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Spectral analysis of sheared nanoribbons

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Abstract

We investigate the spectrum of the Dirichlet Laplacian in a unbounded strip subject to a new deformation of "shearing": the strip is built by translating a segment oriented in a constant direction along an unbounded curve in the plane. We locate the essential spectrum under the hypothesis that the projection of the tangent vector of the curve to the direction of the segment admits a (possibly unbounded) limit at infinity and state sufficient conditions which guarantee the existence of discrete eigenvalues. We justify the optimality of these conditions by establishing a spectral stability in opposite regimes. In particular, Hardy-type inequalities are derived in the regime of repulsive shearing.

Keywords: sheared strips, quantum waveguides, Hardy inequality, Dirichlet Laplacian.

MSC (2010): Primary: 35R45; 81Q10; Secondary: 35J10, 58J50, 78A50.

1 Introduction

With advances in nanofabrication and measurement science, waveguide-shaped nanostructures have reached the point at which the electron transport can be strongly affected by quantum effects. Among the most influential theoretical results, let us quote the existence of quantum bound states due to *bending* in curved strips, firstly observed by Exner and Šeba [11] in 1989. The pioneering paper has been followed by a huge number of works demonstrating the robustness of the effect in various geometric settings including higher dimensions, and the research field is still active these days (see [15] for a recent paper with a brief overview in the introduction).

In 2008 Ekholm, Kovařík and one of the present authors [9] observed that the geometric deformation of *twisting* has a quite opposite effect on the energy spectrum of an electron confined to three-dimensional tubes, for it creates an effectively repulsive interaction (see [12] for an overview of the two reciprocal effects). More specifically, twisting the

tube locally gives rise to Hardy-type inequalities and a stability of quantum transport, the effect becomes stronger in globally twisted tubes [4] and in extreme situations it may even annihilate the essential spectrum completely [13] (see also [2]). The repulsive effect remains effective even under modification of the boundary conditions [3].

The objective of this paper is to introduce a new, two-dimensional model exhibiting a previously unobserved geometric effect of *shearing*. Mathematically, it is reminiscent of the effect of twisting in the three-dimensional tubes (in some aspects it also recalls the geometric setting of curved wedges studied in [14]), but the lower dimensional simplicity enables us to get an insight into analogous problems left open in [4] and actually provide a complete spectral picture now. The richness of the toy model is reflected in covering very distinct regimes, ranging from purely essential to purely discrete spectra or a combination of both. We believe that the present study will stimulate further interest in sheared nanostructures.

The model that we consider in this paper is characterised by a positive number d (the transverse width of the waveguide) and a differentiable function $f : \mathbb{R} \to \mathbb{R}$ (the boundary profile of the waveguide). The waveguide Ω is introduced as the set of points in $\mathbb{R}^2$ delimited by the curve $x \mapsto (x, f(x))$ and its vertical translation $x \mapsto (x, f(x) + d)$, namely (see Figure 1),

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 : f(x) < y < f(x) + d\right\}. \tag{1}$$

We stress that the geometry of Ω differs from the curved strips intensively studied in the literature for the last thirty years (see [16] for a review). In the latter case, the segment $(0, d)$ is translated along the curve $x \mapsto (x, f(x))$ with respect to its *normal* vector field (so that the waveguide is delimited by two *parallel* curves), while in the present model the translation is with respect to the *constant* basis vector in the y-direction.

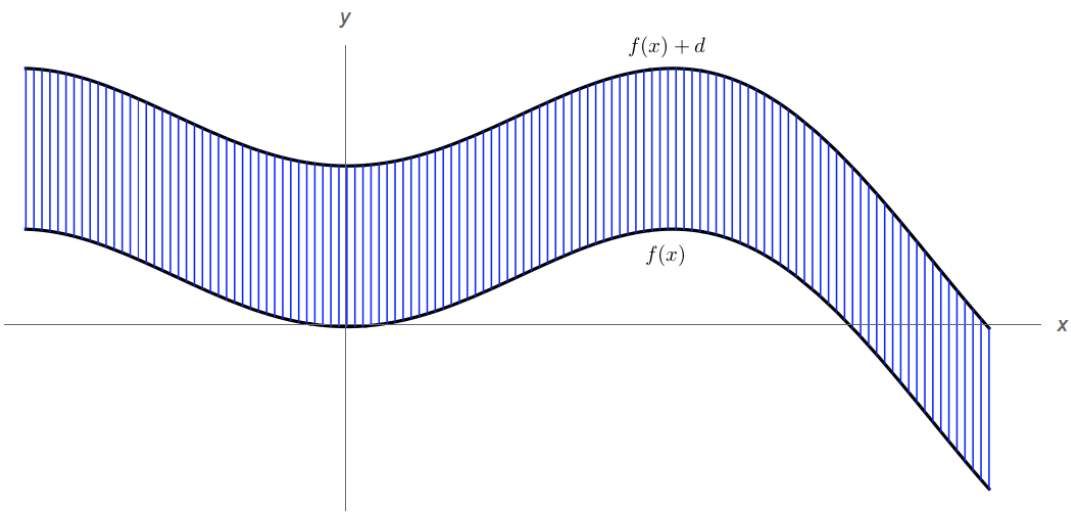

Figure 1: The geometry of a sheared waveguide.

Clearly, it is rather the derivative f' that determines the shear deformation of the straight waveguide $\Omega_0 := \mathbb{R} \times (0, d)$. Our standing assumption is that the derivative f' is

locally bounded and that it admits a (possibly infinite) limit at infinity:

$$\lim_{|x|\to\infty} f'(x) =: \beta \in \mathbb{R} \cup \{\pm\infty\}. \tag{2}$$

If β is finite, we often denote the deficit

$$\varepsilon(x) := f'(x) - \beta\,, \tag{3}$$

but notice that $\varepsilon(x)$ is not necessarily small for finite x.

We consider an effectively free electron constrained to the nanostructure Ω by hard-wall boundaries. Disregarding physical constants, the quantum Hamiltonian of the system can be identified with the Dirichlet Laplacian $-\Delta_D^\Omega$ in $L^2(\Omega)$. The spectrum of the straight waveguide Ω_0 (which can be identified with $f' = 0$ in our model) is well known; by separation of variables, one easily conclude that $\sigma(-\Delta_D^{\Omega_0}) = [(\frac{\pi}{d})^2, \infty)$ and the spectrum is purely absolutely continuous. The main spectral properties of $-\Delta_D^\Omega$ under the shear deformation obtained in this paper are summarised in the following theorems.

First, we locate the possible range of energies of propagating states.

Theorem 1 (Essential spectrum)**.** *Let $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ satisfy* (2)*. Then*

$$\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [E_1(\beta), \infty)\,, \qquad \textit{where} \qquad E_1(\beta) := (1+\beta^2)\left(\frac{\pi}{d}\right)^2. \tag{4}$$

If $\beta = \pm\infty$ (see Figure 2), the result means $\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = \varnothing$, so the spectrum is purely discrete and there are no propagating states. In this case, the distance of points $\mathrm{x} \in \Omega$ to the boundary $\partial\Omega$ tends to zero as $|\mathrm{x}| \to \infty$, so Ω is a quasi-bounded domain (*cf.* Remark 2). This phenomenon is reminiscent of three-dimensional waveguides with asymptotically diverging twisting [13, 2].

Our next concern is about the possible existence or absence of discrete eigenvalues below $E_1(\beta)$ if β is finite. The following theorem together with Theorem 1 provides a sufficient condition for the existence.

Theorem 2 (Attractive shearing)**.** *Let $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ be such that $f'^2 - \beta^2 \in L^1(\mathbb{R})$. Then either of the following conditions*

(i) $\displaystyle\int_{\mathbb{R}} (f'^2(x) - \beta^2)\,\mathrm{d}x < 0\,;$

(ii) $\displaystyle\int_{\mathbb{R}} (f'^2(x) - \beta^2)\,\mathrm{d}x = 0$, *$f'$ is not constant and $f'' \in L^1_{\mathrm{loc}}(\mathbb{R})$;*

implies

$$\inf \sigma(-\Delta_D^\Omega) < E_1(\beta)\,. \tag{5}$$

The theorem indeed implies the existence of discrete spectra under the additional hypothesis (2), because the inequality (5) and the fact that, due to Theorem 1, the essential spectrum of $-\Delta_D^\Omega$ starts by $E_1(\beta)$ ensure that the lowest point in the spectrum of $-\Delta_D^\Omega$ is an isolated eigenvalue of finite multiplicity. Theorem 2 is an analogy of [10], where it was shown that a local "slow down" of twisting in periodically twisted tubes

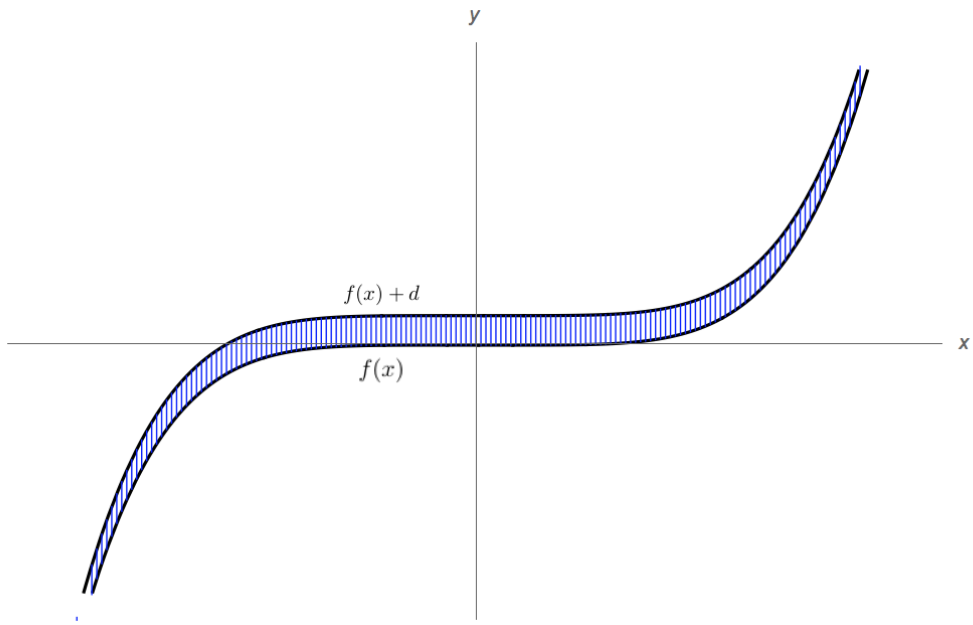

Figure 2: A sheared waveguide with $\beta = +\infty$.

generates discrete eigenvalues (see also [5]). It is not clear from our proof whether the extra regularity assumption in the condition (ii) is necessary.

Recalling (3), notice that $f'^2 - \beta^2 = \varepsilon^2 + 2\beta\varepsilon$, so a necessary condition to satisfy either of the conditions (i) or (ii) of Theorem 2 is that the function $\beta\varepsilon$ is not non-negative. This motivates us to use the terminology that the shear is *attractive* (respectively, *repulsive*) if $\beta\varepsilon$ is non-positive (respectively, non-negative) and ε is non-trivial. In analogy with the conjectures stated in [4], it is expected that the discrete spectrum of $-\Delta_D^\Omega$ is empty provided that the shear is either repulsive or possibly attractive but $\varepsilon^2 \gg -\beta\varepsilon$. The latter is confirmed in the following special setting (see Figure 3).

Theorem 3 (Strong shearing)**.** *Let $f'(x) = \beta + \alpha\varepsilon(x)$, where $\alpha, \beta \in \mathbb{R}$ and $\varepsilon : \mathbb{R} \to \mathbb{R}$ is a function such that* $\operatorname{supp}\varepsilon \subset [0,1]$ *and $c_1 \le \varepsilon(x) \le c_2$ for all $x \in [0,1]$ with some positive constants c_1, c_2. Then there exists a positive number α_0 such that, for all $|\alpha| \ge \alpha_0$,*

$$\sigma(-\Delta_D^\Omega) = [E_1(\beta), \infty)\,. \tag{6}$$

Finally, we establish Hardy-type inequalities in the case of repulsive shear.

Theorem 4 (Repulsive shearing)**.** *Let $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$. If $\beta\varepsilon \ge 0$ and ε is non-trivial, then there exists a positive constant c such that the inequality*

$$-\Delta_D^\Omega - E_1(\beta) \ge \frac{c}{1+x^2} \tag{7}$$

holds in the sense of quadratic forms in $L^2(\Omega)$.

Consequently, if the shear satisfies (2) in addition to the repulsiveness, then (7) and Theorem 1 imply that the stability result (6) holds, so in particular there are no discrete

eigenvalues. Moreover, the spectrum is stable against small perturbations (the smallness being quantified by the Hardy weight on the right-hand side of (7)). From Theorem 4 it is also clear that α does not need to be large if $\alpha\beta \geq 0$ in the setting of Theorem 3.

The organisation of the paper is as follows. In Section 2 we introduce natural curvilinear coordinates to parameterise Ω and express the Dirichlet Laplacian in them. Hardy-type inequalities (and in particular Theorem 4) are established in Section 3. The essential spectrum (Theorem 1) is l in Section 4. The sufficient conditions of Theorem 2 (which in particular imply the existence of discrete eigenvalues in the regime (4)) are established in Section 5. Finally, Theorem 3 is proved in Section 6.

2 Preliminaries

Our strategy to deal with the Laplacian in the sheared geometry Ω is to express it in natural curvilinear coordinates. It employs the identification $\Omega = \mathscr{L}(\Omega_0)$, where $\mathscr{L} : \mathbb{R}^2 \to \mathbb{R}^2$ is the shear mapping defined by

$$\mathscr{L}(s,t) := \big(s, f(s) + t\big). \tag{8}$$

In this paper we consistently use the notations s and t to denote the "longitudinal" and "transversal" variables in the straight waveguide Ω_0, respectively. The corresponding metric reads

$$G := \nabla\mathscr{L} \cdot (\nabla\mathscr{L})^T = \begin{pmatrix} 1 + f'^2 & f' \\ f' & 1 \end{pmatrix}, \qquad |G| := \det(G) = 1, \tag{9}$$

where the dot denotes the scalar product in $\mathbb{R}^2$. Recall our standing assumption that $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$. It follows that $\mathscr{L} : \Omega_0 \to \Omega$ is a local diffeomorphism. In fact, it is a global diffeomorphism, because $\mathscr{L}$ is injective. In this way, we have identified Ω with the Riemannian manifold (Ω_0, G).

The Dirichlet Laplacian $-\Delta^\Omega_D$ is introduced standardly as the self-adjoint operator in the Hilbert space $L^2(\Omega)$ associated with the quadratic form $Q^\Omega_D[u] := \int_\Omega |\nabla u|^2$, $\mathsf{dom}\, Q^\Omega_D := H^1_0(\Omega)$. Employing the diffeomorphism $\mathscr{L} : \Omega_0 \to \Omega$, we have the unitary transform

$$\mathcal{U} : L^2(\Omega) \to L^2(\Omega_0) : \{u \mapsto u \circ \mathscr{L}\}\,.$$

Consequently, $-\Delta^\Omega_D$ is unitarily equivalent (and therefore isospectral) to the operator $H := \mathcal{U}(-\Delta^\Omega_D)\mathcal{U}^{-1}$ in the Hilbert space $L^2(\Omega_0)$. By definition, the operator H is associated with the quadratic form

$$h[\psi] := Q^\Omega_D[\mathcal{U}^{-1}\psi]\,, \qquad \mathsf{dom}\, h := \mathcal{U}\,\mathsf{dom}\, Q^\Omega_D\,.$$

Occasionally, we shall emphasise the dependence of h and H on f' by writing f' as the subscript, *i.e.* $h = h_{f'}$ and $H = H_{f'}$.

Given $u \in C^\infty_0(\Omega)$, a core of Q^Ω_D, it follows that $\psi := u \circ \mathscr{L}$ is also compactly supported and $\psi \in H^1_0(\Omega_0)$. Then it is straightforward to verify that

$$h[\psi] = \|\partial_s\psi - f'\partial_t\psi\|^2 + \|\partial_t\psi\|^2\,, \tag{10}$$

where $\|\cdot\|$ denotes the norm of $L^2(\Omega_0)$ and, with an abuse of notation, we denote by the same symbol f' the function $f'\otimes 1$ on $\mathbb{R}\times(0,d)$. We have the elementary bounds

$$\delta\|\partial_s\psi\|^2+\left(1-\frac{\delta}{1-\delta}\|f'\|^2_{L^\infty(\operatorname{supp}\psi)}\right)\|\partial_t\psi\|^2 \\ \leq h[\psi]\leq 2\|\partial_s\psi\|^2+\left(1+2\|f'\|^2_{L^\infty(\operatorname{supp}\psi)}\right)\|\partial_t\psi\|^2 \quad (11)$$

valid for every $\delta\in(0,1)$. The domain of h coincides with the closure of the set of the functions ψ specified above with respect to the graph topology of h. By a standard mollification argument using (11), it follows that $C_0^\infty(\Omega_0)$ is a core of h. Moreover, it follows from (11) that if f' is bounded, then the graph topology of h is equivalent to the topology of $H^1(\Omega_0)$, and therefore $\mathsf{dom}\, h=H_0^1(\Omega_0)$ in this case. In general (and in particular if (2) holds with $\beta=\pm\infty$), however, the domain of h is not necessarily equal to $H_0^1(\Omega_0)$. Let us summarise the preceding observations into the following proposition.

Proposition 1. *Let $f'\in L^\infty_{\mathrm{loc}}(\mathbb{R})$. Then $C_0^\infty(\Omega_0)$ is a core of h. If moreover $f'\in L^\infty(\mathbb{R})$, then $\mathsf{dom}\, h=H_0^1(\Omega_0)$.*

In a distributional sense, we have $H=-(\partial_s-f'\partial_t)^2-\partial_t^2$, but we shall use neither this formula nor any information on the operator domain of H. Our spectral analysis of H will be exclusively based on the associated quadratic form (10). Notice that the structure of H is similar to twisted three-dimensional tubes [9] as well as curved two-dimensional wedges [14].

3 Hardy-type inequalities

In this section we prove Theorem 4. The strategy is to first establish a "local" Hardy-type inequality, for which the Hardy weight might not be everywhere positive, and then "smeared it out" to the "global" inequality (7) with help of a variant of the classical one-dimensional Hardy inequality. Throughout this section, we assume that $\beta\in\mathbb{R}$ and $\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

Let

$$E_1:=\left(\frac{\pi}{d}\right)^2 \qquad \text{and} \qquad \chi_1(t):=\sqrt{\frac{2}{d}}\,\sin(E_1t)$$

denote respectively the lowest eigenvalue and the corresponding normalised eigenfunction of the Dirichlet Laplacian in $L^2((0,d))$. By the variational definition of E_1, we have the Poincaré-type inequality

$$\forall\chi\in H_0^1((0,d))\,, \qquad \int_0^d|\chi'(t)|^2\,\mathrm{d}t\geq E_1\int_0^d|\chi(t)|^2\,\mathrm{d}t\,. \tag{12}$$

Notice that $E_1=E_1(0)$, where the latter is introduced in (4). Neglecting the first term on the right-hand side of (10) and using (12) for the second term together with Fubini's theorem, we immediately get the lower bound

$$H\geq E_1\,, \tag{13}$$

which is independent of f.

The main ingredient in our approach is the following valuable lemma which reveals a finer structure of the form h. With an abuse of notation, we use the same symbol χ_1 for the function $1 \otimes \chi_1$ on $\mathbb{R} \times (0,d)$, and similarly for its derivative χ_1', while ε also denotes the function $\varepsilon \otimes 1$ on $\mathbb{R} \times (0,d)$.

Lemma 1 (Ground-state decomposition)**.** *For every $\psi \in C_0^\infty(\Omega_0)$, we have*

$$\begin{aligned} h[\psi] - E_1(\beta)\|\psi\|^2 &= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 + \|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 \\ &\quad + \int_{\Omega_0} \beta\,\varepsilon(s)\left[E_1 + \left(\frac{\chi_1'(t)}{\chi_1(t)}\right)^2\right]|\psi(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{14}$$

Proof. Let $\phi \in C_0^\infty(\Omega_0)$ be defined by the decomposition $\psi(s,t) = \chi_1(t)\phi(s,t)$. Writing $2\Re(\phi\partial_t\phi) = \partial_t|\phi|^2$, integrating by parts and using the differential equation that χ_1 satisfies, we have

$$\|\partial_t\psi\|^2 = \|\chi_1'\phi\|^2 + \|\chi_1\partial_t\phi\|^2 + 2\Re(\chi_1'\phi, \chi_1\partial_t\phi) = \|\chi_1\partial_t\phi\|^2 + E_1\|\psi\|^2\,,$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega_0)$. Similarly,

$$\begin{aligned} \|\partial_s\psi - f'\partial_t\psi\|^2 &= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t\phi\|^2 + \beta^2\|\chi_1'\phi\|^2 \\ &\quad - 2\beta\,\Re(\chi_1\partial_s\phi - \varepsilon\chi_1\partial_t\phi - \varepsilon\chi_1'\phi - \beta\chi_1\partial_t\phi, \chi_1'\phi) \\ &= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t\phi\|^2 \\ &\quad + \beta E_1(\varepsilon\chi_1\phi, \chi_1\phi) + \beta(\varepsilon\chi_1'\phi, \chi_1'\phi) + \beta^2 E_1(\chi_1\phi, \chi_1\phi) \\ &= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t\phi\|^2 \\ &\quad + \beta E_1(\varepsilon\psi, \psi) + \beta(\varepsilon\chi_1'\phi, \chi_1'\phi) + \beta^2 E_1\|\psi\|^2\,, \end{aligned}$$

where the second equality follows by integrations by parts in the cross term. Summing up the results of the two computations and subtracting $E_1(\beta)\|\psi\|^2$, we arrive at the desired identity. □

From now on, let us assume that the shear is repulsive, *i.e.* $\beta\varepsilon \geq 0$ and $\varepsilon \neq 0$. If $\beta \neq 0$, then Lemma 1 immediately gives the following local Hardy-type inequality

$$H - E_1(\beta) \geq \beta\varepsilon\left[E_1 + \left(\frac{\chi_1'}{\chi_1}\right)^2\right]. \tag{15}$$

Recall that $E_1(\beta)$ corresponds to the threshold of the essential spectrum if (2) holds (*cf.* Theorem 1), so (15) in particular ensures that there is no (discrete) spectrum below $E_1(\beta)$ (even if $\beta = 0$). Note that the Hardy weight on the right-hand side of (15) diverges on $\partial\Omega_0$. The terminology "local" comes from the fact that the Hardy weight might not be everywhere positive (*e.g.* if $\beta\varepsilon$ is compactly supported).

In order to obtain a non-trivial non-negative lower bound including the case $\beta = 0$, we have to exploit the positive terms that we have neglected when coming from (14) to (15). To do so, let $I \subset \mathbb{R}$ be any bounded open interval and set $\Omega_0^I := I \times (0,d)$. In the Hilbert space $L^2(\Omega_0^I)$ let us consider the quadratic form

$$\begin{aligned} q_I[\psi] &:= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2_{L^2(\Omega_0^I)} + \|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2_{L^2(\Omega_0^I)}\,, \\ \mathsf{dom}\, q_I &:= \{\psi \restriction \Omega_0^I : \psi \in H_0^1(\Omega_0)\}\,. \end{aligned}$$

The form corresponds to the sum of the two first terms on the right-hand side of (14) with the integrals restricted to Ω_0^I.

Lemma 2. *The form q_I is closed.*

Proof. First of all, notice that the form q_I is obviously closed if $\beta = 0$ (one can proceed as in (11)). In general, given $\psi \in \mathsf{dom}\, q_I$, we employ the estimates

$$\begin{aligned}\delta\|\partial_s\psi - \varepsilon\partial_t\psi\|^2_{L^2(\Omega_0^I)} + \left(1 - \frac{\delta}{1-\delta}\,\beta^2\right)\|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2_{L^2(\Omega_0^I)} \\ \leq q_I[\psi] \leq 2\|\partial_s\psi - \varepsilon\partial_t\psi\|^2_{L^2(\Omega_0^I)} + (1+2\beta^2)\|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2_{L^2(\Omega_0^I)}\end{aligned}$$

valid for every $\delta \in (0,1)$ and the identity (easily checked by employing the density of $C_0^\infty(\Omega_0)$ in $H_0^1(\Omega_0)$)

$$\|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2_{L^2(\Omega_0^I)} = \|\partial_t\psi\|^2_{L^2(\Omega_0^I)} - E_1\|\psi\|^2_{L^2(\Omega_0^I)}\,. \tag{16}$$

Instead of the latter, we could also use the fact that $\chi_1^{-1}\psi \in L^2(\Omega_0)$ whenever $\psi \in H_0^1(\Omega_0)$, *cf.* [6, Thm. 1.5.6]. Using in addition the aforementioned fact that q_I is closed if $\beta = 0$, we conclude that q_I is closed for any value $\beta \in \mathbb{R}$. □

Consequently, the form q_I is associated to a self-adjoint operator Q_I with compact resolvent. Let us denote by λ_I its lowest eigenvalue, *i.e.*,

$$\lambda_I := \inf_{\substack{\psi\in\mathsf{dom}\, q_I\\ \psi\neq 0}} \frac{q_I[\psi]}{\|\psi\|^2_{L^2(\Omega_0^I)}}\,. \tag{17}$$

The eigenvalue λ_I is obviously non-negative. The following lemma shows that λ_I is actually positive whenever ε is non-trivial on I.

Lemma 3. *$\lambda_I = 0$ if, and only if, $\varepsilon = 0$ on I.*

Proof. If $\varepsilon \restriction I = 0$, then the function $\psi(s,t) := \chi_1(t)$ minimises (17) with $\lambda_I = 0$. Conversely, let us assume that $\lambda_I = 0$ and denote by ψ_1 the corresponding eigenfunction of Q_I, *i.e.* the minimiser of (17). It follows from (17) together with (16) that

$$\begin{aligned}\|\partial_s\psi_1 - \varepsilon\partial_t\psi_1 - \beta\chi_1\partial_t(\chi_1^{-1}\psi_1)\|^2_{L^2(\Omega_0^I)} = 0\,,\\ \|\partial_t\psi_1\|^2_{L^2(\Omega_0^I)} - E_1\|\psi_1\|^2_{L^2(\Omega_0^I)} = 0\,.\end{aligned} \tag{18}$$

Writing $\psi_1(s,t) = \varphi(s)\chi_1(t) + \phi(s,t)$, where

$$\int_0^d \chi_1(t)\phi(s,t)\,\mathrm{d}t = 0$$

for almost every $s \in I$, we have

$$\|\partial_t\psi_1\|^2_{L^2(\Omega_0^I)} - E_1\|\psi_1\|^2_{L^2(\Omega_0^I)} = \|\partial_t\phi\|^2_{L^2(\Omega_0^I)} - E_1\|\phi\|^2_{L^2(\Omega_0^I)} \geq (E_2 - E_1)\|\phi\|^2_{L^2(\Omega_0^I)}\,,$$

where $E_2 = 4E_1$ is the second eigenvalue of the Dirichlet Laplacian in $L^2((0,d))$. Since the gap $E_2 - E_1$ is positive, it follows from the second identity in (18) that $\phi = 0$. Plugging

now the separated-eigenfunction Ansatz $\psi_1(s,t) = \varphi(s)\chi_1(t)$ into the first identity in (18) and integrating by parts with respect to t in the cross term, we obtain

$$0 = \|\partial_s\psi_1 - \varepsilon\partial_t\psi_1\|^2_{L^2(\Omega_0^I)} = \|\varphi'\chi_1\|^2_{L^2(\Omega_0^I)} + \|\varepsilon\varphi\chi_1'\|^2_{L^2(\Omega_0^I)}\,.$$

Consequently, φ is necessarily constant and $\|\varepsilon\|_{L^2(I)} = 0$. □

Using (17), we can improve (15) to the following local Hardy-type inequality

$$H - E_1(\beta) \geq \beta\varepsilon\left[E_1 + \left(\frac{\chi_1'}{\chi_1}\right)^2\right] + \lambda_I\, 1_{I\times(0,d)}\,, \tag{19}$$

where 1_M denotes the characteristic function of a set M. The inequality is valid with any bounded interval $I \subset \mathbb{R}$, but recall that λ_I is positive if, and only if, I is chosen in such a way that ε is not identically equal to zero on I, *cf.* Lemma 3.

The passage from the local Hardy-type inequality (19) to the global Hardy-type inequality of Theorem 4 will be enabled by means of the following crucial lemma, which is essentially a variant of the classical one-dimensional Hardy inequality.

Lemma 4. *Let $s_0 \in \mathbb{R}$ and $\psi \in C_0^\infty(\Omega_0 \setminus \{s_0\} \times (0,d))$. Then*

$$\|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 + \|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 \geq \frac{1}{4(1+\beta^2)}\int_{\Omega_0}\frac{|\psi(s,t)|^2}{(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t\,. \tag{20}$$

Proof. Denote $\rho(s,t) := (s-s_0)^{-1}$. Given any real number α, we write

$$\begin{aligned}
&\|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi) - \alpha\rho\psi\|^2\\
&= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 + \alpha^2\|\rho\psi\|^2 - 2\alpha\Re(\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi), \rho\psi)\\
&= \|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 + (\alpha^2-\alpha)\|\rho\psi\|^2 + 2\alpha\beta\Re(\chi_1\partial_t(\chi_1^{-1}\psi), \rho\psi)\\
&\leq \left\|\partial_s\psi - \varepsilon\partial_t\psi - \beta\chi_1\partial_t(\chi_1^{-1}\psi)\right\|^2 + (\alpha^2-\alpha+\alpha^2\beta^2)\|\rho\psi\|^2 + \|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2\,,
\end{aligned}$$

where the second equality follows by integrations by parts and the estimate is due to the Schwarz inequality. Since the first line is non-negative, we arrive at the desired inequality by choosing the optimal $\alpha := [2(1+\beta^2)]^{-1}$. □

Now we are in a position to prove the "global" Hardy inequality of Theorem 4.

Proof of Theorem 4. Let $\psi \in C_0^\infty(\Omega_0)$. First of all, since ε is assumed to be non-trivial, there exists a real number s_0 and positive b such that ε restricted to the interval $I := (s_0 - b, s_0 + b)$ is non-trivial. By Lemma 17, it follows that λ_I is positive. Then the local Hardy-type inequality (19) implies the estimate

$$h[\psi] - E_1(\beta)\|\psi\|^2 \geq \lambda_I\|\psi\|^2_{L^2(\Omega_0^I)}\,. \tag{21}$$

Second, let $\eta \in C^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\eta = 0$ in a neighbourhood of s_0 and $\eta = 1$ outside I. Let us denote by the same symbol η the function $\eta \otimes 1$ on $\mathbb{R}\times(0,d)$, and similarly for its derivative η'. Writing $\psi = \eta\psi + (1-\eta)\psi$, we have

$$\int_{\Omega_0}\frac{|\psi(s,t)|^2}{1+(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t \leq 2\int_{\Omega_0}\frac{|(\eta\psi)(s,t)|^2}{(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t + 2\int_{\Omega_0}|((1-\eta)\psi)(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t\,. \tag{22}$$

Since $\eta\psi$ satisfies the hypothesis of Lemma 4, we have

$$\begin{aligned}
&\int_{\Omega_0} \frac{|(\eta\psi)(s,t)|^2}{(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t \\
&\leq 4(1+\beta^2)\left(\|\partial_s(\eta\psi)-\eta\varepsilon\partial_t\psi-\eta\beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2+\|\eta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2\right) \\
&\leq 8(1+\beta^2)\|\partial_s\psi-\varepsilon\partial_t\psi-\beta\chi_1\partial_t(\chi_1^{-1}\psi)\|^2+8(1+\beta^2)\|\eta'\psi\|+4(1+\beta^2)\|\chi_1\partial_t(\chi_1^{-1}\psi)\|^2 \\
&\leq 8(1+\beta^2)(h[\psi]-E_1(\beta)\|\psi\|^2)+8(1+\beta^2)\|\eta'\|_\infty\|\psi\|^2_{L^2(\Omega_0^I)}
\end{aligned}$$

where $\|\eta'\|_\infty$ denotes the supremum norm of η'. Here the last inequality employs Lemma 1 (recall that we assume that the shear is repulsive). At the same time,

$$\int_{\Omega_0} |((1-\eta)\psi)(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t \leq \|\psi\|^2_{L^2(\Omega_0^I)}\,.$$

From (22) we therefore deduce

$$h[\psi]-E_1(\beta)\|\psi\|^2 \geq \frac{1}{16(1+\beta^2)}\int_{\Omega_0}\frac{|\psi(s,t)|^2}{1+(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t-\left(\|\eta'\|^2_\infty+\frac{1}{8(1+\beta^2)}\right)\|\psi\|^2_{L^2(\Omega_0^I)}\,. \tag{23}$$

Finally, interpolating between (21) and (23), we obtain

$$\begin{aligned}
h[\psi]-E_1(\beta)\|\psi\|^2 \geq{}& \frac{\delta}{16(1+\beta^2)}\int_{\Omega_0}\frac{|\psi(s,t)|^2}{1+(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t \\
&+\left[(1-\delta)\lambda_I-\delta\|\eta'\|^2_\infty-\frac{\delta}{8(1+\beta^2)}\right]\|\psi\|^2_{L^2(\Omega_0^I)}
\end{aligned}$$

for every real δ. Choosing (positive) δ in such a way that the square bracket vanishes, we arrive at the global Hardy-type inequality

$$h[\psi]-E_1(\beta)\|\psi\|^2 \geq c'\int_{\Omega_0}\frac{|\psi(s,t)|^2}{1+(s-s_0)^2}\,\mathrm{d}s\,\mathrm{d}t$$

with

$$c' := \frac{\lambda_I}{16(1+\beta^2)(\lambda_I+\|\eta'\|^2_\infty)+2}$$

From this inequality we also deduce

$$h[\psi]-E_1(\beta)\|\psi\|^2 \geq c\int_{\Omega_0}\frac{|\psi(s,t)|^2}{1+s^2}\,\mathrm{d}s\,\mathrm{d}t \tag{24}$$

with

$$c := c'\inf_{s\in\mathbb{R}}\frac{1+s^2}{1+(s-s_0)^2}\,,$$

where the infimum is positive. The desired inequality (7) follows by the unitary equivalence between H and $-\Delta_D^\Omega$ together with the fact that the the coordinates s and x are equivalent through this transformation, *cf.* (8). □

Remark 1. The present proof does not employ the presence of the first term on the right-hand side of (19). The inequality (24) can be consequently improved to

$$H-E_1(\beta) \geq \delta\,\beta\varepsilon(s)\left[E_1+\left(\frac{\chi_1'(t)}{\chi_1(t)}\right)^2\right]+(1-\delta)\,\frac{c}{1+s^2}$$

with any $\delta\in[0,1]$.

4 Location of the essential spectrum

In this section we prove Theorem 1. Since the technical approaches for finite and infinite β are quite different, we accordingly split the section into two subsections. In either case, we always assume $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

4.1 Finite limits

We employ the following characterisation of the essential spectrum, for which we are inspired in [7, Lem. 4.2].

Lemma 5. *A real number λ belongs to the essential spectrum of H if, and only if, there exists a sequence $\{\psi_n\}_{n=1}^\infty \subset \mathsf{dom}\, h$ such that the following three conditions hold:*

(i) $\|\psi_n\| = 1$ *for every* $n \geq 1$,

(ii) $(H-\lambda)\psi_n \to 0$ *as* $n \to \infty$ *in the norm of the dual space* $(\mathsf{dom}\, h)^*$.

(iii) $\operatorname{supp} \psi_n \subset \Omega_0 \setminus (-n,n) \times (0,d)$ *for every* $n \geq 1$.

Proof. By the general Weyl criterion modified to quadratic forms (*cf.* [7, Lem. 4.1] or [17, Thm. 5] where the latter contains a proof), λ belongs to the essential spectrum of H if, and only if, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset \mathsf{dom}\, h$ such that (i) and (ii) hold but (iii) is replaced by

(iii') $\phi_n \to 0$ as $n \to \infty$ weakly in $L^2(\Omega_0)$.

The sequence $\{\psi_n\}_{n=1}^\infty$ satisfying (i) and (iii) is clearly weakly converging to zero. Hence, one implication of the lemma is obvious. Conversely, let us assume that there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset \mathsf{dom}\, h$ satisfying (i), (ii) and (iii') and let us construct from it a sequence $\{\psi_n\}_{n=1}^\infty \subset \mathsf{dom}\, h$ satisfying (i), (ii) and (iii). Since $C_0^\infty(\Omega_0)$ is the form core of H (*cf.* Proposition 1), we may assume that $\{\phi_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega_0)$.

Let us denote the norms of $\mathsf{dom}\, h$ and $(\mathsf{dom}\, h)^*$ by $\|\cdot\|_{+1}$ and $\|\cdot\|_{-1}$, respectively. One has $\|\cdot\|_{\pm 1} = \|(H+1)^{\pm 1/2}\cdot\|$. By writing

$$\phi_n = (H+1)^{-1}(H-\lambda)\phi_n + (\lambda+1)(H+1)^{-1}\phi_n \tag{25}$$

and using (ii), we see that the sequence $\{\phi_n\}_{n=1}^\infty$ is bounded in $\mathsf{dom}\, h$.

Let $\eta \in C^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\eta = 0$ on $[-1,1]$ and $\eta = 1$ on $\mathbb{R}\setminus(-2,2)$. For every $k \geq 1$, we set $\eta_k(s) := \eta(s/k)$ and we keep the same notation for the function $\eta_k \otimes 1$ on $\mathbb{R}\times(0,d)$, and similarly for its derivatives η_k' and η_k''. Clearly, $\operatorname{supp}\eta_k \subset \Omega_0 \setminus (-k,k)\times(0,d)$. For every $k \geq 1$, the operator $(1-\eta_k)(H+1)^{-1}$ is compact in $L^2(\Omega_0)$. By virtue of the weak convergence (iii'), it follows that, for every $k \geq 1$, $(1-\eta_k)(H+1)^{-1}\phi_n \to 0$ as $n \to \infty$ in $L^2(\Omega_0)$. Then there exists a subsequence $\{\phi_{n_k}\}_{k=1}^\infty$ of $\{\phi_n\}_{n=1}^\infty$ such that $(1-\eta_k)(H+1)^{-1}\phi_{n_k} \to 0$ as $k \to \infty$ in $L^2(\Omega_0)$. Consequently, the identity (25) together with (ii) implies that $(1-\eta_k)\phi_{n_k} \to 0$ as $k \to \infty$ in $L^2(\Omega_0)$. It follows that $\eta_k\phi_{n_k}$ can be normalised for all sufficiently large k. More specifically, redefining the subsequence if necessary, we may assume that $\|\eta_k\phi_{n_k}\| \geq 1/2$ for all $k \geq 1$. We set

$$\psi_k := \frac{\eta_k\phi_{n_k}}{\|\eta_k\phi_{n_k}\|}$$

and observe that $\{\psi_k\}_{k=1}^\infty$ satisfies (i) and (iii).

It remains to verify (ii). To this purpose, we notice that, using the duality, (ii) means

$$\|(H-\lambda)\phi_n\|_{-1} = \sup_{\substack{\varphi\in\mathsf{dom}\,h\\ \varphi\neq 0}} \frac{|h(\varphi,\phi_n)-\lambda(\varphi,\phi_n)|}{\|\varphi\|_{+1}} \xrightarrow[n\to\infty]{} 0\,, \tag{26}$$

where $\|\varphi\|_{+1}^2 = h[\varphi]+\|\varphi\|^2$. By the direct computation employing integrations by parts, one can check the identity

$$\begin{aligned} h(\varphi,\eta_k\phi_{n_k}) &- \lambda(\varphi,\eta_k\phi_{n_k}) \\ &= h(\eta_k\varphi,\phi_{n_k}) - \lambda(\eta_k\varphi,\phi_{n_k}) + 2(\partial_s\varphi - f'\partial_t\varphi,\eta_k'\phi_{n_k}) + (\varphi,\eta_k''\phi_{n_k}) \end{aligned}$$

for every test function $\varphi\in C_0^\infty(\Omega_0)$, a core of $\mathsf{dom}\,h$. Using (26), we have

$$\begin{aligned} \sup_{\substack{\varphi\in C_0^\infty(\Omega_0)\\ \varphi\neq 0}} \frac{|h(\eta_k\varphi,\phi_{n_k})-\lambda(\eta_k\varphi,\phi_{n_k})|}{\|\varphi\|_{+1}} &\le \sup_{\substack{\varphi\in C_0^\infty(\Omega_0)\\ \eta_k\varphi\neq 0}} \frac{|h(\eta_k\varphi,\phi_{n_k})-\lambda(\eta_k\varphi,\phi_{n_k})|}{\|\eta_k\varphi\|_{+1}} \\ &= \|(H-\lambda)\phi_{n_k}\|_{-1} \xrightarrow[k\to\infty]{} 0\,. \end{aligned}$$

At the same time, using the Schwarz inequality and the estimates $\|\partial_s\varphi - f'\partial_t\varphi\|^2 \le h[\varphi]\le\|\varphi\|_{+1}^2$, we get

$$\sup_{\substack{\varphi\in C_0^\infty(\Omega_0)\\ \varphi\neq 0}} \frac{|2(\partial_s\varphi - f'\partial_t\varphi,\eta_k'\phi_{n_k})|}{\|\varphi\|_{+1}} \le \|\eta_k'\|_\infty\|\phi_{n_k}\|\,,$$

where $\|\eta_k'\|_\infty$ denotes the supremum norm of η_k'. In view of the normalisation (i) and since $\|\eta_k'\|_\infty = k^{-1}\|\eta'\|_\infty$, we see that also this term tends to zero as $k\to\infty$. Finally, using the Schwarz inequality and the estimate $\|\varphi\|\le\|\varphi\|_{+1}$, we have

$$\sup_{\substack{\varphi\in C_0^\infty(\Omega_0)\\ \varphi\neq 0}} \frac{|(\varphi,\eta_k''\phi_{n_k})|}{\|\varphi\|_{+1}} \le \|\eta_k''\|_\infty\|\phi_{n_k}\| = k^{-2}\|\eta''\|_\infty \xrightarrow[k\to\infty]{} 0\,.$$

Summing up, we have just checked $\|(H-\lambda)(\eta_k\phi_{n_k})\|_{-1}\to 0$ as $k\to\infty$. Recalling $\eta_k\phi_{n_k}\ge 1/2$, the desired property (ii) for $\{\psi_k\}_{k=1}^\infty$ follows. □

Using this lemma, we immediately arrive at the following "decomposition principle" (saying that the essential spectrum is determined by the behaviour at infinity only).

Proposition 2. *If* (2) *holds with* $\beta\in\mathbb{R}$, *then* $\sigma_{\rm ess}(H_{\beta+\varepsilon}) = \sigma_{\rm ess}(H_\beta)$.

Proof. Let $\lambda\in\sigma_{\rm ess}(H_\beta)$. By Lemma 5, there exists a sequence $\{\psi_n\}_{n=1}^\infty$ satisfying the properties (i)–(iii) with H being replaced by H_β. One easily checks the identity

$$h_{\beta+\varepsilon}(\varphi,\psi_n) = h_\beta(\varphi,\psi_n) - (\partial_s\varphi-\beta\partial_t\varphi,\varepsilon\partial_t\psi_n) - (\varepsilon\partial_t\varphi,\partial_s\psi_n-(\beta+\varepsilon)\partial_t\psi_n) \tag{27}$$

for every test function $\varphi\in\mathsf{dom}\,h_\beta = H_0^1(\Omega_0) = \mathsf{dom}\,h_{\beta+\varepsilon}$ (*cf.* Proposition 1). Now we proceed similarly as in the end of the proof of Lemma 5. Using the estimates $\|\partial_s\varphi - \beta\partial_t\varphi\|^2\le h[\varphi]\le\|\varphi\|_{+1}$ and the Schwarz inequality, we have

$$\sup_{\substack{\varphi\in H_0^1(\Omega_0)\\ \varphi\neq 0}} \frac{|(\partial_s\varphi-\beta\partial_t\varphi,\varepsilon\partial_t\psi_n)|}{\|\varphi\|_{+1}} \le \|\varepsilon\partial_t\psi_n\| \le \|\varepsilon\|_{L^\infty(\mathbb{R}\setminus(-n,n))}\|\partial_t\psi_n\| \xrightarrow[n\to\infty]{} 0\,.$$

Recall that $\{\psi_n\}_{n=1}^\infty$ is bounded in $H_0^1(\Omega_0)$, *cf.* (25) and Proposition 1. Similarly, using $\|\partial_t\varphi\| \le \|\varphi\|_{+1}$ and the Schwarz inequality, we obtain

$$\begin{aligned}\sup_{\substack{\varphi\in H_0^1(\Omega_0)\\ \varphi\neq 0}} \frac{|(\varepsilon\partial_t\varphi, \partial_s\psi_n - (\beta+\varepsilon)\partial_t\psi_n)|}{\|\varphi\|_{+1}} &\le \|\varepsilon\partial_s\psi_n\| + \|\beta+\varepsilon\|_\infty\|\varepsilon\partial_t\psi_n\| \\ &\le \|\varepsilon\|_{L^\infty(\mathbb{R}\setminus(-n,n))}(\|\partial_s\psi_n\| + \|\beta+\varepsilon\|_\infty\|\partial_t\psi_n\|) \\ &\xrightarrow[n\to\infty]{} 0\,.\end{aligned}$$

Consequently, since $\|(H_\beta-\lambda)\psi_n\|_{-1} \to 0$ as $n\to\infty$ due to (ii), it follows from (27) and (26) that also $\|(H_{\beta+\varepsilon}-\lambda)\psi_n\|_{-1}\to 0$ as $n\to\infty$. Hence $\lambda\in\sigma_{\rm ess}(H_{\beta+\varepsilon})$.

The opposite inclusion $\sigma_{\rm ess}(H_{\beta+\varepsilon})\subset\sigma_{\rm ess}(H_\beta)$ is proved analogously. □

It remains to determine the (essential) spectrum for the constant shear.

Proposition 3. *One has* $\sigma(H_\beta)=\sigma_{\rm ess}(H_\beta)=[E_1(\beta),\infty)$.

Proof. By performing the partial Fourier transform in the longitudinal variable $s\in\mathbb{R}$, one has the unitary equivalence

$$H_\beta = -(\partial_s-\beta\partial_t)^2-\partial_t^2 \cong -(i\xi-\beta\partial_t)^2-\partial_t^2\,,$$

where $\xi\in\mathbb{R}$ is the dual variable in the Fourier image. Consequently,

$$\sigma(H_\beta)=\bigcup_{\xi\in\mathbb{R}}\sigma(T_\beta(\xi))\,, \tag{28}$$

where, for each fixed $\xi\in\mathbb{R}$, $T_\beta(\xi)$ is the operator in $L^2((0,d))$ that acts as $-(i\xi-\beta\partial_t)^2-\partial_t^2$ and satisfies Dirichlet boundary conditions. The spectral problem for $T_\beta(\xi)$ can be solved explicitly. Alternatively, one can proceed by "completing the square" and "gauging out a constant magnetic field" as follows:

$$\begin{aligned}-(i\xi-\beta\partial_t)^2-\partial_t^2 &= (1+\beta^2)(-\partial_t^2)+2i\xi\beta\partial_t+\xi^2 \\ &= -(1+\beta^2)\left(\partial_t - i\frac{\xi\beta}{1+\beta^2}\right)^2 - \frac{\xi^2\beta^2}{1+\beta^2}+\xi^2 \\ &= -(1+\beta^2)\left(\partial_t - i\frac{\xi\beta}{1+\beta^2}\right)^2 + \frac{\xi^2}{1+\beta^2} \\ &= e^{iFt}\left[-(1+\beta^2)\partial_t^2+\frac{\xi^2}{1+\beta^2}\right]e^{-iFt}\,,\end{aligned}$$

where

$$F:=\frac{\xi\beta}{1+\beta^2}\,.$$

Consequently,

$$\sigma(T_\beta(\xi))=\left\{(1+\beta^2)E_n+\frac{\xi^2}{1+\beta^2}\right\}_{n=1}^\infty\,, \tag{29}$$

where $E_n=n^2E_1$ denotes the $n^{\rm th}$ eigenvalue of the Dirichlet Laplacian in $L^2((0,d))$. As a consequence of (28) and (29), we get the desired claim. □

For finite β, Theorem 1 follows as a direct consequence of Propositions 2 and 3.

4.2 Infinite limits

Let us now prove Theorem 1 in the case $\beta = \pm\infty$. Here the proof is based on the following purely geometric fact.

Proposition 4. *Let $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ satisfy (2) with $\beta = \pm\infty$. Then*

$$\limsup_{\substack{|\mathrm{x}|\to\infty \\ \mathrm{x}\in\Omega}} |B_1(\mathrm{x}) \cap \Omega| = 0\,, \tag{30}$$

where $B_1(\mathrm{x})$ denotes the disk of radius 1 centred at $\mathrm{x} \in \mathbb{R}^2$ and $|\cdot|$ stands for the Lebesgue measure.

Proof. Recall the diffeomorphism $\mathscr{L} : \Omega_0 \to \Omega$ given by (8). For every number $\eta \in \mathbb{R}$, let $\Sigma(\eta)$ denote the set of points intersecting the boundary $\partial\Omega$ with the straight horizontal half-line $(0,\infty)\times\{\eta\}$. Since $f'(s) \to \pm\infty$ as $|s| \to \infty$, it follows that there exists a positive constant s_0 such that f is either strictly increasing or strictly decreasing on (s_0,∞). Let us consider the former situation, the latter can be treated analogously. It follows that there exists a positive constant η_0 such that, for all $\eta \geq \eta_0$, the set $\Sigma(\eta)$ consists of just two points $\mathrm{x}_1(\eta) = \mathscr{L}(s_1(\eta), d)$ and $\mathrm{x}_2(\eta) = \mathscr{L}(s_2(\eta), 0)$ with $1 < s_1(\eta) < s_2(\eta)$. Let $\mathrm{x} = \mathscr{L}(s,t)$ be a point in Ω such that s is positive and the vertical component satisfies the inequality $\eta := f(s)+t \geq 1+\eta_0$ (notice that $|\mathrm{x}| \to \infty$ if, and only if, $|s| \to \infty$). Then we have the crude estimate

$$|B_1(\mathrm{x}) \cap \Omega| \leq 2\,|s_2(\eta+1) - s_1(\eta-1)|\,.$$

By the mean value theorem,

$$2 = f(s_2(\eta+1)) - f(s_1(\eta-1)) = f'(\xi_\eta)\,|s_2(\eta+1) - s_1(\eta-1)|\,,$$

where $\xi_\eta \in (s_1(\eta-1), s_2(\eta+1))$. Since $s_1(\eta-1) \to \infty$ as $s \to \infty$, we also have $\xi_\eta \to \infty$, and therefore $|B_1(\mathrm{x}) \cap \Omega| \to 0$ as $s \to \infty$. The case $s \to -\infty$ can be treated analogously. □

By the Berger-Schechter criterion (*cf.* [8, Thm. V.5.17 & Rem. V.5.18.4]), it follows from (30) that the embedding $W^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Consequently, the Dirichlet Laplacian $-\Delta^\Omega_D$ has no essential spectrum if $\beta = \pm\infty$ and Theorem 1 is proved.

Remark 2. A necessary condition for (30) is that

$$\lim_{\substack{|\mathrm{x}|\to\infty \\ \mathrm{x}\in\Omega}} \mathrm{dist}(\mathrm{x}, \partial\Omega) = 0\,,$$

which means that Ω is a *quasi-bounded* domain ([8, Sec. X.6.1]) in the regime $\beta = \pm\infty$.

5 Existence of discrete spectrum

In view of Theorem 1, the conclusion (5) of Theorem 2 guarantees that any of the conditions (i) or (ii) implies that $-\Delta^\Omega_D$ possesses at least one isolated eigenvalue of finite multiplicity below $E_1(\beta)$. Let us establish these sufficient conditions.

Proof of Theorem 2. Given any positive n, let $\varphi_n : \mathbb{R} \to [0,1]$ be the continuous function such that $\varphi_n = 1$ on $[-n,n]$, $\varphi_n = 0$ on $\mathbb{R}\setminus(-2n,2n)$ and linear in the remaining intervals. Set

$$\psi_n(s,t) := \varphi_n(s)\chi_1(t)$$

and observe that $\psi_n \in \mathsf{dom}(h)\cap H^1_0(\Omega_0)$. Moreover, the identity of Lemma 1 remains valid for ψ_n by approximation. Indeed, it is only important to notice that $\chi_1^{-1}\psi_n \in L^2(\Omega_0)$ as a consequence of [6, Thm. 1.5.6]. Consequently,

$$\begin{aligned} h[\psi_n] - E_1(\beta)\|\psi_n\|^2 &= \|\partial_s\psi_n - \varepsilon\partial_t\psi_n\|^2 + \int_{\Omega_0} \beta\,\varepsilon(s)\left[E_1 + \left(\frac{\chi_1'(t)}{\chi_1(t)}\right)^2\right]|\psi_n(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \|\varphi_n'\|^2_{L^2(\mathbb{R})} + E_1\int_{\mathbb{R}}\left[\varepsilon(s)^2 + 2\beta\varepsilon(s)\right]|\varphi_n(s)|^2\,\mathrm{d}s\,, \end{aligned}$$

where the second equality follows by the special form of ψ_n (the cross term vanishes due to an integration by parts with respect to t) and the formula $\|\chi_1'\|^2_{L^2((0,d))} = E_1$ together with the normalisation of χ_1. Noticing that $\|\varphi_n'\|^2_{L^2(\mathbb{R})} = 2n^{-1}$ and using the dominated convergence theorem, we get

$$h[\psi_n] - E_1(\beta)\|\psi_n\|^2 \xrightarrow[n\to\infty]{} E_1\int_{\mathbb{R}}\left[\varepsilon(s)^2 + 2\beta\varepsilon(s)\right]\mathrm{d}s\,,$$

where the right-hand side is negative by the hypothesis (i). Consequently, there exists a positive n such that $h[\psi_n] - E_1(\beta)\|\psi_n\|^2 < 0$, so the desired result (5) follows by the variational characterisation of the lowest point in the spectrum of H.

Now assume (ii), in which case the shifted form $h_1[\psi_n] := h[\psi_n] - E_1(\beta)\|\psi_n\|^2$ converges to zero as $n\to\infty$. Then we modify the test function ψ_n by adding a small perturbation:

$$\psi_{n,\delta}(s,t) := \psi_n(s,t) + \delta\,\phi(s,t) \qquad \text{with} \qquad \phi(s,t) := \xi(s)\,t\,\chi_1(t)\,,$$

where δ is a real number and $\xi \in C_0^\infty(\mathbb{R})$ is a real-valued function to be determined later. Writing

$$\begin{aligned} \lim_{n\to\infty} h_1[\psi_{n,\delta}] &= \lim_{n\to\infty}\big(h_1[\psi_n] + 2\delta\,h_1(\psi_n,\phi) + \delta^2\,h_1[\phi]\big) \\ &= 2\delta\lim_{n\to\infty} h_1(\psi_n,\phi) + \delta^2\,h_1[\phi]\,, \end{aligned} \tag{31}$$

it is enough to show, in order to establish (5), that the limit of $h_1(\psi_n,\phi)$ as $n\to\infty$ is non-zero for a suitable choice of ξ. Indeed, it then suffices to choose δ sufficiently small and of suitable sign to make the second line of (31) negative, and subsequently choose n sufficiently large to make the whole expression $h_1[\psi_{n,\delta}]$ negative. Employing Lemma 1 and the fact that $\varphi_n = 1$ on $\operatorname{supp}\xi$ for all sufficiently large n, we have

$$\begin{aligned} \lim_{n\to\infty} h_1(\psi_n,\phi) = &\int_{\Omega_0}\Big\{\chi_1(t)\chi_1'(t)\left[\varepsilon(s)^2\xi(s) + \beta\varepsilon(s)\xi(s)\right] - \chi_1(t)\chi_1'(t)t\,\varepsilon(s)\xi'(s) \\ &+ \chi'(t)^2 t\left[\varepsilon(s)^2\xi(s) + \beta\varepsilon(s)\xi(s)\right] + \chi_1(t)^2 t\,E_1\beta\varepsilon(s)\xi(s)\Big\}\,\mathrm{d}s\,\mathrm{d}t \\ = &\frac{1}{2}\int_{\mathbb{R}}\left\{\varepsilon(s)\xi'(s) + E_1\,d\left[\varepsilon(s)^2 + 2\beta\varepsilon(s)\right]\xi(s)\right\}\mathrm{d}s \\ = &\frac{1}{2}\int_{\mathbb{R}}\left\{-\varepsilon'(s) + E_1\,d\left[\varepsilon(s)^2 + 2\beta\varepsilon(s)\right]\right\}\xi(s)\,\mathrm{d}s\,. \end{aligned}$$

Here the last equality follows by an integration by parts employing the extra hypothesis that the derivative ε' exists as a locally integrable function. Now let assume, by contradiction, that the last integral equals zero for all possible choices of ξ. Then ε must solve the differential equation $-\varepsilon'(s)+E_1\,d\,[\varepsilon(s)^2+2\beta\varepsilon(s)]=0$, which admits the explicit one-parametric class of solutions

$$\varepsilon_c(s)=\frac{2\beta}{c\,e^{-2E_1 d\beta s}-1}\,,\qquad c\in\mathbb{R}\,.$$

The solutions ε_c for non-zero c are not admissible because $f'^2-\beta$ is either a positive (if $c>0$) or negative (if $c<0$) function, so its integral cannot be equal to zero (for positive c the function ε_c additionally admits a singularity at $s=-(2E_1 d\beta)^{-1}\log c$), while ε_0 is excluded by the requirement that f' is not constant. Hence, there exists a compactly supported ξ such that the limit of $h_1(\psi_n,\phi)$ as $n\to\infty$ is non-zero and the proof is concluded by the argument described below (31). □

6 Strong shearing

In this last section we prove Theorem 3. Accordingly, let us assume that the shear admits the special form $f'(x)=\beta+\alpha\varepsilon(x)$, where $\alpha,\beta\in\mathbb{R}$ and $\varepsilon:\mathbb{R}\to\mathbb{R}$ is a function such that $\operatorname{supp}\varepsilon\subset[0,1]$ and $c_1\le\varepsilon(x)\le c_2$ for all $x\in[0,1]$ with some positive constants c_1,c_2. We divide the proof into several subsections.

6.1 Preliminaries

If β is non-negative (respectively, non-positive), then the desired stability result (6) obviously holds for all $\alpha\ge 0$ (respectively, $\alpha\le 0$) due to the repulsiveness of the shear (*cf.* Theorem 4). On the other hand, the shear becomes attractive if β is positive (respectively, negative) and α is small in absolute value and negative (respectively, positive), so (6) cannot hold in these cases (*cf.* Theorem 2). The non-trivial part of Theorem 3 therefore consists in the statement that (6) holds again in the previous regimes provided that α becomes large in absolute value.

Without loss of generality, it is enough to prove the remaining claim for β positive, α negative and the shear function

$$f(x):=\begin{cases}\beta x & \text{if}\quad x<0\,,\\ \beta x+\alpha\int_0^x\varepsilon(s)\,\mathrm{d}s & \text{if}\quad 0\le x\le 1\,,\\ \beta x+\alpha\int_0^1\varepsilon(s)\,\mathrm{d}s & \text{if}\quad 1<x\,.\end{cases}\tag{32}$$

As above, we define Ω by (1) and do not highlight the dependence on α. We use the notation $(x,y)\in\mathbb{R}^2$ for the Cartesian coordinates in which we describe Ω.

6.2 Subdomains decomposition

Our strategy is to decompose Ω into a union of three open subsets $\Omega_{\rm ext}$, $\Omega_{\rm ver}$, $\Omega_{\rm int}$ and of two connecting one-dimensional interfaces $\Sigma_{\rm ext}$, $\Sigma_{\rm int}$ on which we impose extra Neumann

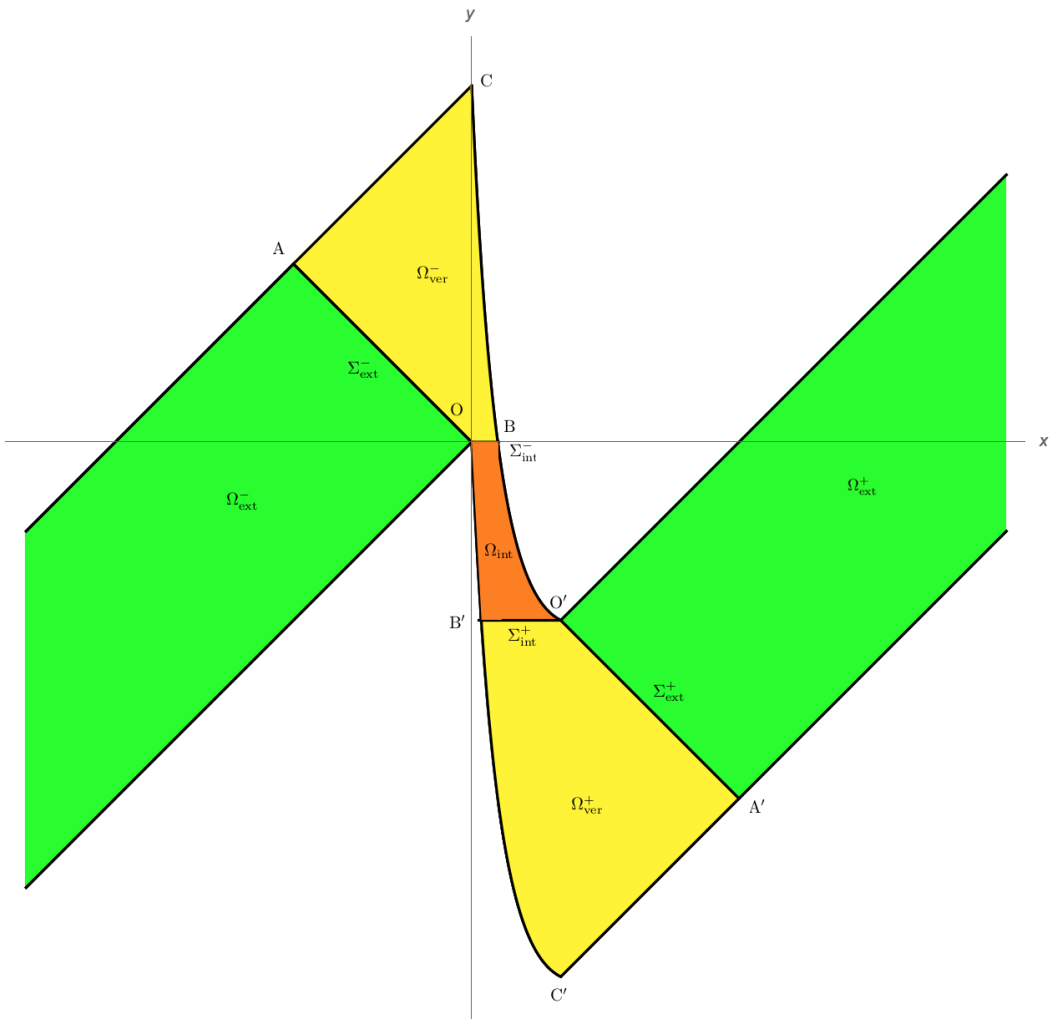

Figure 3: The subdomains decomposition for Ω corresponding to (32).

boundary conditions later on. The definition of the (α-dependent) sets given below is best followed by consulting Figure 3.

Let $A \in \mathbb{R}^2$ be the orthogonal projection of the origin $O := (0,0) \in \mathbb{R}^2$ on the half-line $\{(x, f(x) + d) : x \leq 0\}$ and let Ω^-_{ext} be the unbounded open subdomain of Ω contained in the left half-plane and delimited by the open segment Σ^-_{ext} connecting O with A. Similarly, let A' be the orthogonal projection of the point $O' := (1, f(1) + d)$ on the half-line $\{(x, f(x)) : x \geq 1\}$ and let Ω^+_{ext} be the unbounded open subdomain of Ω contained in the right half-plane and delimited by the open segment Σ^+_{ext} connecting O' with A'. Set $\Omega_{\text{ext}} := \Omega^-_{\text{ext}} \cup \Omega^+_{\text{ext}}$ and $\Sigma_{\text{ext}} := \Sigma^-_{\text{ext}} \cup \Sigma^+_{\text{ext}}$.

Let $x_0 \in (0,1)$ be the unique solution of the equation $f(x_0)+d = 0$ and set $B := (x_0, 0)$. Similarly, let $x_0' \in (0,1)$ be the unique solution of the the equation $f(x_0') = f(1) + d$ and set $B' := (x_0', f(1) + d)$. Let us also introduce $C := (0, f(0) + d)$ and $C' := (1, f(1))$. Let Ω^-_{ver} (respectively, Ω^+_{ver}) be the quadrilateral-like subdomain of Ω determined by the vertices O, A, C and B (respectively, O', A', C' and B'). Let Σ^-_{int} (respectively, Σ^+_{int}) be the open segment connecting O with B (respectively, O' with B'). Set $\Omega_{\text{ver}} := \Omega^-_{\text{ver}} \cup \Omega^+_{\text{ver}}$

and $\Sigma_{\rm int} := \Sigma_{\rm int}^- \cup \Sigma_{\rm int}^+$.

Finally, we define $\Omega_{\rm int} := \Omega \setminus (\Omega_{\rm ext} \cup \Omega_{\rm ver} \cup \Sigma_{\rm ext} \cup \Sigma_{\rm int})$, which is the parallelogram-like domain determined by the vertices $OBO'B'$.

6.3 Neumann bracketing

Notice that the majority of the sets introduced above depend on α, although it is not explicitly highlighted by the notation. In particular, the parallelogram-like domain determined by the vertices $OCO'C'$ (subset of $\Omega_{\rm ver} \cup \Omega_{\rm int}$) converges in a sense to the half-line $\{0\} \times (-\infty, 0)$ as $\alpha \to -\infty$. Hence it is expected that this set is spectrally negligible in the limit and the spectrum of $-\Delta_D^\Omega$ converges as $\alpha \to -\infty$ to the spectrum of the Dirichlet Laplacian in $\{(x,y) \in \Omega : x < 0\}$.

Since we only need to show that the spectral threshold of $-\Delta_D^\Omega$ is bounded from below by $E_1(\beta)$ for all sufficiently large α, in the following we prove less. Instead, we impose extra Neumann boundary conditions on $\Sigma_{\rm ext} \cup \Sigma_{\rm int}$ (*i.e.* no boundary condition in the form setting) and show that the spectral threshold of the Laplacian with combined Dirichlet and Neumann boundary conditions in the decoupled subsets $\Omega_{\rm ext}$, $\Omega_{\rm ver}$, $\Omega_{\rm int}$ is bounded from below by $E_1(\beta)$. More specifically, by a standard Neumann bracketing argument (*cf.* [18, Sec. XIII.15]), we have

$$\inf \sigma(-\Delta_D^\Omega) \geq \min \left\{ \inf \sigma(-\Delta_{DN}^{\Omega_{\rm ext}}), \inf \sigma(-\Delta_{DN}^{\Omega_{\rm ver}}), \inf \sigma(-\Delta_{DN}^{\Omega_{\rm int}}) \right\},$$

where $-\Delta_{DN}^{\Omega_{\rm ext}}$ (respectively, $-\Delta_{DN}^{\Omega_{\rm ver}}$; respectively, $-\Delta_{DN}^{\Omega_{\rm int}}$) is the operator in $L^2(\Omega_{\rm ext})$ (respectively, $L^2(\Omega_{\rm ver})$; respectively, $L^2(\Omega_{\rm int})$) that acts as the Laplacian and satisfies Neumann boundary conditions on $\Sigma_{\rm ext}$ (respectively, on $\Sigma_{\rm ext} \cup \Sigma_{\rm int}$; respectively, on $\Sigma_{\rm int}$) and Dirichlet boundary conditions on the remaining parts of the boundary. It remains to study the spectral thresholds of the individual operators.

6.4 Spectral threshold of the decomposed subsets

6.4.1 Exterior sets

Notice that $\Omega_{\rm ext}$ consists of two connected components, each of them being congruent to the half-strip $(0, \infty) \times (0, d/\sqrt{1+\beta^2})$. Consequently, $-\Delta_{DN}^{\Omega_{\rm ext}}$ is isospectral to the Laplacian in the half-strip, subject to Neumann boundary conditions on $\{0\} \times (0, d/\sqrt{1+\beta^2})$ and Dirichlet boundary conditions on the remaining parts of the boundary. By a separation of variables, it is straightforward to see that $\sigma(-\Delta_{DN}^{\Omega_{\rm ext}}) = [E_1(\beta), \infty)$. In particular, the spectral threshold of $-\Delta_{DN}^{\Omega_{\rm ext}}$ equals $E_1(\beta)$ for all α.

6.4.2 Interior set

Given any $y \in [f(1)+d, 0]$, let $x_1(y)$ (respectively, $x_2(y)$) be the unique solution of the equation $f(x_1(y)) = y$ (respectively, $f(x_2(y)) + d = y$). Notice that $x_1(y) < x_2(y)$. In particular, $x_1(0) = 0$ and $x_2(0) = x_0$, while $x_1(f(1)+d) = x_0'$ and $x_2(f(1)+d) = f(1)+d$.

To estimate the spectral threshold of $-\Delta_{DN}^{\Omega_{\rm int}}$, we write

$$\int_{\Omega_{\rm int}} |\nabla\psi(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \geq \int_{\Omega_{\rm int}} |\partial_x\psi(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y$$
$$\geq \int_{\Omega_{\rm int}} \left(\frac{\pi}{x_2(y)-x_1(y)}\right)^2 |\psi(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y\,,$$

where the second inequality follows by a Poincaré inequality of the type (12) and Fubini's theorem. Subtracting the equations that $x_1(y)$ and $x_2(y)$ satisfy, the mean value theorem yields $d = f(x_1(y)) - f(x_2(y)) = -f'(\xi)[x_2(y)-x_1(y)]$ with $0<\xi<x_0<1$. Since $f'(x) = \beta + \alpha\varepsilon(x) \geq \beta - c_2|\alpha|$ for $x\in[0,1]$, it follows that, for all $\alpha<\beta/c_2$, we have the uniform bound $x_2(y)-x_1(y) \geq d/(c_2|\alpha|-\beta)$. Consequently, $\inf\sigma(-\Delta_{DN}^{\Omega_{\rm int}}) \to \infty$ as $\alpha\to-\infty$. In particular, the spectral threshold of $-\Delta_{DN}^{\Omega_{\rm int}}$ is greater than $E_1(\beta)$ for all negative α with sufficiently large $|\alpha|$.

6.4.3 Verge sets

The decisive set $\Omega_{\rm ver}$ consists of two connected components $\Omega_{\rm ver}^{\pm}$. We consider $\Omega_{\rm ver}^{-}$, the argument for $\Omega_{\rm ver}^{+}$ being analogous. Let us thus study the spectral threshold of the operator $-\Delta_{DN}^{\Omega_{\rm ver}^-}$ in $L^2(\Omega_{\rm ver}^-)$ that acts as the Laplacian and satisfies Neumann boundary conditions on $\Sigma_{\rm ext}^-\cup\Sigma_{\rm int}^-$ and Dirichlet boundary conditions on the remaining parts of the boundary $\partial\Omega_{\rm ver}^-$. Since the spectrum of this operator is purely discrete, we are interested in the behaviour of its lowest eigenvalue.

As $\alpha\to-\infty$, the set $\Omega_{\rm ver}^-$ converges in a sense to the open right triangle T determined by the vertices OAC. More specifically, $|\Omega_{\rm ver}^-\setminus T| = O(|\alpha|^{-1})$ as $\alpha\to-\infty$. Using the convergence result [1, Thm. 29], it particularly follows that the lowest eigenvalue of $-\Delta_{DN}^{\Omega_{\rm ver}^-}$ converges to the lowest eigenvalue $\lambda_1(T)$ of the Laplacian in T, subject to Neumann boundary conditions on the segment OA and Dirichlet boundary conditions on the other parts of the boundary. It remains to study the latter.

By a trivial-extension argument, $\lambda_1(T)$ is bounded from below by the lowest eigenvalue of the Laplacian in the rectangle $(0,|OA|)\times(0,|AC|)$, subject to Neumann boundary conditions on $(0,|OA|)\times\{0\}$ and Dirichlet boundary conditions on the other parts of the boundary. That is, $\lambda_1(T)\geq(1+\beta^{-2})\,E_1(\beta)$.

Summing up, we have established the result

$$\inf\sigma(-\Delta_{DN}^{\Omega_{\rm ver}}) \xrightarrow[\alpha\to-\infty]{} \lambda_1(T) \geq (1+\beta^{-2})\,E_1(\beta)\,.$$

Consequently, there exists a negative α_0 such that, for every $\alpha\leq\alpha_0$, the spectral threshold of $-\Delta_{DN}^{\Omega_{\rm ver}}$ is also bounded from below by $E_1(\beta)$. This concludes the proof of Theorem 3.

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