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Josep Gascón[1] and Pedro Nicolás[2]

[1]Universitat Autònoma de Barcelona, España; [2]Universidad de Murcia, España pedronz@um.es

*We use Hintikka's analysis of the notion of* inquiry *to point out some obstacles that the implementation of inquiry-based learning might find, and to suggest a way to overcome these difficulties via the use of the Reference Epistemological Models developed in the framework of the Anthropological Theory of the Didactic.*

*Keywords: inquiry-based learning; teacher's and students' practices; novel approaches to teaching*

## INQUIRY-BASED LEARNING

Inquiry-based learning (IBL), roughly conceived as a teaching method in which students are invited to learn in a way similar to that of a scientist or a mathematician, has been suggested, with different levels of precision, by several researchers, institutions and pedagogical and didactic approaches. Artigue and Blomhøj (2013) include as instances of IBL the teaching proposals made from the following approaches in Mathematics Education: problem-solving, theory of didactical situations, realistic mathematics education, modelling perspectives, anthropological theory of the didactic, and dialogical and critical approaches. For its part, European Union, through reports prepared by experts (see, for instance, (Rocard, Csermely, Jorde, Lenzen, Walberg-Henriksson, & Hemmo, 2007)) and projects (see, for instance, PRIMAS, http://www.primas-project.eu), has supported also the implementation of IBL in educative european institutions.

It would be incorrect to believe that these IBL proposals have been suggested as a means to reach the very same educational end. Let us mention some proposal aiming different ends. The Theory of Didactical Situations proposes the IBL, embodied in the notion of *situation*, as a means to achieve a *real* knowledge of mathematics (Brousseau, 1997, p. 22). For its part, the Anthropological Theory of the Didactic proposes the IBL, via the notion of *Study and Research Path*, as a means to transform the cognitive *ethos* of our society, that is to say, to provide new attitudes and habits with respect to the acquisition of knowledge (Chevallard, 2015). European Union, in reports and projects like the ones mentioned above, stands up for IBL as a means to remedy the declining interest of youth in science, confirmed by some OECD reports, and the subsequent lack of technological innovation in Europe. According to this point of view, IBL would contribute to change science and mathematics learning into a motivating activity. On the other hand, Europen Union also promotes IBL, for instance in the PRIMAS project, as a means to prepare students for a future in which "it is no longer sufficient (…) to learn facts" but "to be able to solve non-routine problems, to analyse data, to discuss with colleagues, to communicate their result and to work autonomously" (Maaß & Reitz-Koncebovski, 2013, p. 10). Therefore, the ends

to which different proposers aim to arrive through IBL are as varied as the very ends of regulated education: the acquisition of knowledge, habits, attitudes, values, etc.

## KINDS OF KNOWLEDGE

To analyse possible uses of IBL in the acquisition of knowledge we have to wonder: are there different kinds of knowledge? In Epistemology it is customary to distinguish between three kinds of knowledge (Ichikawa, Steup, 2017; Fantl, 2017): knowledge by acquaintance, knowledge how and propositional knowledge. *Knowledge by acquaintance* is the kind of knowledge you have when you can identify something or someone (for instance, the label *criterion of divisibility by 3*) and the corresponding name, description, formulation, etc. (in this case: a natural number is divisible by 3 if and only if the result of adding all its digits is divisible by 3). *Knowledge-how* is the kind of knowledge you have when you carry out a series of intentional actions (for example, to apply the criterion of divisibility by 3) towards the attainment of an end (to know whether a given number is divisible by 3, in this case). Finally, *propositional knowledge* is the kind of knowledge you have when you know *why* a certain proposition is *true* (for example, when you know why it is true the criterion of divisibilty by 3). Notice that in the knowledge by acquaintance you also seem to know the truth of a proposition (for instance, the proposition asserted by the statement "The criterion of divisibility by 3 is the statement *A natural number is divisible by 3 if and only if the result of adding all its digits is divisible by 3*"). But in this case this is just a contingent truth, existing by convention.

## INQUIRY AND PROPOSITIONAL KNOWLEDGE

In order to analyse possible obstacles to the implementation of IBL, we shall use a theoretical model of the notion of inquiry, namely, the one developed by the logician and philosopher Jakko Hintikka and collaborators in several works. Of course, one can also find other theoretical models of the notion of inquiry in some approaches to Mathematics Education. Anyway, here we choose the one provided by Hintikka due to its enlightening use of Logic, which seems to be an essential ingredient for a true comprehension of an inquiry process.

### Interrogative Model of Inquiry

To get a better understanding of the concept *inquiry*, as claimed by Hintikka (1982),

> is not enough to study individual acts instantiating (…) whatever the concept in question may be. We also have to study the more complicated rule-governed behavioral complexes in which their "logical home" is.

For this, Hintikka has developed a game-theoretical model, the so-called *Interrogative Model of Inquiry* (IMI), which presents a formal approach to inquiry. A good reference for this model, which has been explained here and there, is (Hintikka, Halonen, & Mutanenet, 2002). In few words, inquiry is regarded as a game with two players: *Inquirer* and *Nature*. The game starts with a pair, (T, Q), where T is a given theoretical premise, and Q is a question. The game finishes when the Inquirer finds

an answer to Q. Along the game the Inquirer is allowed to make two kinds of moves: *questions* to Nature, and *deductions*. The only moves of Nature are *answers* to Inquirer's questions. We will not explain here all the details of Hintikka's IMI, but at least we will point out two of its main features:

1) The principal goal of an inquiry is the acquisition of propositional knowledge.
2) Inquiry is a process in which a complex dialectics between deductions and questions takes place.

We emphasize here these two aspects because, as we hope to show below, they seem to be crucial for practical didactic considerations about IBL.

**The goal of inquiry is to acquire propositional knowledge**

The inquiry finishes if we can *deduce* (based on the premise and on Nature's answers to our questions) an answer A to the initial question Q (Hintikka et al., 2002, Theorem 1). Of course, the deduction procedure, which is carried out through a series of familiar rules (Hintikka et al., 2002, § 2), has to do with semantics. Indeed, these rules are such that if the premise T and Nature's answers are *true* in a given *model* M, then the answer A is also true in this model M. This is why we can read in (Hintikka, 1996, p. 38) that what the winning of an "interrogative game" shows is "knowledge of truths".

Later, we will speak of the implications of the presence of the model M in the use of IBL for teaching. For the moment, I would like to stress the fact that, according to Hintikka's IMI, the result of an inquiry is a proposition. Inquiry is a quest for propositional knowledge, which is the kind of knowledge linked to non-contingent truth and falsehood. This does not prevent, in the course of an inquiry, the acquisition of knowledge-how and knowledge by acquaintance. But we would not consider an inquiry finished if, along this inquiry, we would have found a successful technique (knowledge-how), whose success remains unexplained. In other words, in the course of an inquiry we can get knowledge-how, but we then would pursue the corresponding propositional knowledge able to explain the success (and even its limitations, portability to other contexts, etc.) of this knowledge-how.

**Dialectics between deductions and questions**

In a game of inquiry not all the questions can be asked at any moment. On the contrary, one is forced to pay attention to the *presuppositions* of that questions. As explained by Cross and Roelofsen (2016), there are different *logical* kinds of questions: whether-questions, which-questions, why-questions, etc. Each kind of questions determines its own kind of presuppositions. For example, the presuppositions of *which*-questions (e.g., "Which is the smallest prime number bigger than 7?") are existential statements (e.g., "There exists a smallest prime number bigger than 7").

According to Genot and Gulz (2015), based in turn on (Hintikka & Hintikka, 1989), the *Inquirer's range of attention*, at a given moment of inquiry, is the set of questions such that:

- The corresponding presuppositions are already available or can be obtained by a deduction move.
- Inquirer is aware of those presuppositions.

The main problem of interrogative inquiry is the mismatch between Inquirer's range of attention and available Nature's answers, since sometimes Inquirer might ask a question which has no answer, or fails to ask a relevant question whose answer is available.

In the literature on IBL it is usual to distinguish between several kinds of inquiries depending on two parameters: how open is the initial question Q, and how guided is the inquiry (PRIMAS, 2011, pp. 11-12).

Concerning the first parameter, a serious study of types of questions and degrees of openness in relation to the development of the subsequent inquiry is still to be done. Our point is that IMI provides a suitable theoretical framework to carry out such a study.

Concerning the second parameter, we refer to (Hintikka, 1982) in order to learn about all the possible moves the teacher can do in a game broader than an inquiry-game, in which the Student-Inquirer and Nature are but two players among many others (the Teacher being one of them). In particular, a kind of move the Teacher can do in order to guide the inquiry is to manipulate Inquirer's range of attention so that to avoid the mentioned mismatch. Among the possible kinds of manipulation one finds: to ask whether we can deduce a certain proposition from the available information (in order to use it further as a presupposition of a question), to attract attention to a certain (already proved) proposition and to incite to consider it as a presupposition of a question, etc. One might wonder to what extent the Teacher can avoid this manipulation in a IBL process channeled to the acquisition of some propositional knowledge. Concerning this, Genot and Gulz (2015) proved that "a trade-off between success [of the inquiry] and autonomy is unavoidable" (p. 1). Indeed, in one hand, with no manipulation of Inquirer's range of attention success might not be achieved. On the other hand

> The IMI [Interrogative Model of Inquiry] does however warrant the following conclusion: a guaranty that an inquiry learner will be able to solve interrogatively a problem can always be obtained by manipulating the learner's range of attention. (Genot & Gulz, 2015, p. 18)

## OBSTACLES TO IBL

Admittidly, to plan for and support IBL is difficult due to the presence of obstacles of different nature: political, cultural, concerning teacher's view of her own profession, concerning teacher's training, epistemological, etc. Let us use the IMI as a microscope to inspect here two of them.

### Propositional knowledge and models

According to (PRIMAS, 2011, p. 22):

> (…) many teachers find that IBL comes into conflict with the way they learnt science and mathematics in school and at the university, and even with the way they have been teaching science and mathematics for many years. That is, with their beliefs about the nature of mathematics/science and/or their beliefs about teaching of mathematics/science. Probably this will be one of major obstacle you will find.

This is strongly related to the fact that content taught in formal education is, typically, knowledge by acquaintance and knowledge-how, with a bleak lack of genuine propositional knowledge. Therefore, in agreement with the analysis of inquiry previously exposed, this content, not being propositional knowledge, can not be learnt through a proper inquiry. This is the case of the knowledge by acquaintance consisting in knowing the statement of theorems (Pythagorean theorem, Thales theorem, etc.) without proof. It is also the case of the following examples of knowledge-how, which lives just in the realm of syntax, of manipulation of symbols, without being supported by meanings: all kind of *algorithms* of addition, subtraction, multiplication and division of several kind of numbers, algorithms to calculate the greatest common divisor and least common multiple, divisibility criteria for natural numbers, etc. *Summarizing, for a piece of mathematics to be learnt in an inquiry-based fashion, it must be presented as propositional knowledge and, insofar as this kind of knowledge is concerned with true propositions, the presence of models fixing meanings and guiding the inquiry is unavoidable. In other words, without models there are no meanings, without meanings there are no truths, without truths there is no propositional knowledge, and so there is no inquiry.*

One might argue that, after all, these mathematical objects do have a meaning. For instance, the fraction 2/3, applied to an object, refers to any portion of this object equivalent (with respect to a certain, previously fixed, magnitude: volume, mass, area, etc.) to the one obtained after performing the following steps:

i) Split this object into three parts so that they were equivalent with respect to the fixed magnitude.

ii) Take any two of these three parts.

We would eventually agree that some objects, like natural numbers, positive fractions, etc., are attached to a meaning in regular teaching. But I claim that this meaning is not used later to support techniques. This makes the difference: whereas an unexplained technique (for example, to multiply fractions) is just mechanical knowledge-how, a justified one becomes propositional knowledge. Let us illustrate my claim with the case of the product of fractions.

Typically, we say that the product of two fractions, a/b and c/d, is the fraction (a·c)/(b·d), whose numerator (respectively, denominator) is the product of the two numerators (respectively, denominators). Notice that this mirrors the common definition in formal mathematics, where a fraction is defined as an ordered pair (a, b) of integers, with b different from zero, and the product of two fractions, (a, b) and (c, d), is (a·c, b·d). This formal definition, taking part in the praiseworthy human enterprise of

founding Mathematics in Set Theory, does not need any further proof. Actually, being a definition, it can not be proved. But, for the existence of a proof, there is a more important obstacle other than the fact that it is a definition: there is no model-interpretation neither of the term *fraction* nor of the term *product* referring to fractions.

Concerning this we would like to underly two points:

- Once you have a model-interpretation of these terms you can (at least) try 'to prove your definition', namely, you can try to ask the question "Is it true that, according to the fixed meaning of the terms *fraction* and *product*, the product of two fractions is calculated by following the former procedure?"
- The way you answer the question strongly relies on the model-interpretation of the terms.

Imagine, for instance, that your interpretation of the term *fraction* is the one above: the denominator indicates the number of equivalent (according to a fixed magnitude) parts into which a given object has been split, and the numerator indicates the number of these parts you are considering. You still have to give an interpretation of the term *product*. This is a hard task. To beging with, we can say that a *product* is the result of a *multiplication*. Now, what is a *multiplication*? In natural numbers, a multiplication is what we do to calculate an amount of magnitude which has been expressed as *a whole amount of a whole amount*, for instance, to calculate the cardinal of a set which results from the union of 27 sets, each of which has 63 elements. Similarly, we can say that the multiplication of fractions is what we do to calculate an amount of magnitude which has been expressed as a *fraction amount of a fraction amount*. For instance, to know which is the total fraction we are considering when we calculate 2/3 of 4/5 of some amount of a given magnitude? I would know how to answer it if I knew how to answer in the case of 1/3 of 4/5. Similarly, this will not be a problem if I knew how to calculate 1/3 of 1/5. But it is not difficult to calculate that if each of the 5 fifths were divided into 3 parts, then the initial amount would be divided into 15 parts. Thus, 1/3 of 1/5 is 1/15. Now, since 1/3 of 4/5 is 1/3 of 4 times 1/5, we get 4 times 1/15, which is 4/15. And 2/3 of 4/5 is 2 times 1/3 of 4/5, that is to say, 2 times 4/15, which is 8/15. One can see, in this and other examples, that the numerator (respectively, the denominator) of the final fraction can be directly obtained from the first two fractions just by multiplying their numerators (respectively, denominators).

These considerations show that only after having an interpretation of the term *product* (again, it is the result of a *multiplication*, and a *multiplication* is what you do to calculate an amount of magnitude which has been expressed as a *fraction amount of a fraction amount*) you can *prove* the truth of the following proposition: "the product of the fractions a/b and c/d is $(a \cdot c)/(b \cdot d)$". The good news is that models allow to go from knowledge-how to propositional knowledge. The bad news is that the acquisition of this propositional knowledge strongly relies on the chosen model. This is not a minor issue. What to do if we have many possible interpretations of our theoretical terms? Which of them should be considered? Is there any didactic criterion (for ins-

tance, to overcome some learning obstacle) to choose among the different possible interpretations? For instance, the interpretation given above to the term *fraction* is the most usual, but it is not the only one. Should we also consider the others interpretations of the term *fraction*? If so, how to prove the multiplication formula for these interpretations? We face the problem to choose criteria for attaching meanings such that: 1) they help us to fight against undesirable didactic phenomena specific to the usual teaching of fractions; 2) they are compatible with the meanings attached to other numerical fields, like negative or real numbers.

**How to plan and support inquiries for students?**

Even if all the questions raised in the previous paragraph about the implementation of semantics in mathematics syntax are answered, we still have to deal with further difficult problems.

A very first one is: what could be the initial question Q of the inquiry? This question seems to be very difficult to find as it is intended to initiate an inquiry through which many kinds of mathematic entities would appear. Among them, notably, concepts. In this direction, we find the following teacher's claim which figures in (Maaß & Reitz-Koncebovski, 2013, p. 12):

> What about conceptual knowledge -surely students cannot be expected to reinvent mathematical or scientific concepts for themselves?

In terms of Hintikka's IMI, to search for Q amounts to looking for a question such that, together with the premises T (student's previous knowledge) and Nature's answers (derived from mathematical examples possibly examined by students), allows to deduce (as a final product but also as something obtained in the course of inquiry) the aimed mathematical propositions.

But still, even if Q is already clear, as it is said in Anderson's study (as cited in PRIMAS, 2011, p. 20), teachers have difficulties with managing "the challenges of new teacher roles and new student roles". According to Walker's work (cited in PRIMAS, 2011, pp. 20 - 22),

> Teacher loses control: although it depends on the degree of freedom teacher gives to students, it is clear that in IBL students should take control of the lesson.

Also, although the next quotation refers to IBL of science, it is perfectly translatable to mathematics:

> Inquiry based lessons might not "work": there is the risk that experiments do not work, that students collect wrong data and that they will get a wrong idea. In the classical use of experiments, these are carefully planned so that they always work and offer the right exemplification of the phenomena that is at stake.

Therefore, teachers have questions concerning their role in and control of students' inquiries. In terms of Hintikka's IMI, the question is: which would be the dialectics between deductions and questions in the expected inquiry? To have this dialectics relatively planned contributes to prepare the teacher in her duty of manipulating stu-

dent's range of attention, which, in turn, grants the success of the inquiry in order to provide the acquisition of the aimed propositional knowledge.

## PROPOSAL FOR TEACHING IBL TO PRE-SERVICE TEACHERS

In order to solve the problem of finding, given a piece of mathematics to be taught, a good model for the theoretical terms of this content, a good initial question Q and a way to handle the corresponding inquiry, PRIMAS has the so-called *professional development modules* [1]. There is, for instance, a module with examples of questions to initiate an inquiry. There is a module with strategies to promote students' questioning. There is, also, a module with examples of ways of acquiring concepts. But still there is a need of *complete* examples, including all together an initial question, the relevant moves students should do in the corresponding inquiry game, and moments in which the teacher could manipulate well enough student's range of attention in this very game.

As we said at the beginning of this work, the Anthropological Theory of the Didactic (ATD) is one of the proposers of IBL. ATD suggests the implementation of the so-called *paradigm of questioning the world* (Chevallard, 2015). According to it, formal education would be carried out by means of *study and research paths* (SRP). Succinctly, a SRP is the process you follow to find the answer A to a question Q. ATD emphasizes that, along this process, you are involved in different kinds of activities: studying possible (perhaps partial) answers to Q, formulating new auxiliar questions, etc. Although less philosophically informed, ATD's analysis of the notion of *inquiry*, via the notion of SRP, runs almost parallel with that of Hintikka, via the IMI.

There is a continuous spectrum of types of SRP, the extremes of which are what we could call *open* and *closed* SRP. The open ones are those in which the teacher is not specially interested in leading the students towards a particular piece of knowledge O. In contrast, the closed ones are those in which the question has been selected with the intention of leading to the natural emergence, in the course of the SRP, of a certain piece of knowledge O previously selected.

In formal mathematics this knowledge O is not expressed as the output of an inquiry, but as a series of axioms, definitions, theorems, examples and standard techniques. Therefore, closed SRP demand, at least, to reorganise the piece of knowledge O to be found along the inquiry. The corresponding reorganisations are what ATD calls *Reference Epistemological Models* (REM) (see, for instance, Sierra, 2006).

Typically a REM is expressed in terms of *praxeologies* (Chevallard, 2006), that is to say, in terms of: *types of tasks*, *techniques* devoted to face these types of tasks, a *techno-logical* considerations about each technique (a detailed description, a justification, a study of its scope and reliability, possible enhacements) and, possibly, also some *theoretical* considerations about the situations under study (our metaphysical description of them: basic entities, basic properties, etc.). It is a key feature of a REM that essentially everything in it appears motivated by the study of the types of tasks. It is worth mentioning that normally the construction of a REM on a piece of know-

ledge O is not only guided by the aim of using it as a basis of a SRP, but also by the intention of counteracting some undesirable didactic phenomena specific to the usual teaching of O (Gascón, Nicolás, in press).

It is still an open question, but our point is that each REM [2] *implicitly* provides a *complete* example of a possible inquiry, and so a solution to the obstacles to the implemenation of the IBL previously mentioned, namely, the need of interpretations-meanings, the need of a good initial question and the need of knowledge about when and how to guide the students in their inquiry. More precisely, we think each REM *implicitly* proposes: a *model* for the theoretical terms and syntax appearing in the mathematical content O to be studied, a *question* Q to initiate an inquiry, a *proof* (based on Logic and Game Theory) of the fact that the corresponding inquiry would fully cover O, and a set of *moments* of the inquiry at which teacher should evaluate, and possibly manipulate, Inquirer's range of attention. In future works we will try to provide evidences for this point via a logical analysis (IMI-like, in terms of deductions, questions and answers) of some of the published REM.

## NOTES

1. Available at http://www.primas-project.eu/artikel/en/1221/Professional+development+modules/view.do

2. For the moment, there are, among others, published REM on natural numbers, integer numbers, decimal numbers, proportionality, algebra and differential calculus. See http://www.atd-tad.org/grupo-tad/

## REFERENCES


Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM Mathematics Education, 45*, 797—810.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.) *Proceedings of the 4th Conference of the European Society for Research in Mathematics Education* (CERME 4) (pp. 21-30). Barcelona: FUNDEMI-IQS.

Chevallard Y. (2015). Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In: S. Cho (Ed.) *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 173-187). Springer, Cham

Cross, C., & Roelofsen, F. (2016). *Questions*, The Stanford Encyclopedia of Philosophy (Spring 2016 Edition), Edward N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/spr2016/entries/questions/>.

Fantl, J. (2017). *Knowledge How*, The Stanford Encyclopedia of Philosophy (Fall 2017 Edition), Edward N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/fall2017/entries/knowledge-how/>.

Gascón, J., & Nicolás, P. (in press). Economía, ecología y normatividad en la teoría antropológica de lo didáctico [Special issue]. *Educaçao Matemática Pesquissa*.

Genot, E., & Gulz, A. (2015). *The interrogative model of inquiry and inquiry learning*. In B. Can (Ed.), Perspectives on Interrogative Models of Inquiry: Developments in Inquiry and Questions Springer.

Hintikka, J. (1982). A dialogical model of teaching. *Synthese, 51*, 39 - 59.

Hintikka J., Hintikka M. B. (1989) Reasoning about Knowledge in Philosophy: The Paradigm of Epistemic Logic. In: *The Logic of Epistemology and the Epistemology of Logic*. Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science), vol 200. Springer, Dordrecht

Hintikka, J. (1996). *The Principles of Mathematics Revisited*. Cambridge: Cambridge University Press.

Hintikka J., Halonen, I., & Mutanen, A. (2002) Interrogative Logic as a General Theory of Reasoning. In D. M. Gabbay, R. H. Johnson, H. J. Ohlbach, and J. Woods (Eds.), *Handbook of the Logic of Argument and Inference,* vol 1 (295 - 337). Elsevier

Ichikawa, J. J., & Steup, M. (2017). *The Analysis of Knowledge*, The Stanford Encyclopedia of Philosophy (Fall 2017 Edition), Edward N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/fall2017/entries/knowledge-analysis/>.

Maaß, K., & Reitz-Koncebovski, K. (Eds.) (2013) *Primas (Promoting Inquiry in Mathematics and Science Education Across Europe). Inquiry-based learning in maths and science*. Final report prepared by PRIMAS, an international project within the Seventh Framework Program of the European Union. (http://www.primas-project.eu)

PRIMAS (2011). PRIMAS guide for professional development providers. Retrieved from http://www.primas-project.eu/artikel/en/1300/professional-development/view.do

Rocard, M., Csermely, P., Jorde, D., Lenzen, D., Walberg-Henriksson, H., & Hemmo, V. (2007). *L'enseignement scientifique aujourd'hui: une pédagogie renouvelée pour l'avenir de l'Europe*. Commission Européenne, Direction général de la recherche, Science, économie et société.

Sierra, T. A. (2006). *Lo matemático en el diseño y análisis de organizaciones didácticas*. Tesis Doctoral, Universidad Complutense de Madrid.