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A TDS analytical framework to study students' mathematical activity

An example: linear transformations at University

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Our research focuses on the teaching of linear transformations in “Classes Préparatoires aux Grandes Écoles”. The theory of didactical situations, jointly with Peirce's semiotics, constitute the main theoretical framework of our works and allow us to analyse student's reasoning in situations of oral evaluation. We illustrate the use and utility of this framework with the study of student's mathematical activity when they are faced to situations involving complex concepts such as linear transformations in polynomial spaces.

Keywords: linear algebra, semiotics, theory of didactical situations, transition, tertiary level

INTRODUCTION

Our work deals with a double object: linear transformations as a structuring concept of the teaching of linear algebra and a particular institution at the undergraduate level, the Classes Préparatoires aux Grandes Écoles. Grounded on didactical motivations, our epistemological analysis allows us to exemplify the crucial role of linear transformations for the emergence of linear algebra concepts. This epistemological part of our research, mainly based on the works of Dorier and Moore, leads us to the use of Peirce's semiotic and to enrich the analytical model of Bloch and Gibel (2011) which is rooted in the Theory of Didactical Situations (TDS). The purpose of this paper is twofold:

- give some responses to the following question: which reasoning forms are actually produced by a student during the different stages of a situation of oral evaluation?
- show the utility of our framework to analyse the signs and arguments produced and thus take part to the development and enrichment of this model.

Within the French didactic tradition, we remind briefly the theoretical tools used in the elaboration of the framework. Then, we expose our model, using the terminology of Bloch and Gibel. Equipped with these tools, we use the model to analyse the arguments produced by students. But, at first, we succinctly introduce the institution of Classes Préparatoires aux Grandes Écoles by highlighting the differences with the University, especially regarding the transition phenomenon with secondary level.

I. THE CLASSES PRÉPARATOIRES AUX GRANDES ÉCOLES: A FRENCH POST-SECONDARY LEVEL INSTITUTION

The Classes Préparatoires aux Grandes Écoles, which we can translate as Higher School Preparatory Classes, are part of the French tertiary education system for over two centuries. They consist in two really intensive years which act as a preparatory course to train undergraduate students for their further enrolment in one of the French graduate schools called *Grandes Écoles*, such as École Polytechnique, École Normale Supérieure, École des Hautes Études Commerciales, also known as HEC School of Management ... The enrolment in one of these *Grandes Écoles* depends on the rating obtained in national competitive and demanding examinations.

We summarize the main differences between the Classes Préparatoires and the University in the following table. Thus, we highlight some facts that Winslow (2007) showed to think about the study of the transition phenomena from the secondary to the post-secondary level.

Classes Préparatoires	University
Full-time teacher	Part-time teacher, part-time researcher
One teacher by class	Several teachers for one class
One class per teacher	Several classes for one teacher
Non adoption of semesters	Adoption of semesters
Common national curriculum	Local curriculum
Non degree course	Degree course
In High School	At University
Class councils	No class council
Report cards	No report card
Selection of students	No selection of students

Table 1: Comparison of CPGE and University

As Winslow (2007), Castela (2011) and more recently Farah (2015) wrote, these differences have deep didactic implications, relative to the theoretical knowledge and praxeologies, to the problem solving approaches, to the evaluations and to the personal homework just to name a few. In our experimental work, we studied some arguments produced by second year students from one of these Classes Préparatoires during an oral evaluation on linear algebra.

As noted by Bloch and Gibel (2016) for calculus, in order to recognize and analyse the reasoning forms actually produced by a student,

it is necessary to classify the objects, signs and reasoning processes they have to cope with during resolution of calculus problems. (Bloch & Gibel, 2016, p. 44)

II. A MODEL TO ANALYSE STUDENTS' REASONING

We then need a tool to modelize the students' productions, more precisely to identify, while in a mathematical situation, which is the knowledge they rely on during their activity; we want to identify the structure and functions of students' reasoning processes in this situation. Briefly, this model should allow us to seize the complexity of the reasoning processes a student has to cope with during the resolution of mathematical problems. By reasoning process, we mean valid or erroneous ones according to the work of Gibel (2015) about reasoning.

This model takes its origins in the Theory of Didactical Situations (Gonzalez-Martin, Bloch, Durand-Guerrier, and Maschietto 2014) and in the semiotics of Peirce. We briefly recall the main theoretical elements of TDS and semiotic used in the elaboration of the model.

The TDS theory

TDS relies on a two basic premises concerning the mathematical activity and the learning of mathematical knowledge. For TDS, the mathematical activity consists of distinct stages: a situation of action, followed by a situation of formulation and then a situation of validation phase. To take the learning activity into account, TDS adds two more stages: the phase of devolution and the phase of institutionalization. TDS defines three fields to construct and analyse such situations. The theoretical field which the domain of elaboration of fundamental situations relative to a knowledge. The a priori experimental field which envisages a situation at a specific level of teaching, taking into account the didactic repertory as defined by Gibel (Gibel, 2004). The a priori analysis of the situation, which checks if the conditions of devolvement are fulfilled, takes place at this second level. The third field is the field of eventuality where the situation is actually implemented. In short, TDS is a didactical framework which tries to implement situations with adidactical parts and offers tools to analyse the teaching and learning activities. These adidactical stages allow students to face a heuristic phase of research and then, through a confrontation to the elements of an adequate milieu, to test, validate or invalidate their conjectures. The notion of milieu appears to have a central role. TDS organizes situations with up to seven logically successive phases, but in our work we will mainly work with three of them: a heuristic one, grounded in a problem, and a formulation and validation one, composing the adidactical moments of the situation, and then the institutionalization by the teacher or with his/her help, which is the didactical moment. The dynamic of the nesting of the situations with the paired levels of milieu illustrates the dynamic of the learning processes involved. The following chart (Bloch 2006, Bloch and Gibel 2016) sums up the levels of milieu paired to the different phases of situations corresponding to the experimental situation

M1 Didactical milieu	E1: reflexive subject	P1: P. planner	S1: situation of project
M0 Learning milieu : institutionalization	E0: generic student	P0: professor teaching	S0: Didactical situation

M-1 Reference milieu: Formulation and validation	E-1: The subject as learner	P-1: Professor Regulator	S-1: Learning situation
M-2 Heuristic milieu: action, research	E-2: The subject as an actor	P-2: P devolves and observes	S-2: Situation of reference
M-3 Material milieu	E-3: epistemological subject		S-3: Objective situation

Table 2: Structuration of the didactical milieu

The levels M-2 and M-1 are the ones that will allow us to identify, describe and analyze the elements (signs, processes ...) associated to the emergence of an argumentation within the proof process. More precisely, as Bloch and Gibel (2016) write

The place where we hope to see the expected reasoning processes appear and develop is located at the articulation between the heuristic milieu and the reference milieu. (*ibidem*, p.46)

Thus, the TDS theoretical framework allows us to consider not only the reasoning processes produced, but their functions within the situation and the levels of argumentation they rely on: it already gives us a glimpse of the multidimensionality of our model.

The semiotic tools

To take into account the semantic dimension, *i.e.* the meaning of the signs and arguments produced, with certain accuracy, we need semiotic tools. The signs a student produce during an adidactical situation, whether formal or informal, are the only observable phenomena that can sustain our semiotic analysis: roughly speaking, the signs produced (syntactic aspect) are in relation to an object (semantic aspect) creating a instantiated sign (pragmatic aspect) relative to the milieu, the didactical repertoire and the repertoire of representations as defined by Gibel (2015). This triadic relation, linking the sign, the object and the student's instantiated sign, led us to use Peirce's semiotics. In this paper, we only analyse the relation between a sign and its object, its content. But in his thesis, one the authors (Lalaude-Labayle 2016) conducted a semiotic study, relying on the full Peirce's triadic classification of signs as 'put in algebra' by Marty (1990). Applying the universal categories to the relation between a sign and its object refers to the notions of icon, index and symbol, describing the abstraction level of this relation. An icon is a sign which stands for an object because of its physical resemblance: a drawing, a triangular matrix represented with triangles ... An index is a sign physically connected to its object: the columns of the pivots in a reduced echelon form of a matrix M is an index that points to the basis of $Im(f)$, where M is canonically associated to f ... A symbol is a sign that refers to its object by virtue of a law: ker is a symbol whose object is the concept of kernel of a linear transformation.

The logical inferences

In our work we understand the term ‘reasoning’ in its broadest sense. More precisely, by reasoning we mean a sequence of representations, following some internal and potentially explainable rules that lead to reach some explicit goal. Postulating, as it is the case within the TDS theory, some rationality of the students, we need to define these rules or inferences. Peirce saw mathematics as the science of drawing necessary conclusions, studying what is and what is not logically possible. But, since one does not think about logical propositions but about and with signs, Peirce broadened the notion of inference. He then distinguishes three kinds of rational inferences: abduction, induction and deduction. Deduction, or necessary reasoning, deduces a proposition B from a proposition A , where B is a formal logical consequence of A . Induction goes from the particular to the general; it allows inferring B from A . Abduction allows inferring A as a probable explanation of B . So, deduction proves that something must be, induction shows how something effectively operates and abduction suggests that something could be.

The framework to analyse students’ productions

To analyse students’ processes of reasoning, Bloch and Gibel (2011) develop a multidimensional model. They focus their didactical analysis on three main axes. The first axis is related to the level of milieu and so to the phase of the situation in which the student produces his/her reasoning (*cf.* Table 2). The second axis of the model is linked to the notions of didactical repertoire, of organizational system and of a repertoire of representations. It studies the functions of the reasoning produced and is in close relation to the first axis. As Bloch and Gibel (2016) state it, they aim at

linking these two axes, showing how the reasoning functions are linked specifically to the levels of milieu and how these functions also *manifest* these levels of milieu. (*ibidem*, p.47)

Semiotic analysis of observable signs constitutes the third axis of the model. Marc Lalaude-Labayle enriched the model by adding a fourth axis about the forms of inference applied. This logical axis links the second and third axis by setting out and clarifying the organization of the reasoning signs and their functions. This fourth axis helps to ‘make visible’ the organization within the system of representations and its actualization. We sum up this model in the following table

	Milieu M-2 Heuristic level	Milieu M-1 Formulation, validation	Milieu M0 Institutionalization
Nature and functions of reasoning	R1.1 SEM - Decision of a working frame (DOO) - Decision of transformation (semiotic register)	R1.2 SYNT/SEM - Generic calculations - Formulation of underpinned conjectures (right or wrong)	R1.3 SYNT - organization of the signs - formalization and certification of validations

	<ul style="list-style-type: none"> - Decision of calculation - Heuristic tools; errors - Exhibition of an example /a counter ex. - pattern research/identification 	<ul style="list-style-type: none"> - Decision on a mathematical object 	<ul style="list-style-type: none"> - Formalization of proofs within the mathematics involved theory
Level of use of symbols	R2.1 SEM Icons or indices depending on the context (schemas, intuitions...)	R2.2 SYNT/SEM Local or more generic arguments: indices, calculations	R2.3 SYNT Formal and specific arguments related to the chosen frame
Actualisation of the repertoire	R3.1 SYNT/SEM - Ancient knowledge - Enrichment at the heuristic level(patterns, praxeologies ...)	R3.2 SYNT/SEM Enrichment at the argumental level: - statements - organizational system	R3.3 SYNT - Formalized proofs - Signs within the relevant theory - theoretical elements
Forms of reasoning	R4.1 - deductive - inductive - abductive	R4.2 - deductive - inductive	R4.3 - deductive

Table 3 – A model to analyse situations

SEM signifies that the formulations are made on a semantic mode whereas SYNT is for syntactic mode. This model emphasizes the fact that the mathematical activity, with its reasoning processes, appears in the heuristic and reference milieu (*cf.* Table 2). These two milieus, and the articulation between them, will thus be of particular interest for our work, even if the situation is an ordinary one.

Let us insist on the fact that the use of this model relies on a precise *a priori* mathematical analysis of the situation and of its components, *e.g.* the problem to be solved. Within the TDS this step appears to be necessary to clarify the didactical *a priori* analysis.

III. A PRACTICAL USE OF THE MODEL IN A LINEAR ALGEBRA ORAL EXAMINATION

We analyse the productions of a second year student of Classes Préparatoires in the context of an oral examination. The students of the Classes Préparatoires, by group of three, pass such an oral exam once every two weeks. It lasts one hour during which the teacher asks to the three students to solve different mathematical problems. They work simultaneously and individually on a large blackboard on the problem they just discovered, in front of the teacher. The teacher helps the students by giving advices or clues. Taking into account the writings noted on the chalkboard, she/he can ask for some explanation or clarification. This can give rise to an oral or written answer and possibly to some discussion to deepen and enrich the repertoire

of the representations of the student. The oral exam we analyse deals with linear algebra, and more specifically linear transformations and matrices.

The student is confronted to the following problem:

Let n be a integer, greater than or equal to 2 and let φ defined on $\mathbf{R}_n[X]$ by: for all polynomial P from $\mathbf{R}_n[X]$, $\varphi(P)=P(X+1)-P(X)$.

1. Show that φ is an endomorphism of $\mathbf{R}_n[X]$.
2. Is φ injective? Surjective? Bijective?

In France, $\mathbf{R}_n[X]$ symbolises the vector space of real polynomials of degree at most n .

As briefly explained earlier, our model provides a framework for investigating mathematical and didactic activities in terms of milieu, focusing on the reasoning processes, signs and their dynamics and on the conditions that enable their development during the situation, be it ordinary or not. As is done within TDS, our didactic analysis is divided in several stages: a detailed and structured *a priori* mathematical and didactic analysis, enriched with a specific *a priori* analysis of the reasoning involved; follow then an *a posteriori* analysis organized in our model.

A priori analysis

From the mathematical point of view, showing that φ is an endomorphism of $\mathbf{R}_n[X]$ can be approached in different ways, engaging several frames: indeed the stability of $\mathbf{R}_n[X]$ under φ can be done in a purely algebraic frame using the degree and the composition rule, in a functional frame using the decomposition $P = \sum a_i X^i$ or in an algebraic frame using the linearity and showing that for all integer i between 0 and n , $\varphi(X^i) \in \mathbf{R}_n[X]$. To study the injectivity and surjectivity of φ , the student can again choose between several frames and several registers of semiotic representation (Duval, 2017): she/he can use an algebraic frame with an example $\varphi(1) = \vec{0}$ and then applying the theorem linking rank and kernel dimension; but she/he can also try to find precisely $\text{Ker } \varphi$, that is find a basis. To do this, she/he can use the functional decomposition of polynomials, she/he can solve a linear system or she/he can determine the matrix of φ in the canonical basis of $\mathbf{R}_n[X]$.

From a didactic point of view, the situation studied here is said to contain an adidactic dimension. Most of the actions, of the frames and of registers of semiotic representation are devolved to the student. So, a first difficulty that occurs for the student can be the control she/he has on the objects involved: a circular application of the definition of linearity of φ to prove its linearity, the complexity of formulas to write down $\varphi(P)$ for a general P , the non operability of φ with wrong calculations of $\varphi(1)$, $\varphi(X)$, $\varphi(X^2)$ for example. For the injectivity of φ , the student can “forget” the structure of the space $\mathbf{R}_n[X]$ he is working on and try to check whether $\varphi(P)=\varphi(Q)$ implies that $P=Q$. All the reasoning processes and objects involved are part of the didactic repertoire of the class the student belongs to.

***A posteriori* analysis: analysis of a student's production**

In the following, we extract some translated excerpts of the student's answer, the whole solution can be found in Lalaude-Labayle (2016).

To solve the first part of the problem, which is an ordinary situation in undergraduate level, the student is mainly confronted to the milieu of reference, articulating objects and processes involved in his repertoire of representations. Here the heuristic milieu is not really requested. He starts by showing the stability $\mathbf{R}_n[X]$ under φ and that writes changing X with $X+1$ doesn't change the degree of P . He then proves the linearity in an algebraic frame. Doing so, he makes a formalization of proofs within the required theory and thus reaches the level R1.3. The semiotic analysis shows that he uses generic arguments (R2.2) and more formal one (R2.3). These arguments and signs don't give any hint to how φ operates on $\mathbf{R}_n[X]$. Its argumentation validates its use of the didactic repertoire, and reveals some implicit assumptions: $\varphi(P)$ is a polynomial is here implicit, as is its use linking degree and composition of polynomials. He uses mainly hypothetical-deductive inferences. But, as an introduction to his argumentation, the student asks himself whether $\deg(\varphi(P)) \leq n$: he formalizes here the start of an abductive reasoning.

To study the injectivity of φ , the student applies a transformation of register of semiotic representations to formalize the link with $\ker \varphi$, starts within an algebraic frame then uses the decomposition of P to study $\ker \varphi$, but without success: he cannot make φ operate on P and says not to have any clue to study the kernel. During this phase, the student tried to use some deductive reasoning involving objects from the reference milieu: the lack of articulation between the heuristic milieu and the reference milieu confirm the difficulties encountered in R1.3 with an aimless organisation of the signs. The semiotic analysis underlines the lack of quantifiers which leads to a incomplete apprehension of the objects (R2.2) and reinforces the feeling of lack of goal in the reasoning. The teacher asks then the student to consider the tools he has got in his repertoire to "calculate" objects in linear algebra. With no answer, he asks the student to consider the matrix of φ in the canonical basis. Doing so, he tries to force the student to face the heuristic milieu and tries to maintain an adidactic dimension in the situation. The calculations produced confirm the fact that the student doesn't know how to compute φ : he obtains the identity matrix. This matrix doesn't appear to be an index to control the reasoning (articulation R1.2 and R1.3). With a new oral intervention of the teacher, the student writes the right matrix. Some misinterpretations are following: \mathbf{R}_n instead of $\mathbf{R}_n[X]$, surjectivity is meant instead of injectivity (R3.1), $\varphi(1)=0$ is used as an symbol for $\ker \varphi = \text{vect}(1)$ instead of a simple index of it (R2.1). The student uses the theorem of the rank (R3.2), relying then on a deductive form of reasoning. But some of his deductions rely on the preceding explicit calculations (articulation R4.2 R4.3).

CONCLUSION

As stated in the introduction, the purpose of this paper is twofold: try to determine the reasoning produced by a student in a specific mathematical situation and show the utility of our framework to analyse the signs and arguments produced in this situation.

Regarding the first question, our analysis allows us to say that this particular student has difficulties to reach fully the institutionalization milieu (level R3): the reasoning and the articulation of the objects involved do not ease his control over his arguments and eventually lead him to aimless computations. Moreover he seems to get stuck in the reference milieu (level R2) and to hypothetical-deductive inferences. The student does not rely on reasoning made in a heuristic milieu (level R1) that would be appropriate to linear transformations and polynomials. The problem we analyze in this work contains an adidactic dimension but fails in asking the student to make effectively operate φ on $\mathbf{R}_n[X]$. In semiotic terminology, we can postulate that the at least incomplete pragmatic dimension in the reasoning leads to some confusion and lack of pertinent association between the syntactic and semantic dimensions (Bloch and Gibel, 2016).

Regarding the second question, our work seems to confirm that the model used within TDS constitutes an efficient framework, as stated in Bloch and Gibel (2011, 2016). It helps specifying the reasoning and signs on which it relies both for the *a priori* and *a posteriori* analysis and highlights the obstacles contained within the mathematical notions. Our future work should be working more specifically on the abductive reasoning and on situations encouraging students to adopt a heuristic approach.

REFERENCES

- Bloch, I. (2003). Teaching functions in a graphic milieu: what forms of knowledge enable students to conjecture and prove? *Educational Studies in Mathematics*, **52**, 3-28.
- Bloch, I. (2006). *Quelques apports de la théorie des situations à la didactique des mathématiques dans l'enseignement secondaire et supérieur. Note de synthèse présentée en vue de l'Habilitation à Diriger des Recherches*, Université Paris 7.
- Bloch, I., Gibel, P. (2011). Un modèle d'analyse des raisonnements dans les situations didactiques: étude des niveaux de preuves dans une situation d'enseignement de la notion de limite, [A model to analyse reasoning processes in didactic situations: a study about levels of proofs in a situation concerning the teaching of limits] *Recherches en Didactique des Mathématiques*, **31-2**, 191-227.

- Bloch, I., Gibel, P. (2016). A model to analyse the complexity of calculus knowledge at the beginning of University course. Two examples: parametric curves and differential equations. *Proceedings of INDRUM, Montpellier*, Nardi, Winslow & Hausberger, ed. ISSN: 2496-1027 (online) p. 43-52.
- Castela C. (2011). *Des mathématiques à leurs utilisations, contribution à l'étude de la productivité praxéologique des institutions et de leurs sujets / Le travail personnel au cœur du développement praxéologique des élèves en tant qu'utilisateurs de mathématiques*, Note de synthèse présentée en vue de l'Habilitation à Diriger des Recherches, Université Paris Diderot. Paris : Irem 7
- Duval, R. (2017). *Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representations*, Springer International Publishing
- Farah L. (2015), *Étude et mise à l'étude des mathématiques en classes préparatoires économiques et commerciales : point de vue des étudiants, point de vue des professeurs*, Thèse de doctorat, Université Paris Diderot (Paris 7)
- Gibel, P. (2004). *Fonctions et statuts des différentes formes de raisonnement dans la relation didactique en classe de mathématiques*, Doctorat de l'Université de Bordeaux 2.
- Gibel, P. (2015). Mise en œuvre d'un modèle d'analyse des raisonnements en classe de mathématiques à l'école primaire. [A model to analyze reasoning processes at primary school]. *Éducation et Didactique*, **9-2**, 51-72, PUR.
- González-Martín, A., Bloch, I., Durand-Guerrier, V., Maschietto, M. (2014). Didactic Situations and Didactical Engineering in University mathematics: cases from the study of Calculus and proof. *International Journal of Research in Mathematics Education* 16(2), 117-134.
- Lalaude-Labayle, M. (2016). *L'enseignement de l'algèbre linéaire au niveau universitaire : Analyse didactique et épistémologique. [The teaching of linear algebra at tertiary level: A didactic and epistemological analysis]*. Université de Pau et des Pays de l'Adour.
- Marty R. (1990), *L'Algèbre des signes, Essai de sémiotique scientifique d'après C. S. Peirce*, Amsterdam, John Benjamins Publishing (Foundations of Semiotics Series 24)
- Winslow C. (2007), Les problèmes de transition dans l'enseignement de l'analyse et la complémentarité des approches diverses de la didactique, *Annales de Didactique et de Sciences Cognitives*, Vol. 12 , 189-204