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► To cite this version:

Imène Ghedamsi, Thomas Lecorre. Students' understanding of ϵ -statements involving equality and limit. Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics, V. Durand-Guerrier, R. Hochmuth, S. Goodchild & N.M Hogstad, Apr 2018, Kristiansand, Norway. pp.34-44. hal-01849943

HAL Id: hal-01849943

<https://hal.science/hal-01849943>

Submitted on 10 Aug 2018

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Students' understanding of ε -statements involving equality and limit

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The purpose of this paper is to explore how students may understand the link between the formalisations through ε -statements of infinite processes in the Archimedean continuum. These processes illustrate either equality or limit. In particular, we focus on the extent to which students perceive the formalisation of the infinite closeness notion in the two processes. The data is collected from an extensive design research carried out at the transition between Calculus course and Analysis course. TDS construct of milieu is deployed to build and to analyse exploratory teaching-experiments. The results put forward how ε -statements may assist students to reconsider their informal understanding of limit.

Keywords: ε -statement, process, limit, equality, milieu.

INTRODUCTION

In the transition between Calculus and Analysis courses, formal definition of the limit is needed not only to establish precise definitions of fundamental notions such as differential, integral, and series, but more importantly, to build up and use formal statements for making formal proofs. Yet, the key question of how to create a rigorous understanding of infinite processes and initiate the use of formal statements remains a challenging issue for researchers in the field of Calculus education.

Considerable research has been conducted on students' difficulties to encapsulate the infinite processes of limit into the formal limit (Tall & Vinner, 1981; Przenioslo, 2004; Roh, 2008; Oehrtman, 2009). Most of this research highlights the impact of students' previous use of informal statements to represent infinite processes both graphically and numerically. Those statements usually involve expressions related to successive computation of terms and closeness such as: *the more is x close to infinity, the more is $f(x)$ close to l* , and inversely. Several other studies have explored the complex structure of the formal limit and have shown multiple aspects that may not help students develop efficient interpretations of formal statements (Cottrill et al., 1996; Durand-Guerrier & Arsac, 2005; Mamona-Downs, 2001; Roh, 2010; Oehrtman et al., 2014). Those aspects fundamentally refer to the role of quantifiers and their order, the arbitrariness of ε and its relation to the other parameter, and the connection between the statements expressing changes in the variables. Some other research have designed tasks to assist students connecting informal and formal statements related to limit (for an overview of the concerned literature, see Bressoud et al, 2016). Specifically, Swinyard (2011) has demonstrated that students are able to reinvent limit using formal statements. Drawing on this study and on the *genetic decomposition of limit* of Cottrill et al. (1996), Swinyard & Larsen (2012) develop a six steps model of how students come to understand the formal definition of limit. This model provides consistent arguments of how students reason about two infinite processes: 1) the process of *finding limit* which

is associated to the first three steps (*as x gets closer to a , $f(x)$ gets closer to L*), and 2) the process of *validating limit* which is described by the formal limit and encapsulated in the last step through the formalisation of the infinite closeness notion (*gets closer to*) via the concept of arbitrary closeness. Swinyard & Larsen hypothesized that limit at infinity may assist students to focus first on variation of the dependent variable and to shift to the validating process. In addition, the focus on the variation of a single variable may improve students' reasoning on the infinite closeness notion. Although we agree with those hypotheses, the empirical data does not outline how students connect informal statements of the first process to the formal statement of the second process. These processes encompass the cornerstone notion of infinite closeness, so why students do not feel the need to formalise the finding process and to emphasize its difference with the validating process and by the way, to understand why quantifiers should be described in such a way? However, Swinyard & Larsen call for research that could investigate students' formalisation of infinite processes in the context of whole-class teaching experiments and beyond the context of reinvention (p.492).

In this paper, we focus on the formalisation of two infinite processes in the Archimedean continuum by using formal statements that we call ϵ -statements. These processes involve the formalisation of infinite closeness notion and illustrate either equality or limit of function at infinity¹. The research of Swinyard & Larsen has served to structure our thought and to rigorously address our central question: to what extent the formalizations of the infinite closeness involved in these two processes and their link may assist students' understanding of formal limit? The aim of this paper is to give some insights on the potency of this link in way that somewhat guarantee students' making sense of formal limit beyond restrictive contexts; this is why we deploy the Theory of Didactic Situations (TDS) constructs to conduct exploratory whole-class teaching experiments in the transition between Calculus and Analysis courses.

THEORETICAL FRAME

The TDS is a model of learning mathematical notions founded on an optimization of the interactions taking place within the system of relationships between students, a teacher, and a mathematical milieu which includes mathematical knowledge (preconstructed tasks, tools, graphs, symbols, etc.), students' prior knowledge, and students' informal understanding. The situation refers to the actual implementation in a classroom of this ideal model (noted Situation with capital S) in accordance with a targeted mathematical notion. The students' work and the teacher management are modeled at several levels according to the nature of the milieu. The expected interactions are materialized through the role of both students and teacher specifically in three particular levels: milieu for action, milieu for formulation and milieu for validation. The efficiency of the interactions among peers is ensured by teacher's enrichment of the mathematical milieu. Depending on the complexity of the targeted

¹ In this paper, the formal statement related to limit of function at infinity is: *The limit of a function f is L at infinity if for all $\epsilon > 0$, there exists $A > 0$ such that for all $x > A$, $L - \epsilon < f(x) < L + \epsilon$.*

notion, the teacher may ask questions and provide some others auxiliary mathematical knowledge without minimizing students' responsibility in producing knowledge (this is the case for example of the formal limit). As mentioned by González-Martin et al. (2014): "*It is important to stress that the central object of TDS is not the **cognizing individual**, but the **Situation**, which shapes and constrain the adaptive processes students can develop, and thus the mathematical knowing which can be constructed.*" (p.118). However, the robustness of a Situation depends fundamentally on the mathematical milieu. The elaboration of this milieu is based upon a consistent epistemological analysis of the targeted notion; this analysis should allow students to experiment motivating questions and to reconsider their informal understanding – test and make conjectures, provide examples and non-examples, and refute formulations. The aim of this research is to design situations to explore how students understand the formalization of two infinite processes that are strongly connected to the natural root of limit idea, and the extent to which the link between those processes may provide some insights on the formal limit. The starting point for the building of the mathematical milieu is the fundamental historical idea of validating equalities by means of infinite processes. The use of those processes provides results (for example, the area of parabolic segments, the sum of infinite geometric progressions, etc.) that would now be dealt with by means of limits and initiates the shift towards the formal limit. If we look to the nowadays structure of equality: $a = b$ if for all $\varepsilon > 0$, $-\varepsilon < a - b < \varepsilon$, we may notice that the link between this structure and formal limit is modest. But, this equality can be applied to a function f using the property P: *There exists $A > 0$, for all $\varepsilon > 0$ such that for all $x > A$, $L - \varepsilon < f(x) < L + \varepsilon$.* If f verifies P then for all $x > A$, $f(x) = L$. However, if we exchange the quantifiers in P, we obtain Q: *For all $\varepsilon > 0$, there exists $A > 0$ such that for all $x > A$, $L - \varepsilon < f(x) < L + \varepsilon$.* Yet, if f verifies Q then the limit of f in plus infinity is L . In this Situation, we focus on the role of ε in ε -statements: it leads "at most" to equality and "at least" to limit depending whether the involved statement contains *there exists $A > 0$ for all $\varepsilon > 0$* or *for all $\varepsilon > 0$ there exists $A > 0$* . In the following, we explain how the constructed milieu concentrates on the formalization of infinite closeness in order to help students to recognize the utilities of formalizing infinite processes through ε -statements. In this milieu, the use of finite limit at infinity helps students to focus on the dependent variable and on the specific role of quantifiers in each ε -statement.

METHODOLOGY

Whole-class teaching experiments

This study is based on extensive design research carried out from 2013 to 2015 at the last year of secondary school in France involving a succession of eight situations related to the limit notion (Lecorre, 2016). The teaching experiments were conducted by one of the two authors serving both as the classes teacher (this author was the official mathematics instructor of those classes) and as a researcher. The teaching experiments took place inside classes' allowed time; each class contains about thirty 17-18 years old students. The teacher-researcher provided the whole-class with preconstructed

tasks and gradually enriched the mathematical milieu asking questions to assist the progression of students' work and giving tools that would help students to address problems. Data consisted of audiotape recordings and copies of students' written work. In this paper, we focus on the transcripts of four successive class sessions that are related to the fifth and sixth situations of the whole design; each class session lasted two hours. The fifth situation is based upon the graph of *the monster* (fig.1) and it is supposed to destabilize students' informal understanding of the infinite closeness notion in the limit process and to trigger the need to formalize this process using ε -statements. The sixth situation is designed in way that students face: 1) the problems of validating equalities and limits candidates using two infinite processes; and 2) the individual subtle formalization via ε -statement of each process depending on whether the statement contains *there exists...for all* or **contains** *for all...there exists*. Prior to taking part in the selected class sessions, the students had participated to the preceding teaching experiments concerning the first four situations of the whole design, and they were already familiar with whole-class discussions. Specifically, the students had constructed informal understanding of limit of function at infinity. Building on the graphs of paradigmatic functions (for example $1/x$), they had investigated limits at infinity by using expressions such as *close to infinity* and *gets closer to*. In addition, they had explored double quantified statements and that double quantifications should be differenced according to the order of the quantifiers and to the convention of interpretation. However, the formal definition of limit is still not introduced to them.

A priori analysis of the monster situation

Students' previous work on the statement $f(x)$ is upper bounded by $g(x)$ "in infinity" led the teacher-researcher to formalize "in infinity" by means of *There exists $A > 0$ such that for all $x > A$* . This formalization which is one of the fundamental elements of the mathematical milieu of this situation is part of prior students' knowledge. The central element of the mathematical milieu is *the monster* (fig.1).

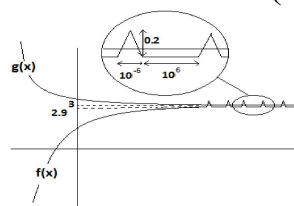


Figure 1: The monster

The monster is the graphical representations of two functions f and g such that f remains below g (g which "soon" becomes a constant) except in rare but regular peaks (every 10^6) where f is over g on small intervals (less than 10^{-6}). In addition, this milieu contains the conjecture C3: *Given two functions f and g having no infinite limits in infinity. If the limit of f is strictly less than the limit of g in plus infinity then there exists $A > 0$ such that for all $x > A$, $f(x) < g(x)$* . The students are asked to answer the core question of this situation: *The monster is an example, a counter-example, a non-example of C3?* The use of graphs allows students to create ideas about the limit process; it also provides them with helpful feedbacks -even if not formal- that may

contribute to reconsider their informal understanding (p.124). The graphs of paradigmatic functions are given by the teacher-researcher during the debate to reinforce the doubt about the meanings that students give to the infinite closeness in limit process. Students' formulations are based on their informal understanding of this closeness. The class discussion may lead to broad agreement about the validity of C3 but some students may remain uncertain considering that it has not been proven yet. This inquiry is not exactly a request for the formalization of the infinite closeness in limit process, but it is the beginning of the awareness that prior understanding about limit process are fragile and have to be formally structured.

A priori analysis of the ε -statements situation

The arbitrariness of ε is the keystone idea of the ε -statements; it founds the equality process and the limit process through the decreasing of ε towards 0. This situation contains three phases, they are planned in way that: the first phase focuses on the formalization of infinite closeness using statements with the only ε ; the second phase deals with the formalization of infinite closeness using statements with ε and other variables; and the third phase highlights the formalization of infinite closeness involved in the limit process by emphasizing the role of quantifiers.

- 1st phase: The mathematical milieu contains the property P for $A = 50$ and $L = 2$ (P1: *For all $\varepsilon > 0$, for all $x > 50$, $2 - \varepsilon < f(x) < 2 + \varepsilon$*), the conjecture C4-1: *If f verifies P1 then for all $x > 50$, $f(x) = 2$* and the conjecture C4-2: *If f verifies P1 then $\lim_{x \rightarrow +\infty} f(x) = 2$* . The students are firstly asked to say what can be concluded if f verifies

P1. Then, the teacher-researcher has the responsibility to enrich the milieu by asking the students whether or not: The function $f(x) = 2 + 1/x$ is an example, a counter-example, a non-example of C4-1? Depending on the evolution of the debate among the class students, the use of the same function should permit to study C4-2. More precisely, students are familiar with the use of graphs to give examples in order to make or to verify conjectures. Graphical representations of functions may lead to the visualization of the closeness of $f(x)$ to 2 by using several values of ε . It is expected that students' validation of C4-1 via a reductio ad absurdum reasoning permits to focus on the arbitrariness of ε as formalizing infinite closeness involved in P1. The study of the function $f(x) = 2 + 1/x$ which does not fit the hypothesis of C4-1 -instead of verifying *there exists A ($A=50$), for all $\varepsilon > 0$ [...]*, this function verifies *for all $\varepsilon > 0$, there exists $A > 0$ [...]*- should support students' formulations about infinite closeness involved in limit process. The discussion of C4-2 should reinforce those formulations and assists students on thinking about the link between the two processes through the notion of infinite closeness. It is rather probable that the choice of A (50 in P1) will be questioned: does any other $A > 0$ and $A \neq 50$ exist in way that the function $f(x) = 2 + 1/x$ fits the hypothesis? The issue related to the values of A will be discussed in the following phase.

- 2nd phase: The mathematical milieu contains the property P which is given for unknown A (P2: *There exists $A > 0$, for all $\varepsilon > 0$ such that for all $x > A$, $2 - \varepsilon < f(x) < 2 + \varepsilon$*), the conjecture C4-3: *If f verifies P2 then for all $x > A$, $f(x) = 2$* and the conjecture

C4-4: *If f verifies P2 then $\lim_{x \rightarrow +\infty} f(x) = 2$.* The students are firstly asked to give some properties of functions verifying P2. Students' formulations may lead to the establishment and the discussion of C4-3 and C4-4. A validation of C4-3 based on reductio ad absurdum proof is not expected; however, the validation will inevitably highlight the formalization of infinite closeness within statement containing ε and another variable (A in P2). The study of C4-4 is supposed to improve students' formulations about infinite closeness in the limit process using the closeness involved in P2. It isn't expected that at this stage students will feel the need to talk about the role of quantifiers; but, when we inverse P2 into P3: *For all $\varepsilon > 0$, there exists $A > 0$ such that for all $x > A$, $2 - \varepsilon < f(x) < 2 + \varepsilon$,* fruitful discussions about the double quantification statements may arise among students; the third phase deals with this inversion.

- 3rd phase: The milieus of the above phases are planned to bring into focus the use of ε -statement of equality to validate a limit candidate and so to stimulate students thinking about the formalization of limit process. In this phase, the milieu focuses on the ε -statement of limit to validate a limit candidate. This milieu contains P3, the conjecture C5-1: *If f verifies P3 then there exists $B > 0$ for all $x > B$ such that $f(x) = 2$* and the conjecture C5-2: *If f verifies P3 then $\lim_{x \rightarrow +\infty} f(x) = 2$.* The students are firstly asked to say whether $f(x) = 2 + 1/x$ is an example, a counter-example, neither an example nor a counter-example of C5-1. Then the teacher-researcher has in charge to add C5-2 and to ask the following question: what do you think about this conjecture? It is expected that the starting point of class discussion concerns the question related to whether $f(x) = 2 + 1/x$ verifies or not the hypothesis of C5-1. Students' formulations may concentrate on the finding of the target A given a specific value of ε ; the validation emerges from the necessity to generalize this argument for each ε . The discussion of C5-1 emphasizes the need to elucidate the link between the formalizations of infinite closeness in both P2 and P3. It is expected that the use of $f(x) = 2 + 1/x$ helps students to catch the subtleties of this link through the inversion of quantifiers. Students' formulations about C5-2 are supposed to concentrate on the formalization of infinite closeness involved in limit process.

Brief description of data analysis method

In the TDS frame, the a priori analysis is important not only to control the data analysis of the experimental situations but mainly to highlight what does not happen as expected specifically by focusing on how students' understanding assist them to progress or not as planned by the situation. In the case of this research, the data analysis is conducted in this spirit and it is organized through two major levels. The global level of the data analysis involved reviewing transcripts paying attention to the potency of the situations to tackle the research question. The global data analysis shows that the situations give students the opportunity to enter on the problem and to test their understanding through actions, formulations and even validations. The planned milieus incite students to express and share their understanding of limit process and to progress towards formal understanding. The social dimension of these situations succeeds to stimulate valuable discussions among students who acted to convince their peers or to be convinced by

them². We take advantage of those discussions to engage on the local level of analysis which focus on students' understanding of infinite closeness and its formalizations in the equality and limit processes. This study is based on the evolution of students' work through the three levels of the milieu (action, formulation and validation) and on the arguments they used to explain their work. In the following section, some results of the local level are exposed and exemplified by generic³ students' utterances that are translated verbatim from French. We mainly focus on students' shared understanding; however, the individual student's understanding is underlined when it is awkward and deep.

RESULTS

Students' understanding of infinite closeness in the limit process

As expected, class's discussion about *the monster* put forward the diversity of students' informal understanding of limit process. Students' actions involve the use of expressions related to closeness such that *approaching more and more, from below, from above, gets closer to*. Yet, Students' argumentations strengthen the need to give more precisions about those expressions. Building on the graphics of prototypical functions (fig.2), the following formulation gains broad agreement about how the infinite closeness should be stated in the limit process: *For this kind of sinus curve no limit, the second, it is sometimes above and sometimes under [...] always going closer to the limit [...] the third function [...] the peaks are shrinking and the values of the function are getting closer to the limit each time*.

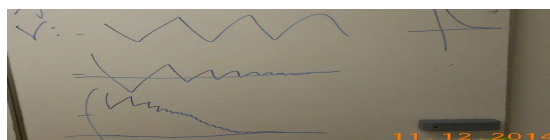


Figure 2: Infinite closeness through 3 graphics

The visualization of infinite closeness through graphical representations helps students not only to share the same meanings but also to get aware of the fragility of their informal understanding. Of course, this is not enough to ensure their engagement in the formalization of closeness in limit process mostly because they have not yet felt the need to validate their limit candidates.

Students' understanding of the formalization of infinite closeness in equality process

To examine students' understanding of the formalization of infinite closeness in equality, we mainly focus on the data analysis related to students' work on C4-1 and C4-3. In the following, the results are organized into two steps depending on whether the statements used refer to the only ε (P1) or not (P2). In both cases, students' work concentrates simultaneously on the process involved in the statement as a way to verify equality as well as a way to validate equality.

² Due to space constraints, the results of the global level of analysis are limited to this description.

³ By generic we mean that it is representative of whole class utterances.

Students' work involving ε -statement with the only ε

The use of several graphical representations assists students' formulations about the role of ε in the statement: *little epsilon, change the value of epsilon*, etc. At this stage, the arbitrariness of ε as formalizing infinite closeness is strongly highlighted and it constitutes the starting point for the shift towards validating the equality. The validation is based upon a graphical *reductio ad absurdum* starting naturally from $x > 50$ (fig.3): *To show that it's true... show that $f(x)$ can't be different from two [...]*.

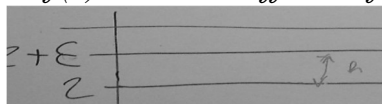


Figure 3: Graphical *reductio ad absurdum*

Students' understanding of the formalization of infinite closeness involved in P1 (fixed $A = 50$) is aided by graphical arguments and emerges from the necessity of both verifying and validating equality.

Students' work involving ε -statement with ε and A

Students' work on P2 (any A) leads to the discussion of C4-3. The students argue on the validity of this conjecture on the basis of the graphical *reductio ad absurdum* specified for a fixed A (50): *It is exactly the same statement with A instead of 50*. This generalization is not yet a proof that students' understanding of the formalization of infinite closeness in equality takes account of quantifiers in P2 statement.

Students' understanding of the infinite closeness in limit process through its formalization in equality process

To study students' understanding of infinite closeness in limit through its formalization in equality, we mainly focus on the data analysis related to students' work on C4-2, C4-4 and the case of $f(x) = 2 + 1/x$. Students' work about whether this function fits or not P1 and P2 is supposed to pave the way for linking equality and limit processes as well as to underline the quantifiers and their order in P2 statement.

Students' work involving ε -statement with the only ε

Students' formulations about whether $f(x) = 2 + 1/x$ fits or not P1 are based on numerical computations and lead soon to the necessity to invalid this example by using the case of $x = 51$ and $\varepsilon = 0.001$. Students' actions on C4-2 are mostly based on the already stated validation of C4-1: *f equals 2 and this result does not give information about the limit of f*. The use of $f(x) = 2 + 1/x$ reinforces the doubt on the validity of C4-2 and some students' formulations about this case permit progressively to highlight the specificities of the relationship between ε and $A = 50$ in P1: *here for all ε there is the same A equal to 50 from which $f(x)$ equal 2 thus $f(x)$ is between $2 - \varepsilon$ and $2 + \varepsilon$ [...] and so the limit is two*. Yet, the involved argument does not provide successful feedbacks among peers. However, the necessity to validate a limit candidate through the use of the equality ε -statement (P1) compels students to reorganize their understanding of the infinite closeness in limit process by taking into account the arbitrariness of ε .

Students' work involving ε -statement with ε and A

Students' validation of the statement $f(x) = 2 + 1/x$ is a non-example of C4-4 is based on a numerical argumentation which is expressed as follows: *I would like to ask those*

who think it is true, to choose an A , any A , and I will be able each time to find a counter example (an epsilon in fact). This argument emphasizes the order of quantifiers in the equality process but students' work on C4-4 is inconclusive mostly because they do not succeed to draw upon the arbitrariness of ε to formalize the limit process. However, their understanding of the infinite closeness is enhanced by the use of P2 as an ε -statement firmly consent with the limit process.

Students' understanding of the role of quantifiers in the formalization of infinite closeness in limit process

These results are mainly based on students' work on C5-1, C5-2 and the case of $f(x) = 2 + 1/x$. They are splitted into two sections: 1) students' interpretations of quantifiers' orders; and 2) the potential sum up of limit process into the formalized ε -statement P3.

Interpretations of quantifiers' orders

Students' work on whether $f(x) = 2 + 1/x$ verifies P3 or not highlights their difficulties to perceive the distinction between P2 and P3 and progressively emphasizes the necessity to take care of quantifiers' orders. Students' firstly act as for P2 to interpret the quantifiers in P3 before focusing on a peer intervention: [...] *the question is written as for all there exists he must give us an epsilon and we have to find an A*. Students' discussions highlight the inversion of quantifiers issue and the need for convention of interpretations. The teacher-researcher intervenes in order to help students finding the targeted A for every given ε and to confirm the invalidity of C5-1.

Sum up of limit process via ε -statement

During the debate concerning quantifiers, students' work with ε -statements is strongly connected to the necessity to answer those questions: given epsilon, how to give A ? Given A , how to give epsilon? In addition, students' work on the validation of the limit candidate involved in C5-2 leads to the formalization of the infinite closeness in limit process. Students' further formulations put forward the need to explore additional question: to what extent this formalization is sufficient to sum up the formal limit?

CONCLUSION

This study examines students' understanding of the formalisations of two infinite processes in the Archimedean continuum by using ε -statements. These formalizations concern with the infinite closeness notion and refer to equality and limit of function at infinity. The aim of this paper is to give data on how these formalizations and their link may support students' understanding of formal limit. We deploy TDS constructs to design situations in which the milieu is built on students' informal understanding of limit and concentrates on the role of quantifiers to differentiate the formalizations given to the infinite closeness in the process of equality and limit, respectively. This study highlights three main results concerning students' understanding of ε -statements: 1) It is possible to organize a milieu that leads students to question their informal understanding of limit process: in this study, the doubt emerges through several ways used by students to perceive the infinite closeness in the limit process; 2) The only formalization of the infinite closeness in the equality process cannot provide insights on its formalization in the limit process. The focus on the quantifiers' orders is crucial

to achieve this formalization; 3) The formalization of the infinite closeness in limit process does not ensure students' sum up of limit process into the formalized ε -statement, this issue needs further investigation. The social dimension of TDS helps students to progressively construct meanings that will constitute the bricks of the meaningful argument which tends to be collectively adopted. In the end, students can admit the irrefutability of the reasoning when all their reluctances are taken into account by their peers.

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