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# On soft capacities, quasi-stationary distributions and the pathwise approach to metastability

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## Abstract

Motivated by the study of the metastable stochastic Ising model at subcritical temperature and in the limit of a vanishing magnetic field, we extend the notion of  $(\kappa, \lambda)$ -capacities between sets, as well as the associated notion of soft-measures, to the case of *overlapping* sets. We recover their essential properties, sometimes in a stronger form or in a simpler way, relying on weaker hypotheses. These properties allow to write the main quantities associated with reversible metastable dynamics, e.g. asymptotic transition and relaxation time, in terms of objects satisfying two-sided variational principles. We also make the connection with the classical “pathwise approach”.

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*Keywords:* Soft capacity, soft measure, quasi-stationary measure, restricted ensemble, metastability, potential theory, relaxation time.

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## 1 Model and main results

We consider a Markovian model for metastable dynamics with, for the sake of simplicity, only one metastable state or two local equilibria. We also assume to be in some asymptotic regime, meaning that the dynamics depends on some parameters, like temperature, volume or magnetic field going to some finite or infinite limit, and we will write  $f \ll g$  whenever  $0 \leq f \leq g$  and the ratio  $f/g$  goes to zero in the considered asymptotic regime. Let then  $X$  be an irreducible and reversible continuous time Markov process on a finite configuration space  $\mathcal{X}$  with generator  $\mathcal{L}$  defined by

$$(\mathcal{L}f)(x) = \sum_{y \neq x} w(x, y)[f(y) - f(x)], \quad x \in \mathcal{X}, \quad f : \mathcal{X} \rightarrow \mathbb{R}. \quad (1)$$

Our assumption about the unicity of the metastable state amounts to say that on the long time scale the dynamics can be described in some sense by a Markov process between two states, i.e., two probability measures on  $\mathcal{X}$ ,  $\tilde{\mu}$  and  $\mu$ , the latter being the equilibrium measure of the system, and such that  $\tilde{\mu}$  and  $\mu$  are concentrated on very different parts of the configuration space  $\mathcal{X}$ . For example we can think of  $\tilde{\mu}$  as the *restricted ensemble*

$$\mu_{\mathcal{R}} = \mu(\cdot | \mathcal{R})$$

for some subset  $\mathcal{R}$  of  $\mathcal{X}$  such that  $\mu(\mathcal{R}) \ll 1$ . We will also consider situations where  $\mu(\mathcal{R})$  is of order 1, in which case we will deal with “two local equilibria”.

Such a phenomenology with only one metastable state is expected in the case of the kinetic Ising model at subcritical temperature  $T$  and under a small magnetic field  $0 < h \ll 1$  in a box  $\Lambda \subset \mathbb{Z}^2$  with volume  $(C/h)^2$  for a large enough constant  $C > 0$ . This is illustrated by Figure 1, in the case of a square box with periodic boundary conditions, where we show a dynamic started from the “all minus configuration”. In

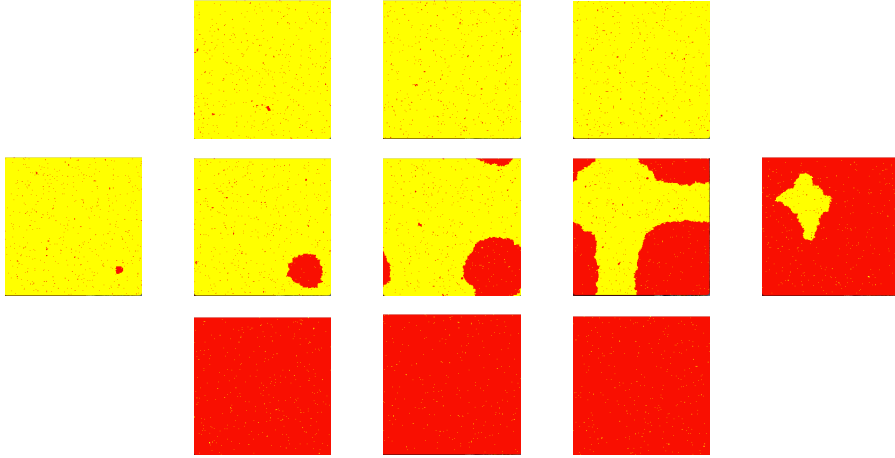
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Figure 1: Snapshots of a kinetic Ising model at temperature  $T = 1.5$  and under magnetic field  $h = 0.14$  in a  $256 \times 256$  box with periodic boundary conditions. Minus spins are yellow and plus spins are red. We took pictures of the first line at times 471, 7482 and 13403, at times 14674, 15194, 15432, 15892 and 16558 for the second line, and times 17328, 23645 and 40048 for the last line.



this figure the first three pictures can be seen as samples from a restricted ensemble  $\mu_{\mathcal{R}}$  concentrated on configurations where minus spins dominate, the last three ones as samples from the equilibrium measure  $\mu$  where plus spins prevail. The time needed to nucleate the “supercritical droplet”, which triggers the transition from the metastable state  $\mu_{\mathcal{R}}$  to the stable one  $\mu$ , is expected to asymptotically follow an exponential law in the vanishing magnetic field regime  $h \ll 1$ . The beautiful paper of Schonmann and Shlosman [SS98] makes rigorous a very large part of this picture. By introducing a suitable subset  $\mathcal{R}$  of the configuration space they compute the critical rate  $\lambda_c > 0$  such that, for a large class of starting distributions and in the regime  $h \ll 1$ , the distribution of the system at time  $t = e^{\lambda/h}$  is close to the restricted ensemble  $\mu_{\mathcal{R}}$  for  $\lambda < \lambda_c$  and close to the equilibrium measure  $\mu$  for  $\lambda > \lambda_c$ . The only missing part is the asymptotic exponentiality of the transition time. (They actually study the infinite volume dynamic for which, for reasons outside the scope of this paper, the asymptotic exponentiality is *not* expected. But it *is* expected in our finite and diverging volume  $(C/h)^2$ , and the first step to prove the absence of exponentiality in infinite volume would be to prove asymptotic exponentiality in this finite volume case.)

## 1.1 Soft measures and capacities

One way to establish this asymptotic exponentiality would be to follow the approach presented in [BG16]. In this paper we introduced, for a given  $\mathcal{R}$ , soft measures  $\mu_{\mathcal{R},\lambda}^*$  as a continuous interpolation, for  $\lambda$  in  $[0, +\infty]$ , between the restricted ensemble  $\mu_{\mathcal{R}}$  and the quasi-stationary measure  $\mu_{\mathcal{R}}^*$  defined by the “Yaglom limit”

$$\mu_{\mathcal{R}}^* = \lim_{t \rightarrow +\infty} \mathbb{P}_{\mu_{\mathcal{R}}} (X(t) = \cdot \mid T_{\mathcal{X} \setminus \mathcal{R}} > t),$$

with the convention that  $T_{\mathcal{A}}$  stands for the hitting time of any  $\mathcal{A} \subset \mathcal{X}$ . Under a suitable set of hypotheses ( $H_0$ ) which we will soon make precise, all these soft measures  $\mu_{\mathcal{R},\lambda}^*$  are close in total variation distance and they all represent a metastable state associated with  $\mathcal{R}$ . To define them one has to consider the process  $X$  killed at rate  $\lambda$  in  $\mathcal{X} \setminus \mathcal{R}$  and its trace on  $\mathcal{R}$ :  $\mu_{\mathcal{R},\lambda}^*$  is then the associated quasi-stationary measure (we will be more precise later). The killing time  $T_{\lambda}$  is in this case the sum of the local time in  $\mathcal{R}$

$$T_{\lambda}^{\mathcal{R}} = \ell_{\mathcal{R}}(T_{\lambda})$$

(which is written  $\tau_{\mathcal{X} \setminus \mathcal{R},\lambda}$  in [BG16]) and an independent exponential variable

$$\sigma_{\lambda} = \ell_{\mathcal{X} \setminus \mathcal{R}}(T_{\lambda})$$

of rate  $\lambda$ . We proved in [BG16] that, under suitable hypotheses the distribution of  $X(T_{\lambda})$  is close to equilibrium and we also established for  $T_{\lambda}^{\mathcal{R}}$ , when  $X$  starts in  $\mathcal{R}$ , an asymptotic exponential law with

rate  $\phi_{\mathcal{R},\lambda}^*$  defined by

$$\frac{1}{\phi_{\mathcal{R},\lambda}^*} = \mathbb{E}_{\mu_{\mathcal{R},\lambda}^*} [T_{\lambda}^{\mathcal{R}}].$$

(Starting from  $\mu_{\mathcal{R},\lambda}^*$  the local time  $T_{\lambda}^{\mathcal{R}}$  has actually an exact exponential distribution with rate  $\phi_{\mathcal{R},\lambda}^*$ , and we use the upper index  $*$  to denote objects like exit rates and quasi-stationary measures associated with killed processes.) Under the same hypotheses one can choose the killing rate  $\lambda$  in such a way that  $1/\lambda$  is small with respect to the mean exit time  $1/\phi_{\mathcal{R},\lambda}^*$ , so that the same asymptotic exponential law also holds for  $T_{\lambda}$  itself.

We also introduced in [BG16]  $(\kappa, \lambda)$ -capacities  $C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})$ , which we will call *soft capacities* in this paper, to describe the asymptotic behaviour of  $\phi_{\mathcal{R},\lambda}^*$ . Under hypotheses  $(H_0)$  it holds

$$\phi_{\mathcal{R},\lambda}^* \sim \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})}$$

for well chosen  $\kappa$  and  $\lambda$  (the symbol  $\sim$  means that the ratio of the two quantities goes to one in the considered asymptotic regime). These soft capacities are naturally associated to the process killed at rate  $\lambda$  in  $\mathcal{X} \setminus \mathcal{R}$  and at rate  $\kappa$  in  $\mathcal{R}$ . They make the mean transition time  $1/\phi_{\mathcal{R},\lambda}^*$  “easy to estimate” since they satisfy a two-sided variational principle: as we will make it precise, any test function and test flow will give upper and lower bound respectively. We finally stress the fact that, under hypotheses  $(H_0)$  and despite the notation, provided that  $\kappa$  and  $\lambda$  are chosen in suitable large windows there is essentially no asymptotic dependence of the associated objects in  $\kappa$  and  $\lambda$ : soft-measures are all close together and soft capacities all have the same asymptotic behaviour. In particular we proved in [BG16] that, provided  $\kappa$  and  $\lambda$  are chosen in the right windows, the relaxation time  $1/\gamma$  of the Markov process  $X$  can also be asymptotically expressed in terms of the  $(\kappa, \lambda)$ -capacities:

$$\frac{1}{\gamma} \sim \frac{\mu(\mathcal{R})\mu(\mathcal{X} \setminus \mathcal{R})}{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}.$$

To make precise hypotheses  $(H_0)$  we have to introduce the restricted process  $X_{\mathcal{R}}$  on  $\mathcal{R}$ , which is obtained from  $X$  by suppressing all jumps from  $\mathcal{R}$  to  $\mathcal{X} \setminus \mathcal{R}$ . This is the Markov process with generator  $\mathcal{L}_{\mathcal{R}}$  defined by

$$(\mathcal{L}_{\mathcal{R}}f)(x) = \sum_{y \in \mathcal{R} \setminus \{x\}} w(x, y)[f(y) - f(x)], \quad x \in \mathcal{R}, \quad f : \mathcal{R} \rightarrow \mathbb{R}. \quad (2)$$

We assume it to be irreducible and we call  $1/\gamma_{\mathcal{R}}$  its relaxation time: the spectral gap  $\gamma_{\mathcal{R}}$  is the convergence rate of the law of  $X_{\mathcal{R}}(t)$  towards the equilibrium measure in  $\mathcal{R}$ , for  $X_{\mathcal{R}}$  started in any generic distribution. As a consequence of our reversibility hypothesis this equilibrium measure is the restricted ensemble  $\mu_{\mathcal{R}}$ . Next we call  $\phi_{\mathcal{R}}^*$  the exit rate from  $\mathcal{R}$  starting from the quasi-stationary measure  $\mu_{\mathcal{R}}^*$ . In other words we set  $\phi_{\mathcal{R}}^* = \phi_{\mathcal{R},+\infty}^*$ . We will assume in  $(H_0)$  that the relaxation time  $1/\gamma_{\mathcal{R}}$  of the restricted process is small with respect to the mean exit time  $1/\phi_{\mathcal{R}}^*$ . In other words, calling  $\epsilon_{\mathcal{R}}^*$  the ratio between these two quantities, we will assume

$$\epsilon_{\mathcal{R}}^* = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} \ll 1.$$

We define in the same way  $\gamma_{\mathcal{X} \setminus \mathcal{R}}$ ,  $\phi_{\mathcal{X} \setminus \mathcal{R}}^*$  and  $\epsilon_{\mathcal{X} \setminus \mathcal{R}}^*$ , assuming that  $\mathcal{X} \setminus \mathcal{R}$  is irreducible too. Let us summarize and add one more assumption to define the hypothesis set  $(H_0)$ :

**Hypotheses  $(H_0)$ :** We assume both  $\mathcal{R}$  and  $\mathcal{X} \setminus \mathcal{R}$  to be such that the associated restricted processes are both irreducible. We assume  $\epsilon_{\mathcal{R}}^* \ll 1$  and  $\epsilon_{\mathcal{X} \setminus \mathcal{R}}^* \ll 1$ . We assume in addition  $\mu(\mathcal{X} \setminus \mathcal{R}) \geq \mu(\mathcal{R})$  and  $1/\gamma_{\mathcal{X} \setminus \mathcal{R}} \ll 1/\phi_{\mathcal{R}}^*$ .

Hypotheses  $(H_0)$  amount to assume that local relaxation times in  $\mathcal{R}$  and  $\mathcal{X} \setminus \mathcal{R}$  are small with respect to the corresponding mean exit times and that the local relaxation time in the larger  $\mathcal{X} \setminus \mathcal{R}$  is small with respect to the mean exit time from  $\mathcal{R}$ . Under hypothesis  $(H_0)$  it is possible to choose  $\kappa$  and  $\lambda$  in such a way that  $\phi_{\mathcal{R}}^* \ll \kappa \ll \gamma_{\mathcal{R}}$  and  $\phi_{\mathcal{R}}^*, \phi_{\mathcal{X} \setminus \mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{X} \setminus \mathcal{R}}$ . These are the large windows we were referring to.

It is important at this point to stress that to *check* hypotheses  $(H_0)$  one need *lower bounds* on mean exit times as well as *upper bounds* on local relaxation times, and that the former are generally much easier to get than the latter. As we will see, the escape rate  $\phi_{\mathcal{R}}^*$  satisfies a variational principle

$$\phi_{\mathcal{R}}^* = \min_{\substack{f \neq 0, \\ f|_{\mathcal{X} \setminus \mathcal{R}} = 0}} \frac{\mathcal{D}(f)}{\|f\|^2}$$

with  $\mathcal{D}$  the Dirichlet form associated with  $\mathcal{L}$  —which is defined by

$$\mathcal{D}(f) = \frac{1}{2} \sum_{x,y \in \mathcal{X}} c(x,y) [f(x) - f(y)]^2, \quad f : \mathcal{X} \rightarrow \mathbb{R},$$

where the conductances  $c(x,y)$  are given by

$$c(x,y) = \mu(x)w(x,y), \quad x,y \in \mathcal{X}$$

— and with  $\|\cdot\|$  the norm in  $\ell^2(\mu)$ :

$$\|f\|^2 = \sum_{x \in \mathcal{X}} \mu(x) f(x)^2, \quad f : \mathcal{X} \rightarrow \mathbb{R}.$$

As a consequence any test function will give an upper bound on  $\phi_{\mathcal{R}}^*$ , i.e., a lower bound on the mean exit time  $1/\phi_{\mathcal{R}}^*$ . The spectral gap  $\gamma_{\mathcal{R}}$  can also be defined by a variational principle,

$$\gamma_{\mathcal{R}} = \min_{\substack{f \in \ell^2(\mu_{\mathcal{R}}), \\ \text{Var}_{\mu_{\mathcal{R}}}(f) \neq 0}} \frac{\mathcal{D}_{\mathcal{R}}(f)}{\text{Var}_{\mu_{\mathcal{R}}}(f)}, \quad (3)$$

with  $\mathcal{D}_{\mathcal{R}}$  the Dirichlet form associated with  $\mathcal{L}_{\mathcal{R}}$ . But this variational principle does not provide any simple upper bound on the local relaxation time  $1/\gamma_{\mathcal{R}}$ , bounding it from above can be a hard task.

This difficulty can depend in a critical way of the choice of  $\mathcal{R}$ , which is part of the problem in metastability studies. To achieve such a goal in the case of our kinetic Ising model at subcritical temperature in the vanishing magnetic field limit, we use flow, or random path, techniques in [GMV18]. These random paths have to describe the natural evolution of our system relaxing in  $\mathcal{R}$ , which implies that  $\mathcal{R}$  should be in some way closed under a natural evolution inherited from  $\mathcal{L}$ . This means that, for this kind of techniques to succeed,  $\mathcal{R}$  has to be chosen “large enough”. But hypotheses  $(H_0)$  do not regard  $\mathcal{R}$  only, they also involve its complementary. Since a “large”  $\mathcal{R}$  would go with a “small”  $\mathcal{X} \setminus \mathcal{R}$ , the same kind of techniques are likely to fail when dealing with  $\mathcal{X} \setminus \mathcal{R}$ . But we can use them in [GMV18] when replacing  $\mathcal{X} \setminus \mathcal{R}$  by a larger set  $\mathcal{S}$ . Extending the results of [BG16] when dealing with a pair of subsets  $\mathcal{R}, \mathcal{S} \subset \mathcal{X}$  such that  $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$  but  $\mathcal{R} \cap \mathcal{S} \neq \emptyset$ , instead of dealing with a partition  $\mathcal{X} = \mathcal{R} \cup \mathcal{X} \setminus \mathcal{R}$ , is the first goal of this paper.

To state the first main results of this paper let us then write  $\mathcal{X}$  as the union of two sets  $\mathcal{R}$  and  $\mathcal{S}$  and introduce the analogous

**Hypotheses (H):** We assume  $\mathcal{R}$  and  $\mathcal{S}$  to be such that the associated restricted processes  $X_{\mathcal{R}}, X_{\mathcal{R} \setminus \mathcal{S}}, X_{\mathcal{S}}$  and  $X_{\mathcal{S} \setminus \mathcal{R}}$  are all irreducible. We assume  $\phi_{\mathcal{R} \setminus \mathcal{S}}^* \ll \gamma_{\mathcal{R}}$  and  $\phi_{\mathcal{S} \setminus \mathcal{R}}^* \ll \gamma_{\mathcal{S}}$ . We assume in addition  $\mu(\mathcal{S}) \geq \mu(\mathcal{R})$  and  $\phi_{\mathcal{R} \setminus \mathcal{S}}^* \ll \gamma_{\mathcal{S}}$ .

(It will be clear later that the irreducibility hypotheses on  $X_{\mathcal{R} \setminus \mathcal{S}}$  and  $X_{\mathcal{S} \setminus \mathcal{R}}$  are in no way crucial: they will simply make easier some later definitions and proofs.) For  $\lambda \in [0, +\infty]$ , we set

$$\lambda_{\mathcal{S}}(x) = \lambda \mathbb{1}_{\{x \in \mathcal{S}\}}, \quad x \in \mathcal{X}, \quad (4)$$

we consider the process  $X$  killed at rate  $\lambda_{\mathcal{S}}(x)$  in each  $x$  in  $\mathcal{X}$  —i.e., killed at rate  $\lambda$  in  $\mathcal{S}$ — and we call  $T_{\lambda_{\mathcal{S}}}$  the associated killing time. Let  $X_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  and  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  be the trace on  $\mathcal{R}$  of this killed process and the associated quasi-stationary measure. We call  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  the killing rate of this trace process in the quasi-stationary state  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$ :

$$\frac{1}{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*} = \mathbb{E}_{\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*} [T_{\lambda_{\mathcal{S}}}^{\mathcal{R}}]$$

with  $T_{\lambda_{\mathcal{S}}}^{\mathcal{R}} = \ell_{\mathcal{R}}(T_{\lambda_{\mathcal{S}}})$  the local time in  $\mathcal{R}$  up to  $T_{\lambda_{\mathcal{S}}}$ . For  $\kappa \in [0, +\infty]$  we also set

$$\kappa_{\mathcal{R}}(x) = \kappa \mathbb{1}_{\{x \in \mathcal{R}\}}, \quad x \in \mathcal{X},$$

and we call  $T_{\kappa_{\mathcal{R}}}$  the killing time of the process  $X$  killed at rate  $\kappa$  in  $\mathcal{R}$ . With

$$V_{\kappa}^{\lambda}(x) = \mathbb{P}_x(T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}), \quad x \in \mathcal{X},$$

we define, following [BG16],

$$C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S}) = \mathcal{D}(V_{\kappa}^{\lambda}) + \kappa \mu(\mathcal{R}) E_{\mu_{\mathcal{R}}} \left[ (V_{\kappa}^{\lambda} - 1)^2 \right] + \lambda \mu(\mathcal{S}) E_{\mu_{\mathcal{S}}} \left[ (V_{\kappa}^{\lambda} - 0)^2 \right].$$

**Theorem 1.** *Assuming hypotheses (H) and choosing  $\kappa$  and  $\lambda$  such that*

$$\phi_{\mathcal{R}\setminus\mathcal{S}}^* \ll \kappa \ll \gamma_{\mathcal{R}} \quad \text{and} \quad \phi_{\mathcal{R}\setminus\mathcal{S}}^* \vee \phi_{\mathcal{S}\setminus\mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{S}},$$

*it holds*

$$\gamma \sim \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})\mu(\mathcal{S})} \quad \text{and} \quad \phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^* \sim \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})}.$$

**Comments:**

- i. This gives a way to compute the precise asymptotic value of the relaxation time  $1/\gamma$ . As soon as  $\kappa$  and  $\lambda$  are chosen in the right window, the spectral gap  $\gamma$  scales, up to the multiplicative term  $\mu(\mathcal{R})\mu(\mathcal{S})$ , like the soft capacity  $C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})$  that satisfies a two-sided variational principle. As we will see in Section 2, any test function and flow will give upper and lower bound for  $C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})$ . These are then translated in upper and lower bounds for  $\gamma$ .
- ii. We already mentioned that the main difficulty to establish the validity of the hypothesis set (H) usually consists in bounding  $\gamma_{\mathcal{R}}$  and  $\gamma_{\mathcal{S}}$  from below. It is often the case that, once such lower bounds are established, there is a natural way to build a suitable flow to get useful lower bounds of the soft capacity: see [BG16] and [GMV18].
- iii. Similarly to what happens in the case  $\mathcal{R} \cap \mathcal{S} = \emptyset$ , the family of measures  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  provides a continuous interpolation between the restricted ensemble  $\mu_{\mathcal{R}}$ , which corresponds to  $\lambda = 0$ , and the quasi-stationary distribution  $\mu_{\mathcal{R}\setminus\mathcal{S}}^*$ , which corresponds to  $\lambda = +\infty$ . We will see in Section 2 that, under hypotheses (H), they are all close together. For  $\lambda$  in the right window each of them can serve as a model for our metastable state, the escape rate of which scales like  $\mu(\mathcal{S})\gamma$ . (We can consider situations where  $\mu(\mathcal{S})$  does not converge to 1: we only assume  $\mu(\mathcal{S}) \geq \mu(\mathcal{R})$ , not  $\mu(\mathcal{S}) \gg \mu(\mathcal{R})$ .)
- iv. The proof of Theorem 1 is deferred to Section 3. In this section we will give explicit upper and lower bounds for  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  and  $\gamma$  that will bound explicitly the multiplicative corrections with respect to the asymptotic values.
- v. About these upper and lower bounds, the latter are the most relevant. Both the exit rate  $\phi_{\mathcal{R}, \lambda}^*$  and the spectral gap  $\gamma$  are indeed *infimums* given by some variational principles, so that upper bound are simpler to get. In particular the lower bound given in Section 3 on  $\gamma$  is a Poincaré inequality that can be generalized to the case of more than one metastable state (cf. Proposition 3.1). The upper bounds indicate however that our lower bounds catch the right orders.
- vi. The windows given in Theorem 1 for  $\kappa$  and  $\lambda$  can in principle be enlarged for the conclusion to hold. For example, as far as the lower bound on  $\kappa$  is concerned, the explicit upper and lower bound of Section 3 will scale like the given asymptotic values for  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*/\kappa \ll 1$ . We will see in Section 2 that  $\kappa \gg \phi_{\mathcal{R}\setminus\mathcal{S}}^*$  is only a sufficient condition for having  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*/\kappa \ll 1$ . Our hypotheses (H) can be relaxed, in the same way, for the conclusion to hold.
- vii. When going from the case  $\mathcal{R} \cap \mathcal{S} = \emptyset$  of [BG16] to the case  $\mathcal{R} \cap \mathcal{S} \neq \emptyset$  of the present paper, some of the arguments of [BG16] are immediately generalized, while others require more work. What is straightforward to extend will be contained in Section 2. In Section 3 we will have to develop different arguments. These new arguments will also improve some bounds from [BG16] in the case  $\mathcal{R} \cap \mathcal{S} = \emptyset$ : we will get a better upper bound for  $\phi_{\mathcal{R}, \lambda}^*$ .

The exit rate  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  is not only associated with the exponential random variable  $T_{\lambda_{\mathcal{S}}}^{\mathcal{R}}$  and the asymptotic exponential time  $T_{\lambda_{\mathcal{S}}}$ , for suitable  $\lambda$ , when  $X$  starts from the metastable equilibrium  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$ . Indeed, we will have, under slightly stronger hypotheses, a “local thermalization property” on a shorter time scale: we will see that in a time of order  $1/(\kappa \wedge \lambda) \ll 1/\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  our system relaxes either to  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  or  $\mu_{\mathcal{S}, \kappa_{\mathcal{R}}}^*$ , the latter being defined as the former with  $\mathcal{S}$  in place of  $\mathcal{R}$  and  $\kappa_{\mathcal{R}}$  in place of  $\lambda_{\mathcal{S}}$ . Hence we will have an asymptotic exponential transition law of mean  $1/\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  for a large class of starting measures or configurations. Also, we can rebuild the full picture, under the same stronger hypotheses, of the classical pathwise approach in which the asymptotic exponential law plays a central role.

## 1.2 The pathwise approach

The pathwise approach to metastability was introduced in [CGOV84]. It consists, on the one hand, in identifying, on the one hand, a stopping time  $T_1$  —the asymptotic exponential law of which will be associated with its expected unpredictability— and, on the other hand, a shorter deterministic time scale  $\theta$  together with a local equilibrium  $\tilde{\mu}$  such that, starting from  $\tilde{\mu}$ , the time averages on time scale  $\theta$  of any observable  $f$ , which are defined at each time  $t \geq 0$  by

$$A_\theta(t, f) = \frac{1}{\theta} \int_t^{t+\theta} f(X(s)) ds,$$

are typically close, in the spirit of the ergodic theorem, to the expected value of  $f$  according to the local equilibrium  $\tilde{\mu}$  before time  $T_1$  —more precisely, up to time  $T_1 - \theta \gg \theta$ — while, at time  $T_1$ , the system is close to the global equilibrium  $\mu$  or to a different local equilibrium. This can be turned in our context into a rigorous mathematical statement by setting  $\tilde{\mu} = \mu_{\mathcal{R}}$  and  $T_1 = T_{\lambda_S}$ , for a suitable  $\lambda$  and under hypotheses that are slightly stronger than (H).

Let us first explain why we need stronger hypotheses. On the one hand, we will have to choose  $\theta$  such that not only  $\theta \ll 1/\phi_{\mathcal{R}, \lambda_S}^*$ , which is of the same order as  $T_1 = T_{\lambda_S}$  when starting from  $\mu_{\mathcal{R}}$ , but also  $\theta \gg 1/\lambda$ , in order to be able to neglect in the time averages on time scale  $\theta$  the excursions in  $\mathcal{S} \setminus \mathcal{R}$  before  $T_{\lambda_S}$ : by construction, their time length is dominated by an exponential variable or rate  $\lambda$ . On the other hand the time spent in  $\mathcal{S}$  up to  $T_{\lambda_S}$  will have to be long enough for  $X(T_{\lambda_S})$  to be close, in total variation at least, to  $\mu_{\mathcal{S}}$ : taking  $1/\lambda$  large with respect to  $1/\gamma_{\mathcal{S}}$ , the relaxation time of  $X_{\mathcal{S}}$ , could be not enough to this purpose. A sufficient condition will be  $1/\lambda \gg (\ln \chi_{\mathcal{S}})/\gamma_{\mathcal{S}}$  with

$$\chi_{\mathcal{S}} = \max_{x \in \mathcal{S}} \frac{1}{\mu_{\mathcal{S}}(x)}$$

(this quantity was denoted by  $1/\zeta_{\mathcal{S}}$  in [BG16]). We will be able to choose a  $\theta$  satisfying the previous conditions if, not only  $1/\gamma_{\mathcal{S}} \ll 1/\phi_{\mathcal{R} \setminus \mathcal{S}}^*$  as ensured by hypotheses (H), but instead  $(\ln \chi_{\mathcal{S}})/\gamma_{\mathcal{S}} \ll 1/\phi_{\mathcal{R} \setminus \mathcal{S}}^*$ .

With such a  $\lambda$  the distribution of  $X(T_{\lambda_S})$  will be close to  $\mu_{\mathcal{S}}$ , which, in turn, is close to the global equilibrium  $\mu$  if  $\mu(\mathcal{R}) \ll 1$ . But if we only have  $\mu(\mathcal{R}) \leq \mu(\mathcal{S})$  and not  $\mu(\mathcal{R}) \ll \mu(\mathcal{S})$ , then  $\mu_{\mathcal{S}}$  can only be considered as a different local equilibrium. In this case we can consider  $T_2$  defined as the killing time at rate  $\kappa_{\mathcal{R}}$  after  $T_1 = T_{\lambda_S}$ :

$$T_2 = T_1 + T_{\kappa_{\mathcal{R}}} \circ \Theta_{T_1},$$

$\Theta_t$  being the usual shift operator, such that, for all  $s \geq 0$ ,  $(\Theta_t(X))(s) = X(t+s)$ , and for all  $i \geq 0$ , with  $T_0 = 0$ ,

$$\begin{aligned} T_{2i+1} &= T_{2i} + T_{\lambda_S} \circ \Theta_{T_{2i}}, \\ T_{2i+2} &= T_{2i+1} + T_{\kappa_{\mathcal{R}}} \circ \Theta_{T_{2i+1}}. \end{aligned}$$

Let us define

$$\chi_{\mathcal{R}} = \max_{x \in \mathcal{R}} \frac{1}{\mu_{\mathcal{R}}(x)} \tag{5}$$

and introduce

**Hypotheses (H')**: *Hypotheses (H) hold together with  $(\ln \chi_{\mathcal{S}})/\gamma_{\mathcal{S}} \ll 1/\phi_{\mathcal{R} \setminus \mathcal{S}}^*$  and  $(\ln \chi_{\mathcal{R}})/\gamma_{\mathcal{R}} \ll 1/\phi_{\mathcal{R} \setminus \mathcal{S}}^*$ .*

We can now make the connection with the time averages of the pathwise approach.

**Theorem 2.** *Assuming hypotheses (H') and choosing  $\kappa$  and  $\lambda$  such that*

$$\phi_{\mathcal{R} \setminus \mathcal{S}}^* \ll \kappa \ll \gamma_{\mathcal{R}}/\ln \chi_{\mathcal{R}} \quad \text{and} \quad \phi_{\mathcal{R} \setminus \mathcal{S}}^* \ll \lambda \ll \gamma_{\mathcal{S}}/\ln \chi_{\mathcal{S}},$$

*it exists a positive  $\theta \ll 1/\phi_{\mathcal{R}, \lambda_S}^*$  such that, for all  $\delta > 0$ , for all  $n > 0$  and for all observable  $f : \mathcal{X} \rightarrow \mathbb{R}$ ,*

$$\mathbb{P}_{\mu_{\mathcal{R}}} \left( \begin{array}{l} \forall i < n, T_{2i+1} > T_{2i} + \theta, T_{2i+2} > T_{2i+1} + \theta, \\ \forall t \in [T_{2i}, T_{2i+1} - \theta], \forall t' \in [T_{2i+1}, T_{2i+2} - \theta], \\ |A_\theta(t, f) - \mu_{\mathcal{R}}(f)| \leq \delta, |A_\theta(t', f) - \mu_{\mathcal{S}}(f)| \leq \delta \end{array} \right) \sim 1.$$

**Comments:**

- i. We will give the proof of this theorem in Section 5. In this section we will make explicit a possible choice for  $\theta$  and we will give explicit estimates of such a probability.
- ii. As in [OV05], if one is interested in convergence, after time rescaling, of the empirical distribution process defined by

$$A_\theta(t) = \frac{1}{\theta} \int_t^{t+\theta} \delta_{X(s)} ds, \quad t \geq 0,$$

this theorem, together with a convergence in law of the rescaled  $T_k$ , is the key to prove convergence in Skorokhod topology of a simple modification  $\tilde{A}_\theta$  of  $A_\theta$ , obtained by setting  $\tilde{A}_\theta(t) = \tilde{A}_\theta(T_k - \theta)$  if  $t$  belongs to some  $[T_k - \theta, T_k[$  and  $\tilde{A}_\theta(t) = A_\theta(t)$  otherwise.

- iii. In the case  $\mu(\mathcal{R}) \ll \mu(\mathcal{S})$ , we will have (recall Theorem 1)

$$\frac{1}{\gamma} \sim \frac{1}{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*} \ll \frac{1}{\phi_{\mathcal{S}, \kappa_{\mathcal{R}}}^*} = \mathbb{E}_{\mu_{\mathcal{S}, \kappa_{\mathcal{R}}}^*} [T_{\mathcal{R}}^{\mathcal{S}}] \sim \mathbb{E}_{\mu_{\mathcal{S}, \kappa_{\mathcal{R}}}^*} [T_{\kappa_{\mathcal{R}}}],$$

where  $T_{\kappa_{\mathcal{R}}}^{\mathcal{S}}$  is defined as the local time up to killing  $T_{\lambda_{\mathcal{S}}}^{\mathcal{R}}$  by replacing  $\lambda$  by  $\kappa$  and exchanging the roles of  $\mathcal{R}$  and  $\mathcal{S}$ , just as  $\mu_{\mathcal{S}, \kappa_{\mathcal{R}}}^*$  was defined similarly to  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$ . The rescaled process on time unit  $1/\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  reaches equilibrium after time  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^* T_1$  of order 1, this equilibrium can be described by  $\mu_{\mathcal{S}}$ , and there is no oscillation on this time scale between the two states  $\mu_{\mathcal{R}}$  and  $\mu_{\mathcal{S}}$ . Our system will go back to  $\mu_{\mathcal{R}}$  only after a much longer time of order  $1/\phi_{\mathcal{S}, \kappa_{\mathcal{R}}}^*$ .

To complete the ‘‘pathwise approach program’’ it remains to establish asymptotic exponential laws for  $T_{\lambda_{\mathcal{S}}}$  or  $T_{\kappa_{\mathcal{R}}}$ , depending on the starting distribution. Starting from  $\mu_{\mathcal{R}}$  it is a consequence of the fact that  $\mu_{\mathcal{R}}$  and  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  are close in total variation distance (see propositions 2.6 and 2.7). Starting from more general initial distributions, such a convergence is a consequence of the ‘‘local thermalization property’’.

### 1.3 Local thermalization and asymptotic exponential laws

As in [BG16] we can use  $T_{\lambda_{\mathcal{S}}}$  and  $T_{\kappa_{\mathcal{R}}}$  to build a stopping time  $T^*$  of order  $1/(\kappa \wedge \lambda)$  at most and such that the law of  $X(T^*)$  is close to  $\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  or  $\mu_{\mathcal{S}, \kappa_{\mathcal{R}}}^*$ . Since we are here mainly interested in convergence in law of the transitions times, closeness in total variation distance will be enough. We will then be able to build  $T^*$  in a simpler way, and with weaker hypotheses, than in [BG16].

**Theorem 3.** *Assuming hypotheses (H<sup>1</sup>) and choosing  $\kappa$  and  $\lambda$  such that*

$$\phi_{\mathcal{R} \setminus \mathcal{S}}^* \ll \kappa \ll \gamma_{\mathcal{R}} / \ln \chi_{\mathcal{R}} \quad \text{and} \quad \phi_{\mathcal{R} \setminus \mathcal{S}}^* \ll \lambda \ll \gamma_{\mathcal{S}} / \ln \chi_{\mathcal{S}},$$

*there are three stopping times  $T_{\mathcal{R}}^*$ ,  $T_{\mathcal{S}}^*$  and  $T^* = T_{\mathcal{R}}^* \wedge T_{\mathcal{S}}^*$  such that, whatever the starting distribution of  $X$ :*

- i. *the expected value of  $T^*$  is smaller than or equal to  $2/(\kappa \wedge \lambda)$ ;*
- ii. *the total variation distance between  $\mu_{\mathcal{R}}$ , respectively  $\mu_{\mathcal{S}}$ , and the law  $\pi^*$  of  $X(T^*)$  conditioned to  $\{T^* = T_{\mathcal{R}}^*\}$ , respectively  $\{T^* = T_{\mathcal{S}}^*\}$ , goes to zero, more precisely*

$$d_{\text{TV}}(\mu_{\mathcal{R}}, \pi^*) \leq \frac{3\kappa}{\gamma_{\mathcal{R}}} \left( 1 + \left[ \ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa} \right]_+ \right),$$

*respectively*

$$d_{\text{TV}}(\mu_{\mathcal{S}}, \pi^*) \leq \frac{3\lambda}{\gamma_{\mathcal{S}}} \left( 1 + \left[ \ln \frac{\gamma_{\mathcal{S}} \sqrt{\chi_{\mathcal{S}}}}{2\lambda} \right]_+ \right),$$

*and  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^* T_{\lambda_{\mathcal{S}}} \circ \Theta_{T^*}$ , respectively  $\phi_{\mathcal{S}, \kappa_{\mathcal{R}}}^* T_{\kappa_{\mathcal{R}}} \circ \Theta_{T^*}$ , converges in law to an exponential variable or parameter 1;*

- iii. *the probability that  $\{T^* = T_{\mathcal{R}}^* = T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}\}$ , respectively  $\{T^* = T_{\mathcal{S}}^* = T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}}\}$ , occurs goes to one when the starting measure  $\nu$  is such that  $\mathbb{P}_\nu(T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}}) \ll 1$ , respectively  $\mathbb{P}_\nu(T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}) \ll 1$ . More precisely,*

$$\mathbb{P}_\nu(T^* = T_{\mathcal{R}}^* = T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}) \geq 1 - 3\mathbb{P}_\nu(T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}}),$$

*respectively*

$$\mathbb{P}_\nu(T^* = T_{\mathcal{S}}^* = T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}}) \geq 1 - 3\mathbb{P}_\nu(T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}}).$$



**Comments:**

- i. We will give the proof of this theorem in Section 4, together with the explicit construction of  $T^*$  and explicit estimates on all error terms.
- ii. This theorem implies an asymptotic exponential law for the rescaled time  $\phi_{\mathcal{R},\lambda_S}^* T_{\lambda_S}$  when  $X$  starts from any measure  $\nu$  such that  $\mathbb{P}_\nu(T_{\lambda_S} < T_{\kappa_{\mathcal{R}}}) \ll 1$ . We will give a direct proof of this fact in Section 2 (see Proposition 2.8). One of the reasons why we needed stronger hypotheses in [BG16] was that we were also interested there in events like  $\mathbb{P}_\nu(T_{\lambda_S} > t)$  for times  $t$  that were large with respect to  $1/\gamma_{\mathcal{R}}$  but of a smaller order than  $1/\phi_{\mathcal{R},\lambda_S}^*$ .
- iii. It is then possible to prove an analogue of Theorem 2 where  $T_0$  is replaced by  $T^*$ ,  $T_1$  by  $T^* + T_{\lambda_S} \circ \Theta_{T^*}$  and  $\mathbb{P}_{\mu_{\mathcal{R}}}$  by  $\mathbb{P}_x(\cdot \mid T^* = T_{\mathcal{R}}^*)$ , for any  $x$  in  $\mathcal{X}$ .

## 2 Basic definitions and properties

### 2.1 Soft measures and killed process

Given our irreducible Markov process  $X$  with generator  $\mathcal{L}$  defined in Equation (1) and assumed to be reversible with respect to a probability measure  $\mu$ , given  $\mathcal{R}$  and  $\mathcal{S}$  such that  $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$  and given a finite  $\lambda \geq 0$ , we build the soft measure  $\mu_{\mathcal{R},\lambda_S}^*$  from the trace  $X_{\mathcal{R},\lambda_S}^*$  on  $\mathcal{R}$  of the process  $X$  killed at rate  $\lambda_S(x)$  in each  $x$  in  $\mathcal{X}$ , with  $\lambda_S$  defined in Equation (4). With  $T_{\mathcal{R}}$  the hitting time of  $\mathcal{R}$  and  $T_{\lambda_S}$  the killing time of  $X$ , namely, the first arrival time  $t$  of an independent Poisson process associated with  $x = X(t)$  of rate  $\lambda_S(x)$ ,  $X_{\mathcal{R},\lambda_S}^*$  is then the Markov process on  $\mathcal{R}$  that, on the one hand, is killed at rate

$$e_{\mathcal{R},\lambda_S}^*(x) = \lambda_S(x) + \sum_{z \neq x} w(x,z) \mathbb{P}_z(T_{\mathcal{R}} > T_{\lambda_S}) = \lambda_S(x) + \sum_{z \notin \mathcal{R}} w(x,z) \mathbb{P}_z(T_{\mathcal{R}} > T_{\lambda_S}), \quad (6)$$

in  $x$  in  $\mathcal{R}$ , where the second term accounts for the rate with which  $X$  escapes from  $\mathcal{R}$  and is killed before returning in it, and that, on the other hand, jumps from  $x$  to a distinct  $y \in \mathcal{R}$  at rate

$$w_{\mathcal{R},\lambda_S}(x,y) = \sum_{z \neq x} w(x,z) \mathbb{P}_z(X(T_{\mathcal{R}}) = y, T_{\mathcal{R}} < T_{\lambda_S}) = w(x,y) + \sum_{z \notin \mathcal{R}} w(x,z) \mathbb{P}_z(X(T_{\mathcal{R}}) = y, T_{\mathcal{R}} < T_{\lambda_S}). \quad (7)$$

It is associated with the sub-Markovian generator  $\mathcal{L}_{\mathcal{R},\lambda_S}^*$  that acts on functions  $f : \mathcal{R} \rightarrow \mathbb{R}$  according to

$$(\mathcal{L}_{\mathcal{R},\lambda_S}^* f)(x) = -e_{\mathcal{R},\lambda_S}^*(x) f(x) + \sum_{y \in \mathcal{R} \setminus \{x\}} w_{\mathcal{R},\lambda_S}(x,y) (f(y) - f(x)), \quad x \in \mathcal{R}.$$

Our irreducibility hypothesis (cf. (H)) on  $X_{\mathcal{R}}$ , the generator of which is defined by Equation (2), implies that of  $X_{\mathcal{R},\lambda_S}^*$ . Then, by Perron-Frobenius Theorem,  $\mathcal{L}_{\mathcal{R},\lambda_S}^*$  has only negative eigenvalues and the smallest eigenvalue  $\phi_{\mathcal{R},\lambda_S}^* > 0$  of  $-\mathcal{L}_{\mathcal{R},\lambda_S}^*$  is non-degenerate and associated with a left eigenvector  $\mu_{\mathcal{R},\lambda_S}^*$  which is a probability on  $\mathcal{R}$ . We call it the *soft measure* associated with the killing rate  $\lambda_S$  and it is a special kind of quasi-stationary measure. We denote by  $\ell_{\mathcal{R}}$  the local time in  $\mathcal{R}$ :

$$\ell_{\mathcal{R}}(t) = \int_0^t \mathbf{1}_{\{X(u) \in \mathcal{R}\}} du, \quad t \geq 0.$$

It is a non-decreasing continuous function of the time  $t$  and we call  $\ell_{\mathcal{R}}^{-1}$  its right-continuous inverse:

$$\ell_{\mathcal{R}}^{-1}(s) = \inf \{t \geq 0 : \ell_{\mathcal{R}}(t) > s\}, \quad s \geq 0.$$

We then have  $X_{\mathcal{R},\lambda_S}^*(s) = X \circ \ell_{\mathcal{R}}^{-1}(s)$  for all  $s < T_{\lambda_S}^{\mathcal{R}}$  and the following lemma.

**Lemma 2.1.** *For all  $s \geq 0$  and all  $x$  and  $y$  in  $\mathcal{R}$  it holds*

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} (X \circ \ell_{\mathcal{R}}^{-1}(s) = x \mid T_{\lambda_S}^{\mathcal{R}} > s) &= \mu_{\mathcal{R},\lambda_S}^*(x), \\ \lim_{s \rightarrow \infty} \mathbb{P}_x (X \circ \ell_{\mathcal{R}}^{-1}(s) = y \mid T_{\lambda_S}^{\mathcal{R}} > s) &= \mu_{\mathcal{R},\lambda_S}^*(y), \end{aligned}$$

and

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} (T_{\lambda_S}^{\mathcal{R}} > s) = e^{-s \phi_{\mathcal{R},\lambda_S}^*}.$$

The exit rate is also given by

$$\phi_{\mathcal{R},\lambda_S}^* = \mu_{\mathcal{R},\lambda_S}^* (e_{\mathcal{R},\lambda_S}^*).$$

These formulas are consequences of the fact that, for all  $s \geq 0$ , every function  $f$  and every probability  $\nu$  on  $\mathcal{R}$ ,

$$\mathbb{E}_\nu \left[ f(X_{\mathcal{R},\lambda_S}^*(s)) \mathbb{1}_{\{T_{\lambda_S}^{\mathcal{R}} > s\}} \right] = \nu \left( e^{s\mathcal{L}_{\mathcal{R},\lambda_S}^*} f \right).$$

We refer to Lemma 2.12 in [BG16] for more details.

As in [BG16] (Lemma 2.13) and assuming hypotheses (H), when  $\lambda$  goes from 0 to  $+\infty$  we get a continuous interpolation between the restricted ensemble  $\mu_{\mathcal{R}}$  and the quasi-stationary measure  $\mu_{\mathcal{R}\setminus\mathcal{S}}^*$ . Indeed, when  $\lambda = 0$  our soft measure  $\mu_{\mathcal{R},\lambda_S}^*$  is nothing but the invariant measure of the trace process on  $\mathcal{R}$ , which coincides with  $\mu(\cdot|\mathcal{R})$  as a consequence of our reversibility hypothesis. And, if  $\lambda = +\infty$  we are led to define  $X_{\mathcal{R},\lambda_S}^*$  as  $X_{\mathcal{R}\setminus\mathcal{S}}^*$ , obtained from  $X$  by killing it instantaneously in  $\mathcal{S}$ . This process is associated with the sub-Markovian generator  $\mathcal{L}_{\mathcal{R}\setminus\mathcal{S}}^*$  defined by

$$\left( \mathcal{L}_{\mathcal{R}\setminus\mathcal{S}}^* f \right) (x) = -e_{\mathcal{R}\setminus\mathcal{S}}^*(x) f(x) + \sum_{\substack{y \in \mathcal{R}\setminus\mathcal{S}, \\ y \neq x}} w(x,y) (f(y) - f(x)), \quad f : \mathcal{R}\setminus\mathcal{S} \rightarrow \mathbb{R}, \quad x \in \mathcal{R},$$

with, for all  $x$  in  $\mathcal{R}\setminus\mathcal{S}$ ,

$$e_{\mathcal{R}\setminus\mathcal{S}}^*(x) = \sum_{z \in \mathcal{S}} w(x,z). \quad (8)$$

Since  $X_{\mathcal{R}\setminus\mathcal{S}}^*$  is assumed to be irreducible, the quasi-stationary measure  $\mu_{\mathcal{R}\setminus\mathcal{S}}^*$  is unambiguously defined. Contrarily to  $X_{\mathcal{R}\setminus\mathcal{S}}^*$ , which cannot penetrate  $\mathcal{S}$ ,  $\mu_{\mathcal{R},\lambda_S}^*$  and  $X_{\mathcal{R},\lambda_S}^*$  are associated with a process which can visit the killing zone  $\mathcal{S}$ . This is the explanation for the terminology *soft* measure.

We denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the standard norm and scalar product in  $\ell^2(\mu)$ . For any  $\mathcal{A} \subset \mathcal{X}$  we denote by  $\mu_{\mathcal{A}}$  the conditional probability measure  $\mu_{\mathcal{A}} = \mu(\cdot|\mathcal{A})$ , and by  $\|\cdot\|_{\mathcal{A}}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  the standard norm and scalar product in  $\ell^2(\mu_{\mathcal{A}})$ . We also denote by  $e_{\mathcal{A}}^*$ , as in Equation (8), the escape rate

$$e_{\mathcal{A}}^*(x) = \sum_{z \notin \mathcal{A}} w(x,z)$$

for  $x$  in  $\mathcal{A}$ .

**Proposition 2.2.** *When  $\lambda$  grows to  $+\infty$ ,*

$$\phi_{\mathcal{R},\lambda_S}^* = \min_{\substack{f: \mathcal{R} \rightarrow \mathbb{R} \\ f \neq 0}} \frac{\langle f, -\mathcal{L}_{\mathcal{R},\lambda_S}^* f \rangle_{\mathcal{R}}}{\|f\|_{\mathcal{R}}^2} = \min_{f > 0} \frac{\langle f, -\mathcal{L}_{\mathcal{R},\lambda_S}^* f \rangle_{\mathcal{R}}}{\|f\|_{\mathcal{R}}^2} \quad (9)$$

*continuously grows to  $\phi_{\mathcal{R}\setminus\mathcal{S}}^*$ . In particular, for all  $\lambda \geq 0$ , it holds*

$$\phi_{\mathcal{R},\lambda_S}^* \leq \phi_{\mathcal{R}\setminus\mathcal{S}}^* = \min_{\substack{f: \mathcal{R}\setminus\mathcal{S} \rightarrow \mathbb{R} \\ f \neq 0}} \frac{\langle f, -\mathcal{L}_{\mathcal{R}\setminus\mathcal{S}}^* f \rangle_{\mathcal{R}\setminus\mathcal{S}}}{\|f\|_{\mathcal{R}\setminus\mathcal{S}}^2}. \quad (10)$$

*Proof:* The first equality in Equation (9) comes from the fact that  $\mathcal{L}_{\mathcal{R},\lambda_S}^*$  is a self-adjoint operator. The last equality in Equation (9) is a consequence of Perron-Frobenius Theorem. Continuity follows from Lemma 2.1 and continuity properties of  $\mu_{\mathcal{R},\lambda_S}^*$  and  $e_{\mathcal{R},\lambda_S}^*$ . Finally, for any positive function  $f$  on  $\mathcal{R}$  it holds

$$\begin{aligned} \langle f, -\mathcal{L}_{\mathcal{R},\lambda_S}^* f \rangle_{\mathcal{R}} &= - \sum_{x \in \mathcal{R}} \sum_{y \in \mathcal{R} \setminus \{x\}} \mu_{\mathcal{R}}(x) f(x) w_{\mathcal{R},\lambda_S}(x,y) f(y) \\ &\quad + \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left( e_{\mathcal{R},\lambda_S}^*(x) + \sum_{y \in \mathcal{R} \setminus \{x\}} w_{\mathcal{R},\lambda_S}(x,y) \right) f^2(x). \end{aligned}$$

The first term, minus sign included, which accounts for the cross product, is clearly increasing in  $\lambda$ , since

$w_{\mathcal{R},\lambda_S}(x, y)$  decreases with  $\lambda$  and  $f$  is positive. As far as the second term is concerned we compute

$$\begin{aligned}
e_{\mathcal{R},\lambda_S}^*(x) + \sum_{y \in \mathcal{R} \setminus \{x\}} w_{\mathcal{R},\lambda_S}(x, y) &= \lambda_S(x) + \sum_{z \notin \mathcal{R}} w(x, z) \mathbb{P}_z(T_{\mathcal{R}} > T_{\lambda_S}) \\
&+ \sum_{y \in \mathcal{R} \setminus \{x\}} w(x, y) + \sum_{y \in \mathcal{R} \setminus \{x\}} \sum_{z \notin \mathcal{R}} w(x, z) \mathbb{P}_z(X(T_{\mathcal{R}}) = y, T_{\mathcal{R}} < T_{\lambda_S}) \\
&= \lambda_S(x) + \sum_{z \notin \mathcal{R}} w(x, z) \mathbb{P}_z(T_{\mathcal{R}} > T_{\lambda_S}) + \sum_{y \in \mathcal{R} \setminus \{x\}} w(x, y) \\
&+ \sum_{z \notin \mathcal{R}} w(x, z) \left(1 - \mathbb{P}_z(T_{\mathcal{R}} > T_{\lambda_S}) - \mathbb{P}_z(X(T_{\mathcal{R}}) = x, T_{\mathcal{R}} < T_{\lambda_S})\right) \\
&= \lambda_S(x) + w(x) - \sum_{z \notin \mathcal{R}} w(x, z) \mathbb{P}_z(X(T_{\mathcal{R}}) = x, T_{\mathcal{R}} < T_{\lambda_S}).
\end{aligned}$$

This coefficient is then increasing in  $\lambda$  and the monotonicity of  $\phi_{\mathcal{R},\lambda_S}^*$  follows.  $\square$

Using simple test functions in such variational principles we have then the following upper bounds on  $\phi_{\mathcal{R},\lambda_S}^*$ .

**Lemma 2.3.** *It holds, for all  $\lambda \geq 0$ ,*

$$\phi_{\mathcal{R},\lambda_S}^* \leq \phi_{\mathcal{R} \setminus S}^* \leq \mu_{\mathcal{R} \setminus S}(e_{\mathcal{R} \setminus S}^*), \quad (11)$$

$$\phi_{\mathcal{R},\lambda_S}^* \leq \lambda \mu_{\mathcal{R}}(\mathcal{R} \cap S) + \mu_{\mathcal{R}}(e_{\mathcal{R}}^*), \quad (12)$$

$$\mathbb{E}_{\mu_{\mathcal{R}}}[T_{\lambda_S}^{\mathcal{R}}] \leq \frac{1}{\phi_{\mathcal{R},\lambda_S}^*}. \quad (13)$$

*Proof:* Formula (11) is obtained from Formula (10) with the constant function equal to 1 on  $\mathcal{R} \setminus S$  used as a test function. We use the constant function  $\mathbb{1}_{\mathcal{R}}$  equal to 1 on  $\mathcal{R}$  in Equation (9) and we bound by 1 the last probability appearing in Equation (6) to get Inequality (12). Finally, to get Inequality (13), we observe that  $1/\phi_{\mathcal{R},\lambda_S}^*$  is the largest eigenvalue of the Green kernel  $\mathcal{G}_{\lambda_S}^{\mathcal{R}} = (-\mathcal{L}_{\mathcal{R},\lambda_S})^{-1}$ , so that, writing  $\ell_y$  for the local time in  $y$ ,

$$\frac{1}{\phi_{\mathcal{R},\lambda_S}^*} = \max_{\|f\|_{\mathcal{R}}=1} \langle f, \mathcal{G}_{\lambda_S}^{\mathcal{R}} f \rangle_{\mathcal{R}} \geq \langle \mathbb{1}_{\mathcal{R}}, \mathcal{G}_{\lambda_S}^{\mathcal{R}} \mathbb{1}_{\mathcal{R}} \rangle_{\mathcal{R}} = \sum_{x,y \in \mathcal{R}} \mu_{\mathcal{R}}(x) \mathbb{E}_x[\ell_y(T_{\lambda_S})] = \mathbb{E}_{\mu_{\mathcal{R}}}[\ell_{\mathcal{R}}(T_{\lambda_S})].$$

$\square$

**Remark:** While  $\phi_{\mathcal{R},\lambda_S}^*$  and  $\phi_{\mathcal{R} \setminus S}^*$  are the means of escape rates according to quasi-stationary measures that can be difficult to compute, the upper bounds given in (11) and (12) refer only to the reversible measure  $\mu$ .

## 2.2 Restricted dynamics

For any finite  $\lambda \geq 0$ , the *restricted dynamics*  $X_{\mathcal{R},\lambda_S}$  is obtained from  $X_{\mathcal{R},\lambda_S}^*$  or  $X$  by removing all killing moves or killing excursions outside  $\mathcal{R}$ . It is associated with the generator  $\mathcal{L}_{\mathcal{R},\lambda_S}$  that acts on functions  $f : \mathcal{R} \rightarrow \mathbb{R}$  according to

$$(\mathcal{L}_{\mathcal{R},\lambda_S} f)(x) = \sum_{y \in \mathcal{R} \setminus \{x\}} w_{\mathcal{R},\lambda_S}(x, y)(f(y) - f(x)), \quad x \in \mathcal{R},$$

with  $w_{\mathcal{R},\lambda_S}(x, y)$  defined by Equation (7) for distinct  $x$  and  $y$  in  $\mathcal{R}$ . In particular this process is reversible with respect to the restricted measure  $\mu_{\mathcal{R}}$  and it inherits irreducibility from the restricted process  $X_{\mathcal{R}}$ . We call  $\gamma_{\mathcal{R},\lambda_S}$  its spectral gap

$$\gamma_{\mathcal{R},\lambda_S} = \min_{\substack{f: \mathcal{R} \rightarrow \mathbb{R}, \\ \text{Var}_{\mu_{\mathcal{R}}}(f) \neq 0}} \frac{\mathcal{D}_{\mathcal{R},\lambda_S}(f)}{\text{Var}_{\mu_{\mathcal{R}}}(f)}, \quad (14)$$

with  $\mathcal{D}_{\mathcal{R},\lambda_S}$  the *Dirichlet form* defined by

$$\mathcal{D}_{\mathcal{R},\lambda_S}(f) = \langle f, -\mathcal{L}_{\mathcal{R},\lambda_S} f \rangle_{\mathcal{R}} = \frac{1}{2} \sum_{\substack{x,y \in \mathcal{R} \\ x \neq y}} c_{\mathcal{R},\lambda_S}(x, y) [f(y) - f(x)]^2, \quad f : \mathcal{R} \rightarrow \mathbb{R},$$

where the *conductances*  $c_{\mathcal{R},\lambda_S}(x, y)$  are given by

$$c_{\mathcal{R},\lambda_S}(x, y) = \mu_{\mathcal{R}}(x)w_{\mathcal{R},\lambda_S}(x, y) = \mu_{\mathcal{R}}(y)w_{\mathcal{R},\lambda_S}(y, x).$$

For  $\lambda = +\infty$  we define  $X_{\mathcal{R},\lambda_S}$  as the limiting process  $X_{\mathcal{R}}$ , rather than by removing all killing moves from the limiting killed process  $X_{\mathcal{R}\setminus S}^*$ . In this case we will also write  $\mathcal{D}_{\mathcal{R}}$  and  $c_{\mathcal{R}}$  in place of  $\mathcal{D}_{\mathcal{R},\lambda_S}$  and  $c_{\mathcal{R},\lambda_S}$ . From Equation (14) as well as the continuity and monotonicity in  $\lambda$  of the conductances  $c_{\mathcal{R},\lambda_S}$ , which is inherited from that of the rates  $w_{\mathcal{R},\lambda_S}(x, y)$ , we get

**Proposition 2.4.** *When  $\lambda$  goes to  $+\infty$ ,  $\gamma_{\mathcal{R},\lambda_S}$  continuously decreases to  $\gamma_{\mathcal{R}}$ . In particular, it holds*

$$\gamma_{\mathcal{R},\lambda_S} \geq \gamma_{\mathcal{R}}$$

for all  $\lambda \geq 0$ .

We have from propositions 2.2 and 2.4 that the ratio

$$\epsilon_{\mathcal{R},\lambda_S}^* = \frac{1/\gamma_{\mathcal{R},\lambda_S}}{1/\phi_{\mathcal{R},\lambda_S}^*} = \frac{\phi_{\mathcal{R},\lambda_S}^*}{\gamma_{\mathcal{R},\lambda_S}}$$

is decreasing in  $\lambda$ . When  $\lambda = +\infty$  we will also write

$$\epsilon_{\mathcal{R},S}^* = \frac{\phi_{\mathcal{R}\setminus S}^*}{\gamma_{\mathcal{R}}} \tag{15}$$

in place of  $\epsilon_{\mathcal{R},\lambda_S}^*$  and we have, for all  $\lambda \geq 0$ ,

$$\epsilon_{\mathcal{R},\lambda_S}^* \leq \epsilon_{\mathcal{R},S}^*. \tag{16}$$

Hence  $\epsilon_{\mathcal{R},\lambda_S}^*$  tends to be negligible under hypotheses (H).

In this case all our soft measures are close together. Let us introduce some more notation to make this precise. We denote by  $h_{\mathcal{R},\lambda_S}^*$  the density of  $\mu_{\mathcal{R},\lambda_S}^*$  with respect to  $\mu_{\mathcal{R}}$ , i.e.,

$$h_{\mathcal{R},\lambda_S}^*(x) = \frac{\mu_{\mathcal{R},\lambda_S}^*(x)}{\mu_{\mathcal{R}}(x)}, \quad x \in \mathcal{R}.$$

In the special case  $\lambda = +\infty$  we write  $h_{\mathcal{R}\setminus S}^*$  for  $h_{\mathcal{R},\lambda_S}^*$ :

$$h_{\mathcal{R}\setminus S}^* : x \in \mathcal{R} \mapsto \frac{\mu_{\mathcal{R}\setminus S}^*(x)}{\mu_{\mathcal{R}}(x)},$$

which is supported on  $\mathcal{R}\setminus S$ . In particular, we have for any finite  $\lambda$ ,

$$\phi_{\mathcal{R},\lambda_S}^* = \frac{\langle h_{\mathcal{R},\lambda_S}^*, -\mathcal{L}_{\mathcal{R},\lambda_S}^* h_{\mathcal{R},\lambda_S}^* \rangle_{\mathcal{R}}}{\|h_{\mathcal{R},\lambda_S}^*\|_{\mathcal{R}}^2},$$

and

$$\phi_{\mathcal{R}\setminus S}^* = \frac{\langle h_{\mathcal{R}\setminus S}^*, -\mathcal{L}_{\mathcal{R}\setminus S}^* h_{\mathcal{R}\setminus S}^* \rangle_{\mathcal{R}}}{\|h_{\mathcal{R}\setminus S}^*\|_{\mathcal{R}}^2}.$$

Since  $\mathcal{L}_{\mathcal{R}\setminus S}^*$  acts on functions of  $\mathcal{R}\setminus S$ , to make sense of  $\mathcal{L}_{\mathcal{R}\setminus S}^* h_{\mathcal{R}\setminus S}^*$  we used in the previous equation

**Convention 2.5.** *Any function  $f$  on  $\mathcal{A} \subset \mathcal{X}$  is identified with the function on its support  $\mathcal{B} \subset \mathcal{A}$ . In particular it also identified, for any  $\mathcal{C} \supset \mathcal{B}$ , with the function  $g$  on  $\mathcal{C}$  that coincides with  $f$  on  $\mathcal{B}$  and has zero value in  $\mathcal{C}\setminus\mathcal{B}$ .*

We can now compare our soft measures in  $\ell^2(\mu_{\mathcal{R}})$  and  $\ell^1(\mu_{\mathcal{R}})$  norms.

**Proposition 2.6.** *For all  $\lambda \in [0, +\infty]$ , if  $\epsilon_{\mathcal{R},\lambda_S}^* < 1$ , then*

$$\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R},\lambda_S}^*) = \|h_{\mathcal{R},\lambda_S}^* - \mathbf{1}_{\mathcal{R}}\|_{\mathcal{R}}^2 \leq \frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}.$$

In particular, with  $d_{\text{TV}}$  the total variation distance,

$$d_{\text{TV}}(\mu_{\mathcal{R},\lambda_S}^*, \mu_{\mathcal{R}}) \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}}. \tag{17}$$

*Proof:* Using  $h_{\mathcal{R},\lambda_S}^*$  as a test function in the variational principle given by Equation (14), we get, for any finite  $\lambda$ ,

$$\begin{aligned} \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R},\lambda_S}^*) &\leq \frac{1}{\gamma_{\mathcal{R},\lambda_S}} \mathcal{D}_{\mathcal{R},\lambda_S}(h_{\mathcal{R},\lambda_S}^*) = \frac{1}{\gamma_{\mathcal{R},\lambda_S}} \langle h_{\mathcal{R},\lambda_S}^*, -\mathcal{L}_{\mathcal{R},\lambda_S} h_{\mathcal{R},\lambda_S}^* \rangle_{\mathcal{R}} \\ &= \frac{1}{\gamma_{\mathcal{R},\lambda_S}} \left( \langle h_{\mathcal{R},\lambda_S}^*, -\mathcal{L}_{\mathcal{R},\lambda_S}^* h_{\mathcal{R},\lambda_S}^* \rangle_{\mathcal{R}} - \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) e_{\mathcal{R}}^*(x) h_{\mathcal{R},\lambda_S}^{*2}(x) \right) \\ &\leq \frac{1}{\gamma_{\mathcal{R},\lambda_S}} \phi_{\mathcal{R},\lambda_S}^* \mu_{\mathcal{R}}(h_{\mathcal{R},\lambda_S}^{*2}) = \frac{\phi_{\mathcal{R},\lambda_S}^*}{\gamma_{\mathcal{R},\lambda_S}} \left( 1 + \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R},\lambda_S}^*) \right). \end{aligned}$$

In the limiting case  $\lambda = +\infty$  we have a similar estimate:

$$\begin{aligned} \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R} \setminus \mathcal{S}}^*) &\leq \frac{1}{\gamma_{\mathcal{R}}} \mathcal{D}_{\mathcal{R}}(h_{\mathcal{R} \setminus \mathcal{S}}^*) \\ &= \frac{1}{\gamma_{\mathcal{R}}} \left( \frac{1}{2} \sum_{x,y \in \mathcal{R} \setminus \mathcal{S}} c_{\mathcal{R}}(x,y) [h_{\mathcal{R} \setminus \mathcal{S}}^*(x) - h_{\mathcal{R} \setminus \mathcal{S}}^*(y)]^2 + \sum_{\substack{x \in \mathcal{R} \setminus \mathcal{S} \\ y \in \mathcal{S} \cap \mathcal{R}}} c_{\mathcal{R}}(x,y) [h_{\mathcal{R} \setminus \mathcal{S}}^*(x)]^2 \right) \\ &\leq \frac{1}{\gamma_{\mathcal{R}}} \left( \sum_{x \in \mathcal{R} \setminus \mathcal{S}} \mu_{\mathcal{R}}(x) h_{\mathcal{R} \setminus \mathcal{S}}^*(x) \sum_{y \in \mathcal{R} \setminus \mathcal{S}} w(x,y) [h_{\mathcal{R} \setminus \mathcal{S}}^*(x) - h_{\mathcal{R} \setminus \mathcal{S}}^*(y)] \right. \\ &\quad \left. + \sum_{x \in \mathcal{R} \setminus \mathcal{S}} \mu_{\mathcal{R}}(x) \sum_{y \in \mathcal{S}} w(x,y) [h_{\mathcal{R} \setminus \mathcal{S}}^*(x)]^2 \right) \\ &= \frac{1}{\gamma_{\mathcal{R}}} \langle h_{\mathcal{R} \setminus \mathcal{S}}^*, -\mathcal{L}_{\mathcal{R} \setminus \mathcal{S}}^* h_{\mathcal{R} \setminus \mathcal{S}}^* \rangle_{\mathcal{R}} \\ &= \frac{1}{\gamma_{\mathcal{R}}} \phi_{\mathcal{R} \setminus \mathcal{S}}^* \mu_{\mathcal{R}}(h_{\mathcal{R} \setminus \mathcal{S}}^{*2}) = \frac{\phi_{\mathcal{R} \setminus \mathcal{S}}^*}{\gamma_{\mathcal{R}}} \left( 1 + \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R} \setminus \mathcal{S}}^*) \right). \end{aligned}$$

Rearranging the terms, we get the desired inequality. Equation (17) follows then by Jensen inequality.  $\square$

As a consequence we have an asymptotic exponential law for  $T_{\lambda_S}$  when starting from  $\mu_{\mathcal{R}}$  as well as  $\mu_{\mathcal{R},\lambda_S}^*$  as soon as  $\lambda \gg \phi_{\mathcal{R},\lambda_S}^*$ , for example if  $\lambda \gg \phi_{\mathcal{R} \setminus \mathcal{S}}^*$  (recall (11)). If  $\epsilon_{\mathcal{R},\lambda_S}^* \ll 1$ , then the random time  $\phi_{\mathcal{R},\lambda_S}^* T_{\lambda_S}$  converges in law to an exponential variable of parameter 1.

**Proposition 2.7.** *For all  $\lambda \geq \phi_{\mathcal{R} \setminus \mathcal{S}}^*$ , or any  $\lambda$  such that  $\lambda \geq \phi_{\mathcal{R},\lambda_S}^*$ , it holds*

$$e^{-t} \leq \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} \left( T_{\lambda_S} > \frac{t}{\phi_{\mathcal{R},\lambda_S}^*} \right) \leq e^{-t} \left\{ \exp \left( \sqrt{\phi_{\mathcal{R},\lambda_S}^* / \lambda} \right) + \exp \left( t - \sqrt{\lambda / \phi_{\mathcal{R},\lambda_S}^*} \right) \right\}$$

as soon as  $\sqrt{\phi_{\mathcal{R},\lambda_S}^* / \lambda} \leq t$ , and, if  $\epsilon_{\mathcal{R},\lambda_S}^* < 1$ ,

$$\left| \mathbb{P}_{\mu_{\mathcal{R}}} \left( T_{\lambda_S} > \frac{t}{\phi_{\mathcal{R},\lambda_S}^*} \right) - \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} \left( T_{\lambda_S} > \frac{t}{\phi_{\mathcal{R},\lambda_S}^*} \right) \right| \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}}.$$

**Remark:** Like in [BG16] we write into brace parenthesis quantities that go to one under suitable hypotheses.

*Proof of the proposition:* The first inequality is a consequence of the fact that  $T_{\lambda_S} \geq T_{\lambda_S}^{\mathcal{R}}$  and that, starting from  $\mu_{\mathcal{R},\lambda_S}^*$ , the latter is an exponential variable of parameter  $\phi_{\mathcal{R},\lambda_S}^*$ . As far as the second one is concerned, it holds, with  $\ell_{\mathcal{S}}$  the local time in  $\mathcal{S}$ ,

$$T_{\lambda_S} \leq T_{\lambda_S}^{\mathcal{R}} + \ell_{\mathcal{S}}(T_{\lambda_S})$$

so that, for any  $0 \leq \beta \leq 1$ ,

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} \left( T_{\lambda_S} > \frac{t}{\phi_{\mathcal{R},\lambda_S}^*} \right) &\leq \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} \left( T_{\lambda_S}^{\mathcal{R}} > \frac{(1-\beta)t}{\phi_{\mathcal{R},\lambda_S}^*} \right) + \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*} \left( \ell_{\mathcal{S}}(T_{\lambda_S}) > \frac{\beta t}{\phi_{\mathcal{R},\lambda_S}^*} \right) \\ &= e^{-(1-\beta)t} + e^{-\lambda \beta t / \phi_{\mathcal{R},\lambda_S}^*}. \end{aligned}$$

Choosing

$$\beta = \frac{1}{t} \sqrt{\frac{\phi_{\mathcal{R},\lambda_S}^*}{\lambda}} \leq 1,$$

this gives the desired result. Finally, by considering an optimal coupling between two random variables  $X(0)$  and  $X'(0)$  of law  $\mu_{\mathcal{R}}$  and  $\mu_{\mathcal{R},\lambda_S}^*$ , such that  $X(0) \neq X'(0)$  with probability (recall Proposition 2.6)

$$d_{\text{TV}}(\mu_{\mathcal{R}}, \mu_{\mathcal{R},\lambda_S}^*) \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}}$$

it holds

$$\mathbb{P}_{\mu_{\mathcal{R}}}(\phi_{\mathcal{R},\lambda_S}^* T_{\lambda_S} > t) \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(\phi_{\mathcal{R},\lambda_S}^* T_{\lambda_S} > t)$$

and

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(\phi_{\mathcal{R},\lambda_S}^* T_{\lambda_S} > t) \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + \mathbb{P}_{\mu_{\mathcal{R}}}(\phi_{\mathcal{R},\lambda_S}^* T_{\lambda_S} > t).$$

□

It follows that we also have an asymptotic exponential law for  $T_{\lambda_S}$  when starting from any probability measure  $\nu$  such that the total variation distance between  $\mu_{\mathcal{R}}$  and  $\nu$  goes to zero. An example of such a distribution  $\nu$  is given by the law of  $X(T_{\kappa_{\mathcal{R}}})$  if  $\kappa \ll \gamma_{\mathcal{R}} / \ln \chi_{\mathcal{R}}$  (recall Equation (5)). Hence, if a probability measure  $\pi$  is such that  $\mathbb{P}_{\pi}(T_{\kappa_{\mathcal{R}}} > T_{\lambda_S}) \ll 1$ , then  $T_{\lambda_S}$  has an asymptotic exponential law of parameter  $\phi_{\mathcal{R},\lambda_S}^*$ .

**Proposition 2.8.** *For all  $x \in \mathcal{X}$  and all  $\kappa > 0$  it holds*

$$d_{\text{TV}}(\mu_{\mathcal{R}}, \mathbb{P}_x(X(T_{\kappa_{\mathcal{R}}}) = \cdot)) \leq \frac{\kappa}{\gamma_{\mathcal{R}}} \left( 1 + \left[ \ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa} \right]_+ \right). \quad (18)$$

Also, for any probability  $\pi$  on  $\mathcal{X}$ , by setting

$$\delta = \mathbb{P}_{\pi}(T_{\lambda_S} < T_{\kappa_{\mathcal{R}}})$$

and

$$\eta = \delta + \frac{\kappa}{\gamma_{\mathcal{R}}} \left( 1 + \left[ \ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa} \right]_+ \right) + \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}},$$

it holds, for all  $\lambda \geq \phi_{\mathcal{R},\lambda_S}^*$ , or any  $\lambda \geq \phi_{\mathcal{R},\lambda_S}^*$ ,

$$e^{-t} \{1 - \eta e^t\} \leq \mathbb{P}_{\pi} \left( T_{\lambda_S} > \frac{t}{\phi_{\mathcal{R},\lambda_S}^*} \right) \leq e^{-t} \left\{ \exp \left( 2 \sqrt{\frac{\phi_{\mathcal{R},\lambda_S}^*}{\kappa \wedge \lambda}} \right) + \left[ \eta + 2 \exp \left( - \sqrt{\frac{\kappa \wedge \lambda}{\phi_{\mathcal{R},\lambda_S}^*}} \right) \right] e^t \right\}.$$

*Proof:* Let us denote by  $X^{\mathcal{R}}$  the trace of  $X$  on  $\mathcal{R}$ . Note that we already gave to this process an heavier notation:

$$X^{\mathcal{R}} = X_{\mathcal{R},0_S}.$$

Since  $X(T_{\kappa_{\mathcal{R}}}) \in \mathcal{R}$ , the law of  $X(T_{\kappa_{\mathcal{R}}})$  is that of  $X^{\mathcal{R}}(T_{\kappa_{\mathcal{R}}}^{\mathcal{R}})$  with

$$T_{\kappa_{\mathcal{R}}}^{\mathcal{R}} = \ell_{\mathcal{R}}(T_{\kappa_{\mathcal{R}}}).$$

Since the relaxation time of  $X^{\mathcal{R}}$  is smaller than or equal to that of  $X_{\mathcal{R}}$  (recall Proposition 2.4), by using reversibility to estimate the  $\ell^2(\mu_{\mathcal{R}})$  distance between (the densities with respect to  $\mu_{\mathcal{R}}$  of) the law of  $X^{\mathcal{R}}(s)$  and  $\mu_{\mathcal{R}}$ , then Jensen inequality to compare the  $\ell^1$  and  $\ell^2$  distances, we have

$$d_{\text{TV}} \left( \mathbb{P}_x \left( X^{\mathcal{R}}(s) = \cdot \right), \mu_{\mathcal{R}} \right) \leq \frac{\sqrt{\chi_{\mathcal{R}}}}{2} e^{-\gamma_{\mathcal{R}} s}.$$

Since  $T_{\kappa_{\mathcal{R}}}^{\mathcal{R}}$  is an exponential variable of parameter  $\kappa$ , it then holds, for all non-negative  $s$ ,

$$d_{\text{TV}} \left( \mathbb{P}_x \left( X(T_{\kappa_{\mathcal{R}}}) = \cdot \right), \mu_{\mathcal{R}} \right) \leq \kappa s + \frac{\sqrt{\chi_{\mathcal{R}}}}{2} e^{-\gamma_{\mathcal{R}} s}.$$

Optimizing in  $s \geq 0$ , we get (18).

Next, denoting by  $\nu$  the law of  $X(T_{\kappa\mathcal{R}})$  with starting distribution  $\pi$ , it holds, for all  $t \geq 0$ ,

$$\mathbb{P}_\nu \left( T_{\lambda\mathcal{S}} > \frac{t}{\phi_{\mathcal{R},\lambda\mathcal{S}}^*} \right) \leq \mathbb{P}_\pi (T_{\lambda\mathcal{S}} < T_{\kappa\mathcal{R}}) + \mathbb{P}_\pi \left( T_{\lambda\mathcal{S}} > \frac{t}{\phi_{\mathcal{R},\lambda\mathcal{S}}^*} \right).$$

By using Proposition 2.7 it follows that

$$\mathbb{P}_\pi \left( T_{\lambda\mathcal{S}} > \frac{t}{\phi_{\mathcal{R},\lambda\mathcal{S}}^*} \right) \geq e^{-t} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda\mathcal{S}}^*}{1 - \epsilon_{\mathcal{R},\lambda\mathcal{S}}^*}} - \frac{\kappa}{\gamma\mathcal{R}} \left( 1 + \left[ \ln \frac{\gamma\mathcal{R}\sqrt{\chi\mathcal{R}}}{2\kappa} \right]_+ \right) - \delta.$$

Finally, for all  $\theta > 0$ , we have

$$\mathbb{P}_\pi \left( T_{\lambda\mathcal{S}} > \frac{t}{\phi_{\mathcal{R},\lambda\mathcal{S}}^*} \right) \leq \mathbb{P}_\pi (T_{\lambda\mathcal{S}} < T_{\kappa\mathcal{R}}) + \mathbb{P}_\pi (T_{\lambda\mathcal{S}} \wedge T_{\kappa\mathcal{R}} > \theta) + \mathbb{P}_\nu \left( T_{\lambda\mathcal{S}} > \frac{t}{\phi_{\mathcal{R},\lambda\mathcal{S}}^*} - \theta \right).$$

By using Proposition 2.7 and the fact that  $T_{\kappa\mathcal{R}} \wedge T_{\lambda\mathcal{S}}$  is dominated by an exponential variable of parameter  $\kappa \wedge \lambda$  we get

$$\begin{aligned} \mathbb{P}_\pi \left( T_{\lambda\mathcal{S}} > \frac{t}{\phi_{\mathcal{R},\lambda\mathcal{S}}^*} \right) &\leq \delta + e^{-(\kappa \wedge \lambda)\theta} + \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda\mathcal{S}}^*}{1 - \epsilon_{\mathcal{R},\lambda\mathcal{S}}^*}} + \frac{\kappa}{\gamma\mathcal{R}} \left( 1 + \left[ \ln \frac{\gamma\mathcal{R}\sqrt{\chi\mathcal{R}}}{2\kappa} \right]_+ \right) + \exp \left[ -\sqrt{\lambda/\phi_{\mathcal{R},\lambda\mathcal{S}}^*} \right] \\ &\quad + e^{-t} \exp \left[ \phi_{\mathcal{R},\lambda\mathcal{S}}^* \theta + \sqrt{\phi_{\mathcal{R},\lambda\mathcal{S}}^* / \lambda} \right], \end{aligned}$$

which gives our last desired inequality by choosing  $\theta = (\phi_{\mathcal{R},\lambda\mathcal{S}}^* (\kappa \wedge \lambda))^{-1/2}$ .  $\square$

### 2.3 Soft capacities

To define our soft capacities we extend the conductance network  $(\mathcal{X}, c)$ , with conductances

$$c(x, y) = c(y, x) = \mu(x)w(x, y), \quad x, y \in \mathcal{X}, \quad x \neq y,$$

which is associated with the process  $X$ , into a larger network  $(\tilde{\mathcal{X}}, \tilde{c})$  by attaching dangling edges  $(r, \bar{r})$  and  $(s, \check{s})$  to each  $r \in \mathcal{R}$  and  $s \in \mathcal{S}$ . Given  $\kappa$  and  $\lambda$  we set to

$$\tilde{c}(r, \bar{r}) = \kappa\mu(r) \quad \text{and} \quad \tilde{c}(s, \check{s}) = \lambda\mu(s)$$

the conductances of these new edges. We call  $\bar{\mathcal{R}}$  and  $\check{\mathcal{S}}$  the collections of these extra nodes:

$$\bar{\mathcal{R}} = \{\bar{r} : r \in \mathcal{R}\}, \quad \check{\mathcal{S}} = \{\check{s} : s \in \mathcal{S}\}.$$

For any positive  $\tilde{\kappa}$  and  $\tilde{\lambda}$ , by extending the probability measure  $\mu$  defined on  $\mathcal{X}$  into a measure  $\tilde{\mu}$  defined on

$$\tilde{\mathcal{X}} = \mathcal{X} \cup \bar{\mathcal{R}} \cup \check{\mathcal{S}}$$

by

$$\tilde{\mu}(\tilde{x}) = \begin{cases} \mu(x) & \text{if } \tilde{x} = x \text{ for some } x \in \mathcal{X}, \\ \kappa\mu(r)/\tilde{\kappa} & \text{if } \tilde{x} = \bar{r} \text{ for some } r \in \mathcal{R}, \\ \lambda\mu(s)/\tilde{\lambda} & \text{if } \tilde{x} = \check{s} \text{ for some } s \in \mathcal{S}, \end{cases}$$

the extended network  $(\tilde{\mathcal{X}}, \tilde{c})$  is then associated with the process  $\tilde{X}$  with rates  $\tilde{w}(\tilde{x}, \tilde{y})$  given, for distinct  $\tilde{x}$  and  $\tilde{y}$  in  $\tilde{\mathcal{X}}$ , by

$$\tilde{w}(\tilde{x}, \tilde{y}) = \begin{cases} w(x, y) & \text{if } \tilde{x} = x \text{ and } \tilde{y} = y \text{ for some } x \text{ and } y \text{ in } \mathcal{X}, \\ \kappa & \text{if } \tilde{x} = r \text{ and } \tilde{y} = \bar{r} \text{ for some } r \text{ in } \mathcal{R}, \\ \tilde{\kappa} & \text{if } \tilde{x} = \bar{r} \text{ and } \tilde{y} = r \text{ for some } r \text{ in } \mathcal{R}, \\ \lambda & \text{if } \tilde{x} = s \text{ and } \tilde{y} = \check{s} \text{ for some } s \text{ in } \mathcal{S}, \\ \tilde{\lambda} & \text{if } \tilde{x} = \check{s} \text{ and } \tilde{y} = s \text{ for some } s \text{ in } \mathcal{S}, \\ 0 & \text{otherwise,} \end{cases}$$

and which is reversible with respect to  $\tilde{\mu}$ . (Note that  $\tilde{\mu}$  is not a probability measure.)

Our soft capacities are then the  $(\kappa, \lambda)$ -capacities of [BG16].

**Definition 2.9.** The  $(\kappa, \lambda)$ -capacity  $C_\kappa^\lambda(\mathcal{R}, \mathcal{S})$  is the capacity between the sets  $\bar{\mathcal{R}}$  and  $\check{\mathcal{S}}$  in the network  $(\tilde{\mathcal{X}}, \tilde{c})$ . According to Dirichlet principle, it is given by

$$\begin{aligned} C_\kappa^\lambda(\mathcal{R}, \mathcal{S}) &= \min \left\{ \frac{1}{2} \sum_{\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}} \tilde{c}(\tilde{x}, \tilde{y}) \left( \tilde{f}(\tilde{x}) - \tilde{f}(\tilde{y}) \right)^2 : \tilde{f}|_{\bar{\mathcal{R}}} = 1, \tilde{f}|_{\check{\mathcal{S}}} = 0 \right\} \\ &= \min_{f: \mathcal{X} \rightarrow \mathbb{R}} \left\{ \mathcal{D}(f) + \kappa \sum_{r \in \mathcal{R}} \mu(r) (f(r) - 1)^2 + \lambda \sum_{s \in \mathcal{S}} \mu(s) (f(s) - 0)^2 \right\} \\ &= \min_{f: \mathcal{X} \rightarrow \mathbb{R}} \left\{ \mathcal{D}(f) + \kappa \mu(\mathcal{R}) \mathbb{E}_{\mu_{\mathcal{R}}} \left[ (f|_{\mathcal{R}} - 1)^2 \right] + \lambda \mu(\mathcal{S}) \mathbb{E}_{\mu_{\mathcal{S}}} \left[ (f|_{\mathcal{S}} - 0)^2 \right] \right\}. \end{aligned} \quad (19)$$

**Remarks:**

- i. Since this definition depends on the conductances only, while it is naturally associated with the Markov process  $\tilde{X}$ , it does not depend on the choice of  $\tilde{\kappa}$  and  $\tilde{\lambda}$ .  $C_\kappa^\lambda(\mathcal{R}, \mathcal{S})$  depends on two parameters only,  $\kappa$  and  $\lambda$ , in which it is increasing.
- ii. Our soft capacities satisfy a two-sided variational principle. On one side they are given by Definition 2.9 as the infimum of some functional, and any test function will provide an upper bound. On the other side they are, by Thomson principle, the supremum of another functional on unitary flows from  $\bar{\mathcal{R}}$  to  $\check{\mathcal{S}}$ , which are antisymmetric functions of oriented edges with null divergence in  $\mathcal{X}$  and total divergence in  $\bar{\mathcal{R}}$  and  $\check{\mathcal{S}}$  equal to 1 and -1 respectively, i.e., on antisymmetric functions  $\tilde{\psi}: \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \rightarrow \mathbb{R}$  such that  $\tilde{\psi}(\tilde{x}, \tilde{y}) = 0$  as soon as  $\tilde{c}(\tilde{x}, \tilde{y}) = 0$  and

$$\operatorname{div}_x \tilde{\psi} = \sum_{\tilde{x} \in \tilde{\mathcal{X}}} \tilde{\psi}(x, \tilde{x}) = 0, \quad x \in \mathcal{X},$$

while

$$\sum_{r \in \mathcal{R}} \operatorname{div}_r \tilde{\psi} = \sum_{r \in \mathcal{R}} \tilde{\psi}(\bar{r}, r) = 1 = - \sum_{s \in \mathcal{S}} \operatorname{div}_s \tilde{\psi} = \sum_{s \in \mathcal{S}} \tilde{\psi}(s, \check{s}).$$

With

$$\tilde{\mathcal{D}}(\tilde{\psi}) = \frac{1}{2} \sum_{\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}} \frac{\tilde{\psi}(\tilde{x}, \tilde{y})^2}{\tilde{c}(\tilde{x}, \tilde{y})}$$

the energy dissipated in  $(\tilde{\mathcal{X}}, \tilde{c})$  by the flow  $\tilde{\psi}$  and with  $\tilde{\Psi}_1(\bar{\mathcal{R}}, \check{\mathcal{S}})$  the set of unitary flows from  $\bar{\mathcal{R}}$  to  $\check{\mathcal{S}}$ , we have

$$C_\kappa^\lambda(\mathcal{R}, \mathcal{S}) = \max_{\tilde{\psi} \in \tilde{\Psi}_1(\bar{\mathcal{R}}, \check{\mathcal{S}})} \tilde{\mathcal{D}}(\tilde{\psi})^{-1}. \quad (20)$$

Any test flow provides then a lower bound on  $C_\kappa^\lambda(\mathcal{R}, \mathcal{S})$ .

- iii. By referring to  $\tilde{X}$ , the stopping time  $T_{\lambda_{\mathcal{S}}}$  can also be defined as the hitting time of  $\check{\mathcal{S}}$ . By defining  $T_{\kappa_{\mathcal{R}}}$  in a symmetric way, the minimum in (19) is realized by the equilibrium potential  $V_\kappa^\lambda$  given by

$$V_\kappa^\lambda(x) = \mathbb{P}_x(T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}), \quad x \in \mathcal{X},$$

whereas the maximum in (20) is realized by the associated normalized current.

As an application of Dirichlet principle we have an upper bound on our soft capacities, or of the asymptotic given for the spectral  $\gamma$  in Theorem 1. Let us define

$$\phi_\kappa^\lambda = \frac{C_\kappa^\lambda(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})\mu(\mathcal{S})}$$

and recall Equation (15).

**Lemma 2.10.** If  $\mu(\mathcal{S}) \geq \mu(\mathcal{R})$  and  $\epsilon_{\mathcal{R}, \mathcal{S}}^* \leq 1$ , then, for all  $\kappa \geq 0$ , it holds

$$\phi_\kappa^\lambda \leq \mu(\mathcal{S})^{-1} \phi_{\mathcal{R} \setminus \mathcal{S}}^* \left\{ \frac{1 + \kappa/\gamma_{\mathcal{R}}}{1 - \epsilon_{\mathcal{R}, \mathcal{S}}^*} \right\} \quad \text{and} \quad \mu(\mathcal{S})^{-1} \leq 2. \quad (21)$$

In particular, under the hypotheses of Theorem 1 it holds  $\phi_\kappa^\lambda = O(\phi_{\mathcal{R} \setminus \mathcal{S}}^*)$ .



*Proof:* By taking  $h_{\mathcal{R}\setminus\mathcal{S}}^*$  as test function in Equation (19), and following Convention 2.5, we have

$$C_\kappa^\lambda(\mathcal{R}, \mathcal{S}) \leq \mathcal{D}(h_{\mathcal{R}\setminus\mathcal{S}}^*) + \kappa\mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}\setminus\mathcal{S}}^*).$$

The second term is straightforwardly bounded by using Proposition 2.6 which reads

$$\kappa\mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}\setminus\mathcal{S}}^*) \leq \mu(\mathcal{R})\phi_{\mathcal{R}\setminus\mathcal{S}}^* \frac{\kappa/\gamma_{\mathcal{R}}}{1 - \epsilon_{\mathcal{R},\mathcal{S}}^*}.$$

As far as the first term is concerned we write

$$\begin{aligned} \mathcal{D}(h_{\mathcal{R}\setminus\mathcal{S}}^*) &= \frac{1}{2} \sum_{x,y \in \mathcal{R}\setminus\mathcal{S}} c(x,y) [h_{\mathcal{R}\setminus\mathcal{S}}^*(x) - h_{\mathcal{R}\setminus\mathcal{S}}^*(y)]^2 + \sum_{\substack{x \in \mathcal{R}\setminus\mathcal{S}, \\ y \in \mathcal{S}}} c(x,y) h_{\mathcal{R}\setminus\mathcal{S}}^{*2}(x) \\ &= \mu(\mathcal{R}) \langle h_{\mathcal{R}\setminus\mathcal{S}}^*, -\mathcal{L}_{\mathcal{R}\setminus\mathcal{S}}^* h_{\mathcal{R}\setminus\mathcal{S}}^* \rangle_{\mathcal{R}} = \mu(\mathcal{R}) \phi_{\mathcal{R}\setminus\mathcal{S}}^* \|h_{\mathcal{R}\setminus\mathcal{S}}^*\|_{\mathcal{R}}^2 = \mu(\mathcal{R}) \phi_{\mathcal{R}\setminus\mathcal{S}}^* (1 + \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}\setminus\mathcal{S}}^*)) \\ &\leq \mu(\mathcal{R}) \phi_{\mathcal{R}\setminus\mathcal{S}}^* \left(1 + \frac{\epsilon_{\mathcal{R},\mathcal{S}}^*}{1 - \epsilon_{\mathcal{R},\mathcal{S}}^*}\right) = \mu(\mathcal{R}) \phi_{\mathcal{R}\setminus\mathcal{S}}^* \frac{1}{1 - \epsilon_{\mathcal{R},\mathcal{S}}^*}, \end{aligned}$$

where the inequality is again an application Proposition 2.6. Observing that  $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$  and  $\mu(\mathcal{S}) \geq \mu(\mathcal{R})$  imply  $\mu(\mathcal{S}) \geq 1/2$ , this leads to (21). The last comment is only based on the fact that hypotheses (H) include  $\epsilon_{\mathcal{R},\mathcal{S}}^* \ll 1$  and  $\kappa \ll \gamma_{\mathcal{R}}$  is assumed in Theorem 1.  $\square$

**Remark 2.11.** All the estimates of Section 2 are “symmetric” in  $\mathcal{R}$  and  $\mathcal{S}$ . This means that the same results hold when the role of  $\mathcal{R}$  and  $\mathcal{S}$  are exchanged as well as those of  $\kappa$  and  $\lambda$ .

We note however that there is an asymmetry in hypotheses (H). It will start to play a role in the next section.

## 3 Proof of Theorem 1

### 3.1 Spectral gap estimates

The statement contained in Theorem 1 concerning the spectral gap  $\gamma$  is a consequence of the quantitative Theorem 4 together with Lemma 2.10, estimates (11) and (16) as well as Remark 2.11. Note also that hypotheses (H) include  $\phi_{\mathcal{R}\setminus\mathcal{S}}^* \ll \gamma_{\mathcal{S}}$ . This is where the asymmetry of (H) matters.

**Theorem 4.** The spectral gap  $\gamma$  satisfies

$$\gamma \geq \phi_\kappa^\lambda \left\{ 1 + \frac{\kappa + \phi_\kappa^\lambda (1 + \mu(\mathcal{R} \cap \mathcal{S}))}{\gamma_{\mathcal{R}}} + \frac{\lambda + \phi_\kappa^\lambda (1 + \mu(\mathcal{R} \cap \mathcal{S}))}{\gamma_{\mathcal{S}}} \right\}^{-1}. \quad (22)$$

In addition, when  $\epsilon_{\mathcal{R},\lambda\mathcal{S}}^*$  and  $\epsilon_{\mathcal{S},\kappa\mathcal{R}}^*$  are smaller than 1, we have the converse inequality

$$\gamma \leq \phi_\kappa^\lambda \left\{ 1 - \frac{\phi_{\mathcal{R},\lambda\mathcal{S}}^*}{\kappa} - \frac{\phi_{\mathcal{S},\kappa\mathcal{R}}^*}{\lambda} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda\mathcal{S}}^*}{1 - \epsilon_{\mathcal{R},\lambda\mathcal{S}}^*}} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{S},\kappa\mathcal{R}}^*}{1 - \epsilon_{\mathcal{S},\kappa\mathcal{R}}^*}} \right\}^{-2} \quad (23)$$

as soon as the braced sum is positive.

*Proof of the lower bound:* We recall that  $\gamma$  satisfies the variational principle

$$\gamma := \min_{\substack{f: \mathcal{X} \rightarrow \mathbb{R}, \\ \text{Var}_\mu(f) \neq 0}} \frac{\mathcal{D}(f)}{\text{Var}_\mu(f)}, \quad (24)$$

We look for a Poincaré inequality, i.e. an inequality of the form

$$\forall f \in \ell^2(\mu), \quad \text{Var}_\mu(f) \leq k\mathcal{D}(f),$$

which is equivalent, by mean of Equation (24), to a lower bound on the spectral gap. Let  $Z, Z'$  two i.i.d. random variables with same law  $\mu$ . The variance of  $f$  can be written as

$$\text{Var}_\mu(f) = \mathbb{E}_\mu[f^2] - \mathbb{E}_\mu^2[f] = \frac{1}{2} \mathbb{E}[(f(Z) - f(Z'))^2]. \quad (25)$$

From the r.h.s. of (25) we get

$$\begin{aligned}
\text{Var}_\mu(f) &\leq \frac{1}{2}\mathbb{E}\left[(f(Z) - f(Z'))^2 \mathbf{1}_{\{Z, Z' \in \mathcal{R}\}}\right] + \frac{1}{2}\mathbb{E}\left[(f(Z) - f(Z'))^2 \mathbf{1}_{\{Z, Z' \in \mathcal{S}\}}\right] \\
&\quad + \mathbb{E}\left[(f(Z) - f(Z'))^2 \mathbf{1}_{\{Z \in \mathcal{R}, Z' \in \mathcal{S}\}}\right] \\
&= \mu^2(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \mu^2(\mathcal{S})\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}}) \\
&\quad + \mu(\mathcal{R})\mu(\mathcal{S})\left(\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}^2] + \mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}^2] - 2\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}]\mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]\right) \\
&= \mu(\mathcal{R})\left(\mu(\mathcal{R}) + \mu(\mathcal{S})\right)\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \mu(\mathcal{S})\left(\mu(\mathcal{R}) + \mu(\mathcal{S})\right)\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}}) \\
&\quad + \mu(\mathcal{R})\mu(\mathcal{S})\left(\mathbb{E}_{\mu_{\mathcal{R}}}^2[f|_{\mathcal{R}}] + \mathbb{E}_{\mu_{\mathcal{S}}}^2[f|_{\mathcal{S}}] - 2\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}]\mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]\right) \\
&= \mu(\mathcal{R})\left(\mu(\mathcal{R}) + \mu(\mathcal{S})\right)\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \mu(\mathcal{S})\left(\mu(\mathcal{R}) + \mu(\mathcal{S})\right)\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}}) \\
&\quad + \mu(\mathcal{R})\mu(\mathcal{S})\left(\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}] - \mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]\right)^2. \tag{26}
\end{aligned}$$

As a test function in Dirichlet principle (19) we pick

$$\tilde{f} = \frac{f - \mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]}{\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}] - \mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]}, \tag{27}$$

which is such that  $\mathbb{E}_{\mu_{\mathcal{R}}}[\tilde{f}|_{\mathcal{R}}] = 1$  and  $\mathbb{E}_{\mu_{\mathcal{S}}}[\tilde{f}|_{\mathcal{S}}] = 0$ . By plugging (27) in (19) we get

$$\begin{aligned}
C_\kappa^\lambda(\mathcal{R}, \mathcal{S}) &\leq \mathcal{D}(\tilde{f}) + \kappa\mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(\tilde{f}|_{\mathcal{R}}) + \lambda\mu(\mathcal{S})\text{Var}_{\mu_{\mathcal{S}}}(\tilde{f}|_{\mathcal{S}}) \\
&= \left(\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}] - \mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]\right)^{-2} \left(\mathcal{D}(f) + \kappa\mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \lambda\mu(\mathcal{S})\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}})\right)
\end{aligned}$$

which actually is an upper bound on  $\left(\mathbb{E}_{\mu_{\mathcal{R}}}[f|_{\mathcal{R}}] - \mathbb{E}_{\mu_{\mathcal{S}}}[f|_{\mathcal{S}}]\right)^2$ . Bounding from above the last term in (26) we then get

$$\begin{aligned}
\text{Var}_\mu(f) &\leq \mu(\mathcal{R})\left(\mu(\mathcal{R}) + \mu(\mathcal{S})\right)\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \mu(\mathcal{S})\left(\mu(\mathcal{R}) + \mu(\mathcal{S})\right)\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}}) \\
&\quad + \frac{\mu(\mathcal{R})\mu(\mathcal{S})}{C_\kappa^\lambda(\mathcal{R}, \mathcal{S})} \left(\mathcal{D}(f) + \kappa\mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \lambda\mu(\mathcal{S})\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}})\right) \\
&= \frac{\mathcal{D}(f)}{\phi_\kappa^\lambda} + \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) \left(\frac{\kappa}{\phi_\kappa^\lambda} + (1 + \mu(\mathcal{R} \cap \mathcal{S}))\right) \\
&\quad + \mu(\mathcal{S})\text{Var}_{\mu_{\mathcal{S}}}(f|_{\mathcal{S}}) \left(\frac{\lambda}{\phi_\kappa^\lambda} + (1 + \mu(\mathcal{R} \cap \mathcal{S}))\right) \\
&\leq \frac{\mathcal{D}(f)}{\phi_\kappa^\lambda} + \frac{\mu(\mathcal{R})\mathcal{D}_{\mathcal{R}}(f|_{\mathcal{R}})}{\gamma_{\mathcal{R}}} \left(\frac{\kappa}{\phi_\kappa^\lambda} + (1 + \mu(\mathcal{R} \cap \mathcal{S}))\right) + \frac{\mu(\mathcal{S})\mathcal{D}_{\mathcal{S}}(f|_{\mathcal{S}})}{\gamma_{\mathcal{S}}} \left(\frac{\lambda}{\phi_\kappa^\lambda} + (1 + \mu(\mathcal{R} \cap \mathcal{S}))\right) \\
&\leq \frac{\mathcal{D}(f)}{\phi_\kappa^\lambda} + \frac{\mathcal{D}(f)}{\gamma_{\mathcal{R}}} \left(\frac{\kappa}{\phi_\kappa^\lambda} + (1 + \mu(\mathcal{R} \cap \mathcal{S}))\right) + \frac{\mathcal{D}(f)}{\gamma_{\mathcal{S}}} \left(\frac{\lambda}{\phi_\kappa^\lambda} + (1 + \mu(\mathcal{R} \cap \mathcal{S}))\right)
\end{aligned}$$

where, in the second inequality, the variational characterization of  $\gamma_{\mathcal{R}}$  and  $\gamma_{\mathcal{S}}$  given by Equation (3) has been used, whereas in the third one we exploited the following fact:

$$\mathcal{D}(f) \geq \mu(\mathcal{R})\mathcal{D}_{\mathcal{R}}(f|_{\mathcal{R}}), \quad \mathcal{D}(f) \geq \mu(\mathcal{S})\mathcal{D}_{\mathcal{S}}(f|_{\mathcal{S}}).$$

Factorising  $\mathcal{D}(f)/\phi_\kappa^\lambda$  we get the Poincaré inequality

$$\text{Var}_\mu(f) \leq \frac{\mathcal{D}(f)}{\phi_\kappa^\lambda} \left(1 + \frac{\kappa + \phi_\kappa^\lambda(1 + \mu(\mathcal{R} \cap \mathcal{S}))}{\gamma_{\mathcal{R}}} + \frac{\lambda + \phi_\kappa^\lambda(1 + \mu(\mathcal{R} \cap \mathcal{S}))}{\gamma_{\mathcal{S}}}\right).$$

□

Before giving the proof of the upper bound, we observe that the proof of this Poincaré inequality carries over to the case of any finite covering of irreducible sets  $\mathcal{X} = \cup_{i < m} \mathcal{R}_i$ .

**Proposition 3.1.** *If  $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{m-1}$  are subsets of  $\mathcal{X}$  such that  $X_{\mathcal{R}_i}$  is irreducible for each  $i < m$ , and  $\kappa_0, \kappa_1, \dots, \kappa_{m-1}$  are any positive numbers then, with*

$$\phi_i^j = \phi_j^i = \frac{C_{\kappa_i}^{\kappa_j}(\mathcal{R}_i, \mathcal{R}_j)}{\mu(\mathcal{R}_i)\mu(\mathcal{R}_j)}, \quad i \neq j, \quad i, j < m,$$

$$\frac{1}{\phi} = \frac{1}{2} \sum_{i,j:i \neq j} \frac{1}{\phi_i^j}$$

and, for each  $i < m$ ,

$$\frac{1}{\phi_i} = \sum_{j \neq i} \frac{1}{\phi_i^j} \leq \frac{1}{\phi},$$

it holds

$$\gamma \geq \phi \left\{ 1 + \sum_{i < m} \frac{\phi}{\gamma_i} \left( \frac{\kappa_i}{\phi_i} + \sum_{j < m} \mu(\mathcal{R}_j) \right) \right\}^{-1} \geq \phi \left\{ 1 + \sum_{i < m} \frac{\kappa_i + \phi \sum_{j < m} \mu(\mathcal{R}_j)}{\gamma_i} \right\}^{-1}.$$

*Proof of the upper bound in Theorem 4:* We want to exploit the variational principle (24), so we look for an upper bound on  $\mathcal{D}(f)$  and a lower bound on  $\text{Var}_\mu(f)$  for a suitable function  $f$ . From (25) we get

$$\begin{aligned} \text{Var}_\mu(f) &\geq \frac{1}{2} \mathbb{E} \left[ (f(Z) - f(Z'))^2 \left( \mathbb{1}_{\{Z \in \mathcal{R}, Z' \in \mathcal{S}\}} + \mathbb{1}_{\{Z \in \mathcal{S}, Z' \in \mathcal{R}\}} - \mathbb{1}_{\{Z, Z' \in \mathcal{R} \cap \mathcal{S}\}} \right) \right] \\ &= \mathbb{E} \left[ (f(Z) - f(Z'))^2 \mathbb{1}_{\{Z \in \mathcal{R}, Z' \in \mathcal{S}\}} \right] - \frac{1}{2} \mathbb{E} \left[ (f(Z) - f(Z'))^2 \mathbb{1}_{\{Z, Z' \in \mathcal{R} \cap \mathcal{S}\}} \right] \\ &= \mu(\mathcal{R})\mu(\mathcal{S}) \left( \mathbb{E}_{\mu_{\mathcal{R}}} [f_{|\mathcal{R}}^2] + \mathbb{E}_{\mu_{\mathcal{S}}} [f_{|\mathcal{S}}^2] \right) - 2\mu(\mathcal{R})\mu(\mathcal{S}) \mathbb{E}_{\mu_{\mathcal{R}}} [f_{|\mathcal{R}}] \mathbb{E}_{\mu_{\mathcal{S}}} [f_{|\mathcal{S}}] \\ &\quad - \mu^2(\mathcal{R} \cap \mathcal{S}) \text{Var}_{\mu_{\mathcal{R} \cap \mathcal{S}}} (f_{|\mathcal{R} \cap \mathcal{S}}) \\ &= \mu(\mathcal{R})\mu(\mathcal{S}) \left( \text{Var}_{\mu_{\mathcal{R}}} (f_{|\mathcal{R}}) + \mathbb{E}_{\mu_{\mathcal{R}}}^2 [f_{|\mathcal{R}}] \right) + \mu(\mathcal{R})\mu(\mathcal{S}) \left( \text{Var}_{\mu_{\mathcal{S}}} (f_{|\mathcal{S}}) + \mathbb{E}_{\mu_{\mathcal{S}}}^2 [f_{|\mathcal{S}}] \right) \\ &\quad - 2\mu(\mathcal{R})\mu(\mathcal{S}) \mathbb{E}_{\mu_{\mathcal{R}}} [f_{|\mathcal{R}}] \mathbb{E}_{\mu_{\mathcal{S}}} [f_{|\mathcal{S}}] - \mu^2(\mathcal{R} \cap \mathcal{S}) \text{Var}_{\mu_{\mathcal{R} \cap \mathcal{S}}} (f_{|\mathcal{R} \cap \mathcal{S}}) \\ &= \mu(\mathcal{R})\mu(\mathcal{S}) \left( \text{Var}_{\mu_{\mathcal{R}}} (f_{|\mathcal{R}}) + \text{Var}_{\mu_{\mathcal{S}}} (f_{|\mathcal{S}}) \right) - \mu^2(\mathcal{R} \cap \mathcal{S}) \text{Var}_{\mu_{\mathcal{R} \cap \mathcal{S}}} (f_{|\mathcal{R} \cap \mathcal{S}}) \\ &\quad + \mu(\mathcal{R})\mu(\mathcal{S}) \left( \mathbb{E}_{\mu_{\mathcal{R}}} [f_{|\mathcal{R}}] - \mathbb{E}_{\mu_{\mathcal{S}}} [f_{|\mathcal{S}}] \right)^2 \end{aligned} \quad (28)$$

To get a suitable lower bound of (28), we write

$$\begin{aligned} \mu^2(\mathcal{R} \cap \mathcal{S}) \text{Var}_{\mu_{\mathcal{R} \cap \mathcal{S}}} (f_{|\mathcal{R} \cap \mathcal{S}}) &= \frac{1}{2} \mathbb{E} \left[ (f(Z) - f(Z'))^2 \mathbb{1}_{\{Z, Z' \in \mathcal{R} \cap \mathcal{S}\}} \right] \\ &\leq \frac{1}{2} \mathbb{E} \left[ (f(Z) - f(Z'))^2 \mathbb{1}_{\{Z, Z' \in \mathcal{R}\}} \right] \\ &= \mu^2(\mathcal{R}) \text{Var}_{\mu_{\mathcal{R}}} (f_{|\mathcal{R}}) \leq \mu(\mathcal{R})\mu(\mathcal{S}) \text{Var}_{\mu_{\mathcal{R}}} (f_{|\mathcal{R}}), \end{aligned} \quad (29)$$

assuming, without loss of generality,  $\mu(\mathcal{R}) \leq \mu(\mathcal{S})$ . Then, by using (29) to bound the second term in (28), dropping one term and taking as test function the equilibrium potential  $V_\kappa^\lambda$ , we get

$$\text{Var}_\mu(V_\kappa^\lambda) \geq \mu(\mathcal{R})\mu(\mathcal{S}) \left( \mathbb{E}_{\mu_{\mathcal{R}}} [V_\kappa^\lambda |_{\mathcal{R}}] - \mathbb{E}_{\mu_{\mathcal{S}}} [V_\kappa^\lambda |_{\mathcal{S}}] \right)^2. \quad (30)$$

Next, we have the following two simple lemmas.

**Lemma 3.2.** *If  $\epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^* < 1$ , then*

$$\mathbb{E}_{\mu_{\mathcal{R}}} [V_\kappa^\lambda |_{\mathcal{R}}] \geq 1 - \frac{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}{\kappa} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}{1 - \epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}}.$$

*Proof:* We have, denoting by  $\sigma_\kappa$  an exponential time of rate  $\kappa$  that is independent of  $X$ ,

$$\mathbb{E}_{\mu_{\mathcal{R}}} [V_\kappa^\lambda |_{\mathcal{R}}] = \mathbb{P}_{\mu_{\mathcal{R}}} (T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}) = \mathbb{P}_{\mu_{\mathcal{R}}} (\ell_{\mathcal{R}} (T_{\kappa_{\mathcal{R}}}) < \ell_{\mathcal{R}} (T_{\lambda_{\mathcal{S}}})) = \mathbb{P}_{\mu_{\mathcal{R}}} (\sigma_\kappa < T_{\lambda_{\mathcal{S}}}^{\mathcal{R}}).$$

Let  $\mathbb{P}^{opt}$  be an optimal coupling at time  $t = 0$  between  $X^{\mu_{\mathcal{R}}}$  and  $X^{\mu_{\mathcal{R},\lambda_S}^*}$  that start with initial distribution  $\mu_{\mathcal{R}}$  and  $\mu_{\mathcal{R},\lambda_S}^*$  and evolve jointly if they have the same starting configuration. By the union bound, Proposition (2.6) and the convexity of  $x \mapsto \frac{1}{1+x}$  we get

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R}}}(\sigma_{\kappa} < T_{\lambda_S}^{\mathcal{R}}) &\geq \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(\sigma_{\kappa} < T_{\lambda_S}^{\mathcal{R}}) - \mathbb{P}^{opt}(X^{\mu_{\mathcal{R}}}(0) \neq X^{\mu_{\mathcal{R},\lambda_S}^*}(0)) \\ &\geq \frac{\kappa}{\kappa + \phi_{\mathcal{R},\lambda_S}^*} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} \geq 1 - \frac{\phi_{\mathcal{R},\lambda_S}^*}{\kappa} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} \end{aligned}$$

□

**Lemma 3.3.** *If  $\epsilon_{S,\kappa_{\mathcal{R}}}^* < 1$ , then*

$$E_{\mu_S}[V_{\kappa}^{\lambda} | \mathcal{S}] \leq \frac{1}{2} \sqrt{\frac{\epsilon_{S,\kappa_{\mathcal{R}}}^*}{1 - \epsilon_{S,\kappa_{\mathcal{R}}}^*}} + \frac{\phi_{S,\kappa_{\mathcal{R}}}^*}{\lambda}.$$

The proof of this lemma is similar to that of the previous one and we omit it. By plugging the two bounds provided by Lemma 3.2 and Lemma 3.3 into (30) we obtain

$$\text{Var}_{\mu}(V_{\kappa}^{\lambda}) \geq \mu(\mathcal{R})\mu(\mathcal{S}) \left\{ 1 - \frac{\phi_{\mathcal{R},\lambda_S}^*}{\kappa} - \frac{\phi_{S,\kappa_{\mathcal{R}}}^*}{\lambda} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} - \frac{1}{2} \sqrt{\frac{\epsilon_{S,\kappa_{\mathcal{R}}}^*}{1 - \epsilon_{S,\kappa_{\mathcal{R}}}^*}} \right\}^2 \quad (31)$$

as soon as the braced quantity is positive. The upper bound on  $\mathcal{D}(V_{\kappa}^{\lambda})$  is straightforward: since the functional associated with the Dirichlet principle, which reaches its minimum  $C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})$  in  $V_{\kappa}^{\lambda}$ , is larger than  $\mathcal{D}$ , we have

$$\mathcal{D}(V_{\kappa}^{\lambda}) \leq C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S}). \quad (32)$$

Inequality (23) follows from Equation (24) together with (31) and (32). □

### 3.2 Exit rate estimates

The last part of Theorem 1 follows in the same way from the quantitative Theorem 5.

**Theorem 5.** *The exit rate  $\phi_{\mathcal{R},\lambda_S}^*$  satisfies*

$$\phi_{\mathcal{R},\lambda_S}^* \geq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})} \left\{ \frac{1 - 2\mu(\mathcal{S})^{-1}\phi_{\mathcal{R},\lambda_S}^*/\lambda}{(1 - \phi_{\mathcal{R},\lambda_S}^*/\lambda)^2} \right\} \left\{ 1 + \frac{\kappa + \phi_{\kappa}^{\lambda}(1 + \mu(\mathcal{R} \cap \mathcal{S}))}{\gamma_{\mathcal{R}}} + \frac{\lambda + \phi_{\kappa}^{\lambda}(1 + \mu(\mathcal{R} \cap \mathcal{S}))}{\gamma_{\mathcal{S}}} \right\}^{-1}.$$

*In addition, when  $\epsilon_{\mathcal{R},\lambda_S}^* < 1$ , we have the converse inequality*

$$\phi_{\mathcal{R},\lambda_S}^* \leq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})} \left\{ 1 + \frac{\phi_{\mathcal{R},\lambda_S}^*}{\lambda} \right\} \left\{ 1 - \frac{\phi_{\mathcal{R},\lambda_S}^*}{\kappa} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} \right\}^{-2}$$

*as soon as the last braced sum is positive.*

*Proof of the lower bound:* We consider the process  $\tilde{X}$  introduced in Section 2.3 in the limiting case  $\tilde{\kappa} = +\infty$ . In this case  $\tilde{\mathcal{X}}$  can be redefined as  $\tilde{\mathcal{X}} = \mathcal{X} \cup \tilde{\mathcal{S}}$  and we call  $\tilde{\mathcal{L}}$  the generator of this process  $\tilde{X}$  on  $\tilde{\mathcal{X}}$ . In according with the notation of Section 2.1, we denote by  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$  and  $\tilde{\mu}_{\mathcal{X},\lambda_S}^*$  respectively the smallest eigenvalue and the corresponding normalized left eigenvector (i.e., the quasi-stationary distribution associated with  $\mathcal{X}$  and  $\lambda_S$ ) of  $-\tilde{\mathcal{L}}_{\mathcal{X},\lambda_S}^*$ , which is defined by

$$(\tilde{\mathcal{L}}_{\mathcal{X},\lambda_S}^* f)(x) = -\lambda_S(x)f(x) + \sum_{y \in \mathcal{X}} w(x,y)[f(y) - f(x)], \quad f : \mathcal{X} \rightarrow \mathbb{R}, \quad x \in \mathcal{X}.$$

For any probability  $\nu$  on  $\mathcal{X}$ ,  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$  is the decay rate of the survival probability  $\mathbb{P}_{\nu}(T_{\lambda_S} > t)$ :

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \ln \mathbb{P}_{\nu}(T_{\lambda_S} > t) = \tilde{\phi}_{\mathcal{X},\lambda_S}^*.$$

Since

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(T_{\lambda_S} > t) \geq \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(\ell_{\mathcal{R}}(T_{\lambda_S}) > t) = \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(T_{\lambda_S}^{\mathcal{R}} > t) = e^{-t\phi_{\mathcal{R},\lambda_S}^*},$$

it follows that

$$\phi_{\mathcal{R},\lambda_S}^* \geq \tilde{\phi}_{\mathcal{X},\lambda_S}^*. \quad (33)$$

To give a lower bound on  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$ , we use its variational representation

$$\tilde{\phi}_{\mathcal{X},\lambda_S}^* = \min_{f:\mathcal{X}\rightarrow\mathbb{R}} \frac{\langle f, -\tilde{\mathcal{L}}_{\mathcal{X},\lambda_S}^* f \rangle}{\|f\|^2} = \min_{f:\mathcal{X}\rightarrow\mathbb{R}} \frac{\mathcal{D}(f) + \|\sqrt{\lambda_S}f\|^2}{\|f\|^2}. \quad (34)$$

Indeed, for any  $f : \mathcal{X} \rightarrow \mathbb{R}$ ,

$$\langle f, -\tilde{\mathcal{L}}_{\mathcal{X},\lambda_S}^* f \rangle = \frac{1}{2} \sum_{x,y \in \mathcal{X}} \mu(x)w(x,y)[f(x) - f(y)]^2 + \sum_{x \in \mathcal{X}} \mu(x)\lambda_S(x)f^2(x).$$

Since the minimum is reached in

$$\tilde{h}_{\mathcal{X},\lambda_S}^* : x \in \mathcal{X} \mapsto \tilde{\mu}_{\mathcal{X},\lambda_S}^*(x)/\mu(x).$$

it holds

$$\tilde{\phi}_{\mathcal{X},\lambda_S}^* \geq \frac{\mathcal{D}(\tilde{h}_{\mathcal{X},\lambda_S}^*)}{\|\tilde{h}_{\mathcal{X},\lambda_S}^*\|^2} = \frac{\|\tilde{h}_{\mathcal{X},\lambda_S}^*\|^2 - 1}{\|\tilde{h}_{\mathcal{X},\lambda_S}^*\|^2} \frac{\mathcal{D}(\tilde{h}_{\mathcal{X},\lambda_S}^*)}{\text{Var}_{\mu}(\tilde{h}_{\mathcal{X},\lambda_S}^*)} \geq \left(1 - \frac{1}{\|\tilde{h}_{\mathcal{X},\lambda_S}^*\|^2}\right) \gamma. \quad (35)$$

Theorem 4 provides a lower bound on  $\gamma$  and it remains to give a lower bound on  $\|\tilde{h}_{\mathcal{X},\lambda_S}^*\|$ . To estimate this norm we restrict the sum to  $\mathcal{X} \setminus \mathcal{S}$  and we use Jensen's inequality:

$$\begin{aligned} \|\tilde{h}_{\mathcal{X},\lambda_S}^*\|^2 &\geq \mu(\mathcal{X} \setminus \mathcal{S}) \sum_{x \in \mathcal{X} \setminus \mathcal{S}} \mu_{\mathcal{X} \setminus \mathcal{S}}(x) \left( \frac{\tilde{\mu}_{\mathcal{X},\lambda_S}^*(x)}{\mu(x)} \right)^2 \\ &\geq \mu(\mathcal{X} \setminus \mathcal{S}) \left( \sum_{x \in \mathcal{X} \setminus \mathcal{S}} \mu_{\mathcal{X} \setminus \mathcal{S}}(x) \frac{\tilde{\mu}_{\mathcal{X},\lambda_S}^*(x)}{\mu(x)} \right)^2 = \frac{(\tilde{\mu}_{\mathcal{X},\lambda_S}^*(\mathcal{X} \setminus \mathcal{S}))^2}{\mu(\mathcal{X} \setminus \mathcal{S})}. \end{aligned} \quad (36)$$

Now, the last identity of Lemma 2.1 reads in this case  $\tilde{\phi}_{\mathcal{X},\lambda_S}^* = \lambda \tilde{\mu}_{\mathcal{X},\lambda_S}^*(\mathcal{S})$ . Therefore, using Inequality (33) again,

$$\tilde{\mu}_{\mathcal{X},\lambda_S}^*(\mathcal{X} \setminus \mathcal{S}) = 1 - \frac{\tilde{\phi}_{\mathcal{X},\lambda_S}^*}{\lambda} \geq 1 - \frac{\phi_{\mathcal{R},\lambda_S}^*}{\lambda}. \quad (37)$$

It follows from (36) and (37) that

$$\begin{aligned} 1 - \frac{1}{\|\tilde{h}_{\mathcal{X},\lambda_S}^*\|^2} &\geq 1 - \frac{1 - \mu(\mathcal{S})}{(1 - \phi_{\mathcal{R},\lambda_S}^*/\lambda)^2} = \frac{\mu(\mathcal{S}) - 2\phi_{\mathcal{R},\lambda_S}^*/\lambda + (\phi_{\mathcal{R},\lambda_S}^*/\lambda)^2}{(1 - \phi_{\mathcal{R},\lambda_S}^*/\lambda)^2} \\ &\geq \mu(\mathcal{S}) \frac{1 - 2\mu(\mathcal{S})^{-1}\phi_{\mathcal{R},\lambda_S}^*/\lambda}{(1 - \phi_{\mathcal{R},\lambda_S}^*/\lambda)^2}. \end{aligned} \quad (38)$$

Finally from (33), (35) and (38) together with (22) we get the desired result.  $\square$

*Proof of the upper bound:* The proof is made of two steps: first we estimate  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$  via a variational principle, then we look for an upper bound of  $\phi_{\mathcal{R},\lambda_S}^*$  in terms of  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$ . By taking as test function in (34) the equilibrium potential  $V_{\kappa}^{\lambda}$  we get

$$\tilde{\phi}_{\mathcal{X},\lambda_S}^* \leq \frac{\mathcal{D}(V_{\kappa}^{\lambda}) + \|\sqrt{\lambda_S}V_{\kappa}^{\lambda}\|^2}{\|V_{\kappa}^{\lambda}\|^2} \leq \frac{C_{\kappa}^{\lambda}(\mathcal{R},\mathcal{S})}{\|V_{\kappa}^{\lambda}\|^2}. \quad (39)$$

The denominator in (39) is lower bounded by

$$\|V_{\kappa}^{\lambda}\|^2 \geq \mu(\mathcal{R}) \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) V_{\kappa}^{\lambda}(x)^2 \geq \mu(\mathcal{R}) \mu_{\mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{R}})^2 \geq \mu(\mathcal{R}) \left\{ 1 - \frac{\phi_{\mathcal{R},\lambda_S}^*}{\kappa} - \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} \right\}^2, \quad (40)$$

where we used Jensen inequality, Lemma 3.2 and the assumption that the braced sum was positive.

To bound  $\phi_{\mathcal{R},\lambda_S}^*$  with  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$ , we write, for any  $0 \leq \beta \leq 1$  and denoting again by  $\sigma_\lambda$  an exponential time of rate  $\lambda$  that is independent of  $X$ ,

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(T_{\lambda_S} > t) &\leq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(T_{\lambda_S}^{\mathcal{R}} + \sigma_\lambda > \beta t + (1-\beta)t) \\ &\leq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(T_{\lambda_S}^{\mathcal{R}} > \beta t) + \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\sigma_\lambda > (1-\beta)t) \\ &= \exp(-\phi_{\mathcal{R},\lambda_S}^* \beta t) + \exp(-\lambda(1-\beta)t). \end{aligned}$$

Recall that  $\tilde{\phi}_{\mathcal{X},\lambda_S}^*$  is the decay rate of the survival probability  $\mathbb{P}_{\mu_{\mathcal{R},\lambda_S}^*}(T_{\lambda_S} > t)$ . If we choose  $\beta$  such that  $\phi_{\mathcal{R},\lambda_S}^* \beta \leq \lambda(1-\beta)$ , then  $\phi_{\mathcal{R},\lambda_S}^* \beta$  is the decay rate of the right-hand side in the previous inequality and we can conclude  $\tilde{\phi}_{\mathcal{X},\lambda_S}^* \geq \beta \phi_{\mathcal{R},\lambda_S}^*$ . We then choose  $\beta = \lambda/(\phi_{\mathcal{R},\lambda_S}^* + \lambda)$ , which gives us

$$\phi_{\mathcal{R},\lambda_S}^* \leq \frac{\tilde{\phi}_{\mathcal{X},\lambda_S}^*}{\beta} = \tilde{\phi}_{\mathcal{X},\lambda_S}^* \left(1 + \frac{\phi_{\mathcal{R},\lambda_S}^*}{\lambda}\right). \quad (41)$$

The thesis follows from (41), (39) and (40).  $\square$

## 4 Proof of Theorem 3

We will use the stopping time

$$\tau = T_{\kappa_{\mathcal{R}}} \wedge T_{\lambda_S}$$

to build  $T^*$ . The laws of  $X(T_{\kappa_{\mathcal{R}}})$  and  $X(T_{\lambda_S})$  are indeed close to  $\mu_{\mathcal{R}}$  and  $\mu_S$ . But conditioning on  $\{\tau = T_{\kappa_{\mathcal{R}}}\}$  or  $\{\tau = T_{\lambda_S}\}$  introduces correlations that can make the law of  $X(\tau)$  in general delicate to control. The law of  $X(T_{\kappa_{\mathcal{R}}})$  can be written as a convex combination of those of  $X(\tau)$  conditioned on  $\{\tau = T_{\kappa_{\mathcal{R}}}\}$  and  $X(T_{\kappa_{\mathcal{R}}})$  conditioned on  $\{T_{\kappa_{\mathcal{R}}} > T_{\lambda_S}\}$ : it holds, for all  $x$  in  $\mathcal{X}$ , and with

$$\alpha(x) = \mathbb{P}_x(T_{\kappa_{\mathcal{R}}} < T_{\lambda_S}),$$

$$\mathbb{P}_x(X(T_{\kappa_{\mathcal{R}}}) = \cdot) = \alpha(x) \mathbb{P}_x(X(\tau) = \cdot \mid \tau = T_{\kappa_{\mathcal{R}}}) + (1 - \alpha(x)) \mathbb{P}_x(X(T_{\kappa_{\mathcal{R}}}) = \cdot \mid T_{\kappa_{\mathcal{R}}} > T_{\lambda_S}).$$

By Proposition 2.8 the first and last of these three distributions are close to  $\mu_{\mathcal{R}}$  for a small enough  $\kappa$ . We can then estimate the total variation distance between  $\mu_{\mathcal{R}}$  and the law of  $X(\tau)$  conditioned on  $\{\tau = T_{\kappa_{\mathcal{R}}}\}$ :

$$d_{\text{TV}}\left(\mu_{\mathcal{R}}, \mathbb{P}_x(X(\tau) = \cdot \mid \tau = T_{\kappa_{\mathcal{R}}})\right) \leq \frac{2 - \alpha(x)}{\alpha(x)} \frac{\kappa}{\gamma_{\mathcal{R}}} \left(1 + \left[\ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa}\right]_+\right),$$

so that, if  $\alpha(x) \geq 1/2$ ,

$$d_{\text{TV}}\left(\mu_{\mathcal{R}}, \mathbb{P}_x(X(\tau) = \cdot \mid \tau = T_{\kappa_{\mathcal{R}}})\right) \leq \frac{3\kappa}{\gamma_{\mathcal{R}}} \left(1 + \left[\ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa}\right]_+\right).$$

This suggests the following construction. We set  $\tau^0 = 0$  and, for all  $i \geq 0$ , we define

$$\tau_{\mathcal{R}}^{i+1} = \tau^i + T_{\kappa_{\mathcal{R}}} \circ \Theta_{\tau^i}, \quad \tau_S^{i+1} = \tau^i + T_{\lambda_S} \circ \Theta_{\tau^i}, \quad \tau^{i+1} = \tau_{\mathcal{R}}^i \wedge \tau_S^i.$$

We call  $I$  the first index  $i > 0$  such that either

$$\tau^i = \tau_{\mathcal{R}}^i \quad \text{and} \quad \alpha(X(\tau^{i-1})) \geq \frac{1}{2}$$

or

$$\tau^i = \tau_S^i \quad \text{and} \quad \alpha(X(\tau^{i-1})) \leq \frac{1}{2}.$$

We finally define

$$T_{\mathcal{R}}^* = \tau_{\mathcal{R}}^I, \quad T_S^* = \tau_S^I, \quad T^* = T_{\mathcal{R}}^* \wedge T_S^*.$$

The random variable  $I$  is stochastically dominated by a geometric random variable of mean 2, each random variable  $(\tau^i - \tau^{i-1})$  is stochastically dominated by an exponential random variable of rate  $\kappa \wedge \lambda$  and it holds

$$T^* = \sum_{i>0} (\tau^i - \tau^{i-1}) \mathbf{1}_{\{I \geq i\}},$$

so that, observing that for  $(\tau^i - \tau^{i-1})$  and  $\{I \geq i\}$  are independent for all  $i > 0$ ,

$$\mathbb{E}_x [T^*] \leq \frac{2}{\kappa \wedge \lambda},$$

for all  $x$  in  $\mathcal{X}$ . We have by construction

$$d_{\text{TV}}(\mu_{\mathcal{R}}, \mathbb{P}_x(X(\tau) = \cdot \mid T^* = T_{\mathcal{R}}^*)) \leq \frac{3\kappa}{\gamma_{\mathcal{R}}} \left( 1 + \left[ \ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa} \right]_+ \right).$$

With

$$\eta = \frac{3\kappa}{\gamma_{\mathcal{R}}} \left( 1 + \left[ \ln \frac{\gamma_{\mathcal{R}} \sqrt{\chi_{\mathcal{R}}}}{2\kappa} \right]_+ \right) + \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}{1 - \epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}},$$

the triangular inequality gives

$$d_{\text{TV}}(\mu_{\mathcal{R}, \lambda_{\mathcal{S}}}^*, \mathbb{P}_x(X(T^*) = \cdot \mid T^* = T_{\mathcal{R}}^*)) \leq \eta$$

and using Proposition 2.7, we have, for all  $\lambda \geq \phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$ , or any  $\lambda \geq \phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$ ,

$$\mathbb{P}_x \left( T_{\lambda_{\mathcal{S}}} \circ \Theta_{T^*} > \frac{t}{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*} \mid T^* = T_{\mathcal{R}}^* \right) \geq e^{-t} \{1 - \eta e^t\}$$

and

$$\mathbb{P}_x \left( T_{\lambda_{\mathcal{S}}} \circ \Theta_{T^*} > \frac{t}{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*} \mid T^* = T_{\mathcal{R}}^* \right) \leq e^{-t} \left\{ \exp \left( \sqrt{\frac{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}{\lambda}} \right) + \left[ \eta + \exp \left( -\sqrt{\frac{\lambda}{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}} \right) \right] e^t \right\}.$$

Finally, if

$$\delta = \mathbb{P}_{\nu}(T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}}),$$

then, by definition of  $\alpha$ ,

$$\frac{1}{2} \mathbb{P}_{\nu} \left( \alpha(X(0)) \leq \frac{1}{2} \right) \leq \mathbb{P}_{\nu} \left( T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}} \mid \alpha(X(0)) \leq \frac{1}{2} \right) \mathbb{P}_{\nu} \left( \alpha(X(0)) \leq \frac{1}{2} \right) \leq \delta$$

and

$$\mathbb{P}_{\nu}(T^* = T_{\mathcal{R}}^* = T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}) \geq 1 - \mathbb{P}_{\nu} \left( \alpha(X(0)) \leq \frac{1}{2} \text{ or } T_{\lambda_{\mathcal{S}}} < T_{\kappa_{\mathcal{R}}} \right) \geq 1 - (2\delta + \delta) = 1 - 3\delta.$$

Since the symmetrical statements hold when exchanging the roles of  $\mathcal{R}$  and  $\mathcal{S}$  as well as those of  $\kappa$  and  $\lambda$ , this concludes the proof.  $\square$

## 5 Proof of Theorem 2

By Proposition 2.8 and Remark 2.11, the law of  $X(T_1) = X(T_{\lambda_{\mathcal{S}}})$  is close to  $\mu_{\mathcal{S}}$ , as the law of  $X(T_2)$  is close to  $\mu_{\mathcal{R}}$ . Hence, it will be sufficient to find  $\theta$  such that our time averages on time scale  $\theta$  are close to the expected values computed with  $\mu_{\mathcal{R}}$  before  $T_1$ . Theorem 2 follows then from the quantitative

**Proposition 5.1.** *If  $\epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^* < 1$  and  $\eta < 1$  is such that*

$$\eta^3 \geq \frac{\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}{\lambda} \quad \text{and} \quad \eta^4 \geq \sqrt{\epsilon_{\mathcal{R}, \lambda_{\mathcal{S}}}^*}, \quad (42)$$

then, setting

$$\theta = \frac{1}{\eta^2} \left( \frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \right) \leq \frac{\eta}{\phi_{\mathcal{R},\lambda_S}^*},$$

it holds

$$\mathbb{P}_{\mu_{\mathcal{R}}} \left( \theta < T_1, \sup_{t < T_1 - \theta} |A_\theta(t, f) - \mu_{\mathcal{R}}(f)| \leq 4\eta \|f\|_\infty \right) \geq \left\{ 1 - 4\eta - \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} \right\} \quad (43)$$

for all  $f : \mathcal{X} \rightarrow \mathbb{R}$ .

*Proof:* We can assume, without loss of generality, that  $|f|$  is bounded by 1 and we will consider four events, each of them with a small probability, such that when none of them occurs  $T_1 > \theta$  and our time averages are indeed close to  $\mu_{\mathcal{R}}(f)$ . The first of these events is  $A = \{T_{\lambda_S}^{\mathcal{R}} \leq \theta\}$ . When its complementary occurs  $\{\theta < T_1\}$  is implied. By Proposition 2.6 we have

$$\mathbb{P}_{\mu_{\mathcal{R}}}(A) \leq \mathbb{P}_{\mu_{\mathcal{R}}}(T_{\lambda_S}^{\mathcal{R}} \leq \theta) \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + 1 - e^{-\phi_{\mathcal{R},\lambda_S}^* \theta} \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + \phi_{\mathcal{R},\lambda_S}^* \theta. \quad (44)$$

Our next event is  $B = \{\ell_{\mathcal{S}}(T_1) > \theta'\}$  with

$$\theta' = \eta\theta$$

being a small fraction of  $\theta$ . When its complementary occurs the excursions of  $X$  in  $\mathcal{R} \setminus \mathcal{S}$  up to  $T_1$ , which cannot be longer than  $\ell_{\mathcal{S}}(T_1)$ , will have a negligible contribution to time averages on time scale  $\theta$ . Since  $\ell_{\mathcal{S}}(T_1)$  is an exponential variable of rate  $\lambda$ , it holds

$$\mathbb{P}_{\mu_{\mathcal{R}}}(B) = \mathbb{P}_{\mu_{\mathcal{R}}}(\ell_{\mathcal{S}}(T_1) > \theta') = e^{-\lambda\theta'}. \quad (45)$$

To control the supremum appearing in (43), we will divide the local time interval  $[0, T_{\lambda_S}^{\mathcal{R}}]$  into intervals of length  $\theta'$ , or smaller as far as the last one is concerned. For  $k_1$  a parameter that we will fix later, our third event is  $C = \{T_{\lambda_S}^{\mathcal{R}} > k_1\theta'\}$ . When its complementary occurs it is sufficient to control the time averages on time scale  $\theta'$  for at most  $k_1$  intervals of this length to estimate time averages on time scale  $\theta$ . By Proposition 2.6 we have

$$\mathbb{P}_{\mu_{\mathcal{R}}}(C) = \mathbb{P}_{\mu_{\mathcal{R}}}(T_{\lambda_S}^{\mathcal{R}} > k_1\theta') \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + e^{-\phi_{\mathcal{R},\lambda_S}^* k_1\theta'}. \quad (46)$$

Our last event, which we will call  $D$ , is that there is a local time interval  $[k\theta', (k+1)\theta']$  with  $k < k_1$  for which the time average associated with the trace  $X^{\mathcal{R}} = X_{\mathcal{R},0_S}$  of  $X$  on  $\mathcal{R}$ ,

$$A_{\theta'}^{\mathcal{R}}(k\theta') = \frac{1}{\theta'} \int_{k\theta'}^{(k+1)\theta'} f(X^{\mathcal{R}}(s)) ds,$$

differs from  $\mu_{\mathcal{R}}(f)$  of more than  $\eta$ . Any time interval  $[t, t + \theta]$  with  $t + \theta < T_1$  can be associated “by remotion of the excursions outside  $\mathcal{R}$ ” with a local time interval  $[\ell_{\mathcal{R}}(t), \ell_{\mathcal{R}}(t + \theta)]$ , which, in turn, can be divided into at most  $\theta/\theta'$  fully covered and two partially covered local time intervals of the form  $[k\theta', (k+1)\theta']$ . Hence, if none of our last three events  $B$ – $D$  occurs, then, for each  $t < T_1 - \theta$  and using  $|f| \leq 1$ , it holds

$$\left| A_\theta(t, f) - \mu_{\mathcal{R}}(f) \right| \leq \frac{\theta' + \frac{\theta}{\theta'} \theta' \eta + 2\theta'}{\theta} = 4\eta.$$

As far as the probability of  $D$  is concerned, since  $X^{\mathcal{R}}$  starts from equilibrium, the expected value of  $A_{\theta'}^{\mathcal{R}}(k\theta')$  is equal to  $\mu_{\mathcal{R}}(f)$  and we can make an exact computation of its variance by writing the decomposition  $f$  in the eigenfunctions basis associated with the generator of  $X^{\mathcal{R}} = X_{\mathcal{R},0_S}$ . By Proposition 2.4 its spectral gap, which is associated with a zero value of  $\lambda$ , is larger than  $\gamma_{\mathcal{R},\lambda_S}$  and we get the upper bound

$$\mathbb{E}_{\mu_{\mathcal{R}}} \left[ \left( A_{\theta'}^{\mathcal{R}}(k\theta') - \mu_{\mathcal{R}}(f) \right)^2 \right] \leq \frac{2\text{Var}_{\mu_{\mathcal{R}}}(f)}{\gamma_{\mathcal{R},\lambda_S} \theta'} \leq \frac{2}{\gamma_{\mathcal{R},\lambda_S} \theta'}.$$

We then have

$$\mathbb{P}_{\mu_{\mathcal{R}}}(D) \leq \frac{2k_1}{\gamma_{\mathcal{R},\lambda_S} \theta' \eta^2}. \quad (47)$$



For (46) and (47) to be useful we need to choose  $k_1$  in such a way that, for  $\eta \ll 1$ ,

$$k_1 \gg \frac{1}{\phi_{\mathcal{R},\lambda_S}^* \theta'} \quad \text{and} \quad k_1 \ll \gamma_{\mathcal{R},\lambda_S} \theta' \eta^2,$$

which is possible if

$$\theta'^2 \gg \frac{1}{\eta^2 \gamma_{\mathcal{R},\lambda_S} \phi_{\mathcal{R},\lambda_S}^*}. \quad (48)$$

We will choose  $\theta'$  and  $\eta$  to ensure this condition and we will set

$$k_1 = \sqrt{\frac{1}{\phi_{\mathcal{R},\lambda_S}^* \theta' \gamma_{\mathcal{R},\lambda_S} \theta' \eta^2}} = \frac{\eta}{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}. \quad (49)$$

For (44) and (45) to be useful we need to choose  $\theta'$  in such a way that, for  $\eta \ll 1$ , (recall that  $\theta = \theta'/\eta$ )

$$\theta' \gg \frac{1}{\lambda} \quad \text{and} \quad \theta' \ll \frac{\eta}{\phi_{\mathcal{R},\lambda_S}^*}.$$

Once combined with (48), this reads

$$\frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \ll \theta' \ll \frac{\eta}{\phi_{\mathcal{R},\lambda_S}^*}. \quad (50)$$

We choose

$$\theta' = \frac{1}{\eta} \left( \frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \right)$$

to satisfy the first inequality in (50); Condition (42) will ensure the last one.

Estimates (44)–(47) give then, recalling (49) and using (42),

$$\begin{aligned} & \mathbb{P}_{\mu_{\mathcal{R}}} \left( T_1 \leq \theta \text{ or } \exists t < T_1 - \theta, \left| A_\theta(t, f) - \mu_{\mathcal{R}}(f) \right| > 4\eta \right) \\ & \leq \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + \frac{\phi_{\mathcal{R},\lambda_S}^*}{\eta^2} \left( \frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \right) + \exp \left\{ -\frac{\lambda}{\eta} \left( \frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \right) \right\} \\ & \quad + \frac{1}{2} \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + \exp \left\{ -\phi_{\mathcal{R},\lambda_S}^* \frac{\eta / \sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta} \left( \frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \right) \right\} + \frac{2\eta / \sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\gamma_{\mathcal{R},\lambda_S} \frac{1}{\eta} \left( \frac{1}{\lambda} \vee \frac{\sqrt{\epsilon_{\mathcal{R},\lambda_S}^*}}{\eta \phi_{\mathcal{R},\lambda_S}^*} \right) \eta^2} \\ & \leq \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + \eta + 2e^{-1/\eta} + 2\eta \leq \sqrt{\frac{\epsilon_{\mathcal{R},\lambda_S}^*}{1 - \epsilon_{\mathcal{R},\lambda_S}^*}} + 4\eta. \end{aligned}$$

□

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