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# **Delineating Aspects of Understanding Eigentheory through Assessment Development**

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*In this report, we share insights we have gained from developing an assessment for documenting students' understanding of eigentheory. We explain the literature and theory that influenced the assessment's development and share question examples. We frame our results in terms of three eigentheory settings ( $A\mathbf{x} = \lambda\mathbf{x}$ ,  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  and eigenspaces) and four interpretations (numeric, algebraic, geometric, and verbal). Results from our analysis include students' reasoning being influenced by setting, insights into students' struggle with understanding eigenspaces, and the importance of making connections between and across various interpretations.*

*Keywords: Teaching and learning of linear and abstract algebra, teaching and learning of specific topics in university mathematics.*

## **INTRODUCTION**

Linear algebra is particularly useful to science, technology, engineering and mathematics (STEM) fields and has received increased attention by undergraduate mathematics education researchers in the past few decades (Dorier, 2000; Artigue, Batanero, & Kent, 2007; Rasmussen & Wawro, 2017). A specifically useful group of concepts in linear algebra is eigentheory, or the study of eigenvectors, eigenvalues, eigenspaces, and other related concepts. Eigentheory is important for many applications in STEM, such as studying Markov chains and modeling quantum mechanical systems; however, research specifically focused on the teaching and learning of eigentheory is a fairly recent endeavor and is far from exhausted.

As part of our ongoing research program analyzing students' understanding of eigentheory (e.g., Watson, Wawro, Zandieh, & Kerrigan, 2017; Wawro, Watson, & Christensen, 2017), we created an assessment instrument focused on the multifaceted and interconnected nature of eigentheory. The purpose of this paper is to describe insights we have gained about students' conceptual understanding of eigentheory as a result of developing, using, and refining this assessment instrument.

## **THEORY AND LITERATURE REVIEW**

We ground our work in the Emergent Perspective (Cobb & Yackel, 1996), which assumes that mathematical development is a process of active individual construction and mathematical enculturation. In this paper we focus on the former by analyzing mathematical conceptions that individual students bring to bear in their mathematical work (Rasmussen, Wawro, & Zandieh, 2015). The literature on the teaching and

learning of eigentheory points to several aspects important to students' conceptual understanding. Here we summarize that literature by highlighting what we found to be important aspects for building a working model for understanding eigentheory.

### **Literature on student understanding of eigentheory**

Thomas and Stewart (2011) found that students struggle to coordinate the two different mathematical processes (*matrix* multiplication versus *scalar* multiplication) captured in the equation  $A\mathbf{x} = \lambda\mathbf{x}$  to make sense of equality as “yielding the same result,” an interpretation that is nontrivial or even novel to students (Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). Furthermore, students have to keep track of multiple mathematical entities (matrices, vectors, and scalars) when working on eigentheory problems, all of which can be symbolized similarly. For instance, the zero in  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  refers to the zero vector, whereas the zero in  $\det(A - \lambda I) = 0$  is the number zero. This complexity of coordinating mathematical entities and their symbolization is something students have to grapple with when studying eigentheory.

Thomas and Stewart (2011) also posit that this struggle to coordinate may prevent them from making the needed symbolic progression from  $A\mathbf{x} = \lambda\mathbf{x}$  to  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ , which is central to determining the eigenvalues and eigenvectors of  $A$ . In their genetic decomposition of eigentheory concepts, Salgado and Trigueros (2015) posit that students need to interpret the procedure of finding eigenvectors and eigenvalues of  $A$  as determining the solution set of the homogeneous system of equations created by the matrix equation  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ . Harel (2000) posits that the interpretation of “solution” in this setting, the set of all vectors  $\mathbf{x}$  that make the equation true, entails a new level of complexity beyond solving equations such as  $c\mathbf{x} = d$ , where  $c$ ,  $\mathbf{x}$ , and  $d$  are real numbers. When considering the notion of eigenspace in particular, Salgado and Trigueros (2015) found that students struggle to coordinate the number of eigenvectors corresponding to a given eigenvalue with the dimension of the space spanned by the eigenvectors. Thus, understanding eigentheory not only involves coordinating mathematical processes and entities but also equations and solution sets.

In addition, students have to make sense of instructors' frequent movement between geometric, algebraic, and abstract modes of description, but this may be challenging (Hillel, 2000). In fact, Thomas and Stewart (2011) found that students in their study primarily thought of eigenvectors and eigenvalues symbolically and were confident in matrix-oriented algebraic procedures, but the majority had no geometric or embodied views. In contrast, other researchers have shown how exploration through dynamic geometry software (Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Nyman, Lapp, St John, & Berry, 2010), geometric interpretations of a linear transformation (Zandieh, Wawro, & Rasmussen, 2017), or real-world contexts (Salgado & Trigueros, 2015) can help students develop conceptual understanding of eigentheory. We similarly agree on the importance of understanding eigentheory concepts in multiple ways and navigating between various interpretations, and we incorporate this complexity in our model of student understanding of eigentheory.

## Working Model of Understanding Eigentheory

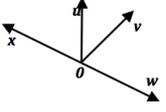
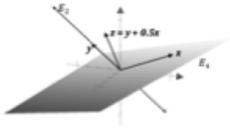
Regarding what it may mean to have a conceptual understanding of eigentheory, our current working model is a network of connections within and across three main settings of how eigentheory is framed. The three sets of relationships that are pertinent are: (1) relationships indicated by the eigen-equation  $A\mathbf{x} = \lambda\mathbf{x}$ ; (2) relationships indicated by the homogeneous form of the eigen-equation  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ ; and (3) relationships indicated by a linear combination of eigenvectors. Within the first two settings, what is most frequently the focus of inquiry is one particular eigenvector  $\mathbf{x}$  for either form of the eigen-equation. However, when considering the collection of all  $\mathbf{x}$  that satisfy either eigen-equation, one arrives at the *eigenspace* of  $A$  associated with  $\lambda$ . The relationships between vectors in the same eigenspace are the focus of the third setting. For instance, if  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are eigenvectors of  $A$  with eigenvalue  $\lambda$ , then all vectors that are a linear combination of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  (i.e.,  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2\} = k_1\mathbf{x}_1 + k_2\mathbf{x}_2$  for scalars  $k_1$  and  $k_2$ ) are also eigenvectors of  $A$  associated with  $\lambda$ . Furthermore, reasoning about relationships in this third setting almost certainly involves reasoning about either the first or second setting as well. Each of these settings includes entities and relationships between those entities that may be realized in various ways. We organize this variability in our model according to four main interpretations: graphical, numeric, symbolic, and verbal.

## THE EIGENTHEORY MCE ASSESSMENT

The development of the Multiple Choice Extended (MCE) assessment instrument for eigentheory grows from our prior work in student understanding of span and linear independence in which we developed the MCE-style question format (Zandieh et al. 2015); questions begin with a multiple-choice element and then prompt students to justify their answer by selecting all statements that could support their choice (see Figure 1). This format was inspired by existing conceptually-oriented assessment instruments in undergraduate mathematics and physics (e.g., Carlson, Oehrtman, & Engelke, 2010; Hestenes, Wells, & Swackhamer, 1992; Wilcox & Pollock, 2013).

Development of the Eigentheory MCE involved multiple steps. First, we compiled a database of questions about eigenvectors, eigenvalues, and related concepts from research on student understanding of eigenvectors and eigenvalues (e.g., Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Thomas & Stewart, 2011), online resources for clicker and classroom voting on linear algebra (Cline & Zullo, 2016), and previous linear algebra homework assignments, exams, and interview protocols used by research team members (e.g., Henderson et al., 2010). Second, we used research results regarding students' understanding of eigentheory from the literature, as well as our own teaching experience and theoretical thinking, to determine which questions seemed to address important aspects of what it may mean to have a conceptual understanding of eigentheory. Third, the most promising questions were edited into the MCE format, which involved moulding the problem into a multiple-choice question and developing six corresponding justification choices that required

students to reason within and between various eigentheory settings and interpretations. Fourth, through multiple rounds of administering the assessment to students, analysing the data, and subsequent refinement, we arrived at the current Eigentheory MCE. It contains six questions, each with six justification choices; five questions are in Figure 1 (the sixth is omitted because of space limitations).

<p>1. The matrix <math>A = \begin{bmatrix} -2 &amp; 4 \\ 2 &amp; 5 \end{bmatrix}</math> has <math>\lambda = 6</math> as one of its eigenvalues. Which of the following vectors is an eigenvector of <math>A</math> with corresponding eigenvalue <math>\lambda = 6</math>?</p> <p>(a) <math>x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}</math> ← CORRECT</p> <p>(b) <math>x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}</math></p> <p>Because ... (select ALL that could justify your choice)</p> <p>(i) This vector <math>x</math> makes <math>Ax = 6x</math> a true statement.</p> <p>(ii) This vector <math>x</math> is the only vector in <math>\mathbb{R}^2</math> for which <math>Ax = 6x</math>.</p> <p>(iii) This vector <math>x</math> makes <math>(A - 6I)x = 0</math> a true statement</p> <p>(iv) Subtracting 6 from the diagonal of <math>A</math> yields this vector <math>x</math> as a column vector of the resulting matrix.</p> <p>(v) The vector <math>Ax</math> is 6 times the magnitude and in the same direction as this vector <math>x</math>.</p> <p>(vi) The matrix <math>A</math> also has <math>\lambda = -3</math> as an eigenvalue.</p>	<p>2. Suppose the vector <math>x \in \mathbb{R}^2</math>, in the two-dimensional sketch below, is an eigenvector of a <math>2 \times 2</math> matrix <math>M</math> with real-valued eigenvalues. Which of the vectors <math>u, v</math> or <math>w</math> illustrated in the figure could be the result of the product <math>Mx</math>?</p>  <p>(a) <math>u</math></p> <p>(b) <math>v</math></p> <p>(c) <math>w</math> ← CORRECT</p> <p>(d) Not enough information is given to know a possible result of the product <math>Mx</math></p> <p>Because ... (select ALL that could justify your choice)</p> <p>(i) This vector is on the same line as <math>x</math>.</p> <p>(ii) This vector and <math>x</math> form a linearly independent set.</p> <p>(iii) This vector is in <math>\text{span}\{x\}</math>.</p> <p>(iv) This vector is a scalar multiple of <math>x</math>.</p> <p>(v) The matrix <math>M</math> needs to be known to determine what <math>Mx</math> could be.</p> <p>(vi) There exists a scalar <math>c</math> such that <math>Mx = cx</math>.</p>	<p>3. Suppose <math>A</math> is a <math>n \times n</math> matrix, and <math>y</math> and <math>z</math> are linearly independent eigenvectors of <math>A</math> with corresponding eigenvalue 2. Let <math>v = 5y + 5z</math>. Is <math>v</math> an eigenvector of <math>A</math>?</p> <p>(a) Yes, <math>v</math> is an eigenvector of <math>A</math> with eigenvalue 2 ← CORRECT</p> <p>(b) Yes, <math>v</math> is an eigenvector of <math>A</math> with eigenvalue 5.</p> <p>(c) No, <math>v</math> is not an eigenvector of <math>A</math>.</p> <p>Because ... (select ALL that could justify your choice)</p> <p>(i) <math>v</math> is a linear combination of eigenvectors that have the same eigenvalue.</p> <p>(ii) The set <math>\{v, y, z\}</math> is linearly dependent.</p> <p>(iii) A linear combination of eigenvectors does not result in another eigenvector</p> <p>(iv) <math>Av = A(5y + 5z) = 5Ay + 5Az = 5 \cdot 2y + 5 \cdot 2z = 2(5y + 5z) = 2v</math>.</p> <p>(v) <math>v</math> is an element of the eigenspace created by the vectors <math>y</math> and <math>z</math>.</p> <p>(vi) <math>Av = A(5y + 5z) = 5Ay + 5Az = 5 \cdot 2y + 5 \cdot 2z = 5(2y + 2z) = 5v</math>.</p>
<p>4. An eigenvalue of the matrix <math>A = \begin{bmatrix} 4 &amp; 2 \\ 1 &amp; 3 \end{bmatrix}</math> is:</p> <p>(a) <math>\lambda = 2</math> ← CORRECT</p> <p>(b) <math>\lambda = 3</math></p> <p>Because ... (select ALL that could justify your choice)</p> <p>(i) The equation <math>(A - \lambda I)x = 0</math> has only the trivial solution <math>x = 0</math> when using this eigenvalue.</p> <p>(ii) This eigenvalue makes <math>(A - \lambda I)</math> noninvertible.</p> <p>(iii) When acted on by matrix <math>A</math>, there is a line of vectors in <math>\mathbb{R}^2</math> that are stretched by the amount of this eigenvalue.</p> <p>(iv) There exists a vector <math>x</math> such that the product of the matrix <math>A</math> and <math>x</math> is the same as the product of this eigenvalue and <math>x</math>.</p> <p>(v) This eigenvalue makes <math>Ax = \lambda x</math> a true statement for <math>x = 0</math>.</p> <p>(vi) This eigenvalue is a solution to <math>\det(A - \lambda) = 0</math>.</p>	<p>5. Suppose a <math>3 \times 3</math> matrix <math>B</math> has two real eigenvalues: for eigenvalue 2 its eigenspace <math>E_2</math> is one-dimensional, and for eigenvalue 4 its eigenspace <math>E_4</math> is two-dimensional. Also suppose that vector <math>x \in \mathbb{R}^3</math> lies on the plane created by the eigenspace <math>E_4</math>, and <math>y \in \mathbb{R}^3</math> lies on the line created by the eigenspace <math>E_2</math>, as illustrated in the graph.</p>  <p>If <math>z = y + 0.5x</math>, which of the following is true?</p> <p>(a) The vector <math>z</math> is an eigenvector of <math>B</math> with an eigenvalue of ____ [fill in the blank]</p> <p>(b) The vector <math>z</math> is not an eigenvector of <math>B</math>. ← CORRECT</p> <p>Because ... (select ALL that could justify your choice)</p> <p>(i) <math>Bz = B(y + 0.5x) = By + 0.5Bx = 2y + 0.5(4x) = 2(y + x) = 2z</math></p> <p>(ii) A linear combination of eigenvectors is always an eigenvector.</p> <p>(iii) <math>Bz</math> is not a scalar multiple of <math>z</math></p> <p>(iv) <math>z</math> does not lie in the plane <math>E_4</math>, or on the line <math>E_2</math>.</p> <p>(v) <math>z</math> is in <math>\text{Span}\{x, y\}</math></p> <p>(vi) The eigenvalue of <math>z</math> is a weighted average of the eigenvalues of <math>x</math> and <math>y</math></p>	

**Figure 1: Questions 1-5 from the Eigentheory MCE**

The MCE questions were created to elicit student thinking about eigentheory within and across the settings and interpretations within our working model of understanding eigentheory. For example, the stem of Question 1 is a numeric interpretation; its given justifications for students to choose as true and relevant, we see that justification (i) is a symbolic interpretation in the  $Ax = \lambda x$  setting, (iii) is a symbolic interpretation in the  $(A - \lambda I)x = 0$  setting, and (v) is a geometric interpretation in the  $Ax = \lambda x$  setting (Figure 1). As students choose justifications that support their answer to the main question, they are prompted to reason about eigenvectors and eigenvalues within and across multiple settings and interpretations.

## METHODS

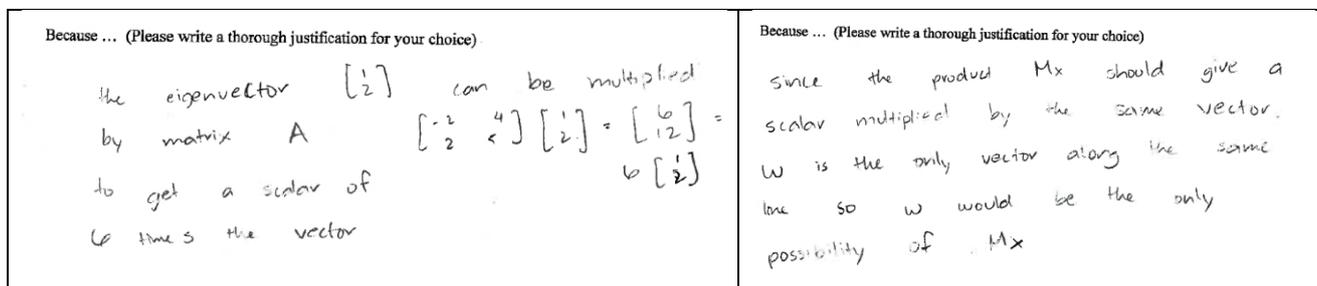
The Eigentheory MCE was given to two introductory linear algebra classes taught by

the same instructor at a large, research-intensive public university in the United States at the end of Spring Semester 2016. The course utilized the Inquiry-Oriented Linear Algebra (<http://iola.math.vt.edu>) curricular materials and Lay (2012) as its textbook. One class (of 29 students) received the MCE with given closed-ended justifications (see Figure 1), and the other class (of 28 students) received an open-ended version where students had to write their own justifications for their multiple-choice answer; we refer to these as Class C and Class O, respectively. Students had 20-25 minutes to work on the assessment. All student work referred to in this paper is labelled “B#.”

Analysis of the closed-ended MCE consisted of entering the data into spreadsheets and looking for trends such as: (a) common sets of justifications that students selected or did not select; (b) how selecting certain justifications may have influenced students’ multiple choice selection; and (c) instances in which we would have expected students to select what we viewed as related justifications, but they did not. We used Grounded Theory (Glaser & Strauss, 1967) to characterize the concepts students brought to bear in their justifications in the open-ended MCE, coding independently and discussing our results as a team to find emerging themes. Finally, we compared students’ responses across questions and across classes to discover further insight into student understanding of eigentheory.

## RESULTS

We include four insights into student understanding of eigentheory discovered from our MCE data analysis. These selected results are organized by settings (sections 1-2) and interpretations (section 3) from our working model of understanding eigentheory.



**Figure 2: B65’s reasoning within the  $Ax = \lambda x$  setting.**

### Reasoning about relationships within $Ax = \lambda x$ or $(A - \lambda I)x = 0$

We found that as students respond to an MCE question, they seem to situate it within a particular setting, perhaps the setting they are most familiar or comfortable with, regardless of the setting in which the question was initially written. Furthermore, a student’s chosen setting can lead to different ways of reasoning about a problem. We present two illustrations of this from Class O: B65’s justifications for Q1 and Q2, and B66’s justifications for Q1 and Q3. First, in Figure 2, B65 seemed to situate both problems within the  $Ax = \lambda x$  setting. On Q1, B65 explained that multiplying the vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  by the matrix  $A$  resulted in six times that vector. On Q2, B65 explained that

$Mx$  needed to be a scalar multiple, and thus the only possible vector would be one along the same line as the vector  $x$ , namely the vector  $w$ . In both cases B65 emphasized that for an eigenvector, multiplying by the matrix yields a multiple of the original vector, thus working within the  $Ax = \lambda x$  setting.

Second, in Figure 3, B66 seemed to situate both problems in the  $(A - \lambda I)x = \mathbf{0}$  setting. On Q1, we infer this student first found the matrix  $(A - \lambda I)$ , multiplied each vector from the MCE question by it, and chose the vector that was mapped to the zero vector. Then, on Q3, although what s/he actually writes is idiosyncratic, we can infer s/he was still reasoning with the homogeneous equation, imagining the vectors  $y$  and  $z$  being mapped to zero by the matrix  $(A - \lambda I)$ , and thus the vector  $v$  would also map to zero. In both Q1 and Q3, B66 emphasized an action on the eigenvectors to produce the zero vector, seemingly invoking the  $(A - \lambda I)x = \mathbf{0}$  setting.

<p>1. The matrix <math>A = \begin{bmatrix} -2 &amp; 4 \\ 2 &amp; 5 \end{bmatrix}</math> has <math>\lambda = 6</math> as one of its eigenvalues. Which of the following vectors is an eigenvector of <math>A</math> with corresponding eigenvalue <math>\lambda = 6</math>?</p> <p>(a) <math>x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}</math>    <math>\begin{bmatrix} -2-6 &amp; 4 \\ 2 &amp; 5-6 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}</math></p> <p>(b) <math>x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}</math></p> <p>Because ... (Please write a thorough justification for your choice)</p> <p><math>\begin{bmatrix} -2-6 &amp; 4 \\ 2 &amp; 5-6 \end{bmatrix} = \begin{bmatrix} -8 &amp; 4 \\ 2 &amp; -1 \end{bmatrix}</math></p> <p><math>\begin{bmatrix} -8 &amp; 4 \\ 2 &amp; -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark</math></p>	<p>3. Suppose <math>A</math> is a <math>n \times n</math> matrix, and <math>y</math> and <math>z</math> are linearly independent eigenvectors of <math>A</math> with corresponding eigenvalue 2. Let <math>v = 5y + 5z</math>.</p> <p>Is <math>v</math> an eigenvector of <math>A</math>?</p> <p>(a) Yes, <math>v</math> is an eigenvector of <math>A</math> with eigenvalue 2.</p> <p>Because ... (Please write a thorough justification for your choice)</p> <p><i>v is a linear combination of y &amp; z</i>  <i>Since the value 2 already causes y &amp; z to equal 0, adding a multiple to it will not change that</i></p> <p><math>v</math> is a linear combination of <math>y</math> &amp; <math>z</math>          Since the value 2 already causes <math>y</math> &amp; <math>z</math> to equal 0, adding a multiple to it will not change that</p>
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**Figure 3: B66’s reasoning with the  $(A - \lambda I)x = \mathbf{0}$  setting.**

We note that the stems of Q1, Q2, and Q3 are not written so as to elicit student reasoning within a particular setting. This allows for use of the open-ended assessment to measure a student’s preferred setting or for the closed-ended assessment to measure whether students can interpret the problem in either setting

### Reasoning about Eigenspaces

The previous section provides examples of the relationships involved in the first two settings:  $Ax = \lambda x$  to  $(A - \lambda I)x = \mathbf{0}$ . In this section we attend to the eigenspace setting, which focuses on the relationships involved with scalar multiples or linear combinations of eigenvectors. An eigenspace, like any vector space, is closed under addition and scalar multiplication; thus, a linear combination of vectors in an eigenspace is also an eigenvector with the same eigenvalue as the other vectors in that eigenspace. When asked about eigenspaces, students may draw on these facts and/or may work within one of the previous two settings to derive these principles.

For Q3, only six (of 28) students in Class O circled the correct answer (a) that  $v$  is an eigenvector with eigenvalue 2, five chose (b) an eigenvalue of 5, and 16 chose (c) that  $v$  was not an eigenvector. Approximately half the students in each group used the phrase “is a linear combination of” as part of their justification (3 for (a), 3 for (b), 7 for (c)). Sample justifications using “is a linear combination of” are given in Figure 4.

Justification given with choosing (a)	Justification given with choosing (c)	Justification given with choosing (b)	Justification given with choosing (b)
$\mathbf{v}$ is a linear combination of $\mathbf{y}$ and $\mathbf{z}$ which have the same eigenvalue.	No, because $\mathbf{v}$ is a linear combination of the two vectors.	$\mathbf{v}$ is a linear combination of $\mathbf{y}$ and $\mathbf{z}$ . Both $5\mathbf{y}$ and $5\mathbf{z}$ are scalar multiples of their previous form so the resultant vector will be an eigenvector as well	Since it is a linear combination of the other eigenvectors, it would also be an eigenvector.

NOTE: Typed versions are used here to improve readability of students' handwritten justifications

#### Figure 4. Example of four students' open-ended justifications for Question 3

Very few students in Class O gave justifications that brought in the relationship between eigenvectors and eigenvalues described in the first two settings. One such student was B66, described in the above section. In Class C, however, 13 (of 29) students chose justification (iv) (symbolic  $A\mathbf{x} = \lambda\mathbf{x}$ ), 11 of which correctly selected answer (a). Because a much higher percentage of Class C circled (a) than in Class O, it is possible that this justification served as a hint that helped some students choose the correct answer. On the other hand, this MCE option allowed us to test whether students recognized the relevance of this set of relationships for the given question.

Although some students who answered (c) used the phrasing "linear combination," their arguments focused more on the linear independence of the vectors. The answers students gave for (c) include: "Eigenvectors must be linearly independent from each other so if  $\mathbf{v}$  is a linear combination of  $\mathbf{y}$  and  $\mathbf{z}$  then it cannot be an eigenvector," [B58], and "Because they all correspond to the same eigenvalue they all must have unique eigenvectors and  $\mathbf{v}$  is a linear combination of  $\mathbf{y}$  and  $\mathbf{z}$  and therefore not unique and not an eigenvector of  $A$ " [B79]. These answers focus on eigenvectors as necessarily being linearly independent or unique. This focus may come from students remembering that eigenvectors of distinct eigenvalues are linearly independent or that textbook solutions often give an eigenspace basis as the final answer, which may explain students thinking there are only finitely many eigenvectors for an eigenspace.

In Q5, eigenspaces were represented geometrically, and students who completed it were rather successful in selecting the correct multiple-choice answer (14/21 in Class C and 21/26 in Class O). However, many students still focused on finite numbers of vectors. On Q5, reasons given by some students to support the correct choice (b) similarly focused on finite numbers of eigenvectors: "Matrix  $B$  already has 3 eigenvectors so there's no room for a 4<sup>th</sup>" [B59], and " $\mathbf{z}$  is a linear combination of  $\mathbf{y}$  and  $\mathbf{x}$ , and there are already 3 eigenvectors for 3 dimensions, so  $\mathbf{z}$  cannot be an eigenvector of  $B$ " [B66]. We conjecture these students may have been conflating the total number of possible eigenvectors (infinite) for a 3x3 matrix with the number of linearly independent vectors needed to create the bases for the 1- and 2-dimensional eigenspaces. Alternatively, B58's justification for Q5 focuses on dimension: "In a 3x3 matrix there can only be 3 dimensions to the eigenspace.  $E_2$  and  $E_4$  together span the entire space of  $\mathbb{R}^3$  so there cannot be another eigenvector of  $B$  besides  $E_2$  and  $E_4$ ." We conjecture grasping the difference between finiteness of dimensions and infiniteness of eigenvectors may be particularly vital for understanding eigenspaces.

## Reasoning Across Interpretations

We conclude our results by discussing various aspects of students reasoning across interpretations and the ways in which the MCE afforded that. In particular, we focus on symbolic versus geometric interpretations of eigentheory. On Q1, as noted in the previous section, a majority of students in Class O wrote at least one equation (symbolic interpretation), but *none* wrote anything geometric in their justifications. This could be an indication that students might favour algebraic reasoning over geometric reasoning when justifying their answers to eigentheory questions, even though the classes used the IOLA curriculum which specifically introduces eigenvectors and eigenvalues geometrically. On the other hand, it could be that the numeric interpretation that Q1 was written in did not elicit geometric interpretations from students in their open-ended justifications, or that students see symbolic justifications as more acceptable to the teacher or the broader math community than geometric ones. In a more direct way of assessing students' ability to see connections to the geometric interpretation, the closed-ended MCE gives students the geometric justification choice (v) on Q1, and roughly half (14/29) of the students in Class C selected it. Furthermore, over 80% of the total students from both classes answered the multiple choice stem of Q1 and Q2 correctly (48/57 and 51/57 respectively), demonstrating some ability to reason both numerically and geometrically about eigenvectors and eigenvalues. Because the wording of the MCE questions and justifications makes use of the four different interpretations from our working model, we are better able to assess students' understanding of the symbolic, numeric, geometric, and verbal interpretations in eigentheory, both within and cross settings.

## DISCUSSION

Research on student thinking often relies on students' written work on mathematics problems as evidence of how students make sense of or reason about particular content. Our research here is no exception, with student work on the MCE revealing a variety of ways that students understand aspects of eigentheory. However, the MCE's closed-ended justifications extend a written question's ability to examine connections between settings and interpretations that students might not have initially considered or felt the need to include in their justifications. For instance on Q1 in Class O, four students wrote some form of  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  as part of their justification, ten wrote some form of  $A\mathbf{x} = \lambda\mathbf{x}$ , four wrote some form of both equations, and ten students did not write either equation. In contrast, on Q1 in Class C, 23 students selected both justifications (i) (symbolic  $A\mathbf{x} = \lambda\mathbf{x}$ ) and (iii) (symbolic  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ ), and only one student selected neither. Hence, when students were forced to consider the two eigentheory settings (i) and (iii), the large majority was able to see both as true and relevant. As other researchers have pointed out the importance of understanding both equations in eigentheory, it is significant that the MCE may give new insight in students' understanding of connections between these two settings.

We do see some potential limitations of the MCE. First, it is more time consuming to

take than a simple multiple-choice test, and this affects the number of questions that can be asked. The MCE can also be cognitively taxing because students must consider each justification to determine its truth and relevance. Third, scoring MCE results can be complicated. We hope that further refinement and use of the MCE, as well as developing possible scoring systems, will continue to broaden and deepen the mathematical community's understanding of how students reason about eigentheory.

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