



# Defining as discursive practice in transition – Upper secondary students reinvent the formal definition of convergent sequences

Alexander Schüler-Meyer

## ► To cite this version:

Alexander Schüler-Meyer. Defining as discursive practice in transition – Upper secondary students reinvent the formal definition of convergent sequences. INDRUM 2018, INDRUM Network, University of Agder, Apr 2018, Kristiansand, Norway. hal-01849923

**HAL Id: hal-01849923**

**<https://hal.science/hal-01849923>**

Submitted on 26 Jul 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# **Defining as discursive practice in transition – Upper secondary students reinvent the formal definition of convergent sequences**

Alexander Schüler-Meyer

TU Dortmund University, Germany

alexander.schueler-meyer@math.tu-dortmund.de

*The investigation of limits is at the heart of analysis at university. Accordingly, it is a worthwhile topic for transition courses. The present study engages upper secondary students in reinventing the definition of convergent sequences. Using a commognitive framework, the central development stages of the definition from experiential to abstract are empirically investigated in terms of activated secondary school discourses. The students' familiarity with secondary school discourses is critical, as it allows them to transition from grasping processes with metaphors towards grasping them as formal and abstract objects. For this, school objects act as intermediate steps. Further studies of transition courses should explicitly address the role of students' secondary notions as resources for reifying processes into abstract objects.*

*Keywords: transition; limits; convergence; practice of defining; design research.*

## **INTRODUCTION**

The investigation of limits is at the heart of analysis on the university level (Cornu, 1991). Accordingly, limits are a worthwhile object of investigation in a transition course. In the German context, students also have previous knowledge about limits in the domain of derivatives from upper secondary school. At school, the students' notions are usually not developed into a formal understanding of limits, as it is not expected in the German curriculum. Students transitioning to university have to develop their formal notion of limits in the regular lecture, which can be difficult. Thus, transition courses located at school have a huge potential for supporting students in developing a more compatible formal and abstract understanding of limits.

Objects in school have an experiential basis, while objects in tertiary mathematics “are specified by formal definitions and their properties reconstructed through logical deductions” (de Guzmán, Hodgson, Robert, & Villani, 1998, p. 753). For different reasons, students in transition often do not use definitions as starting points for their reasoning about objects, as would be expected in tertiary mathematics (Edwards & Ward, 2008, Vinner, 1991). Engaging students in activities of defining objects on a trajectory from experiential to formal approaches might help alleviate these issues.

In this paper, five students in a transition course from secondary to tertiary education in Germany are asked to create a definition of convergent sequences, after having studied them with the model of epsilon strips the week before. The paper illustrates how students' progress from their experiential understanding of convergence towards

a formal definition based on deductions. The qualitative analysis also unfolds how the students use their previous school knowledge for this.

## **DEFINING AS A MATHEMATICAL PRACTICE IN TRANSITION**

### **Students' intuitions about limits**

The transition from secondary to tertiary mathematics has been extensively addressed in terms of changes and obstacles (Thomas et al., 2015). A specific issue of transition in regard to the understanding of limits are the students' intuitions (for typical metaphors: Oehrtmann, 2009). The students' intuitions about limits are not surprising: Historically, mathematicians used their intuitions to think about infinity based on infinitesimals, and this kind of reasoning still permeates modern analysis despite not being accepted as adequate (Cornu, 1991). Students' intuitions are a fruitful starting point for reasoning about limits. Activities of using an epsilon-strip help students to understand the convergence of sequence in terms of neighborhoods, in which are "almost all terms of the sequence" (Przenioslo, 2005, p. 88). They can also help students to understand the logical relations of  $\varepsilon$  and  $N$  (Roh, 2010).

Another issue is the role of pre-formal notions about limits. In upper secondary classrooms in Germany, pre-formal notions of limit are encouraged by teachers and textbooks, as for example  $n \rightarrow \infty$  is commonly referred to as "tends to". By using and extending metaphors of "how many terms make a party" (p. 335) into "an infinite amount of terms will be in that epsilon neighborhood and a finite amount of terms will be outside", this understanding can be developed into a formal understanding of convergence (Dawkins, 2012). It has been illustrated that students can reinvent the formal definition (Swinyard, 2011), but not how students' intuitions and previous knowledge from secondary schools can systematically be activated in this process. Hence, carefully guided activities of investigating sequences with epsilon strips might help students make experiences that connect to their intuitions, but might at the same time be rich enough for students to develop a formal and abstract understanding of limits. While the first aspect has been investigated (see above), the latter aspect of transitioning to the formal and abstract while connecting to previous knowledge and intuitions has not yet been investigated.

### **The practice of defining**

Definitions are the central means for grasping objects on the tertiary level, and the starting point for mathematical reasoning about these objects (Vinner, 1991; Alcock & Simpson, 2002). However, in nearly all mathematical domains, students in transition often rely on their informal understanding of objects, instead of definitions (overview in Thomas et al., 2015). There are several reasons why students have difficulties with defining and definitions. Definitions describe objects in arbitrary ways (Vinner, 1991), i.e. students do not have an experiential basis with the defined objects. Furthermore, students are usually not engaged in practices of defining. Instead, students encounter definitions in processes of proofing or validating (Swinyard & Larsen, 2012). When students can encounter definitions as product of

their own making, it might help them to become aware of the nature of definitions at the university level (Swinyard, 2011).

In sum, students can and should be engaged in practices of defining limits. From a transitional standpoint, practices of defining should be rooted in familiar secondary activities with empirical investigations to create an experiential basis (de Guzmán et al., 1998). Then, practices of defining in a transition course should start with an experiential basis, and let students reinvent an own ‘arbitrary’ but viable definition, and later progress to a more formal and abstract definition.

## **THEORETICAL BASIS: COMMOGNITION AND GUIDED REINVENTION**

### **Commognitive Perspective on Learning and tertiary mathematics**

Within the framework of commognition, students’ participation in discourses is investigated in terms of changes in the ways students discursively realize mathematical objects in their utterances, where such changes constitute learning (Sfard, 2008).

Mathematical discourses can be distinguished from each other by their use of keywords (e.g. “tends to”), of visual mediators (e.g. graphs and symbols), of practices (“routines”, patterned activities like defining or proving) and of narratives (like definitions) (Sfard, 2008, p. 134). Of special interest in the present study is the students’ use of visual mediators and of narratives. Visual mediators are central means for grasping convergent sequences with epsilon strips and with mathematical symbols. Narratives are the means for students to discursively realize relations between objects, or between the facets of the object convergent sequence, like  $\varepsilon$  or  $N$ .

The development of a definition (as a narrative) and its associated mathematical can be characterized through the ways it is discursively realized over time. This development of the definition can be visualized as a “realization tree”. The branches of the realization tree represent the distinct ways in which a definition, in this case about the object convergent sequence, is discursively realized by the students (Sfard, 2008, p. 153), up to a certain point in time in the discourse.

### **Research program of design research and framework of guided reinvention**

The present study follows a guided reinvention approach based on realistic mathematics education. It engages students in context problems in which a mathematical object is the result of own experiences and activities (Gravemeijer & Doorman, 1999). This is a fruitful approach for introducing tertiary mathematical objects (e.g. Dawkins, 2012; and many others). It is located in the research program of design research (Prediger, Gravemeijer, & Confrey, 2015).

The underlying hypothetical learning trajectory builds on the research on the learning of limits. Building on the epsilon-strip activity in the previous session, students are asked to document all facets which are relevant for convergence, for example the height of the strip. Afterwards, the students have to bring these previously found facets into a logical relationship in the form of a narrative. At the same time, the students have to formalize the activity, including the process of finding a limit

candidate. A critical step in this trajectory is the need to progress from an x-first perspective, in which students focus on inputs (x-values) and their respective outputs (y-values), to a y-first perspective. The x-first perspective emphasizes finding a candidate for a limit (Swinyard & Larsen, 2012). In regard to continuity, the y-first perspective and finding a limit candidate is implied in the epsilon-strip activity, as students first choose a strip with a certain height and afterwards arrange it on the sequence. Nevertheless, previous experiences with functions might still guide students to consider an x-first-perspective, so that the y-first perspective needs to be stabilized.

## **Research questions**

In the present study, the students' progression from their experiential notion of convergence with epsilon-strips towards a more the formal, abstract notion of convergence along the outlined trajectory is investigated. The students are engaged in practices of defining, in line with the previous considerations. This encompasses, among others, the following activities: 1. Identifying central elements that should constitute a definition, and baptizing them, and 2. exploring or deducing relations of these elements. Of special interest are the students' ways of building on and connecting with their previous secondary school discourses. The study focuses on the emergent practice of defining, as students' preceding exploration of sequences with epsilon strips has already been studied in other studies (Przenioslo, 2005).

The following research questions will be investigated in the present study:

Q1. What are the crucial steps in the students' learning trajectory from experiential investigations towards a formal definition of convergence?

Q2. How do students activate their previous knowledge from school, and what role does this knowledge play to proceed to a formal, abstract definition?

## **METHODOLOGY**

### **Participants and implementation**

The design research project encompasses three design experiment cycles, the data analyzed in this paper stem from the third cycle in which five students participated (Ludwig, Lawrence, Dominic, Leif and Tanja). These students are highly proficient eleventh-graders, in their penultimate year of upper secondary education. They participate voluntarily in a one-year long transition course, designed by the author for preparing for university STEM-studies. From their regular mathematics classrooms, the students are familiar with an informal understanding of continuity ("Drawing without lifting the pen") and with limits as "tends to". The teaching unit comprised 5 sessions of 90 minutes each, three sessions on convergent sequences, two sessions on continuity. The teaching unit was taught by a Master student with tutoring experiences at university level. All sessions were videotaped and transcribed.

The analysis focuses on Task 2 of Session 2, in which the students attempt to find a formal definition for convergence by drawing upon their activities with the epsilon-

strips (Session 1). Before Task 2, the students recapitulated the activity of using the epsilon-strips with a given convergent sequence. Furthermore, the students have generated a ‘knowledge storage’ in the first session, where they documented the relevant mathematical facets of convergence, and visualized them. Session 2 finishes with the completion of the here investigated task.

### **Analysis of data**

The transcripts of Task 2 in Session 2 were analyzed qualitatively in the framework of commognition (Sfard, 2008). The central steps of the analysis are:

To answer Q1, the collective ways to realize the object convergent sequences over time are analyzed and depicted in form of a realization tree. For that, the transcript is segmented according to the qualitatively different ways in which the students realized the object convergent sequence. As illustrated, changes in the ways of realizing objects with keywords, visual mediators and their relations with narratives in the discourse are indicative of learning. Each branch in the tree hence represents one distinct way of grasping convergence discursively. The progression from one branch of the realization tree to the next one marks a crucial step in students’ learning trajectory, as the students change the discourse about convergence in some substantial regard.

To answer Q2, the transcript is analyzed in regard to episodes in which the students’ discourse builds on school objects, which is indicated by keywords like “function”, “y-value”, and narratives, e.g. the metaphor of “tends to”. These episodes are analyzed in terms of how the students proceed from secondary utterances about school objects towards more tertiary abstract and formal utterances. An object becomes abstract, when its narratives do not refer to empirical phenomena, but to other (previous) mathematical narratives. An object is considered formal, if it is typically realized with a symbol as visual mediator (Sfard, 2008).

## **RESULTS**

### **Different Realizations of convergent sequences and their progression**

After having investigated a convergent sequence for 20 min, the students engage in finding a definition for convergent sequences in Task 2 for the remainder of Session 2. For that, they can build on their “knowledge storage”, in which they documented the central facets of convergence together with their respective graphical representations. The realization tree in Figure 1 illustrates how students progressively realize convergent sequences, from left to right, and achieve an abstract, formal definition (fourth branch). The small rectangles denote the predominant discursive means used. The turn numbers “Tx” under each oval localize the realization branches in the classroom conversation. The conversation lasts 393 Turns.

In regard to the question of proceeding from an experiential notion of convergence towards an abstract definition (Q1), this realization tree reveals several interesting features of the developing practice of defining in transition. At two points in the pro-

cess students specifically go back to the experiential basis of working with the epsilon strip, in the first and third branch of the realization tree. In these instances, the students rely on metaphors to make sense of their experiences. In interaction with the teacher, who explicitly states the rule to not use the metaphor “tends to” after #237, the students adopt the phrase “it exists” from the teacher to form a new narrative about  $A$  in terms of functional relationships, leading to the notion of  $A(m)/N_\varepsilon$ .

In the second and fourth branch of the tree, the students activate their previous school knowledge in order to grasp their experiences in a more formal and objectified way. This will be investigated in more detail in the next section.

### Activation of secondary discourses

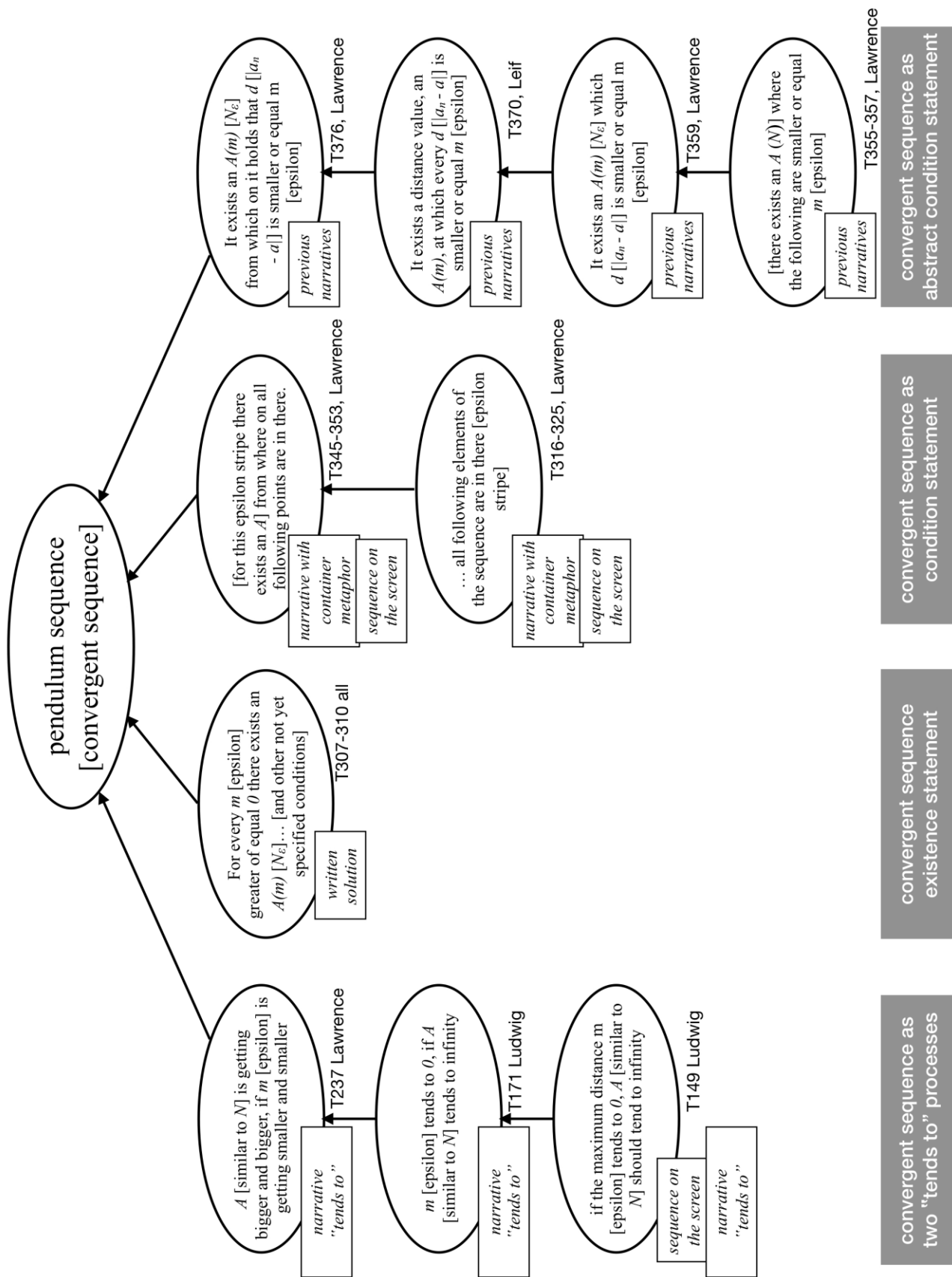
The starting point for the students to define convergence are narratives about limits grounded in “tends to”, resulting in a narrative about two interconnected limit processes: “ $m$  tends to 0 if  $A$  tends to infinity. And we need a target value” (#171-172). In this first branch of the realization tree (Fig. 1), the students rely heavily on the familiar school narratives of “tends to”. Now, the teacher establishes the discursive rule that the students should avoid using “tends to” (#213-230). The students attempt to follow this rule, which leads to a change in the discourse (second branch in Fig. 1). The following conversation occurs right in the beginning of these attempts:

- |     |          |   |
|-----|----------|---|
| 237 | Lawrence | Yes, $A$ is getting smaller and smaller, I mean, bigger, if $m$ is getting smaller and smaller. But we should avoid the tends to. |
| 238 | Leif     | One could try somehow proportional, like proportional-technique to plot it.   |
| 239 | Ludwig   | Dependent from $A$ . Hence, writing in the index.   |
| 240 | Lawrence | $A$ in dependence of $m$ .  |
| 241 | Leif     | I would have said, like, somehow $m$ is proportional to 1 divided by $A$ .  |

In this episode, Lawrence summarizes the result of the previous discussions in terms of “tends to”. Leif proposes to think about the relations in terms of proportionality, and Ludwig and Lawrence pick this up in terms of functional relations.

The students can give up narratives with “tends to” by first replacing the “if...then” relationship with a functional relationship, as indicated by keywords of ‘dependence’. These keywords show that the students’ ideas are rooted in familiar secondary discourses about functional relationships. It seems that these secondary notions are brought into the discourse associatively. Accordingly, the viability of these notions is up to debate, and competing narratives are uttered (#240, 241). This guides Leif to specify his proposal into “ $m$  is proportional to  $1/A$ ”. In this episode, school notions allow the students to collaboratively develop new narratives. They help students to engage in a new discourse in which the discursive rules have changed.

In the following third episode, the students try to decide about the nature of the hypothesized functional relationship.



**Figure 1.** Realizations of the object convergent sequence



- 279 teacher Yes. You already said, what else is dependent on  $m$ ?  
 280 Lawrence Yes,  $A$ .  
 281 Leif We have the dependence of  $m$  and  $A$ .  
 282 Lawrence I write this down now like in short as tends to. [Lawrence writes  $m \rightarrow 0, A \rightarrow \infty$ ]  
 283 teacher What is dependent on what? [...]  
 287 Ludwig  $m$  is the  $y$ -value, thus it is always dependent from the  $x$ -value.  
 288 Lawrence You take, like, the strip and look then, from which value on all of them are in it.  
 289 Leif Yes, but you, you #  
 290 Lawrence We have never taken a value and then looked at how high the strip needs to be. [...]  
 293 Leif Yes, OK, then  $A$  is dependent from  $m$ .  
 294 Lawrence If  $m$  gets smaller,  $A$  gets bigger.

While the students still document their ideas in terms of the narrative of “tends to” (#282), the teacher engages them in thinking again about the functional relationship.

Ludwig treats sequences in terms of properties of functions, namely in terms of  $y$ - and  $x$ -values (#287). Based on this, he concludes that, as with functions,  $m$  has to be dependent from  $A$ , taking a  $x$ -first perspective. Lawrence goes back to activities of using the epsilon strips, and how to choose a strip (#288), proposing an  $y$ -first perspective as implied by the epsilon strip activity. Hence, Ludwig treats sequences as a variation of the familiar secondary objects of functions, but his functional perspective is contested by Lawrence, who enforces his narratives about the epsilon strips and their height and placement (#288, 290). In the end, both narratives are merged as “ $A$  is dependent from  $m$ ” (#293, 294), resulting in an endorsed narrative about  $A(m)$ .

Here, the students’ previous school knowledge about functions and functional relationships is the link between the students’ experiences with epsilon strips and the formal notion of  $A(m)$  as an object. It provides the narrative that  $A$  and  $m$  can be brought together as a combined object  $A(m)$  under a  $y$ -first perspective, and the visual mediator/symbol of  $A(m)$ . However, as Ludwig’s utterance illustrates, treating sequences as functions is at the same time misleading, as it reinforces an  $x$ -first perspective. Above that, there is still an echo of the metaphor “tends to” in  $A(m)$ , as the students use this narrative to summarize their ideas about the functional relationships (#282, #294), suggesting that  $A(m)$  (analogue to  $N_\varepsilon$ ) is an evolution of this narrative.

## Synthesis and Summary

In answer to Q2, secondary discourses seem to have at least two functions in the students’ progression to more abstract discourses (branches of the realization tree in Fig. 1), where abstract means that students more and more endorse narratives *about* narratives instead of narratives about experiences. First, they are the source for the students to develop new discourses with a new set of narratives and visual mediators – in this case about  $A(m)$  [ $N_\varepsilon$ ] – but similar keywords, after the teacher establishes the

discursive rules that “tends to” is not to be used (episode 1). Second, secondary discourses unlock objectified narratives. They allow to transform the narratives about experiences with “tends to” or with “being in the strip” into narratives about abstract objects of  $A(m) [n_\epsilon]$  (episode 2) and about  $d [1/a_n - a/]$  and “ $d \leq m$ ” (Fig. 1). This is an example of *saming*, where “proportionality” (#238) is imported into narratives about experiences with the epsilon strips (Sfard, 2008, p. 170). This step is critical in progressing from an experiential process-notion of relations, which is suggested by the activities of using the epsilon strips, towards understanding these relations as abstract objects. Nevertheless, the teacher is needed to provide the phrases and scaffolds by which the students can engage in new discourses with new narratives, e.g. about logical relations. This is expected, as the students neither have previous knowledge about the typical phrases and narratives from the tertiary level, nor about the practice of defining.

## DISCUSSION

The here presented study contributes to the ongoing investigation of teaching interventions that help students to understand the abstract definition of limits. It especially highlights the role of the students’ previous experiences with secondary mathematical discourses in practices of defining: Students from an upper secondary classroom connect to secondary narratives about limits and other objects (functions) in order to make sense of their experiences with the epsilon strips, and to formulate a definition. The students’ secondary school notions are critical for transposing secondary narratives with metaphors (“tends to”) into more abstract narratives *about narratives* that grasp logical relations. As relatively formal objects, school objects mediate between informal experiences of relationships and a more tertiary, abstract understanding of these relationships as objects, by allowing *saming* (Sfard, 2008).

Investigating students’ practices in terms of familiar secondary discourses and of distinct steps of realizations has proven highly insightful to understand students’ resources in transitioning to tertiary mathematical discourses. The results of the present study call for investigating students’ transition to tertiary mathematics not only in terms of difficulties, but also in terms of students’ (secondary school) resources and how these resources are situationally activated when the discursive rules change.

## REFERENCES

- Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics*, 22(2), 28-34.
- Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153-167). Dordrecht, The Netherlands: Kluwer Academic.
- Dawkins, P. C. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *The Journal of Mathematical Behavior*, 31(3), 331-343.

- De Guzmán, M., Hodgson, B. R., Robert, A., & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. In G. Fischer & U. Rehmann (Eds.), *Proceedings of the international Congress of Mathematicians* (pp. 747-762). Berlin: Documenta Mathematica.
- Edwards, B., & Ward, M. B. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 223-232). Washington, DC: Mathematical Association of America.
- Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, 39(1-3), 111-129.
- Oehrtmann, M. (2009). Collapsing Dimensions, Physical Limitation, and Other Student Metaphors for Limit Concepts. *Journal for Research in Mathematics Education*, 40(4), 396-426.
- Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: an overview on achievements and challenges. *ZDM Mathematics Education*, 47(6), 877-891.
- Przenioslo, M. (2005). Introducing the Concept of Convergence of a Sequence in Secondary School. *Educational Studies in Mathematics*, 60(1), 71-93.
- Roh, K. H. (2010). An empirical study of students' understanding of a logical structure in the definition of limit via the  $\varepsilon$ -strip activity. *Educational Studies in Mathematics*, 73(3), 263-279.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.
- Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *The Journal of Mathematical Behavior*, 30(2), 93-114.
- Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of Limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, 43(4), 465.
- Thomas, M. O. J., de Freitas Druck, I., Huillet, D., Ju, M.-K., Nardi, E., Rasmussen, C., & Xie, J. (2015). Key Mathematical Concepts in the Transition from Secondary School to University. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 265-284). Cham: Springer.
- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). New York, NY et al.: Kluwer Academic.