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APOS (Action-Process-Object-Schema) Theory is used to pose and test a conjecture of mental constructions that may be used to understand the relation between integrals of two variable functions over rectangles and corresponding Riemann sums. Interviews with ten students who had just finished a multivariable calculus course showed that the conjectured mental constructions are necessary.

Keywords: Functions of two variables, APOS, integral calculus

Multivariable functions and multivariable Calculus are important in engineering and the natural sciences as a tool for modelling. Their learning has received more attention lately from the Mathematics Education community. Starting with the analysis of students' understanding about two-variable functions (for example: Trigueros and Martínez-Planell, 2010; Martínez-Planell and Trigueros, 2012) researchers have documented students' difficulties and have shown that the transition from one-variable Calculus to multivariable Calculus is far from being smooth. There are few studies in the literature that deal with students' difficulties and understanding of the integral multivariable Calculus (Jones and Dorko, 2015; Martínez-Planell and Trigueros, 2017). In one of these few studies, McGee and Martínez-Planell (2014) showed that a course based on lectures did not promote students' understanding, while activities introducing the use of semiotic chains and the development of synergy among representations helped students understand this concept. The research questions are:

What constructions relating double integrals and Riemann sums are evidenced by students who finished a Multivariable Calculus course based on lectures?

What constructions may be needed to relate double integrals and Riemann sums?

THEORETICAL FRAMEWORK

APOS theory (Arnon et al. 2014) is used in this study to analyse possible mental constructions by students who have already taken a course on multivariable calculus. We only summarize the main structures of this theory. An Action in APOS Theory is a transformation of a previously constructed mathematical object that the individual perceives as external in the sense that students need some guidance and are not able to justify what they do. When an Action is repeated, and the individual reflects on what he or she does, it may be interiorized into a Process. A Process is perceived as internal in the sense that it has meaningful connections to other mathematical

knowledge of the individual. A Process allows the individual to imagine doing the Actions without actually doing them, to omit steps, anticipate results, and to justify the Process. Different Processes may be coordinated to form new Processes. When the individual needs to apply Actions on a Process, it can be encapsulated into an Object. When an individual shows a Process or Object conception of a mathematical notion we say that the individual "understands" the notion. A Schema is a coherent collection of Actions, Processes, Objects, and other Schemas, that the individual uses to work with problems related to some mathematical notions. Schemas are not used in this paper.

To analyse students' work using APOS Theory, a conjecture of those constructions that may be used to understand a specific mathematical notion is designed. This model, called a genetic decomposition (GD), does not pretend to be unique and needs to be tested with research data. The GD may be revised and expanded in successive cycles of research, teaching material development, and implementation. This research cycle makes it possible to use APOS theory to be better suited to future research needs to study the multivariable integral calculus.

GENETIC DECOMPOSITION

We only present a portion of the GD of integrals of functions of two variables over rectangles. Its development is based on mathematics, on the researchers' teaching experience, and data from the research literature: mainly, ideas about representation registers (Duval, 2006) described in the study by McGee and Martínez-Planell (2014) and the ideas of "orienting pre-layer" and "product layer" described by Sealy (2014), which stress the need of attending to the individual meaning of the product $f(x_i)\Delta x$ and its components in the construction of integrals of one-variable functions.

The GD starts with pre-requisite constructions which include: a Process conception of two-variable functions and volume of prisms as Object.

Actions are performed on a given two-variable function in any representation with domain restricted to a rectangle, to produce the geometric representation of the restricted domain as a subset of 3D space. Actions are performed on the same function to obtain values of the function on the given domain and to represent them in the space as points and/or curves in the graph of the function. These Actions are interiorized into a treatment or conversion Process to represent the graph of the function over the given rectangle together with the rectangle so that the student can imagine the relation between function and rectangular domain as a graph in space.

Actions of evaluating the given function of two variables at a specific point of a given sub-rectangle of its domain, multiplying it by the length and width of the rectangle to form a product of the form $f(a,b)\Delta x\Delta y$ are done. These Actions are interiorized into a Process which can be coordinated with conversion Processes between different representations of function, rectangle, and given point, to imagine the product as the volume of a rectangular prism in space.

Given a continuous function in different representations defined on a rectangle, with the function simple enough so that its maximum and minimum values on the rectangle may be recognized without doing any explicit computation, the Action of obtaining an overestimate and an underestimate of the product $f(a,b)\Delta x\Delta y$ is taken. These Actions may be interiorized into a Process that enables to imagine the existence of points (a,b) where underestimate and/or overestimate of the product $f(a,b)\Delta x\Delta y$ are attained. This Process is coordinated with a treatment or conversion Process to draw a rectangular prism corresponding to over and/or underestimate in space. Actions are performed to change the chosen point to construct a prism that better approximates a given exact value of the integral. These Actions are interiorized into a Process that enables the recognition that for such continuous function, there is a point somewhere on the rectangle that will produce the exact value of the volume between the graph of the function and its rectangular domain.

Given two small specific positive integer numbers, *n* and *m*, the Action of subdividing given intervals [a,b] and [c,d] into subintervals of equal length both numerically and geometrically is done to obtain a subdivision of the rectangle $[a,b]\times[c,d]$. These Actions are interiorized into the corresponding Process. Given a continuous function *f* defined on the rectangle, the Action of choosing a prescribed point (x_i, y_j) on each sub-rectangle of the given partition and producing the products $f(x_i, y_j)\Delta x\Delta y$, and the corresponding sum, interpreting this sum geometrically, numerically, symbolically as an extended sum, symbolically using sigma notation, and verbally, may be interiorized into a Process that enables imagining forming such sums in different representations for the collection of sub-rectangles in any partition of any given rectangle.

METHOD

Ten students were chosen by their professor to be interviewed at the end of a multivariable calculus course selecting four over-average, three average, and three under-average. The course was completely based on lectures. The interviews lasted 46 minutes on average. Students answered a set of questions designed in terms of the GD and also related to what was covered during the course, and produced a written response while sharing their thoughts out-loud. The interviews were recorded, transcribed, individually analysed, and results were negotiated by the two researchers. Students' responses were analysed according to the GD, while keeping notes on unexpected responses and other difficult to classify observations. These were the questions used:

1a. The following is the complete graph of function $z = f(x, y)$. Represent the domain of *f* in the figure (Figure 1).

1b. Let $g(x, y) = x^2 + y$ be a function with domain restricted to $0 \le x \le 2$ and $1 \le y \le 2$. Use the coordinate system given in the following figure to represent the domain in three-dimensional space [An empty drawing of the first quadrant was given].

1c. The above functions *f* and *g* are the same [Figure 1 was given again here and in the rest of the problems]. If $\Delta x = 2$ and $\Delta y = 1$, what is the numerical value of $f(0,1)\Delta x\Delta y$? What does it represent geometrically?

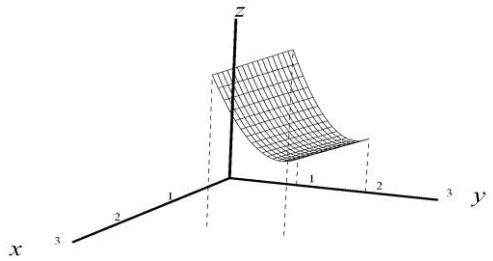

Figure 1: (repeated in each part of problem 1 except 1b)

1d. Let $\Delta x = 2$ and $\Delta y = 1$. How does $f(0,1)\Delta x\Delta y$ compare with $\iint_D f(x,y)dA$? [No numerical computations are needed in parts d, e, f, and g.]

1e. How does $f(2,2)\Delta x\Delta y$ compare with $\iint_D f(x,y)dA$?

1f. Is there any point (*a*,*b*) in the domain *D* of *f* such that $f(a,b)\Delta x\Delta y$ is equal to $\iint_D f(x,y)dA$?

1g. Let $\Delta x = 1$ and $\Delta y = 1/2$. Consider the Riemann sum $f(0,1)\Delta x\Delta y + f(0,1.5)\Delta x\Delta y + f(1,1)\Delta x\Delta y + f(1,1.5)\Delta x\Delta y$ of the integral $\iint_D f(x,y)dA$. What does the Riemann sum represent geometrically and how does its value compare to that of $\iint_D f(x,y)dA$?

Note that problems 1a and 1b are essentially the same in different representations. They both test the portion of the GD dealing with recognition of rectangle and function. Problem 1c gives information on the portion of the GD dealing with forming one term of a Riemann sum. Problems 1d, 1e, and 1f relate to the portions of the GD dealing with underestimate, overestimate, and exact value. Problem 1g gives information on the portion of the GD dealing with a partition and Riemann sum.

RESULTS

On function and domain of a function

Many students showed they had not constructed the concept of two-variable function. They gave evidence of considering these functions in terms of a correspondence rule, and showed difficulty interpreting functions given graphically. Moreover, these students also showed not to have constructed the concept of domain of the function. Most of them considered that the domain of a two-variable function should include

information about the function, since it had to be represented in 3D space. Eight students showed difficulties similar to those of Luis, as exemplified in the following discussion with the interviewer (in Problem 1a):

Luis: I can tell you what the domain is but if I don't have a function I don't think I can tell you the exact point where each of the points in the graph is…

Interviewer: So, is the graph part of the domain?

Luis: No, the domain is obtained from the graph. I can obtain the domain having the function but to do so I have to define the function.

After some discussion:

Interviewer: So the domain, is it only *x* and *y* or may it also include *z*?

Luis: The domain may include the *z*.

This example shows how Luis needs a correspondence rule to determine the domain of the function. It also evidences that he considers the function itself should be part of the domain of the function. Other students showed this difficulty.

Students' responses pointed to a need to pay attention to the different representations of functions in 3D space and to have students do treatments and conversions between representations. In Problem 1a, some of them quickly represented the rectangular domain as part of the given figure in 3D space. However, when the function was given symbolically their notion of domain seemed to change. This shows that recognizing the domain of a two-variable function is a construction that needs the interiorization of Actions on functions given in different representations. These difficulties as well as the counterfactual belief of teachers that students may easily generalize concepts for one-variable functions to multivariable functions have been reported before (Martinez-Planell and Trigueros, 2012).

Area and volume

Students also showed an unexpected confusion between area and volume when they described graphs of functions in 3D-space. This difficulty surfaced in Problem 1c. All students were able to calculate the value of that product; however, they were in trouble when explaining its geometrical meaning. Brian, for example, explained:

Brian: … this part, $f(0,1)$ would be a point in this graph here. Change in *x*, change in *y*, I am not a hundred percent sure… that would be an area then, of the surface, or the entire function…

And later:

Interviewer: Can you tell me what does the double integral of $f(x,y)dA$ represents?

Brian: dA is the area of the function, the area of this figure,

Interviewer: The area of the surface?

Brian: Yes… of the surface on the given domain….

Other students, as Luis, showed confusion:

Luis: The area of the figure, that is, the area of the function which in this case is that

figure [referring to the graph in Figure 1].

Interviewer: Like the area of a surface?

Luis: Exactly

Interviewer: So if it had units would it be like square inches or square centimetres? What units would the double integral have if *x*, *y*, and *z* had units?

Luis: Cubic

Interviewer: Cubic; then, would it be area?

Luis: It would be volume...

After some discussion:

Interviewer: …Let's suppose that this other paper that I am raising here is the graph of the function [He raised a sheet of paper] What volume are we talking about?

Luis: …The volume is the one of this paper…since I have a function and I'm integrating in the values of the function then what I'm going to get are *z*, small *z*'s of what the function is, I'd be getting the volume of the figure.

Interviewer: ... So you pointed to the paper that is floating. But, does it have a volume?

Luis: ... Yes, it has a change in *x*, it has a change in *y*, and the *z* is the one from the function, so I say that it has a volume.

Other five students showed the same confusions. The above excerpt exemplifies that a student can describe the individual components of $f(0,1)\Delta x\Delta y$ but might not be able to do the Action of putting them together to interpret it as the volume of a rectangular prism, even if they can calculate the result of the product by doing the Action of substituting the given values in the expression, as conjectured in the GD. This difficulty is possibly related to the fact that these students have not constructed space as an Object, which does not allow them to imagine what their teachers mean when they talk about a surface in space and the double integral as related to the volume under a surface (Trigueros and Martínez-Planell, 2010).

As considered in the GD, these difficulties make it impossible for students to do the necessary Actions on the function restricted to a rectangle to represent the domain geometrically as a subset of space, and to interiorize the Process to imagine the relation between a restricted region on the domain and the function. The lack of all these constructions becomes an obstacle to understand other related concepts, including double integrals, as will be shown below. Students who do not show these constructions may not follow teachers' explanations; they would be confused and resort to memorization to respond to exam questions.

Only one student, Farid, gave evidence of the pre-requisite constructions described in the GD. He was successful in explaining Riemann sums and double integrals.

Riemann sums, underestimates, overestimates and double integrals

Most students had many difficulties working with problems dealing with Riemann sums and their relation to double integrals. Even after the interviewer explained to some of them that $f(0,1)\Delta x\Delta y$ was a volume and drew it, their difficulties did not enable them to make the whole construction as Brian showed in Problems 1d, 1e, and 1g:

Brian: So the Riemann sum would be the approximation of the area [sic] under this figure [referring to the graph in Figure 1], obviously it wouldn't be as precise as the value of the integral. Let's see... so geometrically 0,1, *x*, *y*, let's draw a square here like this [he is now evaluating and drawing rectangular prisms]... 0, 1.5, maybe another square closer this way, higher... 1,1 we are still at *x* 1 and even higher here… like this, change in *x* change in *y*... change in *y* being 1/2, I don't think we get from 1 to 2 with 1/2 [He seems to believe that since $\Delta y = 1/2$ the prisms will be restricted to the region $1 \le y \le 1.5$. He might think of Δy as "change in *y*" were the "change" is taken from the initial *y* value in point (0,1).], so the integral would give this area [sic] here, a figure more or less like this...

Interviewer: You said area...

Brian: [Interrupting] Volume, I mean volume, sorry... yes, volume of the integral. This would give us something more stepwise... let's see if I can draw it here like this... 1,2,3,4, like this, a series of cubes like this, stepwise, approximating, not all of this, but only this half here… [See Figure 2].

Interviewer: Do you mean the left hand part of the solid?

Brian: Yes, the left hand part of the solid would be what is approximated with this Riemann sum.

Although Brian was able to construct the meaning of volume, his construction was not right, the boxes he drew filled only the left-hand side of the rectangle. It seems that Brian could do the Actions to construct the prisms but he did not interiorize those Actions into the Process that would enable him to imagine all the constructions needed to relate Riemann sums and double integrals.

An interesting result of this experience was that even though students showed many difficulties during the interview, some of them, like Brian, showed evidence of doing some of the expected constructions during the interview. Others reflected during the interview and constructed meaning. This was the case of Victor who had considered $f(0,1)\Delta x\Delta y$ as an area. When discussing Riemann sums, and after being told that this product represents a volume he explained:

Interviewer: So you drew a little box.

Victor: Exactly a little box as we know that delta x would be 2 and delta y 1; a rectangle with width 2, eh, length 2 with 2 and height 1.

Interviewer: Then how do you compare those volumes.

Victor: Ok, now I understand, this f(0,1) delta x delta y is only the volume up to this point, I mean up to a certain height, and then, the double integral on that same area that we put on xy is, let's say, the same box but with a height that varies with the function. Now this is it!

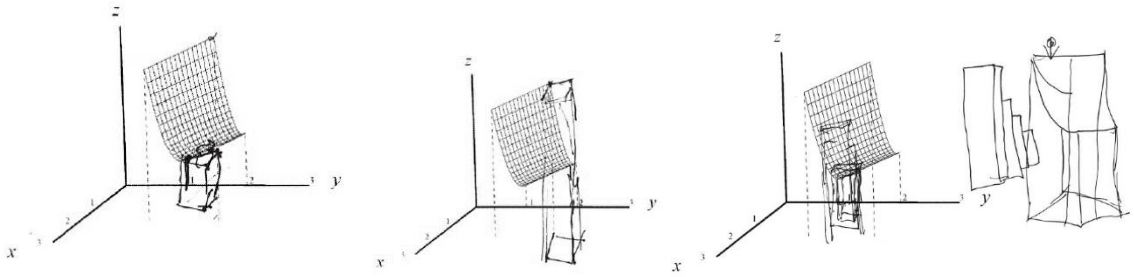

Figure 2: Brian's drawing for Problem 1d, 1e, 1g (respectively)

Interviewer: But which is larger, what is smaller, are they equal?

Victor: No, no, they are completely different, the larger is that obtained from the double integral since the height is higher.

Victor: *f*(2,2) delta *x* delta *y* is the box with dimensions over *D*…and then this is the same equation as before, but *f*(2,2) is higher so the volume there is larger.

Victor could do the Actions needed to compare volumes of prisms obtained from different values of the function. Another student, Farid, evidenced he could imagine forming one term of a Riemann sum, as discussed before. He also showed to have done the constructions necessary to imagine volumes of prisms and their role in Riemann sums. When comparing the volume of the prism in Problem 1d with the double integral:

Interviewer: And what does that represent? [Referring to the double integral.]

Farid: That represents the volume between the surface and the plane, the domain...

Interviewer: Then, how do those two numbers there compare?

After some doubts:

Farid: Represented this part [referring to the product], now, this product would be smaller, than the double integral, because this here is a, I represented it as a cube, given the value of *f* at that point, while this is the double integral of everything, of all the function *x,y* over *D*, so this value seems bigger

Interviewer: Which one?

Farid: The value of the double integral over *D* of $f(x,y)dA$

He was able to compare the volume of the prisms with the double integral. When he had to decide if there would be a prism with volume equal to the value of the double integral, a problem that was impossible for all the other students, he explained:

Farid: … The thing that comes to my mind when thinking on an inequality is the sandwich theorem…that there... must exist then a value for *x* and *y* that could be named *a* and *b* that is equal to the double integral on *dA*.

When discussing Riemann sums with a specific partition, most students could not work with the problem, even with help from the interviewer. As was shown by Brian in the previous example, some students, including Victor, imagined drawing several prisms or boxes that shared the base, and only had different heights. Those students showed they could do the Action of changing the height of a given prism but not that of partitioning the domain into small areas of the same size. Victor could describe the sum of the prisms' volume, at first he said that the Riemann sum was always an approximation to the volume under the surface, although later he reconsidered:

Victor: No, the Riemann sum is an approximation and if you take more subintervals, ah! If you take more subintervals, those were the squares, that one uses, the Riemann sum is a closer approximation and that approximation would be closer with more subintervals, and the double integral is the exact value.

Only Victor and Farid seemed to have interiorized the Action of forming a partition into a Process they coordinated with the Process of selecting heights for each subrectangle into the Process of calculating the volume corresponding to the prisms to approximate the volume under the surface.

DISCUSSION AND CONCLUSION

Results from this experience show that most of these students demonstrate a very limited understanding of two-variable functions and of those concepts associated to the construction of the double integral of a two-variable function and its geometrical interpretation. Only two students showed some understanding, although one of them relied mostly in memorized facts that he could use appropriately in most cases. This student seems to have constructed meaning for some of those facts during the interview. Students' responses show the importance of the predicted constructions included in the Genetic Decomposition. In this investigation we related observed difficulties with specific mental constructions in the GD that students seemed to lack. The importance of the pre-requisite constructions in learning this difficult topic was underscored. Its lack became an insurmountable obstacle to understand even the most basic ideas leading to the learning of the double integral.

A more encompassing understanding of function in different representation registers proved to be indispensable. Results indicate that students who could only perform Actions constructed a confusing network of concepts where the properties learnt about one-variable function are not well differentiated from those of two-variable functions. This inhibits their possibility to make those constructions involved in the understanding of 3D space, functions, and their domains. Fluency in operating within and across different representations plays an important role in the construction of two-variable functions as an Object, instead of considering them as simple correspondence rules containing one or more variables.

These results emphasize once more the importance of spending more time on helping students to construct the notion of two-variable function. But, even when two-variable function has been constructed as a Process, the notions of volume under the surface and the role of the Riemann sum in the construction of the double integral constitute fundamental constructions in the learning of double integrals.

The genetic decomposition proved useful in determining and underscoring those mental constructions that are needed to learn double integrals with meaning. It also reveals the subtleties involved in learning the double integral. After classroom use of specially designed activities, future studies may reformulate the same interview problems and also extend them to explore other ideas of the integral calculus.

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