2} \lambda_2 (H_2^\dagger H_2)^2 + \lambda_3 (H_1^\dagger H_1)(H_2^\dagger H_2) \\
& + \lambda_4 (H_1^\dagger H_2)(H_2^\dagger H_1) + \left\{ \frac{1}{2} \lambda_5 (H_1^\dagger H_2)^2 \right. \\
& \left. + [\lambda_6 (H_1^\dagger H_1) + \lambda_7 (H_2^\dagger H_2)] H_1^\dagger H_2 + \text{H.c.} \right\}. \quad (2.12)
\end{aligned}$$

In order to decouple the additional Higgses, we can, for instance, set  $m_{12}^2 = 0$  and assume  $m_{22}^2 \gg m_W^2$ , or leave  $m_{22}^2$  free, and impose  $m_{12}^2 = \lambda_6 = \lambda_7 = [Y^{(2)}] = 0$ .

At dimension-five in the 2HDM, there are four operators [16]:

$$\begin{aligned}
\delta\mathcal{L} = & + \frac{C_5^{\alpha\beta}}{2\Lambda} (\overline{\ell}_\alpha \varepsilon H_1^*) (\ell_\beta^c \varepsilon H_1^*) + \frac{C_5^{\alpha\beta*}}{2\Lambda} (\overline{\ell}_\beta^c \varepsilon H_1) (\ell_\alpha \varepsilon H_1) \\
& + \frac{C_{21}^{\alpha\beta}}{2\Lambda} ((\overline{\ell}_\alpha \varepsilon H_2^*) (\ell_\beta^c \varepsilon H_1^*) + (\overline{\ell}_\beta^c \varepsilon H_1) (\ell_\alpha^c \varepsilon H_2^*)) \\
& + \frac{C_{21}^{\alpha\beta*}}{2\Lambda} ((\overline{\ell}_\beta^c \varepsilon H_2) (\ell_\alpha \varepsilon H_1) + (\overline{\ell}_\alpha^c \varepsilon H_1) (\ell_\beta \varepsilon H_2)) \\
& + \frac{C_{22}^{\alpha\beta}}{2\Lambda} (\overline{\ell}_\alpha \varepsilon H_2^*) (\ell_\beta^c \varepsilon H_2^*) + \frac{C_{22}^{\alpha\beta*}}{2\Lambda} (\overline{\ell}_\beta^c \varepsilon H_2) (\ell_\alpha \varepsilon H_2) \\
& - \frac{C_A^{\alpha\beta}}{2\Lambda} (\overline{\ell}_\alpha \varepsilon \ell_\beta^c) (H_1^\dagger \varepsilon H_2^*) - \frac{C_A^{\alpha\beta*}}{2\Lambda} (\overline{\ell}_\beta^c \varepsilon \ell_\alpha) (H_2 \varepsilon H_1), \quad (2.13)
\end{aligned}$$

where  $\{C_5, C_{22}, C_{21}\}$  are symmetric on flavor indices (so can contribute to neutrino masses). In the  $\mathcal{O}_{21}$  operator,  $(\overline{\ell}_\alpha \varepsilon H_2^*) (\ell_\beta^c \varepsilon H_1^*) = (\overline{\ell}_\beta^c \varepsilon H_1^*) (\ell_\alpha^c \varepsilon H_2^*)$ , but both terms are retained here because they are convenient in our Feynman rule conventions.<sup>3</sup>

<sup>3</sup>The operator  $\mathcal{O}_{21}$  can also be written as  $2(\overline{\ell}_\beta^c \varepsilon H_1^*) (\ell_\alpha^c \varepsilon H_2^*) + (\overline{\ell}_\beta^c \varepsilon \ell_\alpha^c) (H_1^\dagger \varepsilon H_2^*)$  using the identity (A9), as done in the first reference of [16].

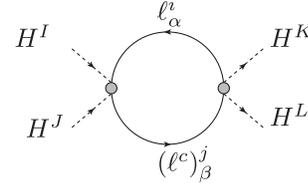


FIG. 2. Two insertions of dimension-five operators can also contribute to dimension-six operators involving four Higgses via this diagram.

Tree-level LFV is often avoided in the 2HDM by imposing a  $Z_2$  symmetry on the renormalizable Lagrangian: if under the  $Z_2$  transformation,  $H_1 \rightarrow H_1$  and  $H_2 \rightarrow -H_2$ , then  $[Y^{(2)}]$ ,  $\lambda_6$  and  $\lambda_7$  are forbidden. We will later discuss this case, but do not impose the  $Z_2$  symmetry from the beginning, because it also forbids the  $C_{21}$  and  $C_A$  coefficients at dimension-five.

### III. THE EFT CALCULATION

#### A. Diagrams and divergences

Diagrams with two insertions of the dimension-five operators are illustrated in Figs. 1 and 2. We focus on the lepton flavor violating diagrams of Fig. 1, and discuss the four-Higgs operators generated by Fig. 2 in Appendix G, because four-Higgs interactions are flavor conserving and arise in the SM.

The Feynman rules arising from the (tree-level) Lagrangian of equations (2.1), (2.4), (2.6) are given in Appendix A. We use them to evaluate, using dimensional regularization in  $4 - 2\epsilon$  dimensions in  $\overline{\text{MS}}$ , the coefficient of the  $1/\epsilon$  divergence of each diagram of Fig. 1. These coefficients can be expressed as a sum of numerical factors multiplying the Feynman rules for the dimension-six operators of Eqs. (2.7) and (2.10) (these Feynman rules are given in Appendix A), and then the EOMs are used to transform the operators of Eq. (2.10) to  $\mathcal{O}_{eH}$  and  $\mathcal{O}_{eH}^\dagger$ . The required counterterm  $\Delta C_{\mathcal{O}}$  for each of the dimension-six operators given in Eq. (2.7) can be identified as  $(-1) \times$  the numerical factor that multiplies its Feynman rule. This counterterm is added in the Lagrangian to the operator coefficient  $C_{\mathcal{O}}$ , resulting in a ‘‘bare’’ coefficient  $C_{\mathcal{O},\text{bare}} = \mu^{2\epsilon} (C_{\mathcal{O}} + \Delta C_{\mathcal{O}})$  that should be independent of the  $\overline{\text{MS}}$  renormalization scale  $\mu$ . Note that the factor  $\mu^{2\epsilon}$  is chosen such that bare Lagrangian remains  $d$ -dimensional.

A more complete and rigorous presentation will be required in the next section, in order to derive the RGEs, so let us replace counterterms by  $Z$  factors in order to minimize notation and introduce the necessary factors of  $\mu^{2\epsilon}$  to obtain the correct dimensions. More details of the formalism and calculations are given in Appendix C.

We allow for multiple operators at both dimension-six and -five, and align the dimension-six coefficients in a row vector  $\vec{C}$ , and the dimension-five coefficients in a row vector  $\vec{C}$ . Then the bare coefficients can be written

$$\vec{C}_{\text{bare}} = \mu^{2\epsilon} \vec{C}[Z], \quad \tilde{C}_{\text{bare}} = \mu^{2\epsilon} [\tilde{C} \hat{Z} + \vec{C}[\tilde{Z}]\vec{C}^\dagger] \quad (3.1)$$

where matrices wearing a hat act on the space of dimension-six coefficients, and matrices in square brackets act in the dimension-five space, so  $\hat{Z}$  represents the renormalization of dimension-six coefficients amongst themselves, and  $[Z]$  represents the renormalization of dimension-five coefficients. The quantity  $[\tilde{Z}]$  renormalizes insertions of two dimension-five operators;  $[\tilde{Z}]_k^{ij}$  is a vector in the dimension-six space with index  $k$ , and a matrix in the dimension-five space with indices  $i, j$ . In the single Higgs model,  $i, j$  correspond to the flavor indices of the Weinberg operator, e.g.,  $i = \alpha\beta, j = \rho\sigma$ . The index  $k$  corresponds to the operator labels and flavor indices of dimension-six operators. The counterterms that renormalize the diagrams of Fig. 1 are then components of the vector  $\vec{C}[\tilde{Z}]\vec{C}^\dagger$ . All terms in the above expressions assume an implicit sum over flavor indices; the explicit flavor dependence is presented in Appendix B.

The first diagram of Fig. 1 has two Higgs and two doublet-lepton legs and so must be renormalized by the operators  $\mathcal{O}_{H\ell(3)}$  and  $\mathcal{O}_{H\ell(1)}$ , and the structures  $\mathcal{S}_{HD\ell(1)}$  and  $\mathcal{S}_{HD\ell(3)}$ . Since these all involve a derivative, the diagram is calculated for finite external momenta. The counterterms that we obtain from this diagram differ from those given in [21]; as discussed in Appendix F, it appears that the authors of [21] dropped one of the terms multiplying the  $1/\epsilon$  divergence. We check our result by attaching an external  $B_\mu$  or  $W_\mu^a$  boson, in all possible ways, to the first diagram of Fig. 1, and verify that our counterterms also cancel the divergences of the 2-Higgs-2-lepton-gauge boson vertices generated by two insertions of the Weinberg operator (this is outlined in Appendix B 3). This diagram can be renormalized using the following counterterms:

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{H\ell(1)}^{\beta\alpha} = -\frac{3}{4} \frac{1}{16\pi^2\epsilon} [C_5 C_5^*]^{\beta\alpha}, \quad (3.2)$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{H\ell(3)}^{\beta\alpha} = +\frac{2}{4} \frac{1}{16\pi^2\epsilon} [C_5 C_5^*]^{\beta\alpha}, \quad (3.3)$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{HD\ell(1)}^{\beta\alpha} = -\frac{3}{4} \frac{1}{16\pi^2\epsilon} [C_5 C_5^*]^{\beta\alpha}, \quad (3.4)$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{HD\ell(3)}^{\beta\alpha} = +\frac{2}{4} \frac{1}{16\pi^2\epsilon} [C_5 C_5^*]^{\beta\alpha}, \quad (3.5)$$

where the last two counterterms represent divergences proportional to the structures  $\mathcal{S}_{HD\ell(1)}$  and  $\mathcal{S}_{HD\ell(3)}$ , which contribute to the renormalization of  $C_{eH}$  through the linear combination given in Eq. (2.10).

The middle diagram of Fig. 1 contributes to  $\mathcal{O}_{eH}^{\beta\alpha}$ , and the divergence it induces can be removed by the counterterm  $(16\pi^2\epsilon)^{-1} [C_5 C_5^* Y]^{\beta\alpha}$  (where the flavor index order is doublet-singlet). Including also the counterterms for  $\mathcal{S}_{HD\ell(1)}^{\beta\alpha}$  and  $\mathcal{S}_{HD\ell(3)}^{\beta\alpha}$  [Eqs. (3.4), (3.5)] gives

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{eH}^{\beta\alpha} = +\frac{3}{4} \frac{1}{16\pi^2\epsilon} [C_5 C_5^* Y]^{\beta\alpha}. \quad (3.6)$$

Since the structures  $\mathcal{S}_{HD\ell(3)}$  and  $\mathcal{S}_{HD\ell(1)}$  are Hermitian, they contribute to the renormalization of both  $\mathcal{O}_{eH}$  and  $\mathcal{O}_{eH}^\dagger$  [see Eq. (2.10)]. Only the contribution to  $\mathcal{O}_{eH}$  is included in (3.6), because the Hermitian conjugate in (2.6) generates a counterterm proportional to  $\mathcal{O}_{eH}^\dagger$  that absorbs the divergence of the ‘‘conjugate’’ process of Fig. 1.

The third diagram of Fig. 1 contributes to the four-lepton operator  $\mathcal{O}_{\ell\ell}^{\rho\alpha\beta\sigma}$ , and the divergence it induces can be removed by the counterterm

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{\ell\ell}^{\rho\alpha\beta\sigma} = -\frac{1}{4} \frac{1}{16\pi^2\epsilon} C_5^{\rho\beta} C_5^{*\sigma\alpha}. \quad (3.7)$$

## B. The 2HDM

In the 2HDM, we consider diagrams analogous to Fig. 1, but with insertions of any of the dimension-five operators given in Eq. (2.13). The external Higgs lines are required to be  $H_1$ , but the internal Higgs lines can be either doublet. The counterterms required to cancel double-insertions of the  $\mathcal{O}_5$  operator, discussed in the previous section, also arise in the 2HDM. In this section, we only list the additional contributions to the counterterms.

We start again with the first diagram of Fig. 1, with  $\mathcal{O}_{21}$  or  $\mathcal{O}_A$  at the vertices. Since by construction, the Feynman rule for  $\mathcal{O}_{21}$  is identical to the rule for  $\mathcal{O}_5$ , double-insertions of  $\mathcal{O}_{21}$  require the same counterterms as given in Eqs. (3.2) to (3.5), but with  $C_5, C_5^*$  replaced by  $C_{21}, C_{21}^*$ . Double insertions of the antisymmetric operator  $\mathcal{O}_A$  require the counterterms:

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{H\ell(1)}^{\beta\alpha} = \frac{1}{4} \frac{1}{16\pi^2\epsilon} [C_A C_A^*]^{\beta\alpha}, \quad (3.8)$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{HD\ell(1)}^{\beta\alpha} = \frac{1}{4} \frac{1}{16\pi^2\epsilon} [C_A C_A^*]^{\beta\alpha}. \quad (3.9)$$

Finally,  $\mathcal{O}_A$  at one vertex and  $\mathcal{O}_{21}$  at the other require the contributions to the counterterms:

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{H\ell(3)}^{\beta\alpha} = \frac{1}{4} \frac{1}{16\pi^2\epsilon} [C_A C_{21}^* - C_{21} C_A^*]^{\beta\alpha}, \quad (3.10)$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{HD\ell(3)}^{\beta\alpha} = \frac{1}{4} \frac{1}{16\pi^2\epsilon} [C_A C_{21}^* - C_{21} C_A^*]^{\beta\alpha}. \quad (3.11)$$

It is straightforward to check, using respectively the anti-symmetry and symmetry of  $C_A$  and  $C_{21}$  on flavor indices, that the combination  $[C_A C_{21}^* - C_{21} C_A^*]$  is Hermitian, as expected for the coefficients of  $\mathcal{O}_{H\ell(3)}$  and  $\mathcal{O}_{HD\ell(3)}$ .

Consider next the middle diagram of Fig. 1. Only the internal Higgs lines can be  $H_2$ , so the additional divergences in the 2HDM will arise from  $\mathcal{O}_A$  or  $\mathcal{O}_{21}$  at the vertex

farthest from the Yukawa coupling, which can be cancelled by the counterterms  $(16\pi^2\epsilon)^{-1}[C_{21}C_5^*Y^{(2)}]^{\beta\alpha}$  and  $-[C_A C_5^* Y^{(2)}]^{\beta\alpha}/(16\pi^2\epsilon)$ . Including also the additional counterterms for  $\mathcal{O}_{HD\ell(1)}^{\beta\alpha}$  and  $\mathcal{O}_{HD\ell(3)}^{\beta\alpha}$  in the 2HDM gives

$$\begin{aligned} \Delta(\vec{C}[\vec{Z}]\vec{C}^\dagger)_{eH}^{\beta\alpha} &= \frac{1}{4} \frac{1}{16\pi^2\epsilon} (4[(C_{21} - C_A)C_5^*Y^{(2)}]^{\beta\alpha} \\ &\quad + [(C_A C_A^* + C_A C_{21}^* - C_{21} C_A^* \\ &\quad - C_{21} C_{21}^*)Y^{(1)}]^{\beta\alpha}). \end{aligned} \quad (3.12)$$

Finally, for the four-lepton operator, there are additional counterterms in the 2HDM to cancel the divergences induced by double-insertions of  $\mathcal{O}_{22}$ , of  $\mathcal{O}_{21}$ , and of  $\mathcal{O}_A$ . (The possible diagrams with an insertion of both  $\mathcal{O}_{21}$  and  $\mathcal{O}_A$  vanish due to antisymmetry.) We obtain:

$$\begin{aligned} \Delta(\vec{C}[\vec{Z}]\vec{C}^\dagger)_{\ell\ell}^{\rho\sigma\beta\alpha} &= -\frac{1}{4} \frac{1}{16\pi^2\epsilon} C_{22}^{\rho\beta} C_{22}^{*\alpha\sigma} - \frac{1}{2} \frac{1}{16\pi^2\epsilon} C_{21}^{\rho\beta} C_{21}^{*\alpha\sigma} \\ &\quad + \frac{1}{2} \frac{1}{16\pi^2\epsilon} C_A^{\rho\beta} C_A^{*\alpha\sigma}. \end{aligned} \quad (3.13)$$

### C. The renormalization group equations

The contribution of dimension-five operators to the renormalization group equations of dimension-six operators, due to double insertions, can be obtained following the discussion of Herrlich and Nierste [31]. The derivation is presented in Appendix C. Here we schematically outline the result.

The bare Lagrangian coefficients are defined at one loop as in Eq. (3.1), where the counterterm for one operator can depend on the coefficients of other operators. Recall that the bare coefficients are independent of the dimensionful parameter  $\mu$ , and that the renormalized  $C$ s are dimensionless. Using  $\vec{C} = \mu^{-2\epsilon} \vec{C}_{\text{bare}}[Z^{-1}]$  allows one to obtain, in  $4 - 2\epsilon$  dimensions:

$$\begin{aligned} (16\pi^2)\mu \frac{d}{d\mu} \vec{C} &= -\vec{C} \left\{ 2\epsilon(16\pi^2) + (16\pi^2) \left[ \mu \frac{d}{d\mu} Z \right] [Z^{-1}] \right\} \\ &\equiv \vec{C}[\gamma] - 2\epsilon(16\pi^2)\vec{C} \end{aligned} \quad (3.14)$$

where  $[\gamma]$  denotes the 4-dimensional anomalous dimension matrix, and we (unconventionally)<sup>4</sup> factor the  $16\pi^2$  out of the anomalous dimension matrices. While the  $-2\epsilon$  term does not contribute in  $d = 4$  dimensions to the mixing of the dimension-five operators, it plays an essential role in the renormalization group equations of the dimension-six operators.

<sup>4</sup>The usual definition [15] is  $\mu \frac{d}{d\mu} C = C\gamma$ , then  $\gamma$  is expanded in loops:  $\gamma = \frac{a_s}{4\pi} \gamma_0 + \dots$ . However, here we only work at one loop, have other subscripts on our  $\gamma$ s and the one loop mixing of dimension-five-squared into dimension-six is not induced by a renormalizable coupling. So we factor out the  $16\pi^2$ .

For the dimension-six coefficients, it is straightforward to obtain from Eq. (3.1):

$$\begin{aligned} \mu \frac{d}{d\mu} \vec{C} &= -\vec{C} \cdot \left\{ \mu \frac{d}{d\mu} \hat{Z} \right\} \hat{Z}^{-1} + 2\epsilon \vec{C} \cdot \vec{Z} \cdot \vec{C}^\dagger \hat{Z}^{-1} \\ &\quad - \vec{C} \cdot \left[ \mu \frac{d}{d\mu} \vec{Z} \right] \cdot \vec{C}^\dagger \hat{Z}^{-1} \\ &\quad - \vec{C} \cdot [Z] \left[ \mu \frac{d}{d\mu} Z^{-1} \right] \cdot [\vec{Z}] \cdot \vec{C}^\dagger \hat{Z}^{-1} \\ &\quad - \vec{C} \cdot [\vec{Z}] \cdot \left[ \mu \frac{d}{d\mu} Z^{-1} \right]^\dagger [Z]^\dagger \vec{C}^\dagger \hat{Z}^{-1}, \end{aligned} \quad (3.15)$$

where terms of  $\mathcal{O}(\epsilon)$  that vanish in 4 dimensions are neglected, and the summation over flavor and operator indices is indicated with a dot. The second line can be dropped, because the first term vanishes at one loop, and the remaining terms are of two-loop order because both  $[\vec{Z}]$  and  $d[Z^{-1}]/d\mu$  arise at one-loop. So the renormalization group equations for the dimension-six coefficients can be written

$$(16\pi^2)\mu \frac{d}{d\mu} \vec{C} = \vec{C} \hat{\gamma} + \vec{C}[\vec{\gamma}]\vec{C}^\dagger, \quad (3.16)$$

where  $\hat{\gamma}$  is the one-loop anomalous dimension matrix for dimension-six operators [19] and  $[\vec{\gamma}] = 2(16\pi^2)\epsilon[\vec{Z}]$  is the anomalous dimension tensor.

We give below the anomalous dimensions describing the one-loop mixing of double-insertions of dimension-five operators into LFV dimension-six operators, in the 2HDM. The single Higgs model can be easily retrieved by setting  $C_{21} = C_A = C_{22} = 0$  in the equations below. The anomalous dimension tensor mixing a pair of dimension-five operators into a dimension-six operator is necessarily a three-index object; below we sum over the two dimension-five indices, and give these summed components of the tensor as elements of a vector in the dimension-six operator space. These anomalous dimensions parametrize the mixing of Fig. 1 in the 2HDM (recall that a factor  $1/16\pi^2$  is scaled out of our anomalous dimensions):

$$\begin{aligned} (\vec{C}[\vec{\gamma}]\vec{C}^\dagger)_{H\ell(1)}^{\beta\alpha} &= -C_5^{\beta\rho} \frac{3\delta_{\rho\sigma}}{2} C_5^{*\sigma\alpha} - C_{21}^{\beta\rho} \frac{3\delta_{\rho\sigma}}{2} C_{21}^{*\sigma\alpha} \\ &\quad + C_A^{\beta\rho} \frac{\delta_{\rho\sigma}}{2} C_A^{*\sigma\alpha} \end{aligned} \quad (3.17)$$

$$\begin{aligned} (\vec{C}[\vec{\gamma}]\vec{C}^\dagger)_{H\ell(3)}^{\beta\alpha} &= C_5^{\beta\rho} \delta_{\rho\sigma} C_5^{*\sigma\alpha} + C_{21}^{\beta\rho} \delta_{\rho\sigma} C_{21}^{*\sigma\alpha} + C_A^{\beta\rho} \frac{\delta_{\rho\sigma}}{2} C_A^{*\sigma\alpha} \\ &\quad - C_{21}^{\beta\rho} \frac{\delta_{\rho\sigma}}{2} C_A^{*\sigma\alpha} \end{aligned} \quad (3.18)$$

$$\begin{aligned} (\vec{C}[\vec{\gamma}]\vec{C}^\dagger)_{eH}^{\beta\alpha} &= C_5^{\beta\rho} \frac{3[Y^{(1)}]_{\eta\alpha} \delta_{\rho\sigma}}{2} C_5^{*\sigma\eta} + 2[(C_{21} - C_A)C_5^*Y^{(2)}]^{\beta\alpha} \\ &\quad + \frac{1}{2} [(C_A C_A^* + C_A C_{21}^* - C_{21} C_A^* - C_{21} C_{21}^*)Y^{(1)}]^{\beta\alpha} \end{aligned} \quad (3.19)$$

$$\begin{aligned}
(\vec{C}[\tilde{\gamma}]\vec{C}^\dagger)_{\ell\ell}^{\rho\sigma\beta\alpha} &= -C_5^{\beta\rho} \frac{1}{2} C_5^{*\sigma\alpha} - C_{22}^{\beta\rho} \frac{1}{2} C_{22}^{*\sigma\alpha} \\
&\quad - C_{21}^{\beta\rho} C_{21}^{*\sigma\alpha} + C_A^{\beta\rho} C_A^{*\sigma\alpha}
\end{aligned} \quad (3.20)$$

where the operator label and flavor indices on the left-hand side refer to the dimension-six operator (the dimension-five indices are summed).

In the next section, we will need the RGEs for dimension-five operators. Recall that in the single Higgs model,  $[\gamma]$  is in principle a  $9 \times 9$  matrix (or  $6 \times 6$ , if one uses the symmetry of  $C_5^{\alpha\beta}$ ), mixing the elements of  $C_5$  among themselves. However, in the basis where the charged leptons are diagonal,  $[\gamma]$  is diagonal, and the anomalous dimension for the coefficient  $C_5^{\alpha\beta}$  of the Weinberg operator is [16]:

$$\begin{aligned}
16\pi^2\gamma &= -\frac{3}{2}([Y_e]_{\alpha\alpha}^2 + [Y_e]_{\beta\beta}^2) + (\lambda - 3g_2 + 2\text{Tr}(3[Y_u]^\dagger[Y_u] \\
&\quad + 3[Y_d]^\dagger[Y_d] + [Y_e]^\dagger[Y_e]))
\end{aligned} \quad (3.21)$$

where the Higgs self-interaction in the SM Lagrangian is  $\frac{\lambda}{4}(H^\dagger H)^2$ , and  $[Y_f]$  are the fermion Yukawa matrices.

#### IV. PHENOMENOLOGY

In order to solve the RGEs, it is convenient to define  $t = \frac{1}{16\pi^2} \ln \frac{\mu}{m_W}$ , in which case the one-loop RGEs for dimension-five and -six operator coefficients can be written as

$$\begin{aligned}
\frac{d}{dt}\vec{C} &= \vec{C} \cdot \hat{\gamma} + \vec{C} \cdot [\tilde{\gamma}] \cdot \vec{C}^\dagger \\
\frac{d}{dt}\vec{C}^\dagger &= \vec{C}^\dagger \cdot [\tilde{\gamma}].
\end{aligned} \quad (4.1)$$

These are among the most familiar of differential equations, whose solutions have the form

$$\vec{C}(t_f) = \vec{C}(0) \exp\{\gamma t_f\} \simeq \vec{C}(0) \left[ 1 + \gamma \frac{1}{16\pi^2} \ln\left(\frac{\Lambda}{m_W}\right) + \dots \right] \quad (4.2)$$

$$\vec{C}^\dagger(t_f) = \left[ \int_0^{t_f} d\tau \vec{C}^\dagger(0) e^{\gamma\tau} [\tilde{\gamma}] [e^{\gamma\tau}]^T \vec{C}^\dagger(0) e^{-\hat{\gamma}\tau} + \vec{C}^\dagger(0) \right] e^{\hat{\gamma}t_f} \quad (4.3)$$

where  $16\pi^2 t_f = \ln(\frac{\Lambda}{m_W})$ . In these solutions, the anomalous dimension matrices were approximated as constant; this is not necessarily a good approximation, because the anomalous dimensions depend on running coupling constants, in particular the Yukawa couplings can evolve significantly above  $m_W$ . Although the impact of the running Yukawa couplings is a higher order effect if the logarithms are not

resummed, their contribution could be amplified by a large logarithm. Therefore, if one wished to perform a precision prediction of lepton flavor violation in the case that  $\Lambda \gg v$ , it would be necessary to incorporate the effect from running Yukawa couplings. However, this mass hierarchy highly suppresses lepton flavor violation, and such a precision prediction is beyond the scope of this work.

A simple solution to Eq. (4.3) can be obtained by expanding the exponentials under the integral, as in Eq. (4.2):

$$\begin{aligned}
\vec{C}(m_W) &= \vec{C}(\Lambda) - \vec{C}(\Lambda) \hat{\gamma} \frac{1}{16\pi^2} \ln \frac{\Lambda}{m_W} \\
&\quad - \vec{C}(\Lambda) [\tilde{\gamma}] \vec{C}^\dagger(\Lambda) \frac{1}{16\pi^2} \ln \frac{\Lambda}{m_W} + \dots
\end{aligned} \quad (4.4)$$

#### A. The single Higgs model

In the SM case where there is only one Higgs doublet, there is only the Weinberg operator at dimension-five: a symmetric  $3 \times 3$  matrix, whose entries are determined by neutrino masses and mixing angles (in the mass basis of charged leptons). We now want to estimate the contribution of double-insertions of this dimension-five operator to lepton-flavor violating processes.

We neglect the ‘‘Majorana phases,’’ suppose that the lightest neutrino mass is negligible, and neglect the lepton Yukawas in the RGEs. Then the RG running of  $C_5^{\alpha\beta}$  between  $m_W$  and  $\Lambda$  can be approximated as a rescaling, with  $\gamma \approx \lambda - 3g_2 + 6y_t^2 \approx 3.5$ :

$$C_5^{\alpha\beta}(\Lambda) = C_5^{\alpha\beta}(m_W) \left[ 1 + 3.5 \frac{1}{16\pi^2} \ln \frac{\Lambda}{m_W} + \dots \right] \quad (4.5)$$

For  $\Lambda \leq 10^{16}$  GeV, the log is  $\leq 32$ .

We can now estimate the contribution of the neutrino mass operator to lepton flavor violating processes from Eq. (4.4). We neglect  $\vec{C}(\Lambda)$  and find that the contribution is  $\frac{1}{16\pi^2} \ln \frac{\Lambda}{m_W} \times$  the coefficients of Eqs. (3.17)–(3.20), that is, of order

$$\vec{C}(m_W) \sim \frac{C_5^2}{16\pi^2} \ln \frac{\Lambda}{m_W}. \quad (4.6)$$

As expected, this is negligibly small, because  $C_5^2/\Lambda^2 \sim m_\nu^2/v^4$ . Notice, however, that the logarithm can be large, and the anomalous dimension is  $\mathcal{O}(1)$ ; the effect is tiny because small neutrino masses imply that  $C_5$  is small or  $\Lambda$  is large.

#### B. The two Higgs doublet model

Experimental neutrino data constrain the dimension-five operator in the one Higgs doublet model, so the lepton flavor violating effects estimated in Eq. (4.6) are

suppressed by the smallness of the neutrino masses. The situation changes in an extended Higgs sector, where more than one dimension-five operator is present. The operator  $\mathcal{O}_A$  cannot contribute to neutrino masses as it is antisymmetric in flavor space and is hence unconstrained. In addition, the neutrino mass contribution of operators  $\mathcal{O}_{21}$  and  $\mathcal{O}_{22}$  is suppressed if the vacuum expectation value of the second Higgs doublet is small. Renormalization group effects [16–18] will in general mix all operators, which could lift these suppression mechanisms at loop level. However the mixing factorizes in the limit where  $\lambda_6, \lambda_7$  and  $Y^{(2)}$  tend to zero: then the operators  $\mathcal{O}_{21}$  and  $\mathcal{O}_A$  will not mix into  $\mathcal{O}_5$  and  $\mathcal{O}_{22}$  and are hence not constrained by the observed neutrino masses. Furthermore, the mixing of  $\mathcal{O}_{22}$  into  $\mathcal{O}_5$  vanishes in the limit where in addition  $\lambda_5$  tends to zero (see [32] for a symmetry argument).

In the following we will study the sensitivity of lepton-flavor violating decays to these additional operators. We assume that the Wilson coefficients of the dimension-five operators are generated at  $\Lambda = 10$  TeV, while all other dimension-six Wilson coefficients are zero at this scale. To avoid constraints from the observed neutrino masses we consider the scenario where the second Higgs doublet has a negligible vacuum expectation value and a mass at the weak scale. The Higgs sector could be assumed to be close to that of an inert two-Higgs doublet model [26–29] and the dangerous couplings  $\lambda_6, \lambda_7$  and  $Y^{(2)}$  are not generated radiatively. Renormalization group running will then generate nonzero Wilson coefficients of several dimension-six operators at  $\mu \sim v$ . Only those dimension-six operators that involve standard model particles are of interest to us, since the vanishing vacuum expectation value of the second Higgs doublet will suppress the contribution of the other operators after spontaneous symmetry breaking. Applying the constraints presented in Appendix E of the Wilson coefficients evaluated using Eq. (4.4) neglecting the small  $\log \ln(m_{22}/m_W)$ , we find the following: the  $\mu \rightarrow 3e$  decays provide the greatest sensitivity to the additional dimension-five Wilson coefficients. In particular the left-handed contribution implies

$$\left| C_{21}^{ee} C_{21}^{e\mu*} + 0.5 C_{22}^{ee} C_{22}^{e\mu*} + 0.1 \sum_{\sigma} (C_A^{e\sigma} - C_{21}^{e\sigma})(C_A^{\sigma\mu*} + C_{21}^{\sigma\mu*}) \right| < \frac{1}{5.2 \ln(\Lambda/m_{22})} \left( \frac{\Lambda}{10 \text{ TeV}} \right)^2, \quad (4.7)$$

where we neglected the mixing of the dimension-five operators amongst themselves, as this would contribute at two-loop order to the lepton flavor violating processes. For the right-handed contribution we find

$$\left| \sum_{\sigma} (C_A^{e\sigma} - C_{21}^{e\sigma})(C_A^{\sigma\mu*} + C_{21}^{\sigma\mu*}) \right| < \frac{1.6}{\ln(\Lambda/m_{22})} \left( \frac{\Lambda}{10 \text{ TeV}} \right)^2, \quad (4.8)$$

which exhibits a weaker sensitivity. The contribution of the  $\mu \leftrightarrow e$  flavor-changing  $Z$  vertex to  $\mu \rightarrow e\gamma$  is relatively suppressed by a loop factor, so is beyond current experimental sensitivity. However, this  $Z$  vertex contributes at tree-level to  $\mu \rightarrow e$  conversion, in interference with vector and scalar 2-quark-2-lepton operators. Indeed, the current sensitivity of  $\mu \rightarrow e$  conversion in gold is  $|C_{H\ell(1)}^{e\mu} + C_{H\ell(3)}^{e\mu}| \approx 1.4 \times 10^{-7} (\Lambda/m_t)^2$ . The resulting constraints on the Wilson coefficients reads:

$$\left| \sum_{\sigma} (C_A^{e\sigma} - C_{21}^{e\sigma})(C_A^{\sigma\mu*} + C_{21}^{\sigma\mu*}) \right| < \frac{1}{6.5 \ln(\Lambda/m_{22})} \left( \frac{\Lambda}{10 \text{ TeV}} \right)^2, \quad (4.9)$$

We also checked that the current experimental situation for  $\tau$  decays does not lead to significant constraints.

## V. SUMMARY

Motivated by neutrino masses and the expected progress in searches for lepton flavor violation, we calculated the leading one-loop contribution of a pair of lepton number violating dimension-five operators to the coefficients of lepton flavor violating dimension-six operators. The diagrams are given in Fig. 1. The dimension-five operators that we considered are the Weinberg operator, constructed out of SM fields and given in Eq. (2.4), and three additional dimension-five operators that can be constructed in the two Higgs doublet model, given in Eq. (2.13). The dimension-six, lepton flavor violating operators of the SMEFT are listed in Appendix D, in the ‘‘Warsaw’’ basis, and the subset of these operators relevant for our calculation is given in Eq. (2.7). A selection of constraints on their coefficients, evaluated at the weak scale, is given in Appendix E.

In Sec. III, we obtain the anomalous dimensions mixing two dimension-five operators into the lepton flavor violating operators of Eq. (2.7). The required counterterms are given in Eqs. (3.2)–(3.7) for the standard model with a single Higgs, and in Eqs. (3.8)–(3.13) for the case of the two Higgs doublet model. Then in Sec. III C, we outline the derivation of the renormalization group equations (Appendices B and C present our calculation and the flavor dependence of our result in more detail), and the resulting anomalous dimensions are listed in Eqs. (3.17)–(3.20). The mixing of two dimension-five operators into the lepton flavor conserving four-Higgs operator, via the diagram of Fig. 2 is given in Appendix G; however, we do not consider mixing into dimension six operators constructed with the second Higgs of the 2 Higgs doublet model. This completes the one loop renormalization group equations of the standard model effective theory, up to operators of dimension six. It is amusing that the insignificant effect we calculate does not involve standard

model couplings, so, in an expansion in terms of SM couplings, our result is the “leading” contribution to the one-loop RGEs of the dimension-six SMEFT.<sup>5</sup>

In the effective field theory constructed with standard model fields, the coefficient of the Weinberg operator is proportional to the neutrino mass matrix. So the lepton flavor changing amplitudes induced by double insertions of the Weinberg operator are  $\propto (m_\nu/m_W)^2 \ln \Lambda/m_W$ , and far below current sensitivities. This is outlined in Sec. IV A. However, the situation is different in the 2 Higgs doublet model, as discussed in Sec. IV B: there are four operators at dimension five, and the neutrino mass matrix only constrains one combination. We evaluated the mixing of the four operators into lepton flavor violating operators of the standard model effective theory, and for a lepton number violating scale of 10 TeV we found that the current experimental value of  $\mu \rightarrow 3e$  is sensitive to the Wilson coefficients of these additional operators.

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## APPENDIX A: FEYNMAN RULES AND IDENTITIES

### 1. Feynman rules

We use Feynman rules of Ref. [25], in order that the fermion traces in loops multiply spinors in the correct order. The Feynman rule for the Weinberg operator of Eq. (2.4) can be obtained reliably by using Lehmann-Symanzik-Zimmermann reduction or Wick’s theorem, which gives the signs for fermion interchange. The fermion fields are expanded as [33]

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E}} (a_k^s u_s(k) e^{-ik \cdot x} + b_k^{s\dagger} v_s(k) e^{+ik \cdot x})$$

so the amplitude  $\mathcal{M}_{fi}$  is

<sup>5</sup>Mixing among dimension six operators occurs via the exchange of a SM particle, so is  $\propto [\text{SM coupling}]^2$ .

$$\begin{aligned} & \langle \ell_{\alpha j} H_I | i \frac{C_5^{\alpha\beta}}{2\Lambda} (\overline{\ell}_{\sigma n} \varepsilon_{nN} H^{N*}) (\ell_{\rho m}^c \varepsilon_{mM} H^{M*}) | \ell_{\beta i}^c H_J^* \rangle \\ &= (-i) i \frac{C_5^{\alpha\beta}}{2\Lambda} (\overline{u}_{\alpha j} P_R u_{\beta i} + \overline{u}_{\beta i} P_R u_{\alpha j}) (\varepsilon_{il} \varepsilon_{jJ} + \varepsilon_{iJ} \varepsilon_{jI}) \\ &= (-i) i \frac{C_5^{\alpha\beta} + C_5^{\beta\alpha}}{2\Lambda} \overline{u}_{\alpha j} P_R u_{\beta i} (\varepsilon_{il} \varepsilon_{jJ} + \varepsilon_{iJ} \varepsilon_{jI}) \\ &= (-i) i \frac{C_5^{\alpha\beta}}{\Lambda} \overline{u}_{\alpha j} P_R u_{\beta i} (\varepsilon_{il} \varepsilon_{jJ} + \varepsilon_{iJ} \varepsilon_{jI}), \end{aligned} \quad (\text{A1})$$

where the SU(2) lepton indices are lower case, Higgs indices are upper case,  $\ell_{\alpha j}$  and  $\ell_{\alpha j}^c$  represent a final state lepton and an initial state anti-lepton respectively. The factor  $i$  is the usual factor for Feynman rules and the factor  $(-i)$  is due to the calculation of  $\mathcal{M}_{fi}$ . This expression agrees with Feynman rule of Ref. [21].

A Feynman-rule to attach a  $W$ -boson to the  $\ell^c$  line also will be needed. With the following identities [25]

$$\ell^c = C \overline{\ell}^T, \quad C = i\gamma_0 \gamma_2, \quad C^{-1} = C^\dagger, \quad C^\dagger \gamma^{\mu T} C = -\gamma^\mu \quad (\text{A2})$$

$$\begin{aligned} \overline{\ell}^c &= [C \gamma_0^T \ell^{*}]^\dagger \gamma_0 = \ell^T \gamma_0 C^\dagger \gamma_0 = \ell^T C^\dagger C \gamma_0 C^\dagger \gamma_0 \\ &= -\ell^T C^\dagger \gamma_0 \gamma_0 = -\ell^T C^{-1} \end{aligned} \quad (\text{A3})$$

one obtains (where the  $(-1)$  is for interchanging fermions)

$$[\overline{\ell}_i \tau_{ij} W / P_L \ell_j]^T = (-1) [-\overline{\ell}_j^c C \tau_{ji}^{*} P_L^T W_\mu^a \gamma_\mu^T C^{-1} \ell^c] \quad (\text{A4})$$

$$= \overline{\ell}^c \tau^{*a} W_\mu^a C \gamma_\mu^T C^{-1} P_R \ell^c \quad (\text{A5})$$

$$= -\overline{\ell}^c \tau^{*a} W_\mu^a \gamma_\mu P_R \ell^c \quad (\text{A6})$$

and recall that  $\tau = \tau^\dagger$ , so  $\tau^* = \tau^T$ . The Feynman rules for the dimension-four, dimension-five and physical dimension-six interactions are given in Figs. 3, 4, and 5 respectively. The Feynman rules for the dimension-six equation of motion vanishing interactions are given in Fig. 6.

### 2. Identities

The following identities are useful:

$$2\varepsilon_{il} \varepsilon_{jJ} = \delta_{ij} \delta_{lJ} - \tau_{ij}^a \tau_{a,lJ} \quad \text{Fierz} \quad (\text{A7})$$

$$\frac{1}{4} \tau_{ij}^a \tau_{a,kl} = \frac{1}{2} \delta_{il} \delta_{kj} - \frac{1}{4} \delta_{ij} \delta_{kl} \quad \text{SU}(N) \quad (\text{A8})$$

$$\varepsilon_{ab} \varepsilon_{cd} + \varepsilon_{bc} \varepsilon_{ad} + \varepsilon_{ac} \varepsilon_{bd} = 0 \quad (\text{A9})$$

$$\varepsilon_{ij} \varepsilon_{kJ} = \delta_{ik} \quad (\text{A10})$$

$$\varepsilon_{ij} S_{jk}^a \varepsilon_{kl} = S_{li}^a \quad (\text{A11})$$

$$0 = \delta_{ij} S_{kl}^a - \delta_{jl} S_{ki}^a + \delta_{kl} S_{ji}^a - \delta_{ik} S_{jl}^a \quad (\text{A12})$$

$$\varepsilon_{ij} \varepsilon_{kl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}, \quad (\text{A13})$$

where

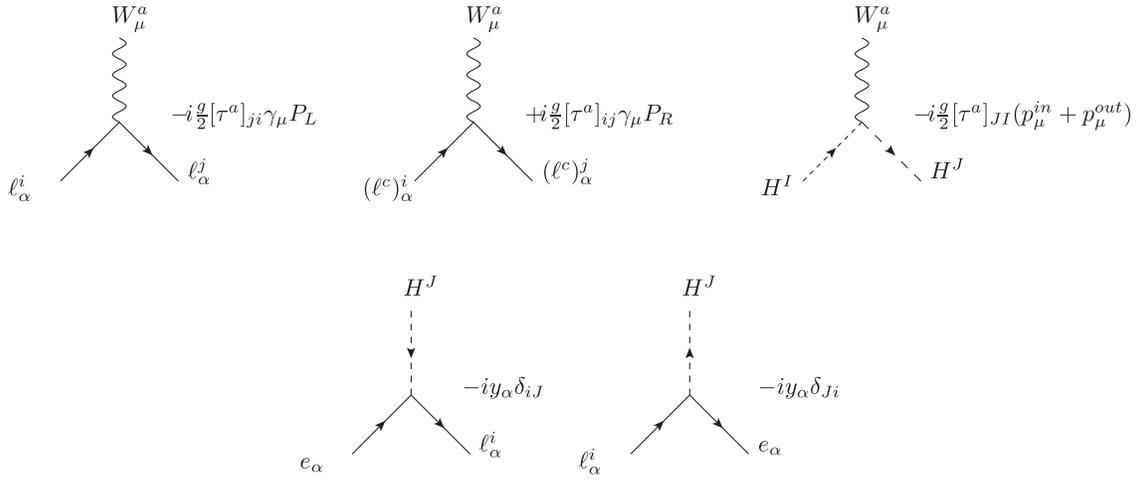
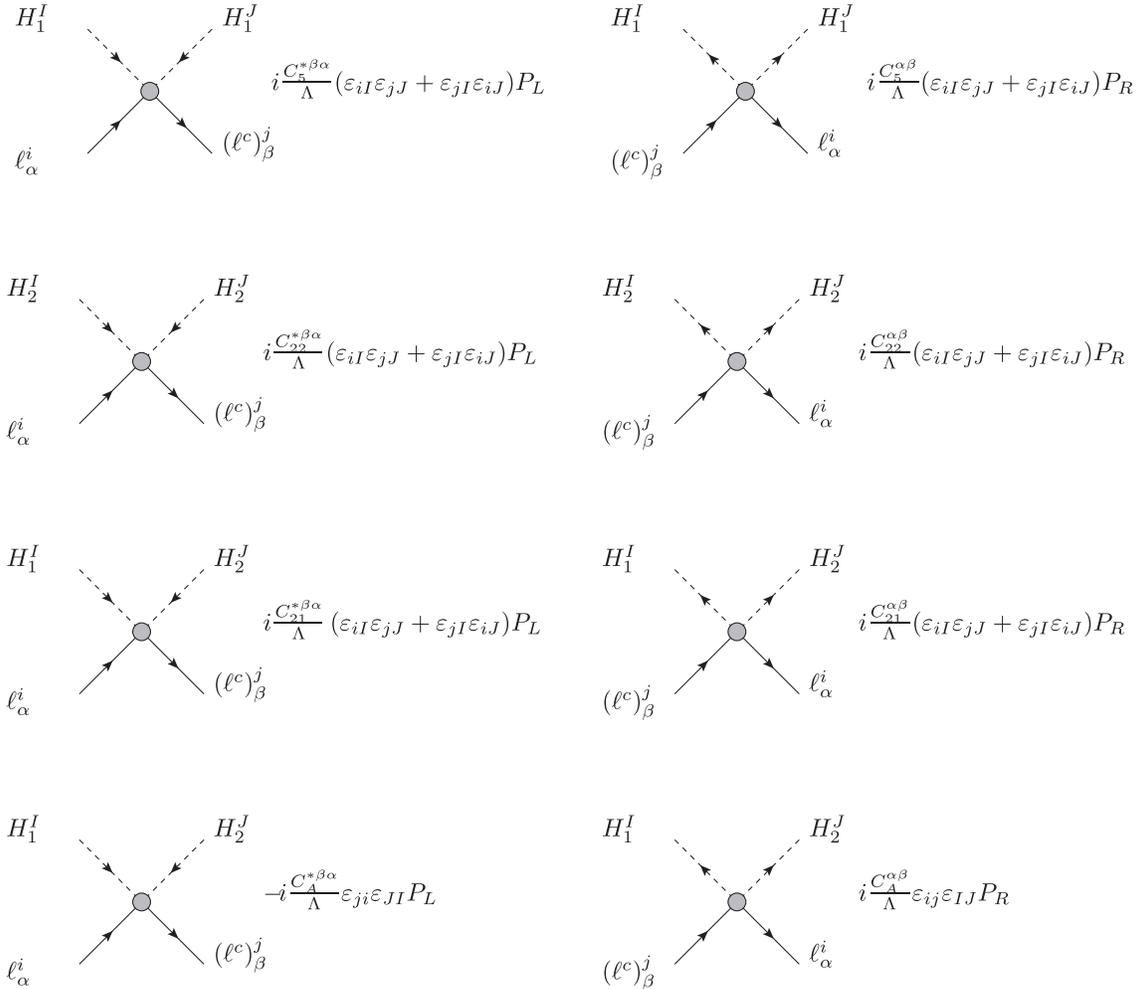
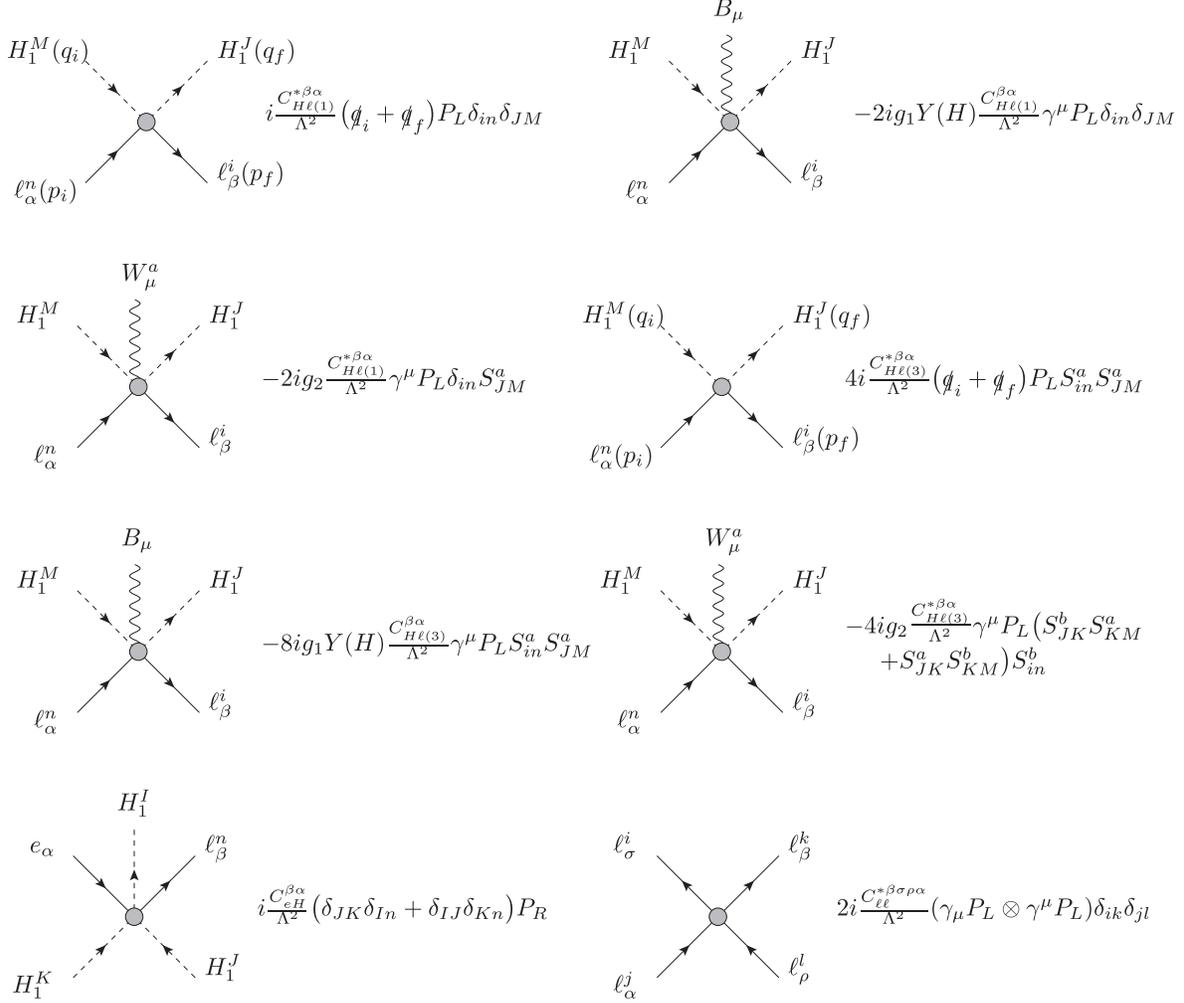


FIG. 3. Feynman rules for dimension-four interactions.


 FIG. 4. Feynman rules for dimension-five interactions, in the single and two Higgs Doublet Models.  $H_1$  is the SM Higgs.  $H_2$  is the second Higgs of the 2HDM, with the same hypercharge as the SM Higgs, opposite to the lepton doublet.


 FIG. 5. Feynman rules for dimension-six operators of the SMEFT using the “Warsaw”-basis [20].  $H_1$  is the SM Higgs.

$$\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \vec{\tau} = \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

and the SU(2) generators are  $S^a = \tau^a/2$ .

## APPENDIX B: THE LOOP CALCULATION

### 1. Flavor dependence

We allow for multiple operators at both dimension-five and -six, and denote a particular Wilson coefficient by  $C_X^\zeta$ , where  $X$  and  $\zeta$  are the operator and flavor labels, respectively. Then the bare Wilson coefficients of the dimension-six standard model effective theory Lagrangian can be written as

$$\begin{aligned} & \sum_{\zeta, X} C_{X, \text{bare}}^\zeta \mathcal{Q}_{X, \text{bare}}^\zeta \\ &= \mu^{2\epsilon} \sum_{\theta, Y} \left( \sum_{\zeta, X} C_X^\zeta Z_{XY}^{\zeta\theta} + \sum_{\zeta, \eta} C_5^\zeta [C_5^\eta]^\dagger Z_{55, Y}^{\zeta\eta\theta} \right) \mathcal{Q}_{Y, \text{bare}}^\theta, \end{aligned} \quad (\text{B1})$$

where  $\zeta$ ,  $\eta$  and  $\theta$  represent generation indices of an operator, and the renormalization constants  $Z_{XY}^{\zeta\theta}$  encode the mixing of dimension-six Wilson coefficients amongst themselves, which can be extracted from the anomalous dimensions of reference [19]. In the standard model, the mixing of two dimension-five Wilson coefficients into a dimension-six coefficient is given by  $Z_{55, Y}^{\zeta\eta\theta}$ . They are induced by the double-insertions of dimension-five operators, as shown in Fig. 1. In the case of a 2HDM effective field theory we extend the summation of the dimension-five flavor indices to a sum over all dimension-five operators and their respective flavor components.

The renormalization constants can be expanded in the number of loops and powers of epsilon. At one-loop in the  $\overline{\text{MS}}$  scheme the counterterms of the physical and EOM-vanishing operators are pure  $1/\epsilon$  poles, and the renormalization of evanescent operators does not play a role. Hence we can expand

$$Z_{55, j}^{\zeta\eta\theta} = \frac{1}{16\pi^2} \frac{1}{\epsilon} \delta Z_{55, j}^{\zeta\eta\theta} \quad (\text{B2})$$

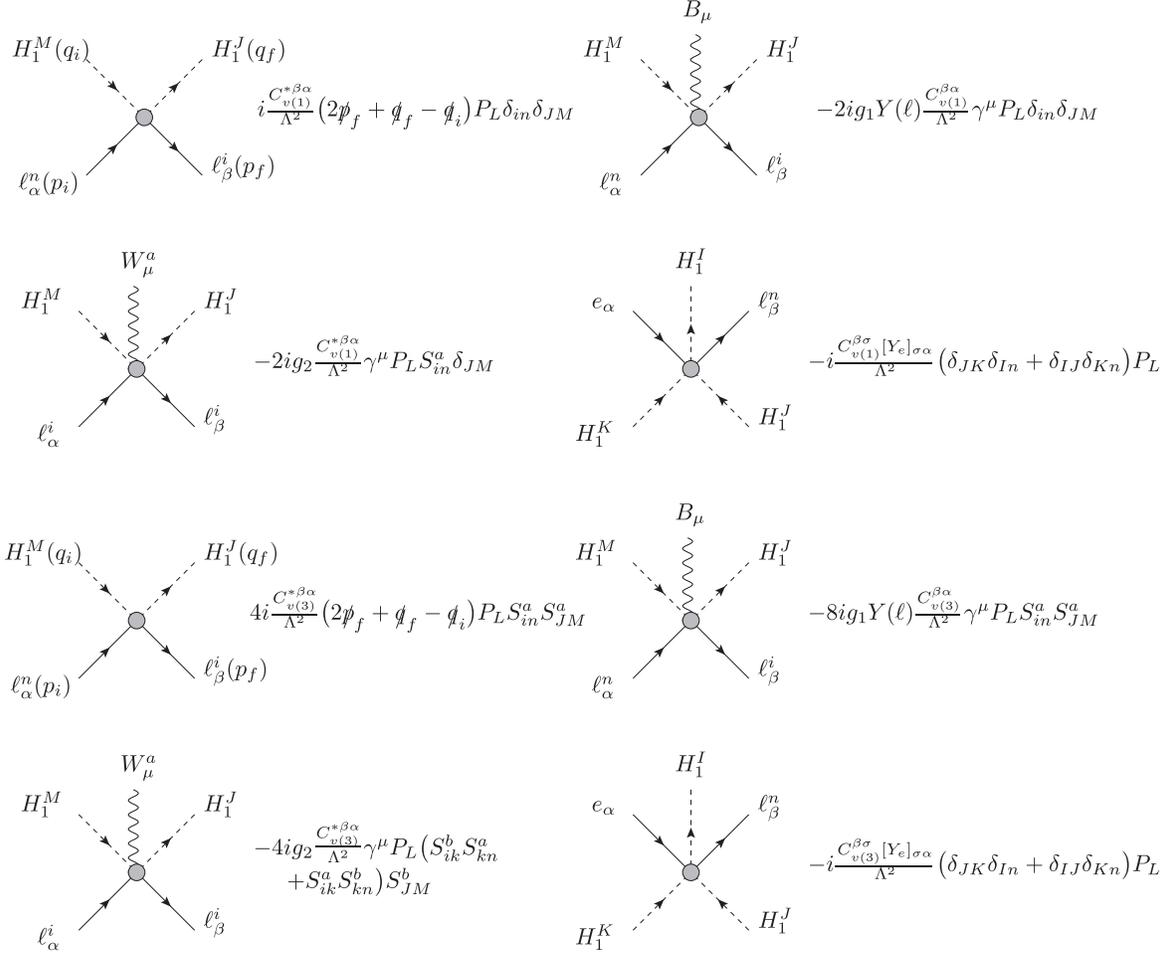


FIG. 6. Feynman rules for dimension-six operators that are vanishing by the equations of motion, in the single Higgs doublet model (SMEFT).  $H_1$  is the SM Higgs. Note that we have chosen a convention for our Feynman rules to eliminate any dependence on the momentum of the incoming lepton,  $p_i$ , since all momenta are not independent.

and write the generation summation in the case of an operator involving four fermions explicitly as:

$$C_5^\zeta C_5^{\eta\dagger} \delta Z_{55,X}^{\zeta\eta\theta} Q_X^\theta = C_5^{\alpha\beta} C_5^{\delta\gamma*} \delta Z_{55,X}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} Q_X^{\rho\sigma\tau\nu}. \quad (\text{B3})$$

The sum over generation indices reduces trivially for operators that involve less fermions. The corresponding renormalization equation ensures that the pole of the one-loop off-shell matrix element of an insertion of two dimension-five operators is cancelled by its counterterm. Factoring out the common overall factor  $C_5^{\alpha\beta} C_5^{\delta\gamma*}$  we write:

$$\langle f | Q_5^{\alpha\beta} (Q_5^{\gamma\delta})^\dagger | i \rangle_{1/\epsilon}^{(1)} + (\delta Z_{55,X}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} \langle f | Q_X^{\rho\sigma\tau\nu} | i \rangle + \text{H.c.}) = 0, \quad (\text{B4})$$

where  $|1/\epsilon\rangle$  denotes the  $1/\epsilon$  pole of a one-loop diagram and  $\langle f |$  and  $|i\rangle$  are arbitrary off-shell final and initial states.

In calculations of the loop diagrams the following generation structures arose:

$$\begin{aligned} T_1^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{2} (\delta_{\delta\sigma} \delta_{\alpha\rho} \delta_{\beta\gamma} + \delta_{\alpha\delta} \delta_{\beta\rho} \delta_{\gamma\sigma}), \\ T_{1A}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{2} (\delta_{\delta\sigma} \delta_{\alpha\rho} \delta_{\beta\gamma} - \delta_{\alpha\delta} \delta_{\beta\rho} \delta_{\gamma\sigma}), \\ T_2^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{2} (\delta_{\alpha\rho} \delta_{\beta\gamma} Y_{\delta\sigma} + \delta_{\alpha\delta} \delta_{\beta\rho} Y_{\gamma\sigma}), \\ T_{2A}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{2} (\delta_{\alpha\rho} \delta_{\beta\gamma} Y_{\delta\sigma}^{(2)} - \delta_{\alpha\delta} \delta_{\beta\rho} Y_{\gamma\sigma}^{(2)}), \\ T_3^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} &= \frac{1}{4} (\delta_{\delta\sigma} \delta_{\gamma\nu} + \delta_{\delta\nu} \delta_{\gamma\sigma}) (\delta_{\alpha\tau} \delta_{\beta\rho} + \delta_{\alpha\rho} \delta_{\beta\tau}), \\ T_{3A}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} &= -\frac{1}{4} (\delta_{\delta\sigma} \delta_{\gamma\nu} - \delta_{\delta\nu} \delta_{\gamma\sigma}) (\delta_{\alpha\tau} \delta_{\beta\rho} - \delta_{\alpha\rho} \delta_{\beta\tau}). \end{aligned} \quad (\text{B5})$$

These structures, when contracted with Wilson coefficients and multiplied by loop factors and the appropriate  $\delta Z$ s, give the results in Eqs. (3.2)–(3.13). Note that in intermediate steps, more general structures than the  $T_{1(A)}$ ,  $T_{2(A)}$  listed here

arise, which show a greater degree of symmetry. These can be obtained by taking the  $T_{1(A)}$  and  $T_{2(A)}$  given here, and symmetrizing/antisymmetrizing over the appropriate indices, such that the structures become symmetric (or antisymmetric) under  $\alpha \leftrightarrow \beta$  and symmetric (or antisymmetric) under  $\gamma \leftrightarrow \delta$ . The structures in Eq. (B5) were matched onto the generation structures of the dimension-six operators (the matching is more subtle for the four-lepton operator  $\mathcal{O}_{\ell\ell}^{\alpha\beta\gamma\delta}$ , where the matching is done via a Fierz-evanescent dimension-six operator  $\mathcal{O}_e^{\alpha\beta\gamma\delta}$ ), and the generation structure therefore extracted from the renormalization constants, which can then be written as a generation structure multiplied by a numerical factor.

At one-loop we find the following nonvanishing mixing into the physical dimension-six operators

$$\begin{aligned}
 \delta Z_{55,H\ell(1)}^{\alpha\beta\gamma\delta,\rho\sigma} &= -\frac{3}{4}T_1^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{21\bar{2}1,H\ell(1)}^{\alpha\beta\gamma\delta,\rho\sigma} &= -\frac{3}{4}T_1^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{AA,H\ell(1)}^{\alpha\beta\gamma\delta,\rho\sigma} &= -\frac{1}{4}T_1^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{55,H\ell(3)}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{2}T_1^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{21\bar{2}1,H\ell(3)}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{2}T_1^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{A\bar{2}1,H\ell(3)}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{4}T_{1A}^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{21\bar{1}A,H\ell(3)}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{4}T_{1A}^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{55,eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{3}{4}T_2^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{21\bar{5},eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= T_2^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{A\bar{5},eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= -T_{2A}^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{AA,eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= -\frac{1}{4}T_2^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{A\bar{2}1,eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{4}T_{2A}^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{21\bar{1}A,eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= \frac{1}{4}T_{2A}^{\alpha\beta\gamma\delta,\rho\sigma}, & \delta Z_{21\bar{2}1,eH}^{\alpha\beta\gamma\delta,\rho\sigma} &= -\frac{1}{4}T_2^{\alpha\beta\gamma\delta,\rho\sigma}, \\
 \delta Z_{55,\ell\ell}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} &= -\frac{1}{4}T_3^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu}, & \delta Z_{22\bar{2}2,\ell\ell}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} &= -\frac{1}{4}T_3^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu}, \\
 \delta Z_{21\bar{2}1,\ell\ell}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} &= -\frac{1}{2}T_3^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu}, & \delta Z_{AA,\ell\ell}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} &= \frac{1}{2}T_{3A}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu}.
 \end{aligned} \tag{B6}$$

$$\begin{aligned}
 \langle f | (\delta Z_{55,\text{eva}}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} \mathcal{O}_{\text{eva,scalar}}^{\rho\sigma\tau\nu} + \text{H.c.}) | i \rangle_{LR} &= \delta Z_{55,\text{eva}}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} (\bar{u}_{\psi i} P_L v_{\omega j}) (\bar{v}_{\phi k} P_R u_{\chi l}) \\
 &\times [\delta_{il} \delta_{jk} (\delta_{\psi\sigma} \delta_{\omega\nu} \delta_{\chi\rho} \delta_{\phi\tau} + \delta_{\omega\sigma} \delta_{\psi\nu} \delta_{\phi\rho} \delta_{\chi\tau}) + \delta_{ik} \delta_{jl} (\delta_{\omega\sigma} \delta_{\psi\nu} \delta_{\chi\rho} \delta_{\phi\tau} + \delta_{\psi\sigma} \delta_{\omega\nu} \delta_{\phi\rho} \delta_{\chi\tau})],
 \end{aligned} \tag{B9}$$

where we have used the hermiticity condition of the renormalization constants.<sup>6</sup> The one-loop contribution to the  $L \times R$  part is then renormalized by the renormalization constant  $\delta Z_{55,\text{eva}}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} = -\frac{1}{2}T_3^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu}$ . As there is no  $(V-A) \times (V-A)$  contribution to the Green's function, the  $(V-A) \times (V-A)$  parts have to cancel between the counterterms

<sup>6</sup>The four-lepton renormalization constants fulfil the hermiticity condition  $Z_{55,\ell\ell}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} = (Z_{55,\ell\ell}^{\delta\gamma\beta\alpha,\sigma\rho\tau\nu})^*$ .

## 2. Four-lepton Green's function

In the following we will explicitly present the renormalization of a Green's function involving four lepton doublets. When we consider double-insertions of dimension-five operators one additional operator that vanishes in the limit  $d \rightarrow 4$ , a so-called evanescent operator, appears in our calculation. The exact definition of the evanescent operator in  $d$  dimensions is not important, but will induce a scheme dependence beyond one-loop. We use

$$\mathcal{O}_{\text{eva}}^{\alpha\beta\gamma\delta} = \frac{1}{2} \delta_{ij} \delta_{kl} (\bar{\ell}_{ia} \ell_{kj}^c) (\bar{\ell}_{l\delta}^c \ell_{j\beta}) - \frac{1}{2} \mathcal{O}_{\ell\ell}^{\alpha\beta\gamma\delta}, \tag{B7}$$

where the first term has a left-right chirality structure and  $i, j, k, l$  are SU(2) indices.

Denoting the flavor and SU(2) component of the final state  $\langle f | = \langle \ell_{k,\phi} \ell_{l,\chi} |$  and the initial state  $| i \rangle = | \ell_{i,\psi} \ell_{j,\omega} \rangle$  by  $\phi, \chi, \psi, \omega$ , and  $i, j, k, l$  respectively, we find for the third diagram of Fig. 1

$$\begin{aligned}
 \langle f | \mathcal{O}_5^{\alpha\beta} (\mathcal{O}_5^{\gamma\delta})^\dagger | i \rangle_{1/e}^{(1)} &= \frac{(\bar{u}_{\psi i} P_L v_{\omega j}) (\bar{v}_{\phi k} P_R u_{\chi l})}{64\pi^2} (\delta_{\psi\delta} \delta_{\omega\gamma} + \delta_{\omega\delta} \delta_{\psi\gamma}) \\
 &\times (\delta_{\chi\alpha} \delta_{\phi\beta} + \delta_{\phi\alpha} \delta_{\chi\beta}) (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}),
 \end{aligned} \tag{B8}$$

which exactly matches the scalar contribution of the evanescent operator  $\mathcal{O}_{\text{eva}}$  at tree level

of  $\mathcal{O}_{\text{eva}}$  and  $\mathcal{O}_{\ell\ell}$ , i.e.,  $\delta Z_{55,\ell\ell}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} = (1/2) \delta Z_{55,\text{eva}}^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu} = -\frac{1}{4} T_3^{\alpha\beta\gamma\delta,\rho\sigma\tau\nu}$ .

## 3. W emission

The values of renormalization constants may be checked by renormalizing other loop processes involving a double-insertion of dimension-five operators, and matching them to the same operator basis  $\mathcal{O}_{H\ell(1)}^{\beta\alpha}$ ,  $\mathcal{O}_{H\ell(3)}^{\beta\alpha}$ ,  $\mathcal{O}_{v(1)}^{\beta\alpha}$  and  $\mathcal{O}_{v(3)}^{\beta\alpha}$ . The internal Higgs and lepton lines of the loop diagram

may couple to  $B_\mu$  or  $W_\mu^a$  bosons of the  $U(1)_Y$  and  $SU(2)_L$  groups respectively. Since the group structure of  $U(1)_Y$  is trivial, we concentrate here on the calculation resulting from emission of a  $W_\mu^a$  boson. The results for emission of  $B_\mu$  emission may be retrieved from these results by replacing the  $SU(2)_L$  generators everywhere by  $U(1)_Y$  generators,  $\frac{1}{2}\tau_{ij}^a \rightarrow Y(H, \ell)\delta_{ij}$  at the beginning of the calculation.

The renormalization equation for the process  $H^M \ell_\alpha^n \rightarrow H^J \ell_\beta^i W_\mu^a$  in  $\overline{\text{MS}}$  is

$$\begin{aligned} 0 = & \langle \ell_\beta^i H^J W_\mu^a | \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger | \ell_\alpha^n H^M \rangle \Big|_\epsilon^{(1)} \\ & + Z_{55, H\ell(1)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] \tau_{JM}^a \delta_{in} \\ & + Z_{55, H\ell(3)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] (\delta_{JM} \tau_{in}^a) \\ & + Z_{55, v(1)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] \delta_{JM} \tau_{in}^a \\ & + Z_{55, v(3)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] (\delta_{in} \tau_{JM}^a). \end{aligned}$$

where the tree-level matrix elements are replaced by their respective amplitudes and the  $SU(2)$  algebra has been simplified.

Two diagrams must be evaluated for the double insertion of dimension-five operators with associated emission of a  $W_\mu^a$  boson, which can couple to either the internal Higgs or internal lepton. These diagrams are denoted by  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , and are shown in Fig. 7.

Calculating the diagrams and isolating the  $1/\epsilon$  poles gives

$$\begin{aligned} \mathcal{D}_1 \Big|_\epsilon = & \frac{1}{\epsilon} \frac{g_2}{64\pi^2} [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] (\delta_{\alpha\kappa} \delta_{\beta\gamma} \delta_{\delta\eta}) \\ & \times (2\delta_{JM} \tau_{in}^a - \delta_{Jn} \tau_{iM}^a - \delta_{iM} \tau_{Jn}^a - \delta_{in} \tau_{JM}^a), \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} \mathcal{D}_2 \Big|_\epsilon = & \frac{1}{\epsilon} \frac{g_2}{64\pi^2} [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] (\delta_{\alpha\kappa} \delta_{\beta\gamma} \delta_{\delta\eta}) \\ & \times (\delta_{iM} \tau_{Jn}^a + \delta_{Jn} \tau_{iM}^a - 3\delta_{JM} \tau_{in}^a), \end{aligned} \quad (\text{B11})$$

where we use the symmetries  $\gamma \leftrightarrow \delta$  and  $\eta \leftrightarrow \kappa$  of the flavor indices of the Weinberg operator to simplify our

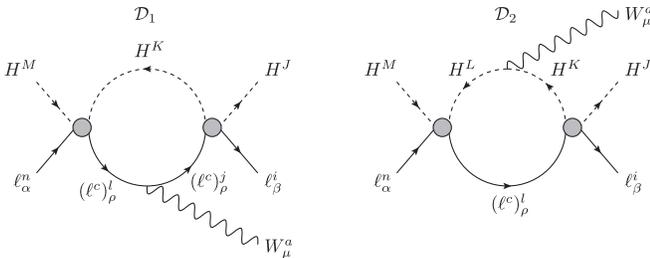


FIG. 7. Double insertions of dimension-five operators with associated emission of  $W_\mu^a$  that mix into dimension-six operators.

expressions here and in the following. The total amplitude of the double-insertion of dimension-five operators is therefore:

$$\begin{aligned} & \langle \ell_\beta^i H^J W_\mu^a | \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger | \ell_\alpha^n H^M \rangle \Big|_\epsilon^{(1)} \\ & = -\frac{1}{\epsilon} \frac{g_2}{64\pi^2} [\bar{u}_{\beta i} \gamma^\mu P_L u_{\alpha n}] (\delta_{\alpha\kappa} \delta_{\beta\gamma} \delta_{\delta\eta}) (\delta_{JM} \tau_{in}^a + \delta_{in} \tau_{JM}^a). \end{aligned} \quad (\text{B12})$$

In this form it is simple to set up simultaneous equations for the renormalization condition by comparing the loop and tree amplitudes,

$$Z_{55, H\ell(1)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) + Z_{55, v(3)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) - \frac{1}{\epsilon} \frac{g_2}{64\pi^2} (\delta_{\alpha\kappa} \delta_{\beta\gamma} \delta_{\delta\eta}) = 0, \quad (\text{B13})$$

$$Z_{55, v(1)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) + Z_{55, H\ell(3)}^{\gamma\delta\eta\kappa, \beta\alpha} (-g_2) - \frac{1}{\epsilon} \frac{g_2}{64\pi^2} (\delta_{\alpha\kappa} \delta_{\beta\gamma} \delta_{\delta\eta}) = 0. \quad (\text{B14})$$

This underconstrained set of equations may be constrained by substituting in solutions for  $Z_{55, v(1)}^{\gamma\delta\eta\kappa, \beta\alpha}$  and  $Z_{55, v(3)}^{\gamma\delta\eta\kappa, \beta\alpha}$  from the momentum-dependent calculation, to verify the solutions

$$\delta Z_{55, H\ell(1)}^{\gamma\delta\eta\kappa, \beta\alpha} = -\frac{3}{4} T_1^{\kappa\beta\gamma\delta, \alpha\eta}, \quad \delta Z_{55, H\ell(3)}^{\gamma\delta\eta\kappa, \beta\alpha} = \frac{1}{2} T_1^{\kappa\beta\gamma\delta, \alpha\eta}. \quad (\text{B15})$$

### APPENDIX C: RENORMALIZATION GROUP EQUATIONS

The bare Wilson coefficients of dimension-five operators can be written as

$$\vec{C}_{X, \text{bare}}^\eta = \mu^{2\epsilon} \vec{C}_Y^\theta(\mu) Z_{YX}^{\theta\eta}(\mu), \quad (\text{C1})$$

where  $\vec{C}_Y^\theta(\mu)$  is the renormalized Wilson coefficient,  $Z_{YX}^{\theta\eta}(\mu)$  is the renormalization matrix, and  $\mu$  is the renormalization scale. The  $\mu^{2\epsilon}$  introduces an additional term proportional to  $\epsilon$  into the  $d$ -dimensional renormalization group equation

$$\mu \frac{d}{d\mu} \vec{C}_X^\eta = -\vec{C}_Y^\theta \left( \mu \frac{d}{d\mu} Z_{YZ}^{\theta\zeta} \right) [Z^{-1}]_{ZX}^{\zeta\eta} - 2\epsilon \vec{C}_X^\eta. \quad (\text{C2})$$

This reduces to the renormalization group equation in  $d = 4$  dimensions

$$(16\pi^2) \mu \frac{d}{d\mu} \vec{C}_X^{d=4} = \vec{C}_Y^{\theta, \theta\eta}, \quad (\text{C3})$$

where the 4-dimensional anomalous dimension matrix

$$\gamma_{YX}^{\theta\eta} = -(16\pi^2) \left( \mu \frac{d}{d\mu} Z_{YZ}^{\theta\zeta} \right) [Z^{-1}]_{ZX}^{\zeta\eta} \quad (\text{C4})$$

is independent of the choice of the overall factor  $\mu^{2\epsilon}$ . Therefore the  $\mu^{2\epsilon}$  term can be neglected when only considering mixing amongst operators of equal dimensions. In the case of mixing between operators of different dimensions a more careful treatment is required.

At loop level, operators of different dimensions can mix via multiple operator insertions [31]. Consider the specific case of loop diagrams involving two dimension-five operators mixing into diagrams with a single dimension-six operator insertion. We denote dimension-six quantities with a tilde, quantities that mix dimension-five and -six with a hat, and dimension-five quantities without a tilde or hat. The bare dimension-six Wilson coefficient is

$$\tilde{C}_{X,\text{bare}}^\eta = \mu^{2\epsilon} \tilde{C}_Y^\theta(\mu) \hat{Z}_{YX}^{\theta\eta}(\mu) + \mu^{2\epsilon} C_A^\zeta(\mu) \tilde{Z}_{AB,X}^{\zeta\theta,\eta}(\mu) [C_B^\theta]^\dagger(\mu), \quad (\text{C5})$$

where  $\tilde{C}_{\text{bare}}^\eta$  is  $\mu$ -independent. Therefore the renormalization group equation is

$$(16\pi^2) \mu \frac{d}{d\mu} \tilde{C}_X^\eta = \tilde{C}_Y^\theta \hat{\gamma}_{YX}^{\theta\eta} + C_A^\zeta \tilde{\gamma}_{AB,X}^{\zeta\theta,\eta} [C_B^\theta]^\dagger, \quad (\text{C6})$$

where  $\hat{\gamma}_{YX}^{\theta\eta}$  is defined analogously to Eq. (C2), and

$$\begin{aligned} \tilde{\gamma}_{AB,X}^{\zeta\theta,\eta} = & (16\pi^2) \left( 2\epsilon \tilde{Z}_{AB,Y}^{\zeta\theta,v} - \mu \frac{d}{d\mu} \tilde{Z}_{AB,Y}^{\zeta\theta,v} \right) [\hat{Z}^{-1}]_{YX}^{\eta\theta} \\ & - (16\pi^2) ([\gamma_{BD}^{\theta\omega}]^\dagger \delta_{AC}^{\zeta\chi} + \gamma_{AC}^{\zeta\chi} \delta_{BD}^{\theta\omega}) \tilde{Z}_{CD,Y}^{\chi\omega,v} [\hat{Z}^{-1}]_{YX}^{\eta\theta} \end{aligned} \quad (\text{C7})$$

where the explicit form in terms of generation indices is  $[\gamma_{AB}^{\alpha\beta\gamma\delta}]^\dagger = [\gamma_{AB}^{\beta\alpha\delta\gamma}]^*$  and  $\delta_{AB}^{\alpha\beta\gamma\delta} = \delta_{AB} \delta_{\alpha\gamma} \delta_{\beta\delta}$ . The terms in the second line of the above equation only contribute beyond one-loop. Furthermore, the contribution to the renormalization tensor  $Z_{AB,Y}^{\zeta\theta,v}$  is  $\mu$  independent at one-loop and only the term proportional to  $2\epsilon$  contributes in our calculation. A comment regarding the sign of the  $2\epsilon$  contribution is in order. The factor in  $\mu^{2\epsilon}$  in (C5) generates a term proportional to  $-2\epsilon$ , while the derivative of the dimension-five Wilson coefficients generates a contribution proportional to  $2 \times 2\epsilon$  from (C2). Hence the one-loop anomalous dimension matrix reads

$$\tilde{\gamma}_{AB,C}^{\zeta\eta,\theta} = 2\delta \tilde{Z}_{AB,C}^{\zeta\eta,\theta} \quad (\text{C8})$$

in terms of the one-loop renormalization constants defined in Eq. (B2). Correspondingly we find  $[\tilde{\gamma}] = 2(16\pi^2)\epsilon[\tilde{Z}]$ .

#### APPENDIX D: OPERATORS

This Appendix lists dimension-six, SM-gauge invariant operators that change lepton flavor. The operators are in the

Buchmuller-Wyler basis, as pruned in Grzadkowski *et al.* [20], commonly referred to as the ‘‘Warsaw’’ basis. All operators are added to the Lagrangian + H.c., as given in Eq. (2.6):

$$\delta\mathcal{L}_6 = \sum_{X,\zeta} \frac{C_X^\zeta}{\Lambda^2} \mathcal{O}_X^\zeta + \text{H.c.}$$

where the flavor indices are represented by  $\zeta$ , and are all summed over all generations. In the conventions of [20] and [19], the hermitian conjugate is not added for ‘‘self-conjugate’’ operators, for which  $\sum_\zeta C_X^\zeta \mathcal{O}_X^\zeta = [\sum_\zeta C_X^\zeta \mathcal{O}_X^\zeta]^\dagger$ . (For instance,  $\mathcal{O}_{\ell\ell}^{\alpha\beta\rho\sigma}$  of Eq. (D11) is Hermitian, because  $[(\bar{e}\gamma^\mu\mu)(\bar{\tau}\gamma_\mu\tau)]^\dagger = (\bar{\mu}\gamma^\mu e)(\bar{\tau}\gamma_\mu\tau)$ ). So we define such operators with a factor 1/2 to avoid this double-counting.

The four-fermion operators involving  $\beta \leftrightarrow \alpha$  flavor change and two quarks are

$$\mathcal{O}_{\ell q}^{(1)\alpha\beta nm} = \frac{1}{2} (\bar{\ell}_\alpha \gamma^\mu \ell_\beta) (\bar{q}_n \gamma_\mu q_m) \quad (\text{D1})$$

$$\mathcal{O}_{\ell q}^{(3)\alpha\beta nm} = \frac{1}{2} (\bar{\ell}_\alpha \gamma^\mu \tau^a \ell_\beta) (\bar{q}_n \gamma_\mu \tau^a q_m) \quad (\text{D2})$$

$$\mathcal{O}_{eq}^{\alpha\beta nm} = \frac{1}{2} (\bar{e}_\alpha \gamma^\mu e_\beta) (\bar{q}_n \gamma_\mu q_m) \quad (\text{D3})$$

$$\mathcal{O}_{\ell u}^{\alpha\beta nm} = \frac{1}{2} (\bar{\ell}_\alpha \gamma^\mu \ell_\beta) (\bar{u}_n \gamma_\mu u_m) \quad (\text{D4})$$

$$\mathcal{O}_{\ell d}^{\alpha\beta nm} = \frac{1}{2} (\bar{\ell}_\alpha \gamma^\mu \ell_\beta) (\bar{d}_n \gamma_\mu d_m) \quad (\text{D5})$$

$$\mathcal{O}_{eu}^{\alpha\beta nm} = \frac{1}{2} (\bar{e}_\alpha \gamma^\mu e_\beta) (\bar{u}_n \gamma_\mu u_m) \quad (\text{D6})$$

$$\mathcal{O}_{ed}^{\alpha\beta nm} = \frac{1}{2} (\bar{e}_\alpha \gamma^\mu e_\beta) (\bar{d}_n \gamma_\mu d_m) \quad (\text{D7})$$

$$\mathcal{O}_{\ell equ}^{\alpha\beta nm} = (\bar{\ell}_\alpha^A e_\beta) \epsilon_{AB} (\bar{q}_n^B u_m) \quad (\text{D8})$$

$$\mathcal{O}_{\ell edq}^{\alpha\beta nm} = (\bar{\ell}_\alpha e_\beta) (\bar{d}_n q_m) \quad (\text{D9})$$

$$\mathcal{O}_{T,\ell equ}^{\alpha\beta nm} = (\bar{\ell}_\alpha^A \sigma^{\beta\nu} e_\beta) \epsilon_{AB} (\bar{q}_n^B \sigma_{\beta\nu} u_m) \quad (\text{D10})$$

where  $\ell$ ,  $q$  are doublets and  $e$ ,  $u$  are singlets,  $n$ ,  $m$  are possibly equal quark family indices, and  $A$ ,  $B$  are SU(2) indices. The operator names are as in [20] with  $\varphi \rightarrow H$ ; the flavor indices are in superscript.

In the case of four-lepton operators, the flavor change can be by one or two units. Notice that in the case of  $\mathcal{O}_{ee}$  and  $\mathcal{O}_{\ell\ell}$ , which are symmetric under interchange of the two bilinears  $(\bar{e}\gamma^\mu\mu)(\bar{\tau}\gamma_\mu\tau) = (\bar{\tau}\gamma_\mu\tau)(\bar{e}\gamma_\mu\mu)$ , there will be two equal coefficients that contribute to the Feynman rule:

$$\mathcal{O}_{\ell\ell}^{\alpha\beta\rho\sigma} = \frac{1}{2}(\bar{\ell}_\alpha\gamma^\mu\ell_\beta)(\bar{\ell}_\rho\gamma_\mu\ell_\sigma) \quad (\text{D11})$$

$$\mathcal{O}_{\ell e}^{\alpha\beta\rho\sigma} = \frac{1}{2}(\bar{\ell}_\alpha\gamma^\mu\ell_\beta)(\bar{e}_\rho\gamma_\mu e_\sigma) \quad (\text{D12})$$

$$\mathcal{O}_{ee}^{\alpha\beta\rho\sigma} = \frac{1}{2}(\bar{e}_\alpha\gamma^\mu e_\beta)(\bar{e}_\rho\gamma_\mu e_\sigma). \quad (\text{D13})$$

Then there are the operators allowing interactions with gauge bosons and Higgses. This includes the dipoles, which are normalized with the muon Yukawa coupling so as to match onto the normalization of Kuno-Okada [2]:

$$\mathcal{O}_{eH}^{\alpha\beta} = (H^\dagger H)(\bar{\ell}_\alpha H e_\beta) \quad (\text{D14})$$

$$\mathcal{O}_{eW}^{\alpha\beta} = y_\beta(\bar{\ell}_\alpha\tau^\alpha H\sigma^{\mu\nu} e_\beta)W_{\mu\nu}^a \quad (\text{D15})$$

$$\mathcal{O}_{eB}^{\alpha\beta} = y_\beta(\bar{\ell}_\alpha H\sigma^{\mu\nu} e_\beta)B_{\mu\nu} \quad (\text{D16})$$

$$\mathcal{O}_{H\ell(1)}^{\alpha\beta} = \frac{i}{2}(H^\dagger\overleftrightarrow{D}_\mu H)(\bar{\ell}_\alpha\gamma^\mu\ell_\beta) \quad (\text{D17})$$

$$\mathcal{O}_{H\ell(3)}^{\alpha\beta} = \frac{i}{2}(H^\dagger\overleftrightarrow{D}_\mu^a H)(\bar{\ell}_\alpha\gamma^\mu\tau^a\ell_\beta) \quad (\text{D18})$$

$$\mathcal{O}_{He}^{\alpha\beta} = \frac{i}{2}(H^\dagger\overleftrightarrow{D}_\mu H)(\bar{e}_\alpha\gamma^\mu e_\beta), \quad (\text{D19})$$

where  $y_\beta$  denotes the Yukawa coupling of a charged lepton  $e_\beta$  in the mass basis, the double derivatives are defined in Eq. (2.8), and we include factors of 1/2 for Hermitian operators as discussed above Eq. (D1).

## APPENDIX E: EXPERIMENTAL BOUNDS ON COEFFICIENTS

The aim of this Appendix is to obtain experimental constraints on the coefficients of the LFV operators of Eq. (2.7), evaluated at the weak scale  $m_W$ . We are interested in this subset of operators because they are generated at one loop by double-insertions of dimension-five, lepton number changing (LNV) operators. Such constraints will allow an estimation of the sensitivity of LFV processes to the coefficients of LNV operators. We neglect the constraints on 2-lepton-2-quark operators, which are beyond the scope of this work, and focus on  $\tau \leftrightarrow e$  and  $\tau \leftrightarrow \mu$  flavor change, because  $\mu \leftrightarrow e$  is discussed in [34,35]. Nonetheless, some  $\mu \leftrightarrow e$  bounds are listed for completeness.

Recall that constraints and sensitivities are different. A constraint is an exclusion, which tells the range of values a coefficient cannot have. For instance, the dipole coefficient (evaluated at the muon mass scale)  $C_{D,R}^{e\mu}(m_\mu)$ , cannot be larger than  $1.05 \times 10^{-8}$  because the branching ratio searched for by the MEG experiment [3] is

$$\text{BR}(\mu \rightarrow e\gamma) = 384\pi^2 \frac{v^4}{\Lambda^4} (|C_{D,L}^{e\mu}|^2 + |C_{D,R}^{e\mu}|^2),$$

and the current experimental search imposes this constraint. Sensitivity is often discussed when an observable depends on many coefficients, and gives the range of values where a coefficient could have been seen. For instance, among the many loop processes that contribute to  $\mu \rightarrow e\gamma$ , there are two-loop diagrams involving flavor-changing Higgs coupling  $C_{eH}^{e\mu}(m_W)$ . Calculating these diagrams and imposing that they saturate the current experimental bound gives

$$\left| \frac{e\alpha_e y_t}{8\pi^3 y_\mu} C_{eH}^{e\mu}(m_W) \right| = 1.05 \times 10^{-8},$$

where the Yukawa eigenvalue of fermion  $f$  is denoted  $y_f$ . Smaller values of  $C_{eH}^{e\mu}$  are allowed (the experiment could not have seen them), but larger values are not excluded by MEG, because many other operator coefficients could contribute to the rate, with possibly cancellations.

The difference between an exclusion and a sensitivity is illustrated in Fig. 8, where the allowed region is the diagonal ellipse. The horizontal variable  $x$  is excluded outside the projection of the ellipse onto the  $x$ -axis (where the axis is thickened). But the experiment is only insensitive to  $x$  inside the intersection of the axis with the ellipse (dashed red line). Values of  $x$  between these two regions are allowed, provided that  $y$  has the appropriately correlated value.

Three ways to relate low-energy experimental bounds to the coefficients of operators at a higher scale are:

- (1) to calculate the sensitivity of an experimental process to a particular operator coefficient. This is usually simple.
- (2) To express an experimental rate as a function of high-scale coefficients. This is slightly more difficult, because more coefficients are involved: each coefficient that contributes at the experimental scale will become a linear combination of high scale coefficients due the renormalization group mixing.

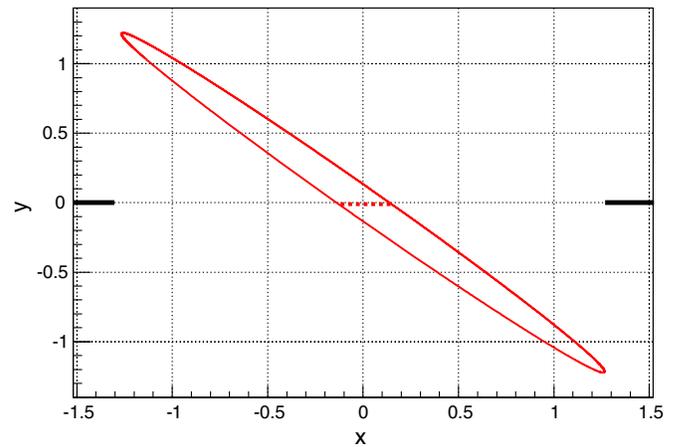


FIG. 8. An illustration of constraints vs sensitivities: the red ellipse represents an experimentally allowed region of parameter space. Parameter  $x$  is excluded outside the projection of the ellipse onto the axis (thick black line). The experiment is insensitive to  $x$  inside the ellipse.

- (3) To obtain constraints on coefficients at the high scale. This is more involved, because a sufficient number of experimental constraints must be combined, in order to obtain a finite allowed region in coefficient space (no “flat directions”). Then the allowed region must be projected onto the various axes, in order to obtain constraints.

The third option is the most useful, but beyond the scope of this work. Instead here, we partially follow the second option, as a contribution to the third: we consider experimental bounds on the dimension-six operators which are generated in RGE evolution by double-insertions of dimension-five operators that change lepton

number. We aim to quote these bounds at  $m_W$ . The processes in question are LFV Higgs and  $Z$  decays (which occur at the weak scale), and flavor-changing lepton decays at low energy (these bounds must be translated to the weak scale via the RGEs of QED and QCD). So we will not succeed in our aim of setting constraints on coefficients at  $m_W$ , because the low-energy experimental bounds depend on many coefficients at the weak scale, and we do not include enough experimental bounds.

In the following sections, we outline the calculations of the various rates, and summarize the experimental constraints on coefficients at  $m_W$  in Table I.

TABLE I. Bound on operator coefficients of the SMEFT, evaluated at  $m_W$ , from the bounds listed in column 2 on the processes of column 1. The bounds on coefficients of Hermitian operators ( $\mathcal{O}_{H\ell(1)}$ ,  $\mathcal{O}_{H\ell(3)}$ ,  $\mathcal{O}_{\ell\ell}$ ,  $\mathcal{O}_{\ell e}$ ) also apply to the conjugate coefficient. All the bounds apply to running coefficients evaluated at  $m_W$ , and are for  $\Lambda = v \simeq m_\tau$ . The combination of coefficients  $C_{\text{penguin}}$  is defined in Eq. (E12) and before Eq. (E23),  $\delta$  is defined after Eq. (E23), and  $g_R^e = 2s_W^2$ ,  $g_L^e = -1 + 2s_W^2$ .

Process	BR <	$\frac{v^2}{\Lambda^2}  \sum C  <$
$Z \rightarrow e^\pm \mu^\mp$	$7.5 \times 10^{-7}$ [4]	$ C_{H\ell(1)}^{e\mu} + C_{H\ell(3)}^{e\mu}  < 1.2 \times 10^{-3}$
$Z \rightarrow \tau^\pm \mu^\mp$	$1.2 \times 10^{-5}$ [5]	$ C_{H\ell(1)}^{\mu\tau} + C_{H\ell(3)}^{\mu\tau}  < 4.6 \times 10^{-3}$
$Z \rightarrow e^\pm \tau^\mp$	$9.8 \times 10^{-6}$ [6]	$ C_{H\ell(1)}^{e\tau} + C_{H\ell(3)}^{e\tau}  < 4.1 \times 10^{-3}$
$h \rightarrow e^\pm \mu^\mp$	$3.5 \times 10^{-4}$ [7]	$ C_{eH}^{\mu e} ,  C_{eH}^{e\mu}  < 2.5 \times 10^{-4}$
$h \rightarrow \tau^\pm \mu^\mp$	$1.5 \times 10^{-2}$ [8]	$ C_{eH}^{\mu\tau} ,  C_{eH}^{\tau\mu}  < 1.6 \times 10^{-3}$
$h \rightarrow e^\pm \tau^\mp$	$6.9 \times 10^{-3}$ [7]	$ C_{eH}^{e\tau} ,  C_{eH}^{\tau e}  < 1.1 \times 10^{-3}$
$\tau \rightarrow ee\bar{e}$	$2.7 \times 10^{-8}$ [9]	$ C_{\ell\ell}^{e\tau ee} + C_{\ell\ell}^{e\tau e\tau} + g_L^e [C_{H\ell(1)}^{e\tau} + C_{H\ell(3)}^{e\tau}] - \delta C_{\text{penguin}}^{e\tau}  < 2.8 \times 10^{-4}$ $ C_{\ell e}^{e\tau ee} + g_R^e [C_{H\ell(1)}^{e\tau} + C_{H\ell(3)}^{e\tau}] - \delta C_{\text{penguin}}^{e\tau}  < 4.0 \times 10^{-4}$
$\tau \rightarrow e\mu\bar{\mu}$	$2.7 \times 10^{-8}$ [9]	$ C_{\ell\ell}^{\tau\mu\mu} + C_{\ell\ell}^{\mu\tau e\tau} + C_{\ell\ell}^{\mu\mu\tau\tau} + C_{\ell\ell}^{\mu\tau e\mu} + g_L^e [C_{H\ell(1)}^{\tau\mu} + C_{H\ell(3)}^{\tau\mu}] - \delta C_{\text{penguin}}^{\tau\mu}  < 4.0 \times 10^{-4}$ $ C_{\ell e}^{\tau\mu\mu} + g_R^e [C_{H\ell(1)}^{\tau\mu} + C_{H\ell(3)}^{\tau\mu}] - \delta C_{\text{penguin}}^{\tau\mu}  < 4.0 \times 10^{-4}$
$\tau \rightarrow \mu e\bar{e}$	$1.8 \times 10^{-8}$ [9]	$ C_{\ell\ell}^{\mu\tau ee} + C_{\ell\ell}^{e\mu\tau e} + C_{\ell\ell}^{e\tau\mu e} + g_L^e [C_{H\ell(1)}^{\mu\tau} + C_{H\ell(3)}^{\mu\tau}] - \delta C_{\text{penguin}}^{\mu\tau}  < 3.2 \times 10^{-4}$ $ C_{\ell e}^{\mu\tau ee} + g_R^e [C_{H\ell(1)}^{\mu\tau} + C_{H\ell(3)}^{\mu\tau}] - \delta C_{\text{penguin}}^{\mu\tau}  < 3.2 \times 10^{-4}$
$\tau \rightarrow \mu\mu\bar{\mu}$	$2.1 \times 10^{-8}$ [9]	$ C_{\ell\ell}^{\mu\tau\mu\mu} + C_{\ell\ell}^{\mu\mu\tau\tau} + g_L^e [C_{H\ell(1)}^{\mu\tau} + C_{H\ell(3)}^{\mu\tau}] - \delta C_{\text{penguin}}^{\mu\tau}  < 2.5 \times 10^{-4}$ $ C_{\ell e}^{\mu\tau\mu\mu} + g_R^e [C_{H\ell(1)}^{\mu\tau} + C_{H\ell(3)}^{\mu\tau}] - \delta C_{\text{penguin}}^{\mu\tau}  < 3.5 \times 10^{-4}$
$\tau \rightarrow ee\bar{\mu}$	$1.5 \times 10^{-8}$ [9]	$ C_{\ell\ell}^{e\tau e\mu} + C_{\ell\ell}^{e\mu e\tau}  < 3.2 \times 10^{-4}$
$\tau \rightarrow \mu\mu\bar{e}$	$1.7 \times 10^{-8}$ [9]	$ C_{\ell\ell}^{\mu\tau\mu e} + C_{\ell\ell}^{\mu e\mu\tau}  < 3.2 \times 10^{-4}$
$\mu \rightarrow 3e$	$1 \times 10^{-12}$ [10]	$ C_{\ell\ell}^{e\mu ee} + C_{\ell\ell}^{ee\mu\mu} + g_L^e [C_{H\ell(1)}^{e\mu} + C_{H\ell(3)}^{e\mu}] - \delta C_{\text{penguin}}^{e\mu}  < 7.1 \times 10^{-7}$ $ C_{\ell e}^{e\mu ee} + g_R^e [C_{H\ell(1)}^{e\mu} + C_{H\ell(3)}^{e\mu}] - \delta C_{\text{penguin}}^{e\mu}  < 1.0 \times 10^{-6}$
$\tau \rightarrow e\gamma$	$3.3 \times 10^{-8}$ [11]	$ C_{e\gamma}^{\tau e*} + \frac{e\alpha_e y_\tau}{8\pi^3 y_\mu} C_{eH}^{\tau e*} + \frac{eg_R^e}{16\pi^2} C_{He}^{\tau e}  < 7.3 \times 10^{-6}$ $ C_{e\gamma}^{\tau e\tau} + \frac{e\alpha_e y_\tau}{8\pi^3 y_\mu} C_{eH}^{\tau e\tau} + \frac{eg_R^e}{16\pi^2} [C_{H\ell(1)}^{\tau e\tau} + C_{H\ell(3)}^{\tau e\tau}]  < 7.3 \times 10^{-6}$
$\tau \rightarrow \mu\gamma$	$4.4 \times 10^{-8}$ [11,12]	$ C_{e\gamma}^{\tau\mu*} + \frac{e\alpha_e y_\tau}{8\pi^3 y_\mu} C_{eH}^{\tau\mu*} + \frac{eg_L^e}{16\pi^2} C_{He}^{\tau\mu}  < 8.1 \times 10^{-6}$ $ C_{e\gamma}^{\tau\mu\tau} + \frac{e\alpha_e y_\tau}{8\pi^3 y_\mu} C_{eH}^{\tau\mu\tau} + \frac{eg_R^e}{16\pi^2} [C_{H\ell(1)}^{\tau\mu\tau} + C_{H\ell(3)}^{\tau\mu\tau}]  < 8.1 \times 10^{-6}$
$\mu \rightarrow e\gamma$	$4.2 \times 10^{-13}$ [3]	$ C_{e\gamma}^{\mu e*} + \frac{e\alpha_e y_\tau}{8\pi^3 y_\mu} C_{eH}^{\mu e*} + \frac{eg_L^e}{16\pi^2} C_{He}^{\mu e}  < 1.05 \times 10^{-8}$ $ C_{e\gamma}^{\mu e\mu} + \frac{e\alpha_e y_\tau}{8\pi^3 y_\mu} C_{eH}^{\mu e\mu} + \frac{eg_R^e}{16\pi^2} [C_{H\ell(1)}^{\mu e\mu} + C_{H\ell(3)}^{\mu e\mu}]  < 1.05 \times 10^{-8}$

## 1. Rates and calculations

### a. $Z \rightarrow l_\alpha \bar{l}_\beta$ decay

When the Higgs gets a vev, the ‘‘penguin’’ operators  $\mathcal{O}_{H\ell(1)}$  and  $\mathcal{O}_{H\ell(3)}$  generate a vertex involving the  $Z$  and two charged leptons. If the flavor-changing  $Z$ -fermion vertex is written in a SM-like form :  $-\bar{l}_\alpha Z^\mu \frac{g}{2c_W} \gamma_\mu (g_V - g_A \gamma_5) l_\beta$ , then

$$g_V = g_A = -(C_{H\ell(1)} + C_{H\ell(3)}) \frac{v^2}{\Lambda^2} \quad (\text{E1})$$

(for  $v \sim m_t$ ).

The branching ratio can be written

$$\text{BR}(Z \rightarrow l_\alpha \bar{l}_\beta) = \frac{M_Z}{2.5 \text{ GeV}} \frac{g^2}{48\pi c_W^2} (|g_V|^2 + |g_A|^2) \quad (\text{E2})$$

where 2.5 GeV is the  $Z$  width in the SM. Since  $\mathcal{O}_{H\ell(1)}$  and  $\mathcal{O}_{H\ell(3)}$  are Hermitian, the conjugate process  $Z \rightarrow l_\beta \bar{l}_\alpha$  necessarily occurs at the same rate, so the BR to the experimental final state is

$$\begin{aligned} \text{BR}(Z \rightarrow l_\alpha^\pm l_\beta^\mp) &= \text{BR}(Z \rightarrow l_\alpha \bar{l}_\beta) + \text{BR}(Z \rightarrow l_\beta \bar{l}_\alpha) \\ &= \frac{M_Z}{2.5 \text{ GeV}} \frac{g^2}{12\pi c_W^2} |C_{H\ell(1)}^{\alpha\beta} + C_{H\ell(3)}^{\alpha\beta}|^2 \frac{v^4}{\Lambda^4} \end{aligned} \quad (\text{E3})$$

and the bounds we obtain on the operator coefficients, evaluated at  $\sim m_W$ , are given in Table I.

### b. $h \rightarrow \ell_\alpha^+ e_\beta^-, e_\alpha^+ \ell_\beta^-$ decays

The flavor-changing Higgs decays occur via the non-Hermitian operator  $\mathcal{O}_{eH}$ . When the Higgs has a vev, it induces the Feynman rules for a flavor-changing Higgs vertex with two fermions:

$$C_{eH}^{\alpha\beta} \mathcal{O}_{eH}^{\alpha\beta} \rightarrow i \frac{3C_{eH}^{\alpha\beta} v^2}{\sqrt{2}\Lambda^2} P_R, \quad C_{eH}^{\beta\alpha*} \mathcal{O}_{eH}^{\beta\alpha*} \rightarrow i \frac{3C_{eH}^{\beta\alpha*} v^2}{\sqrt{2}\Lambda^2} P_L. \quad (\text{E4})$$

We calculate the flavor-changing branching ratio by comparing to  $\text{BR}(h \rightarrow b\bar{b}) = 0.575 \pm 0.32$  (from the Appendix of the Higgs Working Group Report [36], for  $m_h = 125.1$  GeV), assuming the Feynman rule for  $h b\bar{b}$  is  $-\frac{i}{\sqrt{2}} y_b(m_h) P_{L,R}$ . We use a one-loop approximation [15] for the running  $b$  mass

$$y_b(m_h) v = m_b(m_b) \left[ \frac{\alpha(m_h)}{\alpha(m_b)} \right]^{\gamma_m^{(0)}/2\beta^{(0)}} \simeq 3.0 \text{ GeV} \quad (\text{E5})$$

where  $\alpha(m_h) \simeq 0.12$ ,  $\alpha(m_b) \simeq 0.23$ ,  $\gamma_m^{(0)} = 8$ ,  $\beta^{(0)} = 23/3$  and  $m_b(m_b) = 4.2$  GeV.

The operator  $\mathcal{O}_{eH}$  is not hermitian, but is always included in the Lagrangian + H.c.. So  $C_{eH}^{\alpha\beta} \mathcal{O}_{eH}^{\alpha\beta} + \text{H.c.}$  will induce both  $h \rightarrow e_L \bar{\mu}_R$  and  $h \rightarrow \mu_R \bar{e}_L$  at the same rate:

$$\frac{\text{BR}(h \rightarrow \bar{e}_L \mu_R)}{\text{BR}(h \rightarrow b\bar{b})} = \frac{9|C_{eH}^{\alpha\beta}|^2 v^4}{6y_b^2 \Lambda^4}, \quad (\text{E6})$$

where downstairs there is a 3 for quark color sums, and a 2 from the chiral projectors in the lepton decay. The experimental search sums the  $e_L \bar{\mu}_R$  and  $\mu_R \bar{e}_L$  final states, so we obtain

$$3v^4 \frac{|C_{eH}^{\alpha\beta}|^2}{\Lambda^4}, \quad 3v^4 \frac{|C_{eH}^{\beta\alpha}|^2}{\Lambda^4} \leq y_b^2(m_h) \frac{\text{BR}(h \rightarrow l_\alpha^\pm l_\beta^\mp)}{\text{BR}(h \rightarrow b\bar{b})} \quad (\text{E7})$$

and the resulting constraints are given in Table I.

### c. Including the low energy decays

The flavor-changing  $\tau$  and  $\mu$  decays listed in Table I occur at energies  $\sim m_\mu, m_\tau$ , so the decay rates are usually written in terms of the coefficients of dimension-six operators from the QCD  $\times$  QED invariant basis appropriate at low energies. These ‘‘low energy’’ coefficients, which we denote with a tilde  $\tilde{C}$ , can be expressed in terms of SMEFT coefficients at  $m_W$  by running them up to  $m_W$ , then matching the QCD  $\times$  QED-invariant operator basis onto the SMEFT. This was performed in [34] for  $\mu \rightarrow e\gamma$ , so we use the results of [34] for the radiative decays of Sec. E 1 e. Reference [35] studied the renormalization group evolution, below the weak scale, of the coefficients who mediate  $\mu \rightarrow e\bar{e}e$  (as well those for as  $\mu \rightarrow e\gamma$  and  $\mu \rightarrow e$  conversion); we use these results, combined with the weak-scale matching conditions of [34], for the discussion in Sec. E 1 d of three body leptonic decays of  $\tau$ s and  $\mu$ s. The minor differences between  $\mu$  and  $\tau$  decays are discussed in Sec. E 1 d.

In the EFT below  $m_W$ , we use the basis of lepton-flavor-changing four-fermion operators introduced in [2,34] for  $\mu \leftrightarrow e$  flavor change.<sup>7</sup> The operators and coefficients have as subscript their Lorentz structure ( $V, S, T$ ) and the chiral projection operators of the two fermion bilinears, and the flavor indices of the four fermions as superscript. They wear tildes to distinguish them from the coefficients of SMEFT operators. We restrict to the dipole and vector operators, and neglect the scalars and tensors, which will turn out to be irrelevant for our study of LFV operators generated by double-insertions of LNV operators. So the four-fermion operator basis below  $m_W$  is

$$\begin{aligned} \delta\mathcal{L}_{4f} &= \sum_{\alpha\beta} \sum_f [\tilde{C}_{V,LL}^{\alpha\beta ff} (\bar{e}_\alpha \gamma^\omega P_L e_\beta) (\bar{f} \gamma_\omega P_L f) \\ &\quad + \tilde{C}_{V,LR}^{\alpha\beta ff} (\bar{e}_\alpha \gamma^\omega P_L e_\beta) (\bar{f} \gamma_\omega P_R f)] + \text{H.c.} \\ &\quad + \sum_{\alpha\beta\sigma\rho} [\tilde{C}_{V,LL}^{\alpha\beta\sigma\rho} (\bar{e}_\alpha \gamma^\omega P_L e_\beta) (\bar{e}_\sigma \gamma_\omega P_L e_\rho)] + \text{H.c.} \end{aligned} \quad (\text{E8})$$

<sup>7</sup>In this basis, the flavor indices are written explicitly, so the 2 discussed above Eq. (D11) is absent, and Fierz transformations are used to put the flavor change in one bilinear in the case of  $\Delta L = 1$  four-fermion operators.

where  $\alpha\beta \in \{e\mu, \mu\tau, e\tau\}$ ,  $f \in \{e, \mu, \tau, u, d, s, c, b\}$ , and  $\alpha\beta\sigma\rho \in \{e\tau e\mu, \mu\tau e\mu\}$ . In addition, below  $m_W$  we consider the photon dipole operators

$$\delta\mathcal{L}_{\text{dipole}} = \frac{m_\beta}{\Lambda^2} (C_{D,L}^{\alpha\beta} \bar{e}_R^\alpha \sigma^{\rho\sigma} e_L^\beta F_{\rho\sigma} + C_{D,R}^{\alpha\beta} \bar{e}_L^\alpha \sigma^{\rho\sigma} e_R^\beta F_{\rho\sigma}) + \text{H.c.} \quad (\text{E9})$$

because the SMEFT operators  $\mathcal{O}_{H\ell(1)}$ ,  $\mathcal{O}_{H\ell(3)}$  and  $\mathcal{O}_{eH}$  match onto the dipole at  $m_W$ . The current bounds on  $\mu \rightarrow e\gamma$ ,  $\tau \rightarrow e\gamma$  and  $\tau \rightarrow \mu\gamma$  will give better sensitivity to the coefficients  $C_{H\ell(1)}$ ,  $C_{H\ell(3)}$  and  $C_{eH}$  than Higgs and Z decays.

#### d. $\tau \rightarrow 3l$ and $\mu \rightarrow 3e$

The first step is to translate the experimental bounds into constraints on operator coefficients at the experimental scale. For the three-body leptonic decays of the  $\tau$ , it is convenient to define

$$\widetilde{\text{BR}}(\tau \rightarrow 3l) \equiv \frac{\text{BR}(\tau \rightarrow 3l)}{\text{BR}(\tau \rightarrow \mu\bar{\nu}\nu)} \quad (\text{E10})$$

(where  $\text{BR}(\tau \rightarrow \mu\bar{\nu}\nu) = 0.174$  [1]). Then  $\widetilde{\text{BR}}(\tau \rightarrow 3l)$  can be directly compared to the branching ratio for  $\mu \rightarrow 3e$  [2]:

$$\begin{aligned} \text{BR}(\mu \rightarrow e\bar{e}e) \frac{\Lambda^4}{v^4} &= \frac{|\tilde{C}_{S,LL}^{e\mu ee}|^2 + |\tilde{C}_{S,RR}^{e\mu ee}|^2}{8} + 2|\tilde{C}_{V,RR}^{e\mu ee} + 4e\tilde{C}_{D,L}^{e\mu}|^2 \\ &+ 2|\tilde{C}_{V,LL}^{e\mu ee} + 4e\tilde{C}_{D,R}^{e\mu}|^2 + \left(64\ln\frac{m_\mu}{m_e} - 136\right) \\ &\times (|e\tilde{C}_{D,R}^{e\mu}|^2 + |e\tilde{C}_{D,L}^{e\mu}|^2) \\ &+ |\tilde{C}_{V,RL}^{e\mu ee} + 4e\tilde{C}_{D,L}^{e\mu}|^2 + |\tilde{C}_{V,LR}^{e\mu ee} + 4e\tilde{C}_{D,R}^{e\mu}|^2, \end{aligned} \quad (\text{E11})$$

where  $2\sqrt{2}G_F = 1/v^2$  and the generalization to  $\tau$  decays is straightforward, after accounting for 2s as we now discuss.

We calculate the decay rates in the approximation that all final state fermions are massless. Factors of 2 can arise when there are two identically-flavored fermions in the final state: there will be 2 diagrams, and a factor of 1/2 in the final-state phase space. Then there are two cases:

- if the identical fermions have the same chirality, there is constructive interference between the two diagrams (despite the fact that they have relative minus signs due to Fermi statistics), which doubles the rate. (This is consistent with  $\mu \rightarrow 3e$  rate of Kuno and Okada [2] given above.)
- if the fermions have different chirality, the interference is suppressed by final state masses (which are neglected), so the two for two diagrams cancels the 1/2 from phase space.

We set the dipole coefficients to zero, because they are better constrained by the radiative decays discussed in the

TABLE II. Bounds on some operator coefficients from three-body lepton decays, evaluated at the experimental scale.

Process	$\widetilde{\text{BR}} <$	$\frac{v^2}{\Lambda^2}  C  <$
$\tau \rightarrow ee\bar{e}$	$1.6 \times 10^{-7}$	$\tilde{C}_{V,LL}^{\tau eee} < 2.8 \times 10^{-4}$ , $\tilde{C}_{V,LR}^{\tau eee} < 4 \times 10^{-4}$
$\tau \rightarrow e\mu\bar{\mu}$	$1.6 \times 10^{-7}$	$\tilde{C}_{V,LR}^{\tau e\mu\mu}$ , $\tilde{C}_{V,LL}^{\tau e\mu\mu} < 4 \times 10^{-4}$
$\tau \rightarrow \mu e\bar{e}$	$1.0 \times 10^{-7}$	$\tilde{C}_{V,LR}^{\tau \mu ee}$ , $\tilde{C}_{V,LL}^{\tau \mu ee} < 3.2 \times 10^{-4}$
$\tau \rightarrow \mu\mu\bar{\mu}$	$1.2 \times 10^{-7}$	$\tilde{C}_{V,LL}^{\tau e\mu\mu} < 2.5 \times 10^{-4}$ , $\tilde{C}_{V,LR}^{\tau e\mu\mu} < 3.5 \times 10^{-4}$
$\tau \rightarrow ee\bar{\mu}$	$8.6 \times 10^{-8}$	$\tilde{C}_{V,LL}^{\tau eee} < 3.2 \times 10^{-4}$ ,
$\tau \rightarrow \mu\mu\bar{e}$	$1.0 \times 10^{-7}$	$\tilde{C}_{V,LL}^{\tau \mu ee} < 3.2 \times 10^{-4}$
$\mu \rightarrow ee\bar{e}$	$1.0 \times 10^{-12}$	$\tilde{C}_{V,LL}^{\mu eee} < 7.1 \times 10^{-7}$ , $\tilde{C}_{V,LR}^{\mu eee} < 10^{-6}$

next subsection (see Table I). Then it is clear that each upper bounds on a three-body leptonic decay of the  $\tau$  or  $\mu$ , implies six independent constraints on operator coefficients (evaluated at the experimental scale), those of interest to us are given in Table II.

The operator coefficients  $\tilde{C}_X(m_\tau)$  given in Table II can be expressed in terms of coefficients at  $m_W$  using the one-loop RGEs [34,35]:

$$\begin{aligned} \mu \frac{d}{d\mu} \tilde{C}_I &= \frac{\alpha_e}{4\pi} \tilde{C}_J [\gamma_e]_{JI} \\ \Rightarrow \tilde{C}_I(m_\tau) &= \tilde{C}_J(m_W) \left[ \delta_{JI} - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} [\gamma_e]_{JI} + \dots \right] \end{aligned}$$

where  $[\gamma_e]$  is the one-loop anomalous dimension matrix of QED,  $\ln \frac{m_W}{m_\tau} = 3.85$ ,  $\ln \frac{m_W}{m_\mu} = 6.64$  and the approximate solution neglects the running of  $\alpha_e$ . The one-loop QED corrections involve photon exchange between two legs of the operator, which does not change the flavor or chiral indices, and also ‘‘penguin’’ diagrams, where two legs of the operator are closed in a loop, and a photon is attached, which turns into two external leg fermions. The ‘‘penguins’’ can change the chirality and flavor, and allow 2-lepton-2-quark operators to mix with the four-lepton operators. We therefore need a recipe for dealing with the quark-sector thresholds  $m_b$ ,  $m_c$  and  $\Lambda_{\text{QCD}}$ . We make the simplest approximation, which is to have a single low-energy threshold at  $m_\tau$ , and run from  $m_W \rightarrow m_\tau$  with five flavors of quark, and we use this low-energy scale also for the decays of the  $\mu$ . In this approximation, it is convenient to define the combination of operator coefficients

$$\begin{aligned} \tilde{C}_{\text{penguin}}^{\alpha\beta} &= -\frac{4N_c}{3} \sum_q Q_q (\tilde{C}_{V,LL}^{\alpha\beta qq} + \tilde{C}_{V,LR}^{\alpha\beta qq}) \\ &+ \frac{4}{3} \sum_l ([1 + \delta_{al} + \delta_{\beta l}] \tilde{C}_{V,LL}^{\alpha\beta ll} + \tilde{C}_{V,LR}^{\alpha\beta ll}) \end{aligned} \quad (\text{E12})$$

where  $l \in \{e, \mu, \tau\}$ ,  $q \in \{u, d, s, c, b\}$ , and  $Q_q$  is the electric charge of the quark. Then the coefficients constrained in Table II can be written

$$\begin{aligned}\tilde{C}_{V,LR}^{e\mu ee}(m_\tau) &= \left[1 + 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LR}^{e\mu ee}(m_W) \\ &\quad - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \tilde{C}_{\text{penguin}}^{e\mu}(m_W)\end{aligned}\quad (\text{E13})$$

$$\begin{aligned}\tilde{C}_{V,LL}^{e\mu ee}(m_\tau) &= \left[1 - 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LL}^{e\mu ee}(m_W) \\ &\quad - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \tilde{C}_{\text{penguin}}^{e\mu}(m_W)\end{aligned}\quad (\text{E14})$$

$$\begin{aligned}\tilde{C}_{V,LR}^{e\tau ll}(m_\tau) &= \left[1 + 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LR}^{e\tau ll}(m_W) \\ &\quad - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \tilde{C}_{\text{penguin}}^{e\tau}(m_W)\end{aligned}\quad (\text{E15})$$

$$\begin{aligned}\tilde{C}_{V,LL}^{e\tau ll}(m_\tau) &= \left[1 - 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LL}^{e\tau ll}(m_W) \\ &\quad - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \tilde{C}_{\text{penguin}}^{e\tau}(m_W)\end{aligned}\quad (\text{E16})$$

$$\begin{aligned}\tilde{C}_{V,LR}^{\mu\tau ll}(m_\tau) &= \left[1 + 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LR}^{\mu\tau ll}(m_W) \\ &\quad - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \tilde{C}_{\text{penguin}}^{\mu\tau}(m_W)\end{aligned}\quad (\text{E17})$$

$$\begin{aligned}\tilde{C}_{V,LL}^{\mu\tau ll}(m_\tau) &= \left[1 - 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LL}^{\mu\tau ll}(m_W) \\ &\quad - \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \tilde{C}_{\text{penguin}}^{\mu\tau}(m_W)\end{aligned}\quad (\text{E18})$$

$$\tilde{C}_{V,LL}^{\mu\tau\mu e}(m_\tau) = \left[1 - 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LL}^{\mu\tau\mu e}(m_W) \quad (\text{E19})$$

$$\tilde{C}_{V,LL}^{e\tau e\mu}(m_\tau) = \left[1 - 12 \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau}\right] \tilde{C}_{V,LL}^{e\tau e\mu}(m_W). \quad (\text{E20})$$

Finally, the combinations of coefficients that are constrained by data can be matched at  $m_W$  onto coefficients of SMEFT operators [34]<sup>8</sup>:

$$\begin{aligned}\tilde{C}_{V,LL}^{e\tau e\mu}(m_W) &= C_{\ell\ell}^{e\tau e\mu}(m_W) + C_{\ell\ell}^{\mu e\tau}(m_W) \\ \tilde{C}_{V,LL}^{\mu\tau\mu e}(m_W) &= C_{\ell\ell}^{\mu\tau\mu e}(m_W) + C_{\ell\ell}^{\mu e\mu\tau}(m_W) \\ \tilde{C}_{V,LL}^{\mu\tau ee}(m_W) &= C_{\ell\ell}^{\mu\tau ee}(m_W) + C_{\ell\ell}^{ee\mu\tau}(m_W) + C_{\ell\ell}^{\mu e e\tau}(m_W) + C_{\ell\ell}^{e\tau\mu e}(m_W) + g_L^e [C_{H\ell(1)}^{\mu\tau}(m_W) + C_{H\ell(3)}^{\mu\tau}(m_W)] \\ \tilde{C}_{V,LL}^{\mu\tau\mu\mu}(m_W) &= C_{\ell\ell}^{\mu\tau\mu\mu}(m_W) + C_{\ell\ell}^{\mu\mu\mu\tau}(m_W) + g_L^e [C_{H\ell(1)}^{\mu\tau}(m_W) + C_{H\ell(3)}^{\mu\tau}(m_W)] \\ \tilde{C}_{V,LR}^{\mu\tau ll}(m_W) &= C_{\ell e}^{\mu\tau ll}(m_W) + g_R^e [C_{H\ell(1)}^{\mu\tau}(m_W) + C_{H\ell(3)}^{\mu\tau}(m_W)] \\ \tilde{C}_{V,LL}^{\mu\tau\mu\mu}(m_W) &= C_{\ell\ell}^{\mu\tau\mu\mu}(m_W) + C_{\ell\ell}^{\mu\mu e\tau}(m_W) + C_{\ell\ell}^{e\mu\mu\tau}(m_W) + C_{\ell\ell}^{\mu\tau e\mu}(m_W) + g_L^e [C_{H\ell(1)}^{e\tau}(m_W) + C_{H\ell(3)}^{e\tau}(m_W)] \\ \tilde{C}_{V,LL}^{e\tau ee}(m_W) &= C_{\ell\ell}^{e\tau ee}(m_W) + C_{\ell\ell}^{ee e\tau}(m_W) + g_L^e [C_{H\ell(1)}^{e\tau}(m_W) + C_{H\ell(3)}^{e\tau}(m_W)] \\ \tilde{C}_{V,LR}^{e\tau ll}(m_W) &= C_{\ell e}^{e\tau ll}(m_W) + g_R^e [C_{H\ell(1)}^{e\tau}(m_W) + C_{H\ell(3)}^{e\tau}(m_W)] \\ \tilde{C}_{V,LL}^{e\mu ee}(m_W) &= C_{\ell\ell}^{e\mu ee}(m_W) + C_{\ell\ell}^{eee\mu}(m_W) + g_L^e [C_{H\ell(1)}^{e\mu}(m_W) + C_{H\ell(3)}^{e\mu}(m_W)] \\ \tilde{C}_{V,LR}^{e\mu ee}(m_W) &= C_{\ell e}^{e\mu ee}(m_W) + g_R^e [C_{H\ell(1)}^{e\mu}(m_W) + C_{H\ell(3)}^{e\mu}(m_W)]\end{aligned}\quad (\text{E21})$$

where  $l \in \{e, \mu\}$  in the above equations, and  $g_R^e = 2\sin^2\theta_W$ ,  $g_L^e = -1 + 2\sin^2\theta_W$ . In order to match the ‘‘penguin’’ coefficient of Eq. (E12) onto coefficients of the SMEFT, matching conditions for operators with a quark bilinear are also required:

$$\begin{aligned}\tilde{C}_{V,LL}^{\alpha\beta uu}(m_W) &= C_{\ell q(1)}^{\alpha\beta uu}(m_W) - C_{\ell q(3)}^{\alpha\beta uu}(m_W) + g_L^u [C_{H\ell(1)}^{\alpha\beta}(m_W) + C_{H\ell(3)}^{\alpha\beta}(m_W)] \\ \tilde{C}_{V,LL}^{\alpha\beta dd}(m_W) &= C_{\ell q(1)}^{\alpha\beta dd}(m_W) + C_{\ell q(3)}^{\alpha\beta dd}(m_W) + g_L^d [C_{H\ell(1)}^{\alpha\beta}(m_W) + C_{H\ell(3)}^{\alpha\beta}(m_W)] \\ \tilde{C}_{V,LR}^{\alpha\beta uu}(m_W) &= C_{\ell u}^{\alpha\beta uu}(m_W) + g_R^u [C_{H\ell(1)}^{\alpha\beta}(m_W) + C_{H\ell(3)}^{\alpha\beta}(m_W)] \\ \tilde{C}_{V,LR}^{\alpha\beta dd}(m_W) &= C_{\ell d}^{\alpha\beta dd}(m_W) + g_R^d [C_{H\ell(1)}^{\alpha\beta}(m_W) + C_{H\ell(3)}^{\alpha\beta}(m_W)]\end{aligned}\quad (\text{E22})$$

<sup>8</sup>These equations differ from [34] due to different conventions for operator normalization and signs, and also due to some errors in [34]. The SMEFT basis used here is normalized according to [20], where there are ‘‘redundant’’ flavor-changing four-fermion operators, which are absent from the basis used below  $m_W$  in [34], compare e.g., the left and right hand sides of Eq. (E21). Then, the sign convention used here for the  $g_{V,LR}^f$  and the Z-vertex Feynman rule agrees with the PDG but is opposite to that of [34]. Finally, in [34], there is an incorrect factor of 2 multiplying the penguin coefficients which generate  $s$  and  $t$  channel diagrams; this 2 should not appear, because the four-fermion operator generates the same  $s$  and  $t$  channel diagrams.

where  $\alpha\beta \in \{\mu\tau, e\tau, e\mu\}$ ,  $g_L^\mu = 1 - \frac{4}{3}\sin^2\theta_W$ ,  $g_R^\mu = -\frac{4}{3}\sin^2\theta_W$ ,  $g_L^d = -1 + \frac{2}{3}\sin^2\theta_W$  and,  $g_R^d = \frac{2}{3}\sin^2\theta_W$ . Combining the definition (E12) with the matching conditions of Eq. (E22) allows the definition of a combination of SMEFT coefficients  $C_{\text{penguin}}^{\alpha\beta}(m_W)$ . Then the experimental constraint on, for instance  $\tilde{C}_{V,LL}^{e\tau\mu\mu}(m_\tau)$ , gives

$$|[1 - 12\delta][C_{\ell\ell}^{e\tau\mu\mu} + C_{\ell\ell}^{\mu e\tau} + C_{\ell\ell}^{e\mu\tau} + C_{\ell\ell}^{\mu\tau e\mu}] + g_L^e[C_{H\ell(1)}^{e\tau} + C_{H\ell(3)}^{e\tau}] - \delta C_{\text{penguin}}^{\alpha\beta}| < 4 \times 10^{-4} \quad (\text{E23})$$

where all the coefficients are evaluated at  $m_W$ , and  $\delta = \frac{\alpha_e}{4\pi} \ln \frac{m_W}{m_\tau} \sim 1/400$ . This and other constraints from 3-body  $\tau$  decays are given in Table I, where for compactness,  $[1 \pm 12\delta]$  is approximated as 1.

### e. $l_\beta \rightarrow l_\alpha \gamma$

The radiative decays  $l_\beta \rightarrow l_\alpha \gamma$  provide some of the most restrictive bounds on lepton flavor violation. The branching ratio at  $m_\beta$  can be written

$$\begin{aligned} \widetilde{\text{BR}}(l_\beta \rightarrow l_\alpha \gamma) &\equiv \frac{\text{BR}(l_\beta \rightarrow l_\alpha \gamma)}{\text{BR}(l_\beta \rightarrow l_\alpha \bar{\nu} \nu)} \\ &= 384\pi^2 \frac{v^4}{\Lambda^4} (|C_{D,L}^{\alpha\beta}|^2 + |C_{D,R}^{\alpha\beta}|^2) \\ &\leq \begin{cases} 4.2 \times 10^{-13} & \mu \rightarrow e\gamma \\ 2.0 \times 10^{-7} & \tau \rightarrow e\gamma \\ 2.5 \times 10^{-7} & \tau \rightarrow \mu\gamma \end{cases} \quad (\text{E24}) \end{aligned}$$

where the low energy dipole operators are added to the Lagrangian as in Eq. (E9).

The dipole coefficients evaluated at the experimental scale can be expressed in terms of SMEFT coefficients at the weak scale as [34]

$$\begin{aligned} C_{D,L}^{\alpha\beta}(m_\tau) &= C_{e\gamma}^{\beta\alpha*}(m_W) + \frac{e\alpha_e y_t}{8\pi^3 y_\mu} C_{eH}^{\beta\alpha*}(m_W) \\ &+ \frac{eg_L^e}{16\pi^2} C_{He}^{\alpha\beta}(m_W) + \dots \quad (\text{E25}) \end{aligned}$$

$$\begin{aligned} C_{D,R}^{\alpha\beta}(m_\tau) &= C_{e\gamma}^{\alpha\beta}(m_W) + \frac{e\alpha_e y_t}{8\pi^3 y_\mu} C_{eH}^{\alpha\beta}(m_W) \\ &+ \frac{eg_R^e}{16\pi^2} [C_{H\ell(1)}^{\alpha\beta}(m_W) + C_{H\ell(3)}^{\alpha\beta}(m_W) + \dots] \quad (\text{E26}) \end{aligned}$$

where the contributions of scalar and tensor four-fermion operators were neglected,  $g_R^e$  and  $g_L^e$  are defined after Eq. (E21), and

$$C_{e\gamma}^{\alpha\beta} = c_W C_{eB}^{\alpha\beta} - s_W C_{eW}^{\alpha\beta}. \quad (\text{E27})$$

## APPENDIX F: COMPARISON WITH THE LITERATURE

The standard model calculation has been performed in Ref. [21] in a different operator basis. We disagree with their final results even after transforming our results to their basis. To do this we specify our basis

$$\tilde{\mathcal{O}} = (\mathcal{O}_{H\ell(1)}, \mathcal{O}_{H\ell(3)}, \mathcal{O}_{eH}, \mathcal{O}_{eH}^\dagger, \mathcal{O}_{v(1)}, \mathcal{O}_{v(1)}^\dagger, \mathcal{O}_{v(3)}, \mathcal{O}_{v(3)}^\dagger)^T, \quad (\text{F1})$$

and the one used in Ref. [21]

$$\tilde{\mathcal{Q}} = (\mathcal{Q}_{\phi\ell}^{(-)}, \mathcal{Q}_{\phi\ell}^{(+)}, \mathcal{Q}_{e\phi}, \mathcal{Q}_{e\phi}^\dagger, \mathcal{O}_{v(1)}, \mathcal{O}_{v(1)}^\dagger, \mathcal{O}_{v(3)}, \mathcal{O}_{v(3)}^\dagger)^T, \quad (\text{F2})$$

where the additional operators are defined as

$$\mathcal{Q}_{\phi\ell}^{(-)} = \frac{i}{2} [(H^\dagger D_\mu H)(\bar{\ell}\gamma^\mu \ell) - (H^\dagger D_\mu^a H)(\bar{\ell}\tau^a \gamma^\mu \ell)] \quad (\text{F3})$$

$$\mathcal{Q}_{\phi\ell}^{(+)} = \frac{i}{2} [(H^\dagger D_\mu H)(\bar{\ell}\gamma^\mu \ell) + (H^\dagger D_\mu^a H)(\bar{\ell}\tau^a \gamma^\mu \ell)] \quad (\text{F4})$$

$$\mathcal{Q}_{e\phi} = \mathcal{O}_{eH}. \quad (\text{F5})$$

Here we drop the generation indices and note that the operators  $\mathcal{Q}_{\phi\ell}^{(-)}$  and  $\mathcal{Q}_{\phi\ell}^{(+)}$  are not Hermitian. For this reason we treat the operator  $\mathcal{O}_{eH}$  and the EOM-vanishing operators independent from their Hermitian conjugate in our basis transformation. Writing the resulting linear transformation as

$$\tilde{\mathcal{O}} = \hat{R} \tilde{\mathcal{Q}},$$

only the first two rows of  $\hat{R}$  have entries that are not proportional to an identity transformation. These two rows are determined by the following linear transformation<sup>9</sup>:

$$\begin{pmatrix} \mathcal{O}_{H\ell(1)} \\ \mathcal{O}_{H\ell(3)} \end{pmatrix} = \begin{pmatrix} 2 & 2 & Y & -Y^\dagger & 1 & -1 & 0 & 0 \\ -2 & 2 & Y & -Y^\dagger & 0 & 0 & 1 & -1 \end{pmatrix} \tilde{\mathcal{O}}. \quad (\text{F6})$$

The Wilson coefficients and renormalization constants will consequently fulfill our hermiticity conditions in our basis, but not necessarily in the basis of Ref. [21]. The counterterms of the Wilson coefficients transform in the same way as the respective Wilson coefficients under our change of basis, i.e., as

$$\delta\tilde{c} = \hat{R}^T \delta\tilde{C},$$

<sup>9</sup>To perform the change of basis we have to move covariant derivatives from one term to another. This can be done by noting that the total derivatives  $D_\mu[(H^\dagger H)(\bar{\ell}\gamma^\mu \ell)]$  and  $D_\mu[(H^\dagger \tau^a H)(\bar{\ell}\tau^a \gamma^\mu \ell)]$  are vanishing.

where  $\delta\tilde{C} = (16\pi^2)\epsilon\tilde{C}\tilde{Z}\tilde{C}^\dagger$  represent the counterterms multiplied with  $(16\pi^2)\epsilon$ , while  $\delta\tilde{c}$  correspond to the analogous expression in the  $\tilde{Q}$  basis.

Using the counterterms presented in Eqs. (3.2), (3.3) and (3.6), and the results of (F6) we obtain

$$\delta\tilde{c} = \left( -\frac{5}{2}[C_5C_5^*], -\frac{1}{2}[C_5C_5^*], \frac{1}{2}[C_5C_5^*Y], [Y^\dagger C_5^*C_5] \right)^T, \quad (\text{F7})$$

which fulfil the hermiticity condition of the overall Lagrangian, even though this is not immediately apparent due to the choice of basis. These results are in disagreement with the final results quoted in Ref. [21]. Yet using the results quoted in the individual diagrams in Appendix B of Ref. [21] we find agreement with the expression of Eq. (F7) apart from a global minus sign, which suggests that a different projection was performed. We explicitly checked that the diagrams given in Appendices B1, B3 and B4 of their calculation have the opposite sign compared with our calculation, while we agree with their lepton conserving contribution presented in Appendix B2. Following the explanations of the calculation it appears that part (the  $\delta\delta$  part) of the diagram evaluated in Appendix B.1 of Ref. [21] is projected onto an operator basis where the operators  $\mathcal{Q}_{\phi\ell}^{(\pm)}$  are replaced by  $\mathcal{Q}_{\phi\ell}^{(\pm)'} = \mathcal{Q}_{\phi\ell}^{(\pm)} + (\mathcal{Q}_{\phi\ell}^{(\pm)})^\dagger$ , while another part (the  $\epsilon\epsilon$  part) is projected onto the basis presented in Eq. (F2).

Transforming now to the primed basis, where the Hermitian conjugate is added to the first two operators of Eq. (F2) we find that the nontrivial transformation matrix involves only the first two elements of our and the primed basis. Writing explicitly

$$\begin{pmatrix} \mathcal{O}_{H\ell(1)} \\ \mathcal{O}_{H\ell(3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}_{H\ell}^{(-)'} \\ \mathcal{O}_{H\ell}^{(+)' } \end{pmatrix}, \quad (\text{F8})$$

we find

$$\delta\tilde{c}' = \left( -\frac{5}{4}[C_5C_5^*], -\frac{1}{4}[C_5C_5^*], \frac{3}{4}[C_5C_5^*Y], \frac{3}{4}[Y^\dagger C_5^*C_5] \right)^T. \quad (\text{F9})$$

Again, this result does not agree with Ref. [21]. Finally, note that projecting the results quoted for the individual diagrams in Appendix B of Ref. [21], except the  $\epsilon\epsilon$  part, would give

$$\delta\tilde{c}'_{\text{not}\epsilon\epsilon} = \left( +\frac{1}{4}[C_5C_5^*], +\frac{1}{4}[C_5C_5^*], -\frac{3}{4}[C_5C_5^*Y], -\frac{3}{4}[Y^\dagger C_5^*C_5] \right)^T, \quad (\text{F10})$$

while projecting only the  $\epsilon\epsilon$  part on the non-Hermitian basis yields  $\delta\tilde{c}_{\epsilon\epsilon} = (+2[C_5C_5^*], 0, 0, 0)^T$ . Summing these two terms would reproduce the results of Ref. [21].

## APPENDIX G: FLAVOR CONSERVING CONTRIBUTION

Even though the diagram in Fig. 2 cannot induce lepton flavor violation it contributes to the renormalization of  $\mathcal{O}_{eH}$  and the corresponding operators that involve quarks. We also explicitly checked that the diagrams that involve six external Higgses vanish after summing over them. Denoting the trace over the product of the two dimension-five Wilson coefficients by  $\text{Tr}[C_5C_5^*]$  we find

$$\begin{aligned} (\tilde{C}[\tilde{Z}]\tilde{C}^\dagger)_{eH}^{\beta\alpha} &= -\frac{1}{16\pi^2\epsilon}\text{Tr}[C_5C_5^*[Y_e]_{\beta\alpha}], \\ (\tilde{C}[\tilde{Z}]\tilde{C}^\dagger)_{uH}^{\beta\alpha} &= -\frac{1}{16\pi^2\epsilon}\text{Tr}[C_5C_5^*[Y_u]_{\beta\alpha}], \\ (\tilde{C}[\tilde{Z}]\tilde{C}^\dagger)_{dH}^{\beta\alpha} &= -\frac{1}{16\pi^2\epsilon}\text{Tr}[C_5C_5^*[Y_d]_{\beta\alpha}], \end{aligned} \quad (\text{G1})$$

where  $Y_u$  and  $Y_d$  are defined as  $\Gamma_u$  and  $\Gamma_d$  of Ref. [20]. In addition we also generate the following mixing into operators that only comprise Higgs and gauge fields and write

$$\begin{aligned} (\tilde{C}[\tilde{Z}]\tilde{C}^\dagger)_H &= -2\frac{1}{16\pi^2\epsilon}\text{Tr}[C_5C_5^*]\lambda, \\ (\tilde{C}[\tilde{Z}]\tilde{C}^\dagger)_{HD} &= -2\frac{1}{16\pi^2\epsilon}\text{Tr}[C_5C_5^*], \\ (\tilde{C}[\tilde{Z}]\tilde{C}^\dagger)_{H\Box} &= -\frac{1}{16\pi^2\epsilon}\text{Tr}[C_5C_5^*], \end{aligned} \quad (\text{G2})$$

where the additional operators are defined as:

$$\begin{aligned} \mathcal{O}_{dH}^{\alpha\beta} &= (H^\dagger H)\bar{q}_\alpha H d_\beta \\ \mathcal{O}_{uH}^{\alpha\beta} &= (H^\dagger H)\bar{q}_\alpha \epsilon H^* u_\beta \\ \mathcal{O}_H &= \frac{1}{2}(H^\dagger H)^3 \\ \mathcal{O}_{HD} &= \frac{1}{2}(H^\dagger D^\mu H)^*(H^\dagger D_\mu H) \\ \mathcal{O}_{H\Box} &= \frac{1}{2}(H^\dagger H)\Box(H^\dagger H). \end{aligned} \quad (\text{G3})$$

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