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# Theoretical and empirical description of phases in the proving processes of undergraduates

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*In the presented study we adopt a process-oriented perspective on proving in order to gain further insights into relevant actions and typical obstacles in undergraduates' approaches to proving. The primary aim is to theoretically and empirically describe different phases, understood as bunches of intentionally closely related actions of proving. Therefore, we suggest a theoretical model of the proving process and confirm empirically that it can be used as an analytical tool for proving approaches. Based on this model, several proving processes have been analysed. In this paper we present first findings regarding the contribution of each phase to proof construction as well as the general structure of the proving process.*

*Keywords: proof construction, proving process, phases, proving cycle.*

## THEORETICAL BACKGROUND

It is well known that undergraduates commonly have to deal with great difficulties in constructing proofs (e.g. Weber, 2001; Moore, 1999). In order to gain additional insights into the main obstacles in proving at university level, we focus on the processes of proof construction. Preparing further projects, the presented study serves as a pilot study, which initially aims at investigating the general structure of proving processes on a macroscopic level. Therefore, we first give an overview about the main characteristic of the proving process in contrast to its product. Based on this, we deduce a theoretical model from literature in order to use this model as an analytical tool for undergraduates' approaches to proving.

### The process of proving

Talking about proof construction, there is a consensus that a proof, especially at university level, is a line of reasoning, which is strictly deductive and exclusively based on theorems and axioms. In contrast to its result, the process of proving contains not only deductive reasoning, but also includes inductive and explorative processes (e.g. Hersh, 1993). Thus, developing a key idea and transferring it into a formal proof is a complex and highly demanding process, which occurs in the tense atmosphere of conviction and explanation as well as intuition and formality (Hemmi, 2008). In the present paper we will discuss on a theoretical and empirical basis, how these different perspectives and processes interact in constructing proofs.

Mejia-Ramos and Inglis (2009) distinguish between three distinct types of construction activities, which emerge from different external conditions. Although all of these activities aim at constructing an appropriate proof, each is guided by a specific goal: *Exploration of a problem* consists of working on an open-ended

question in order to discover and at least prove a new statement. However, *estimation of truth* starts with an already prepared conjecture and *justification* begins with a statement estimated to be true. In these cases, proof construction is rather aimed at determining or verifying the truth-value of a statement instead of inferring it. Meyer (2010) differentiates activities associated with mathematical proof construction in a similar way based on the theory of abduction, induction and deduction. According to his approach, each kind of inferring is related to a different type of construction activity. Thus, external conditions and, especially, task designs have a relevant influence on proving processes by stimulating different types of inferring and construction activities. Combining both frameworks, we assume that the activities described above are independent for the reason that they consist of different types of processes and pursue different goals. In particular, justification is not a part of exploration, which follows producing a conjecture, but requires specific cognitive processes.

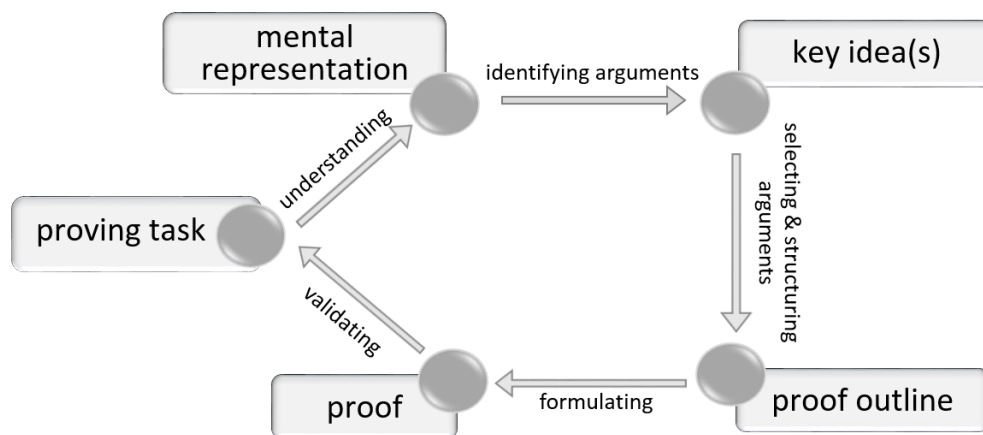
In order to describe those cognitive processes, various models have been developed, which summarise the relevant actions and demands associated with proof construction. These models mainly focus on processes related to the exploration of a problem. As proving tasks at university level often consist of a statement estimated to be true, these models only seem to be partially suitable for analysing undergraduates' proving processes. Hence, we suggest a model of proving, which is mainly following existing models, but focusses on justification.

### **Models of proving processes**

To analyse the cognitive processes of mathematical proof construction, there is a need for abstraction. The complex structure of the process has to be reduced to the relevant actions and, therefore, transferred onto a macroscopic level. Doing so, most process-oriented models use the unit of episodes or phases in order to subsume closely related actions in service of the same goal under a generic activity. Thus, phases are “macroscopic chunks of consistent behaviour” (Schoenfeld, 1985, p. 292), which summarise the relevant processes associated with proof construction.

Existing models of the proving process mainly differ in their amount of suggested phases. The most cited models have been presented by Stein (1984) and Boero (1999). Both models focus on the activity of problem exploration, which means that the proving process is based on an open-ended problem area. In the first phase of proving this problem area is explored regarding relevant conditions and regularities in order to produce a conjecture. When the conjecture has been formulated as a statement, the proving construction contains three phases: Identifying arguments for the correctness of the statement, linking them into a deductive chain and formulating an appropriate proof. As Stein's (1984) model focusses on proving approaches of students at secondary level and Boero (1999) describes the proof construction of mathematicians, quality and formalisation of the proofs intended in the models differ according to the mathematical standards shared in the particular context. Boero

(1999) even adds a further phase, in which mathematicians approach a formal proof. Apart from this last step, which primarily seems to be relevant for experts, the phases of proof construction described by Stein (1984) and Boero (1999) have been taken as a basis for several research projects in mathematics education (e.g. Reiss & Renkl, 2002). According to this, we suppose the phases described in both models to be relevant for undergraduates' proving processes as well. However, proving tasks in the initial phase of studies rather initiate justification instead of exploration. Due to this, we suggest the following variation of Stein's and Boero's model:



**Figure 1: Proving cycle as a model for proof construction**

In contrast to the existing models, the proving cycle starts with a proving task consisting of a statement estimated to be true. Although this statement can be similar to the produced conjecture in other models, it plays another role in the proving process. In case of problem exploration the statement is connected to insights from conjecturing and exploration, but in case of justification no further information about the statement is given. Hence, it is necessary to analyse the given statement, to clarify the terms and conditions and to access previous conceptual or strategic knowledge. Leaning on the approach of a situation model (Kintsch & Greeno, 1985; Reusser, 1990), we suppose undergraduates to develop a mental representation, which refers to their individual understanding of the given statement and guides further processes. Based on this mental representation we suggest similar phases as presented in the models of Stein (1984) and Boero (1999). The second phase aims at exploring the statement area and discovering key ideas. By (re-)constructing relations and objects or applying theorems reasons for the validity of the statement can be identified. If some key ideas have been found, it is necessary to select promising ideas, work out their details and structure single arguments in an appropriate deductive order. This phase results in a proof outline, which contains the main aspects of the proof, but can be fragmentary or difficult to understand for someone else. In order to prepare a final proof, which meets the mathematical standards of one's community, one has to fill gaps and revise the linguistic and formal arrangement of the proof. As a last phase the model includes validating activities, which can be compared to a certain extent to Pólya's stage of looking back

(Pólya, 1945). In this phase the final proof is reviewed regarding content, structure and linguistics. Beyond that, one can consider further (shorter or more elegant) proofs in this phase or reflect on the proving process, as, for example, thinking about key ideas, difficulties and their solutions.

The single phases in the model are arranged as a cycle. In accordance with Stein's (1984) and Boero's (1999) considerations, we assume that proof construction does not necessarily proceed in a linear way. Instead, the proving process is shaped by interruptions and revisions that cause transitions between different phases.

## **RESEARCH QUESTIONS**

The primary aim of the presented study is to develop an analytic tool for proving processes, which makes it possible to describe and compare students' approaches to proof construction on an individual, macroscopic level. Therefore, a proving cycle has been derived from literature consisting of five phases estimated to occur in undergraduates' proving approaches. The unit of a phase seems to be well-suited to describe proving activities in a clear and abstract way without neglecting specific details and differences in the proving processes. Accordingly, the proving cycle may be used to analyse proving approaches by describing the frequency, the duration and the order of different phases occurring in the process. The presented study aims to provide evidence of the utility of the proving cycle as an analytical tool and – if so – to gain more detailed information about the process of proof construction at the level of phases. In detail, the aims of the study described above lead to the following research question:

1. Is the proving cycle an appropriate tool for analysing proving processes? That means, is it possible to reconstruct the different phases stated in the proving cycle empirically? Do further activities exist, which do not fit the theoretical description?
2. How can the process of proof construction be described in general? That means, which phases are taking a relevant share in the process? In which order do undergraduates go through the different phases? Can the cyclic nature of the proving process be confirmed?

## **METHOD**

In accordance with the open-ended character of the research questions, an explorative laboratory study has been designed. In this study proving processes of undergraduates are initiated, observed and finally analysed. The concept of the study and its conditions are described in more detail below.

### **Sampling and data collection**

The study focuses on undergraduates and pre-service mathematics teachers (high school) attending their first year of studies. Performing an informal unstructured

interview, participants are encouraged to work on proving tasks in the field of real analysis. Doing this, they are told to prepare a joint solution that satisfies the requirements of a proof in the initial phase of studies and that would be accepted by a tutor or a lecturer. To encourage the participants to talk about their ideas and approaches, the working processes are organised in pairs. In order to secure the same conditions for all participants and, therefore, the comparability of the observed processes, the interviewer offers no support. Instead, a commonly used textbook of real analysis is provided. For investigation such proving tasks have been chosen that require a one- or two-step proof and can be solved by applying a prominent theorem of real analysis like the intermediate or mean value theorem. In order to explore the proving cycle's applicability, the proving tasks contain universal as well as existential quantifications and can be proved directly or by contradiction.

While working on the proving tasks, the participants are videotaped. That means, their proving approaches are recorded in sound and vision. Additionally, we collected the final solutions as well as the notes of each pair. During the pilot phase of the study seven pairs of undergraduates took part in the interviews and worked each on one or two proving tasks. Processing time for a single task varies from 30 to 75 minutes. However, not all proving approaches were successful. While some participants gave a proof, which was not completely correct, but contained useful approaches, other students could not achieve any solution.

### **Preparation and analysis of data**

To prepare the observed proving approaches for analysis, we transcribed each videotape entirely. For precise investigations the transcripts of the dialog are expanded by further information like non-verbal activities and notes. Combining the transcription of natural conversation and written approaches, the protocols of the proving processes are finally encoded according to Mayring's (2014) structuring content analysis. In accordance with the research questions a deductive category formation with nominal categories has been chosen, which is closely related to the proving cycle described above. The aim of the coding is to identify changes in the participants' behaviour in order to describe the structure of their proving process as a sequence of transitions between different phases. Leaning on Schoenfeld's (1985) method of protocol analysis the coding consists of two steps: First, the proving process is parsed, that is, making decisions regarding dividing lines of phases. Once a proving process is partitioned into phases, each phase is characterised as one of the theoretical stated phases in the proving cycle. The coding results in a macroscopic description of the students' proving processes that combines closely related actions into phases and provides a summary of relevant activities.

## **RESULTS**

In this section the results of analysing data from nine proving processes is presented. For the moment, a case study is introduced to demonstrate the methodological

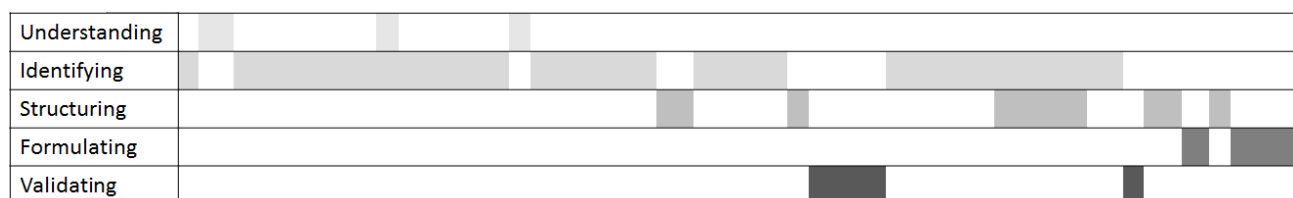
approach as well as possible results on an individual level. Based on this, findings regarding the proving cycle being an appropriate analytic tool are discussed in general. Beyond that, we present first assumptions concerning the frequency, the duration and the order of different phases in undergraduates' approaches to proving.

### Case study of Michael and Leon

Michael and Leon are working on the following task, which can be solved by applying the intermediate value theorem:

Let  $f : [0, 1] \rightarrow [0, 1]$  be a continuous function. Show that  $f$  has a fixed point, that is, there exists a  $x \in [0, 1]$  with  $f(x) = x$ .

Partitioning their working process and characterising the identified phases, the proving process of Michael and Leon can be described by the sequence of phases presented in Figure 2.

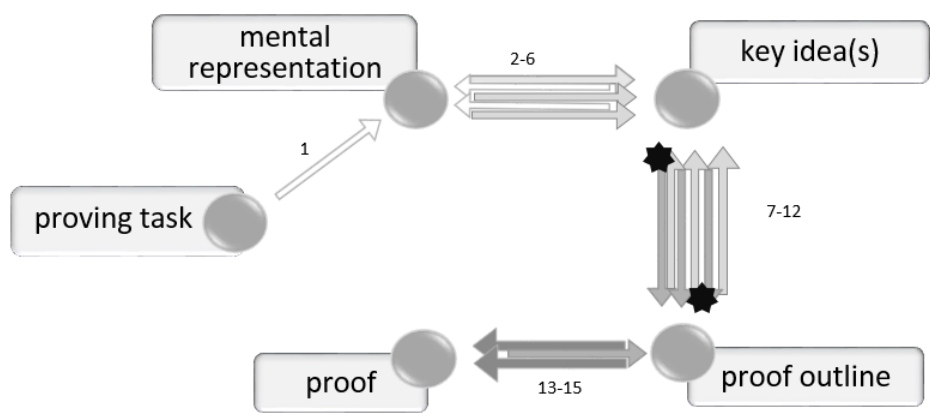


**Figure 2: Parsing of the proving process of Michael and Leon**

After reading the task, Michael and Leon start their working process with a brief brainstorming concerning useful theorems. Because the given function does not meet the required preconditions, they experience difficulties applying these theorems, which makes them reading the given statement a second time more carefully. Doing so, they express confusion about the term *fixed point* and, hence, try to clarify its meaning by looking it up and making a drawing. Based on an enhanced understanding, Michael and Leon review their ideas and add new ones by leafing through the book. Sooner or later, each of these ideas proves to be inadequate. While Michael is reading in the book searching for applicable theorems, Leon makes further drawings in order to visualise the preconditions and assertions given in the task. Thereby, he remembers a strategic approach used in a proof before. Leaning on this approach, he tries to construct an auxiliary function, which turns the original problem of the existence of a fixed point into a similar one of a zero. Michael and Leon work together on this approach until they recognise they have different concepts regarding the connection between the domain and the range of the given function. They resolve this disagreement by referring to the drawings and regarding some exemplary points of the function. For more than half an hour Michael and Leon have been working out the details of the proof now. At times they stop working on new ideas and summarise previous insights in order to structure their arguments and to identify gaps or inconsistencies in the proof outline. One of this structuring activities sends Michael having doubts about the correctness of their formulation.

Leon is able to convince Michael of their approach, but has to admit that some modifications are necessary like regarding an additional case. Later, another phase of validating occurs as a result of identifying arguments. This time they recognise that their auxiliary function is the identity function and that they can prepare their arguments in a more elegant way. Michael and Leon are now able to construct a satisfying proof by formulating and structuring alternately.

During the whole process of proof construction Michael and Leon are progressing continuously until they have achieved a satisfying proof.



They only return to a previous phase, when their proving process comes to a temporary halt and they feel there is a need for reviewing some steps done before in order to specify or improve their mental representation, their key idea or the proof outline. Michael and Leon go through a phase of validating twice, because one of the student voices doubts about the previous considerations being correct. They do not review their final proof to check or improve details.

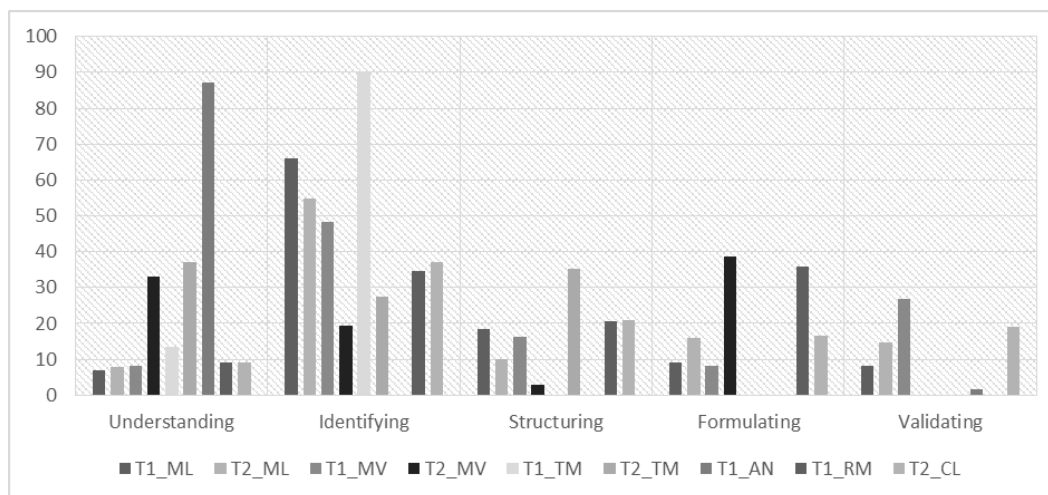
**Figure 3: Proving process of Michael and Leon, validating activities are represented by stars**

As displayed in figure 2, at times it has been inevitable to encode the same period of time with two different categories. This kind of double-coding is necessary, if both students work on their own in service of different goals or if some related activities from different phases are done contemporaneously. The analysis of the proving process presented in this section is quite typical for the sample. In the following section similarities and differences between the individual proving processes are discussed in general.

**General observations**

The analysis of data from nine interviews shows that the categories are applicable to the proving protocols in a satisfactorily objective and reliable way. Interrater reliability in coding is quite high ( $\kappa = .73-.93$ ). Therefore, the proving cycle seems to be an appropriate tool for describing and analysing undergraduates' approaches to proof construction on a general macroscopic level. Regardless of the proving tasks and a direct or non-direct approach to proving, each of the suggested phases could be empirically confirmed in at least five of nine cases (Figure 4). In those cases, where the phases *identifying arguments* and *selecting and structuring arguments* are missing, participants have not been able to establish any serious approach due to a





**Figure 4: Percentages of the different phases occurring in proving processes**

poor mental representation or a lack of key ideas. However, quitting the working process without any solution was only observed twice. More frequently it happens that even in successful proving processes the phases of formulating and validating are omitted. Instead of formulating a precise and clear proof, some participants content themselves with quick but fragmentary solutions. Here, formulating the proof according to mathematical standards does not seem to be as important as gaining insight into the key ideas of a proof and verifying the statement for oneself. This kind of view might be a general attitude towards mathematical formalism, but could also be caused by the laboratory setting. Analysing the phase of validating, we differentiate two kinds of action: Validating activities in service of reviewing and enhancing a final proof could only be observed in one single case. In contrast, activities like checking details and analysing suggestions occur continuously and are connected to several other activities in the proving process. Although validating activities are only listed in five of nine cases, it is reasonable that more validating takes place without being encoded on a macroscopic level because of no sufficient impact on the proving process. Comparing the percentages of different phases, it emerges that in nearly all cases the phase of identifying arguments as a highly creative and demanding action takes a large share in the proving process and, therefore, seems to be one of the most significant parts of mathematical proof construction. In contrast, the percentage of understanding and exploring the given assertion varies greatly. While some participants spend a lot of time on drawings or clarifications, others start with a brief glimpse on the task and continue with applying theorems immediately.

In regard to the composition of proving processes, it has been assumed that the underlying structure is a cycle. Due to that, there must be a high amount of transitions and revisions in the proving approaches. In fact, only one third of the encoded transitions is linear in that way that students move forward to the next phase in the proving cycle. Though, the proving processes does not proceed as cyclic as suggested. As illustrated in the case study of Michael and Leon a large shape of non-

linear transitions is made by transitions between consecutive phases, that is, moving backwards to the phase before. Wider leaps from one phase to another are quite rare. Most of these transitions between non-consecutive phases contain an interaction with validating activities as students switch from any phase to validating and backwards. This finding supports the assumption that validating is an activity, which is closely related to other phases of the proving process. Only in a very few cases the key idea is rejected at some point in the proving process and the participants restart identifying arguments and exploring the given statement cyclically. Accordingly, we suggest proof construction to be less cyclic than a linear process, which is interrupted by several mini-cycles between consecutive phases.

## **DISCUSSION**

In this paper a model for the process of mathematical justification at university level has been derived from literature to develop a macroscopic analytical tool, which describes a proving process as a sequence of phases. In a sample of nine processes the proving cycle has proved suitable for describing undergraduates' approaches to mathematical proof construction. Each of the suggested phases could be empirically confirmed. Analysing similarities and differences between individual proving processes, there have been two key findings: Undergraduates' proof construction mainly proceeds on a straight line basis, which is interrupted at times by transitions into immediately preceding phases in order to specify or improve considerations done before. An exception to this are validating activities. Questioning, reviewing and reflecting seem to be processes, which are rarely performed at the end of proof construction, but are closely connected to other phases of the proving cycle. Hence, initial results indicate that validating is not confined to the final proof, but relates to the mental representation as well as the key ideas and the proof outline.

The presented study prepares a larger project by providing an analytical framework for students' approaches on proof construction. Based on the proving cycle, we intend to analyse a larger sample including participants, who differ in progress and performance. Comparing first-year and advanced students as well as successful and non-successful proving processes might provide new ideas for fostering programs in the introductory phase of studies. On the one hand investigations will remain on the macroscopic level of phases in order to identify effective and less effective patterns of proving processes. Therefore, the occurrence and the duration of a phase in a proving process are compared with the quality of the corresponding proof. On the other hand, further investigations are intended, which gain more inductive insights on a microscopic level. Therefore, we aim at describing typical actions of each relevant phase in detail. By doing so, frequent difficulties and potential obstacles of an individual phase can be identified as well as effective and non-effective proving strategies to overcome these obstacles. This information can help arranging effective, process-oriented fostering programs.

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