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The complexity of knowledge construction in a classroom setting

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We study a class of mathematics education MA students in an introductory course on Chaos and Fractals, as they grapple with the Sierpinski triangle, and in particular with the apparent paradox that its area equals 0, while its perimeter is infinitely long. For this purpose, we network an approach for investigating the construction of knowledge in small groups with one for examining how ideas and ways of reasoning function-as-if-shared in a classroom. Our results show complexities: (i) small group work and whole class discussions mutually influence each other; (ii) ideas may function-as-if-shared in the whole class even if the majority of students have not previously constructed them in their groups; (iii) knowledge constructed in the small groups may or may not later function-as-if-shared in the whole class.

Keywords: Teachers' and students' practices at university level, teaching and learning of analysis and calculus, knowledge construction in classrooms, paradoxes

INTRODUCTION

The research presented here deals with the construction of knowledge in a student centred, inquiry-based classroom, where small group work (SGW) alternates with whole class discussions (WCDs). Construction of knowledge is usually investigated by observing small groups (1 to 4 students) of students. The reason for this is that in larger groups, the density of information for each student is low and does not allow the researcher to interpret their utterances or actions. However, intentional learning more often than not takes place in classrooms with many more than 4 students. We therefore use different approaches for analysing the SGW and the WCDs. The aim of our research is to link the two analyses by following ideas from their emergence in SGW or WCD, via their flow between SGW and WCD settings, until they possibly function-as-if-shared in the class, even though they may not have been constructed by all students. We thus aim at tracing and describing the complexity of knowledge construction across several classroom settings.

THEORETICAL BACKGROUND

The perspective we adopt for analysing the construction of knowledge during SGW is Abstraction in Context (AiC), a theoretical framework for analysing processes of constructing abstract mathematical knowledge (Dreyfus, Hershkowitz, & Schwarz, 2015). AiC methodology begins with an a priori task analysis identifying the new (to the learner) knowledge elements required or useful when solving the task. It then uses a model of three types of epistemic actions – actions pertaining to the knowing of the learners – to analyze their learning processes. The model suggests *constructing* as the central epistemic action of mathematical abstraction. Constructing consists of assembling, interweaving and integrating previous constructs to make a new

construct emerge. It refers to the first time the new construct is expressed or used by the learner. Hence, while the term constructing refers to the process, the term construct refers to the outcome of the action.

The perspective we adopt for analysing WCD episodes is documenting collective activity (DCA). Collective activity of a class refers to the ways of reasoning that function-as-if-shared (FAIS) as students work together to solve problems, explain their thinking, represent their ideas, and so on (Rasmussen & Stephan, 2008). These FAIS ways of reasoning can be used to describe the mathematical activity of a group and may or may not be appropriate descriptions of the characteristics of each individual student in the group. The empirical evidence that a way of reasoning is FAIS is obtained by using Toulmin's (1958) model of argumentation, the core of which consists of Data, Claim, and Warrant. Typically, the data consist of facts or procedures that lead to the claim that is made. To further improve the strength of the argument, speakers often provide more clarification, which serves as a warrant for connecting the data to the claim. Backings provide further support for the core of the argument. For examples, see the data analysis below, e.g., in *WCD 9*. The following three criteria are used to determine when a way of reasoning becomes normative: 1) When the backing and/or warrants for particular claim are initially present but then drop off; 2) when certain parts of an argument shift position within subsequent arguments (e.g., a claim shifts to data); or 3) when a particular idea is repeatedly used as either data or warrant for different claims across multiple arguments.

In earlier studies (Tabach et al., 2014; Hershkowitz et al., 2014), we have shown how DCA and AiC combine to provide an in-depth analysis of knowledge shifts in the classroom and of the knowledge agents that initiate these shifts. In Tabach et al. (2017), we articulate why and how the two approaches are theoretically compatible. In this paper, we analyse a lesson where students dealt with an apparent paradox because of its potential to bring to the fore the complex nature of knowledge constructing processes across social settings in a classroom. Specifically, the paradox is an infinite perimeter that delimits a shape with no area, a phenomenon occurring in fractals. While paradoxes are abundant in the study of infinity, we found only two studies relating to similar ones: Sacristán (2001) examined how the coordination of visual and numerical representations supported a single student's resolution of this apparent paradox. Wijeratne & Zazkis (2015) found that their students were hindered by contextual considerations when attempting to resolve a similar paradox of a solid of revolution with finite volume but infinite surface area. Neither of these studies focused on the construction of knowledge in a classroom community.

METHODOLOGY

The setting for the research was a course on Chaos and Fractals at a US university, which formed part of the mathematics requirement toward a master's degree in mathematics education. Participants were 11 students with an undergraduate degree in mathematics, the teacher, and an instructor/observer who occasionally intervened. The teacher and instructor were both part of the research team. Classes took place

during one semester twice a week for 75-minutes each; typical class periods alternated between SGW and WCD. During SGW, students worked in four stable groups; they were invited to use huddle boards - one table sized white board per group - in order to promote group communication and to facilitate subsequent whole class presentation of their work. The teacher and instructor went from group to group, trying to understand student thinking and attempting to focus students' activity on what they saw as the main issues; they did this mainly by asking questions but did not otherwise intervene in the SGW. The four stable groups will be numbered 1 (Carmen, Jan and Joy); 2 (Kevin, Elise and Mia); 3 (Soo, Kay and Shani); and 4 (Curtis and Sam). All names are pseudonyms. Groups 1 and 2 were video-recorded during SGW; the class was video-recorded during WCDs. In WCDs, groups had the opportunity to use the huddle boards to share their thinking; there were also teacher led discussions and short lectures whose aim it was to facilitate reflection on issues having been discussed by some groups.

On Day 9 (out of 24), class work was based on an activity about the Sierpinski Triangle (ST). As shown in Figure 1, the ST may be produced by a recursive procedure: Draw an equilateral triangle; connect the midpoints of its sides; remove the middle triangle to get three equilateral triangles (of side $\frac{1}{2}$ of the original one); repeat these steps (including the repetition) for each of the three smaller triangles. The ST is obtained by means of the (infinite) recursion.

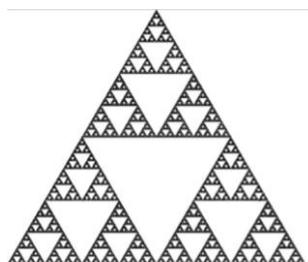


Figure 1. The Sierpinski Triangle (shown after 6 iterations)

The activity was based on a three-part worksheet. In Part A, students were asked to carry out the recursive procedure six times, blackening the removed triangles on the huddle boards. In Part B, they were asked to imagine continuing the recursion forever and to discuss the figure they would obtain, in particular its area and its perimeter. In Part C, they were asked about properties of the resulting figure, especially comparing it to its parts.

The teacher had planned for the students to come up with enough properties in Part C to enable a definition of self-similarity, and as a by-product of learning about self-similarity, to realize that self-similar objects may have finite (or even zero) area and infinite perimeter. However, as will be seen below, the class had its own emergent goals, which led us to investigate student reasoning about area (A) and perimeter (P).

Our task analysis yielded 4 knowledge elements for each of A and P: nature, process, limit, and infinity, denoted A_n , A_p , A_l , A_∞ , and P_n , P_p , P_l , P_∞ . *Nature* refers to the

nature and properties of the region at step k whose area/perimeter is being considered. *Process* refers to the process of removing the triangles and computing the relevant area/perimeter. *Limit* refers to the realization that the process is never-ending and the areas form a sequence converging to zero whereas the perimeters form a diverging sequence. *Infinity* refers to the awareness that the actual figure has zero area and an infinitely long perimeter. The difference between limit and infinity may be construed as the difference between potential and actual infinity. These knowledge elements have been formulated in precise language and operational criteria have been fixed to decide whether a student or group of students has constructed each knowledge element. As an example, P_∞ is defined as follows: Eventually, there actually is a figure whose perimeter is longer than any finite curve. Operationally, we will say that a student has constructed P_∞ if the student explicitly claims that the eventual shape or figure or region has an infinitely long perimeter.

The data used in this paper consisted of transcripts from Day 9, images of the groups' huddle boards, and researcher notes taken in the classroom. SGW was analysed using AiC. We present only constructing actions here. In most cases, these will be attributed to the groups rather than individual students; exceptions will be noted. The WCDs were analysed using DCA; for each argument (numbered as A1, A2, etc.), Claim, Data and Warrant were identified, so that the criteria for ideas that FAIS could be applied.

SMALL GROUP WORK AND WHOLE CLASS DISCUSSIONS

The class on Day 9 started by watching and discussing an excerpt from a video about fractals with real world examples including a cauliflower, mountains, a magnetic pendulum and the coast of Britain; this took about 25 minutes and included WCD 1, SGW 2, and WCD 3. Then students were then asked to start working in groups on the worksheet; they spent about 26 minutes on drawing according to Part A and discussing the meaning of repeating the repetition (SGW 4, WCD 5). They used terms such as “infinite loop” and “zooming in”. The focus of this paper is their work on Part B of the activity during the remaining 24 minutes, split into three episodes of SGW (6, 8, 10) and three WCDs (7, 9, 11). The class did not reach Part C on Day 9.

SGW 6 (area as process) - The teacher invited the students to develop a conjecture about area and perimeter. After a brief discussion about the perimeter showing confusion (Joy: “in one sense it's infinity, because you keep adding a little bit more. But it should approach a number, right?”), Group 1 focused on area, and attempted to compute the area after one repetition. A reminder by the teacher to produce a conjecture led to a seed of the idea of recursion (Carmen: “That's one-fourth of it, so each term maybe three-fourths of it”) thus starting the construction of A_p .

Group 2 quickly came up with a formula (Elise: “So it's three fourths to the n of our A_1 ?”) and spent the remainder of the time discussing what n means and how to denote things (e.g., A_0 for the initial area). We interpret this as having constructed A_p .

WCD 7 (computations) - The teacher asked whether the groups had come up with a conjecture, and the students reacted by presenting computational results. Kevin presented their result as the sequence $(3/4)^n$ and Joy added that they had just started in this direction after computing the area of the first triangle. Student arguments included only claims and hence were not analysed per the DCA approach.

SGW 8 (perimeter as process) - Group 2 focused on computing the perimeter (Kevin: “we have an additional a , we have three halves more a ”, and later Mia: “So, it's like, it's going by a scale of three over two, to the n ”). The group also made attempts at seeing what happens in the long run (Mia: “the perimeter is just keeps getting bigger, and bigger and bigger”; Elise: “Or is there, like, a limit? That it stops?”). While this points in the direction of P_1 , our interpretation is that they have not constructed P_1 yet: In spite of them having identified what an expert might see as a diverging geometric sequence, they question whether it converges or not. We also note that some students may be thinking additively rather than multiplicatively.

Group 1 quickly constructed A_p (Carmen: “And then three-fourths of our three-fourths”; Jan: “It's alright, we got enough... to do the whole”) and somewhat hesitatingly, A_1 (Carmen: “Maybe zero?”; Joy: “No no no, because this is like three-fourths time three-fourths is nine-sixteenths, and after that would be... what? Twenty-seven over sixty-four?”; Carmen: “Is it approach... zero? I think it does”; Joy: “Okay, so you are right, it approaches zero”). There is evidence that they also constructed A_∞ (e.g., Joy: “If you keep filling it in, there's not going to be any white area”). We note that there was no discussion about A_n . However, they then held a long discussion about P_n (e.g., Joy: “So what counts as the perimeter?”; Carmen: “Is it cumulative perimeter?”). Our analysis resulted in the decision that while constructing P_n was under way, it had not yet been achieved. Next, they mentioned aspects of P_p (Joy: “So let's say the perimeter of this is three, we would add in... half of each. So, like, three... times the half”) without completing a constructing process. They were reminded by the instructor of the area tending to zero, which brought tension with respect to the perimeter (e.g., Carmen: “if we keep zooming in, there's no area, so there can be no fence [perimeter]”).

WCD 9 (the controversy) – This discussion in Group 1 prompted the instructor to ask for the teacher's permission to ask Carmen and Joy to present their controversy to the class. Carmen's argument (A2) used as data “there's no area” and claimed “there'd be no perimeter” with warrant “there's nothing to... nothing to put a fence around it”. Joy, on the other hand, argued (A3) for the opposite, using as data “as you zoom in there's more and more to fence”, supported by the warrant that one keeps putting in more fencing material. This brought about a suggestion that when one removed a triangle (i.e. colours it black), the perimeter of this black triangle is added to the existing perimeter. In argument (A4) the claim is that “the perimeter of the, the white is also the same as the perimeter of the, perimeter of the black part” (Curtis), and this is based on the data “When you shade it in, you're adding the perimeter of the black” (Kevin) with the warrant that “the fence is guarding both properties” (Carmen). When

encouraged by the instructor to explain Carmen and Kevin’s thinking, Soo built argument A5: Claim: “So you keep adding the numbers, right?”; Data: “So you have more areas”; Warrant: “You keep zooming in, you’re going to get more triangles forming”. Next, Mia argued (A6): Claim: “I see the perimeter increasing and then this, the unshaded area is what’s left over, and that’s constantly decreasing and going to zero”; Data: “You’re going to have all these shaded triangles, with perimeters”; Warrant, upon Carmen’s question “Is this a cumulative perimeter or a perimeter at a point in time?”: “I see it the first way” (Mia). Several more arguments (A7, A8, A9) in this WCD focused on the area decreasing and tending to zero in an unending process of creation. The analysis of arguments in this WCD resulted in two ideas that FAIS:

FAIS A: *Perimeter of white is also perimeter of black*; this was a claim in argument A4 but a justification in argument A6; the justification was that the perimeters of the shaded (black) triangles cumulatively constitute the perimeter of the remaining, unshaded, white area. Hence this idea satisfies Criterion 2.

FAIS B: *The perimeter is cumulative*; this was a claim in A5 and a warrant in A6 (we will see it serving as justification again in A13), and hence also satisfies Criterion 2.

SGW 10 (*connections*) - Group 1 had a discussion of all four aspects of perimeter, completing the construction of, at least, P_n and P_p . They built their thinking on the fact that they used more and more ink at each stage to draw the additionally generated bits of perimeter, and concluded (Joy): “I thought it went infinitely, because if you zoom in, there's more fencing to put in. And if you zoom in there's more fence to put in”. We have no evidence that they constructed P_1 and P_∞ . In fact, this is unlikely since they only completed constructing P_n and P_p toward the very end of the SGW. Moreover, Carmen, while admitting that the perimeter tends to infinity, insisted that intuitively, no area implies no perimeter.

Group 2 attempted to combine what they knew about the unending processes of area and perimeter. For example, Mia: “There's nothing for the area, but you're still... you're counting the perimeter of what you're taking out” and Kevin: “...as soon as you say - as n approached infinity, that means you're going to computation. So, I think what we want is something general, like - the area is getting smaller, but the perimeter is getting larger, and just leave it at that general statement”. Our interpretation is that they may have started constructing A_∞ and P_∞ but are still far from completing these constructions.

The constructing processes resulting from the AiC analysis are summarised in Table 1. The table lists only constructs that we have evidence for; in other words, the fact that, for example, A_n does not appear does not mean they have not constructed A_n – it only means that A_n has not been discussed during SGW in a manner that lets us as researchers conclude that A_n has been constructed.

Group	SGW6	SGW8	SGW10
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1	(A _p)	A _p A _l A _∞ (P _n) (P _p) (P _∞)	P _n P _p
2	A _p	(P _p) (P _l)	(A _∞) (P _∞)

Table 1. The constructs; parentheses mean (under construction)

WCD 11 (linking area and perimeter processes) - The instructor called on group after group to present their thinking, however tentative. Group 1 used an ink metaphore, the ink being used to draw the additional perimeter bits. In this way, they explained how cutting out further triangles reduces the area while using more ink – thus increasing the perimeter. Carmen, however, added, that for her “that’s not resonating”. The researchers identified two arguments while this group was reporting (A10, A11) and one (A12) while another group was reporting. During their report, Group 2 connected the perimeter process to the area process like Group 1. Elise built the following argument (A13): Claim: “The area is getting smaller and the perimeter is getting bigger”. Data: “You’re adding smaller and smaller pieces”. Warrant: “...but you’re adding those pieces to what you already have”. Backing: “The perimeter is, like, all of this, combined with all of this, combined with all of this, combined...”. We note that the argument focuses almost completely on perimeter, although the claim equally relates to area. Finally, the report of Group 4 included arguments A14 and A15. We only describe A15, produced by Sam: “So the area would be... go to zero... There would be limited amount of areas, so we're going to have a limited number of... perimeters. So, we don't have infinite number of perimeters”. Sam’s claim of a “not infinite perimeter” is based on the data that the area goes to zero and hence there is a limited amount of area, with the “limited number of perimeters” serving as warrant. Based on these arguments, we identified two more ideas that FAIS:

FAIS C: *Unending process of creation*; this meets Criterion 3: Continued use of an idea (e.g., *keep adding*) across multiple arguments to describe the process that is being analysed. This idea is related to potential infinity.

FAIS D: *Area going to zero*; this was repeatedly a claim, including in arguments A6, A7, A8, A9, A13 and became data in argument A15, hence satisfying Criterion 2.

The relationship between SGWs and WCDs

As a preliminary, we note the richness and diversity of students’ ways of reasoning about area and perimeter, which in a less student-centred classroom might have been quickly undermined with an infinite geometric sequence that is decreasing ($r < 1$) and hence tending to zero for area and an infinite geometric sequence that is increasing ($r > 1$) and hence tending to infinity for perimeter. We note that the term geometric was once mentioned briefly with respect to area by Sam during WCD 11 (“it’s geometric, so it’s going to converge. So, the area would be... go to zero”) but this was rather toward the end of class and was not picked up by any of the other students. Generally, students seem to have been satisfied by arguments of the type “the sequence is infinite and decreases, hence it tends to 0” (as in the discussion of Group

1 in SGW 8) or “the sequence is infinite and increases” (e.g., Mia and Elise of Group 2 in SGW 8).

The diversity manifests itself, among others, in the metaphors the students used (fence, ink), in a tendency to use, at least initially, numerical considerations for area and perimeter, in the attempt to link the area process with the perimeter process, and the related discussion about the nature of the perimeter as separating the region that belongs to the ST from the one that doesn't. Little of this was initiated or suggested by the teacher or the worksheet.

This last issue appears as P_n in the AiC analysis of the SGWs and as FAIS A in the DCA analysis of the WCDs. Similarly, there are relationships between the other FAIS ideas and knowledge elements. Table 2 shows these relationships.

FAIS	A	B	C	D
Constructs	P_n	P_p	A_1, P_1	A_1, A_∞

Table 2. Relationship between FAIS and constructs

While FAIS idea A (*the perimeter of white is also perimeter of black*) is related to construct P_n , the relationship between the constructing process of P_n and the arguments establishing A as FAIS is complex. We don't have evidence of P_n having been constructed in either of the two observed groups before the relevant WCD 9; and the first argument establishing idea A as a claim (A4) was initiated by Curtis but immediately supported by Kevin and Carmen. Moreover, the second argument, in which idea A became a justification, A6, was presented by Mia. This may indicate that the beginning P_n construction we identified in Group 1 was substantial, and maybe even that the discussion in Group 2, which on the face of it focused on P_p caused Kevin and Mia to think about P_n . Finally, we ask ourselves to what extent the constructing process of P_n continued during WCD 9 for Kevin, Carmen and Mia.

FAIS B (*the perimeter is cumulative*) – is similar to FAIS C (and intimately related to it from the point of view of the mathematical content). While we hesitated to claim that P_p has been constructed by Group 2, it is Mia from that group who produced A6 and Elise from that same group who produced A13, the two arguments where the element switched position to becoming a justification and thus allowed us, according to DCA to categorize this idea as FAIS.

FAIS C (*unending process of creation*) exhibits a case in which the two analyses connect rather smoothly. Group 1 constructed A_p and P_p , and Group 2 may be assumed to have implicitly constructed A_p and to be progressing in the constructing process of P_p . The frequent use of this knowledge element in many WCD arguments may indicate a similar situation in the other two groups. Soo from the Group 3 produced Argument A5; and Sam from the Group 4 produced Argument A9.

Finally, the relationship between FAIS D (*area going to zero*) and A_1 seems obvious and needs little comment. When this idea functions as if shared in the classroom, it is

possible that some of the students relate to construct A_∞ (as shown above for Carmen and Joy) and others think in terms of A_1 or even in terms of A_p only. While to the expert, thinking in terms of A_1 may be satisfactory at best, and thinking in terms of A_p may be insufficient, such differences are tended to be glossed over in this classroom with respect to the rather basic construct of area, and we may speculate that similar situations pertain to more complex constructs in this and other classrooms.

It becomes obvious that there are many ways, in which SGW and WCD can interact. The relationship is by no means unidirectional from constructing an idea in SGW to this idea FAIS in WCDs. Rather, constructing processes may well be continued or even initiated in WCDs. On the other hand, ideas may FAIS without having been constructed by all or even by a majority of the students in class. For example, we have no evidence for P_1 having been constructed in either of the two observed groups, though Group 2 had started this constructing process; but C (*unending process of creation*) is anyway FAIS in relation to both area and perimeter. Of course, it could have been constructed in the groups we have no data on. This raises the question whether an idea can FAIS if it has not been constructed at least in some group. We can only say that we have no example for this having happened in these data.

On the other hand, notions can be constructed by some students, or in some groups, without ever functioning as if shared. In fact, some of the constructs may not have a chance to come up in any WCD. We have no unequivocal evidence for this happening but we do know that in the first few minutes of the next lesson (Day 10), which took place two days later, Kevin referred to the perimeter as an increasing and hence diverging geometric sequence. We could also point to the fact that A_p has been constructed by both analysed groups but does not appear in the lower row of Table 2. However, this argument is weak since A_p appears indirectly as a component of A_1 .

CONCLUSION

The complexity of knowledge flow in the classroom, even based on this one class session, is far greater than one might imagine. Inquiry-based instruction features students' deep engagement in mathematics and peer to peer interaction. As such instruction increases at the university level, the field is in need of theoretically grounded approaches for analysing individual and collective mathematical progress. This paper makes a contribution in that direction. A strength of AiC is that the approach allows researchers to gain insight into the ideas that individuals or small groups of individuals construct, as long as the number of students remains small. DCA provides a complementary approach that provides researchers insight into the ideas and ways of reasoning that characterize the collective progress of the classroom community. In Tabach et al. (2017), we showed how the two approaches combine theoretically, and the present paper adds to this a detailed analysis of how an AiC analysis interacts with a DCA analysis to express the complexity of knowledge construction in the classroom. An open question, and one that we are currently pursuing with this data, is how to coordinate the small group and classroom level findings with individual interviews conducted shortly after such rich class sessions.

Finally, the analysis presented here also contributes to what we know about how students reason about area and perimeter in a paradoxical situation. In contrast to the findings of Wijeratne and Zazkis (2015), the students in our classroom found ways to profitably use contextual considerations, metaphors, numerical computations, and figural reasoning to support their endeavour to understand the apparent paradox.

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REFERENCES

- Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2015). The nested epistemic actions model for abstraction in context - Theory as methodological tool and methodological tool as theory. In A. Bikner-Ahsbals, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 185-217). Dordrecht: Springer, Advances in Mathematics Education series.
- Hershkowitz, R., Tabach, M., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts in a probability classroom – A case study coordinating two methodologies. *ZDM - The International Journal on Mathematics Education*, 46, 363-387.
- Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of innovative design research in science, technology, engineering, mathematics (STEM) education* (pp. 195 - 215). New York, NY: Routledge.
- Sacristán, A. I. (2001). Students' shifting conceptions of the infinite through computer explorations of fractals and other visual models. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th International Conference for the Psychology of Mathematics Education*, Vol. 4 (pp. 129–136). Utrecht, The Netherlands: PME.
- Tabach, M., Hershkowitz, R., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts in the classroom – A case study. *Journal of Mathematical Behavior*, 33, 192-208.
- Tabach, M., Rasmussen, C., Dreyfus, T., & Hershkowitz, R. (2017). Abstraction in context and documenting collective activity. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Conference of the European Society for Research in Mathematics Education (CERME10)* (pp. 2692-2699). Dublin, Ireland: Dublin City University and ERME.
- Toulmin, S. (1958). *The uses of argument*. Cambridge, UK: Cambridge University Press.
- Wijeratne, C., & Zazkis, R. (2015). On painter's paradox: Contextual and mathematical approaches to infinity. *International Journal of Research in Undergraduate Mathematics Education*, 1, 163-186.