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# Fostering Heuristic Strategies in Mathematics Teacher Education

Peter Stender and Ann Sophie Stuhlmann

University of Hamburg, Germany; [Peter.Stender@Uni-Hamburg.de](mailto:Peter.Stender@Uni-Hamburg.de),

*The “double discontinuity” stated by Felix Klein 1908 is still relevant in the mathematics teacher education at German universities. We are developing a course approach, which is intended to bridge the double discontinuity in a didactic dimension. We offer additional learning opportunities for teacher students, which are characterized by clarifications of mathematical methods that are fundamental to the regular lecture contents. We focus especially on reflections on heuristic strategies, which are the core part of mathematical methods according to George Pólya. Feedback shows that students consider our approach as helpful in the transition from school to university.*

*Keywords: Novel approaches to teaching, teaching and learning of analysis and linear algebra, mathematical methods, heuristic strategies, teacher education, Double discontinuity.*

## INTRODUCTION

In Germany, the first year at university is often associated with great problems for mathematics teacher students. The dropout rates in the study entrance phase are extremely high in mathematics compared to other study programs. (Dieter et al., 2008). Klein’s “double discontinuity” is still an issue in the mathematics teacher education at German universities (Hefendehl-Hebeker, 2013). On the one hand, the double discontinuity can be understood as a discontinuity of content, but Klein also assigns a didactic dimension to it (Allmendiger, 2016). While mathematics teaching at school can be described as intuitive and problem-based, mathematics at the university is characterized by a deductive structure (Klein, 1905). The ideas and strategies behind the findings of mathematics are rarely presented at the university. The mathematics teacher students take part in the same mathematics lectures as regular students who study mathematics as a major subject. Therefore, bridging the double discontinuity is only possible by offering additional learning opportunities for teacher students. A few years ago, we implemented additional seminars for teacher students where we showed connections between school mathematics and university mathematics. Feedback of such seminars showed that students did not consider our approach as helpful for bridging the discontinuity. They said that the regular lectures were challenging and time consuming enough so they did not want to think about additional issues in our seminars. For this reason, we have developed an approach for additional learning opportunities for teacher students, which is characterized by

reflections on intuitions and strategies that are fundamental to the regular lecture contents. Our approach is intended to bridge the didactic discontinuity. On the one hand, we want to support teacher students in the transition from school to university, and on the other hand, knowledge about strategies and intuitions is important in the future teaching activity at school. In the following report, we describe first experiences we have made with our seminars.

## **THEORETICAL FRAMEWORK**

### **Subjects and methods in mathematics**

In the philosophy of science, it is generally accepted that each science has its specific subject and its specific method<sup>i</sup>. As mathematical subjects in university mathematics, we deem the mathematical definitions, theorems and their proofs including mathematical algorithms. The mathematical methods describe how the subjects of mathematics are created or investigated by a mathematician<sup>ii</sup>. Which methods does a mathematician use, when formulating a new definition or a theorem? Furthermore: how does the mathematician come to the ideas when developing a proof? Sometimes the steps of a proof are easy in the sense that a beginner could do them but the decision to do these steps needs a lot of experience. In fact, producing a proof that does not already exist is an act of problem solving according to the definition of Schoenfeld (1985, p.74) or the idea Pólya (1973, p. 3) describes by mentioning “problems to proof”. This point of view corresponds also with the definition of the concept “problem” the German cognitive psychologist Dörner gave in 1976 (p. 10<sup>iii</sup>, own translation):

What a problem is, is easy to define: an individual faces a problem if he or she is in an inner state that is unwanted by any reason, but the individual has not the means in the moment to transform the unwanted situation into the desired one. Three aspects characterize a problem:

1. An unwanted start situation  $s_{\alpha}$ .
2. A desired final situation  $s_{\omega}$ .
3. A *barrier* that prevents the transformation from  $s_{\alpha}$  to  $s_{\omega}$  at the moment.

“We distinguish problems from tasks. Tasks are intellectual challenges for which the methods<sup>iv</sup> are known. [...]. Tasks only need reproductive thinking while in problem solving you have to produce something new.” (1976, p. 10, own translation)

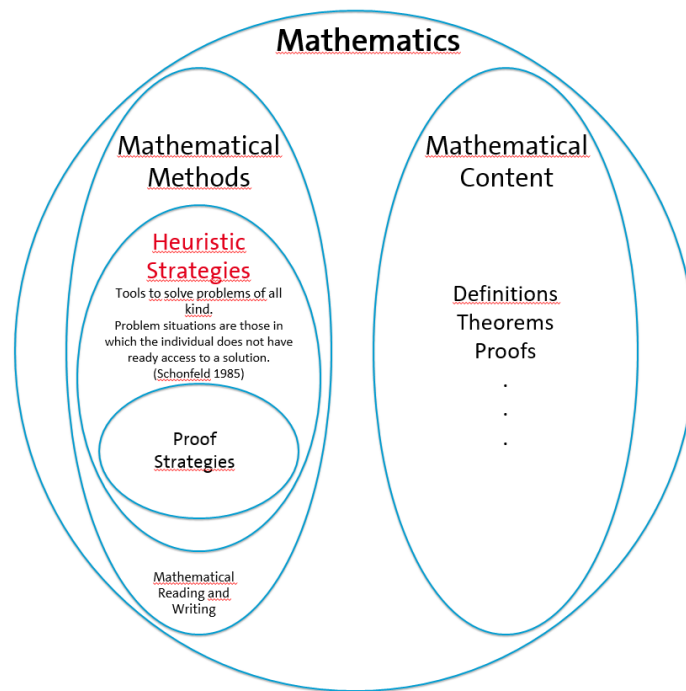
Dörner (1976, S. 10, own translation) emphasised: “Whether a question is a problem or a task depends on the experience of the individual.” If a student just learned induction “Prove the Bernoulli Inequality by induction!” as the first exercise might still be a problem but having done a series of similar proofs, it should be a task.

Based on this definition of a problem Dörner (1976) defines heuristic strategies as „the structure that organises and controls a problem-solving process” (Dörner, 1976, p. 273, own translation), which covers a very wide range of methods and is not bound to mathematics. Following this concept, heuristic strategies cover much of what is meant by the concept of “method” in the philosophy of science once applied to mathematics. Pólya (1973) already underlined this as the subtitle of “How to solve it” is “A new aspect of mathematical methods”.

The following list of heuristic strategies was formulated based on the work of Pólya and others relying on the broad definition by Dörner. These strategies are arranged in groups for a better overview:

- organise your material / understand the problem: change the representation of the situation if useful, trial and error, use simulations with or without computers, discretize situations,
- use the working memory effectively: combine complex items to supersigns, which represent the concept of ‘chunks’, use symmetry, break down your problem into sub-problems,
- think big: do not think inside dispensable borders, generalise the situation,
- use what you know: use analogies from other problems, trace back new problems to familiar ones, combine particular cases to solve the general case, use algorithms where possible,
- functional aspects: analyse special cases or extreme cases, in order to optimise you have to vary the input quantity, discretize the situation,
- organise the work: work backwards and forwards, keep your approach – change your approach – both at the right moment.

Examples and detailed descriptions for each strategy are shown in Stender (2017). These strategies do not cover all the methods in the work of mathematicians but a broad range due to our literature analysis. Many of these strategies are also applicable beyond the field of mathematics<sup>v</sup> and this way of importance as general problem solving strategies. We would call a list of heuristic strategies that are used only in mathematics as they rely on a formal language “proof strategies”. This list contains for example mathematical induction, proof by contradiction, proof by exhaustion, the Invariance Principle (Engel, 1998, p. 1), Cantor's diagonal argument, the box principle (Engel, 1998, p. 59) but also very simple ideas like “use an adequate substitution” or “add zero in an appropriate way.” We do not claim that any of these lists is complete. A third aspect is the use of mathematical language like reading a mathematical text appropriately (for example see Hodds, 2014). We visualised these aspects in figure 1.



**Fig. 1: Mathematical Content and Methods**

**The importance of the methods in mathematics**

The discussion on problem solving since Pólya (1973) shows the importance of heuristic strategies in mathematics and thus (if one follows the argument above) the importance of mathematical methods in the process of learning mathematics. But mathematical methods are not emphasized often in (German) lectures in the mathematics departments. Therefore, we formulate an additional argument here.

We look at a special case of mathematics competence. Imagine a person that knows many mathematical definitions, theorems and even proofs by heart, but has hardly any mathematical methods available. This person would not be able to solve any mathematical problem.

The other extreme would be a person with little declarative knowledge of mathematical facts but competent in applying mathematical methods. He or she just has forgotten most of the mathematical content he or she learned. This person would be able to understand new mathematical content very fast and could fill gaps in the mathematical knowledge quickly if necessary.

This thought experiment shows that a lack of mathematical methods is a much more severe problem while doing mathematics than a lack of knowledge about mathematical content.

A mathematician often uses mathematical methods unconsciously. This is appropriate, as this is faster in the most cases than channelling everything through the restricted working space (see Miller, 1956) in the prefrontal cortex. For a future teacher it is much more important to be aware of his own methods as he or she shall

later explain his actions to children. An explicit reference to his own methods is essential in these explanations.

Furthermore, the methods mentioned are the same in university mathematics and in school mathematics while the content shows little similarities as Klein (1945/1908) pointed out. This way the methods are an important link between school mathematics and university mathematics and give studying abstract mathematics a meaning for students who want to become a teacher.

These considerations lead us to the following goal for our research: we want to implement and evaluate teaching and learning of heuristic strategies in the mathematics teacher education in a close connection to the mathematics content trying to show that learning abstract mathematics means learning mathematical methods which is crucial for teaching mathematics in school.

## **IMPLEMENTED INTERVENTIONS**

The approach we have developed for teacher students includes different additional learning opportunities:

- We offer special tutorials for mathematics teacher students connected to the Linear Algebra and mathematical Analysis lectures.
- Within the scope of Linear Algebra and Analysis lectures students must do weekly exercises. We have implemented a weekly workshop where students get support in creating and formulating proofs.
- All students may choose a preparation lecture before beginning their studies. Part of this lecture is a tutorial. We offer special tutorials for mathematics teacher students too.

These learning opportunities have in common that they do not intervene in the regular lectures and are not obligatory. This way we are free to realise our teaching approach, but this also leads to smaller groups so a standard pre-test-post-test design to evaluate the tutorials is not appropriate. We offered the tutorial over four semesters with 20 participants. This number was quite stable while participants changed: some gave up studying mathematics, some new students joined during the semesters when they heard about the tutorial. Due to the individual timetable of students, some of them couldn't participate over all four semesters. The tutorial was open to students with a major in math too (parallel traditional tutorials exist that only focus on mathematical content) so overall five teacher students participated over the whole time.

As the tutorials fulfil different functions in the learning process, we offer the following aspects for the students:

Among other aspects as answering questions related to the lectures, we point out the appearance of heuristic strategies in the lecture content. Besides students solve small exercises so we can exemplify the application of heuristic strategies. We discuss the application of heuristic strategies in school by means of examples. Furthermore, we give a structured overview over the lecture content several times per semester to help students to network their knowledge. The overview is closely connected to some heuristic strategies as *using analogies* and *generalise the situation*.

An example for the use of heuristic strategies is the use of equivalence classes in linear algebra for example using quotient rings. Quotient rings were presented in the lecture using a formal definition. The lecturer gave  $\mathbb{Z}/m\mathbb{Z}$  as an example but this structure was not familiar to the students. In the tutorial we showed the following ideas to the students:

- Using a special example, we presented the ring  $\mathbb{Z}/4\mathbb{Z}$ .
- For this structure, we gave several different representations – a number line where the numbers of the same equivalence classes were coloured in the same way, the four classes as sets, a representation on a circle and a table that showed the calculations in this structure.
- The importance of the concept of equivalence classes was emphasised: a number of objects is chunked together to one new object according to Miller (1965) so an element of a quotient ring is what we call a supersign according to Kießwetter (1983).
- We pointed out, that similar structures occur in school when dealing with rational numbers, which are classes of pairs of natural numbers (supersigns) and where several representations are necessary while working with fractions. Even if fractions are not implemented in this abstract description in school, students must deal with this mathematical structure.

This way similarities in school mathematics and university mathematics are brought into the focus of the students and showed that students in school dealing with fractions might experience the same difficulties as students in university dealing with quotient rings.

A second example from the tutorial is a short exercise we presented to the students: In the tutorial matrices were used in vector spaces with bases. In the tutorial, we showed how the matrix describing a rotation ( $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ , angle  $\alpha$ ) could be build. Then we asked the students to develop the matrix for a reflection through an axis that has the angel  $\alpha$  with the  $x$ -axis. As this group was already used to heuristic strategies, we gave the following help:

- Change the representation (figure).
- Look for special cases (basis).

- Trace back new problems to familiar ones (use the knowledge from the rotation).
- Break down your problem into sub-problems (work on one element of the basis at a time).
- Use symmetry (to work on  $\sin(2\alpha-180^\circ)$ ).

This example could also be situated in high school, so it breaks down university mathematics to school mathematics and shows in addition how heuristic strategies can be used supporting students work.

In the weekly workshops, students do exercises on their own. Mostly students have to prove small theorems related to the lecture content. They work in groups of three or four and a tutor supports them in developing and formulating proofs. In case students are not able to overcome barriers in proving processes by themselves, the tutor promotes the application of an appropriate heuristic strategy. Following a proving process, students reflect on the strategies they have applied in the process.

## **EVALUATION RESULTS**

As said before the number of students participating in the additional learning opportunities is too small for quantitative designs. We are in the process of design-based research as our interventions are an early approach for the use of heuristic strategies in mathematics education. This means results of efficiency do not make sense in this phase of the research.

We have conducted interviews with five participants of the tutorials and the workshops at the end of each semester. We have analysed these interviews using qualitative content analysis according to Mayring (2010). In the interviews, the students were asked what they remembered from the tutorial and what they think was helpful (not discriminating helpful for studying mathematics or helpful for the later profession). If whole aspects were not mentioned the interviewer stimulated an answer using a key word. The two kinds of answers (with or without stimulation) were separated in the analysis.

Over the four semesters of this process, a slightly increasing awareness of the heuristic strategies in mathematics can be observed in the responses of the students. Over the time, the students gave more examples of the use of heuristic strategies during the interviews. The students gave a positive feedback according to the heuristic strategies as a link between school and university. They said that it delivered a meaning to the fact that the content of the mathematical studies had hardly any connect to school mathematics. One student mentioned that he used heuristic strategies to explain mathematics when he was working in school as a substitute teacher. The students regard the workshops as very helpful for their learning progress. They emphasize the importance of developing proofs by their own



to understand the lecture content. Therefore, they consider it beneficial that the tutor gave only a few strategic hints in case of barriers in the proving process rather than presenting solutions. Especially the heuristic strategy of changing the representation helps students to develop new ideas.

We think that it can be expected that the heuristic strategies integrated in the lecture as metacognition, added to the mathematical content would be far more effective for learning mathematical methods.

## **CONCLUSIONS AND LOOKING AHEAD**

Overall, we can observe that there are effects of the metacognitive input according the heuristic strategies as mathematical methods, but the learning process is very slow. It needs many different examples for each strategy to be able to realise the importance of these strategies and even more to identify them in your own mathematical work or use them explicitly to solve problems. Pólya (1973, p. 208) quotes a fictional traditional professor “A method is a device which you use twice.” Obviously “twice” is far not enough in the learning process – and not enough to make a device so important that it should be taught.

The teaching concept to support students by stressing the mathematical methods particularly by stressing the heuristic strategies seems to have a positive impact but is not realised easily. This corresponds with the difficulty of teaching problem solving and heuristic strategies (Schoenfeld1985, p. 95). The examples we gave to the students during our tutorials and workshops had an impact on realising heuristic strategies in mathematics on the long run. However, we do not know if they were not enough to let the student’s use these strategies in their mathematical work explicitly.

We redesign the tutorial if necessary and expand the number of learning situations where we implement metacognition on the use heuristic strategies. A long-time goal is to include metacognition in the lectures itself but that means to convince mathematics lecturer to change their lectures.

Our next project is developing a tutor training for master students tutoring beginners. In the tutor training, we will use former homework tasks and analyse the solutions according to the use of heuristic strategies. This will be presented to the students in few examples and then done on further tasks by the students working in small groups. In the next step, we will demonstrate how these strategies can be used explaining mathematics and supporting students to overcome barriers during problem solving. The students shall use their own results from the first step to practise this.

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<sup>i</sup> The concept of “method” is used in the philosophy of science in a much broader way than used by e.g. Schoenfeld (1985) when he mentions “This way a method becomes a strategy”. In this paper method is mostly used in this broader meaning of the philosophy of science.

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- <sup>ii</sup> There is a broad discussion whether mathematics is produced by humans or discovered and for both standpoints and for the idea, that mathematics is a mixture of both there are good arguments. For this paper it makes no difference if one changes the verb “producing” by “discovering”.
- <sup>iii</sup> The original is in German language, this is an own translation.
- <sup>iv</sup> Dörner uses the word “method” not in the same way “scientific method” is used in this paper. The word method has more the meaning of an algorithm here.
- <sup>v</sup> Pólya (1973) gave examples from mathematics for the heuristic strategies he mentioned but also examples from outside mathematics for most of the heuristic strategies shown.