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What are solid findings in mathematics education?

Tommy Dreyfus

Relying on earlier studies by an ICMI Study and the Education Committee of the EMS, the question what the term ‘solid finding’ might mean with respect to mathematics education is discussed and criteria are proposed. Examples are provided for solid findings that mathematics education research has produced.

Introduction

Mathematics education as a research community has grown over the past approximately 50 years: ERME, The European Society for Research in Mathematics Education is approaching its 20th anniversary in 2018 – CERME1, the first conference took place in Osnabrück, Germany, in August, 1998. PME, the International Group for the Psychology of Mathematics Education has held its 40th annual conference in 2016 - the first one took place in 1977 in Utrecht, The Netherlands. JRME, the *Journal for Research in Mathematics Education*, is now producing its 48th annual volume, and ESM, *Educational Studies in Mathematics* is currently in its 50th year of publication since Volume 1 appeared in 1968. One of the characteristics of research results in (mathematics) education is that they depend on the context in which the research has been designed and carried out. Nevertheless, after 50 years, one would expect the community to be able to make statements that go beyond “it depends on the context and the learning environment”, which is often implicit in the results of even high quality research articles. Review articles could be expected to remedy this situation to some extent but few review articles are published in the domain.

What are the results of research in mathematics education – ICMI Study 8

The question whether we, as a research community, have obtained results with a certain scope, range or breadth of validity and what these results are, has been approached at least twice, once in the framework of the study conference of ICMI Study 8 in 1996 (ICMI stands for the International Commission on Mathematical Instruction), and a second time in the framework of the Education Committee of the European Mathematical Society (EMS) in 2011.

The task assigned by ICMI to the Study 8 program committee was to discuss *what is research in mathematics education and what are its results*. The title of the book published two years later as outcome of the study is *Mathematics Education as a Research Domain: A Search for Identity* (Sierpinska & Kilpatrick, 1998). Maybe significantly, the word ‘results’ has disappeared in the process. Nevertheless, one of the working groups at the study conference dealt with results (Dreyfus & Becker, 1998). One of the questions the working group dealt with was what counts as result; the term ‘solid’ did not appear. Rather, ‘result’ was interpreted as ‘significant result’.

Working group members agreed that without a question, there can only be facts but no results. Results are more than data: They are based on data collected with questions in mind that have been asked within a theoretical framework, and consist of findings interpreted in that theoretical framework. Effects alone (e.g., statistical differences in achievements between different groups) are not results. In mathematics education, we need to *explain* the differences, not only show them. We

need to identify the variables of the didactic situation in order to combine the different facts into a coherent network of reasons, which informs the circular process of understanding the learning of mathematics and thus improving its teaching. Hence, results are often theoretical as well as experimental. Many of our theoretical frameworks are mathematics-specific (e.g., process, object, procept), and therefore our research questions and results are often domain-specific.

Context was seen as relevant with respect to theory as well as beyond: mathematical context (contents, concepts, symbols, representations and epistemological status), the community, the educational system, among others. Results might be tied not only to the theoretical framework but also to the institution that asked the question. It is not the result itself, but the conditions under which it was obtained, that make it significant.

The contextual nature of results implies that results are neither universal nor eternal, that their validity is situated in space and time, and that we have to be careful when trying to generalize. The validity of a result depends on the interpretation within a theory, and the theory might change with time and place, with mathematical content, learning environment, and so on. Hence results are permanent but their relevance might be ephemeral.

Characterising solid findings

It is on this background that the members of the Education Committee (EC) of the European Mathematical Society (EMS) asked themselves what solid findings mathematics education has produced. While the question was motivated by the intention of the committee to present mathematics education to mathematicians, in particular to EMS members, with an interest in mathematics education, committee members were well aware that the exercise of identifying solid findings contributes to establishing and organizing mathematics education as scientific discipline and to determining where we come from, where we are and where we might go as a research community.

Of course, the first, and possibly most difficult task of the EC was to discuss, agree and explain what they meant by ‘solid findings’. A major difficulty in defining what it means that a result is solid is the context dependence, mentioned above. A second and related difficulty is complexity. As we know well, things are more complex than one might think; we know, for example that the mathematics taught and learned in parallel classes with a similar population according to the same curriculum may be quite different (e.g., Even & Kvatinsky, 2010; Pinto, 2013). A third, and of course also related difficulty is that much of the research in mathematics education is qualitative. Since qualitative empirical research cannot be repeated in a strict sense, reproducibility is replaced by the question how close the results are that one obtains in similar contexts; and the answer to this question of course depends on the metric used to measure closeness. This lack of reproducibility may appear as a serious drawback of mathematics education’s claim to be a scientific discipline; however, reproducibility has recently been shown to be very low even in many hard sciences such as physics, chemistry and engineering (Baker, 2016).

Aware of these difficulties and with the ICMI 8 study characterization of (significant) results as background, the EC has observed that results with the potential of being considered solid usually do not stand alone but have emerged from a line of research consisting of a larger set of related studies.

Solid findings are typically yielded by such a line of studies. Next, the EC has built on three properties of research quality proposed by Schoenfeld (2007 – see there for a much more detailed discussion): trustworthiness, generality and importance. Each of these contributes to the solidity of research results. A characterisation adapted to the purposes of the EC was agreed upon and published in the Newsletter of the EMS (Education Committee of the European Mathematical Society, 2011a). I summarize this characterization here, adapting and supplementing it for the purposes of the present CERME panel.

Trustworthiness includes the explanatory power of research, its rigor and specificity, and whether it makes use of multiple sources of evidence. However, a study may be trustworthy but trivial, in terms of generality or importance.

Generality (or scope) refers to the question: What is the scope or generality of a research result? How widely does this finding, this idea, or this theory apply across content domains, learning contexts, cultures, etc.? For example, did researchers, in different countries and school systems obtain comparable or related empirical results? Do theoretical constructs turn out to be useful beyond the bounds of the individual studies in which they were developed?

Trustworthiness and generality together are expected to impart some predictive power to a result. A result that has no predictive power cannot be considered solid. On the other hand, the difficulties mentioned above, such as context dependence, will usually limit this predictive power. If a result is used to predict an outcome in a new context, and the prediction failed, a trustworthy explanation of the failure may in fact increase the solidity of the result.

Importance addresses the question: Does it matter? What is the (actual or potential) contribution of the research to theory and practice. Of course, importance is to a large extent a value judgment. As in any other field of study, beliefs about what is essential and what is peripheral are not static but change over decades, reflecting trends both within and beyond the discipline. Hence recognition of the significance of the result by experts contributes essentially to the solidity of a result.

The term ‘solid’ may remind the reader of the term ‘robust’ often used in related situations. ‘Robust’ often has a technical meaning that refers to a finding having been repeatedly observed or confirmed in many studies reporting the same or similar results leading to the same (general) conclusions (see Primi, below). The term solid has been chosen intentionally, to refer to results rather than findings, and imply that robustness in the technical sense is not possible, nor maybe desirable, in mathematics education.

While robustness can be defined and hence (dis-)proved, solidity cannot. The above is a characterization or description – not a definition. Hence, solidity cannot be proved but it can definitely be argued by on the basis of the above criteria of significance, trustworthiness, generality and adaptability to context.

Examples

The second major task of the EC with respect to solid findings was to provide a variety of examples of findings that are solid according to the EC’s characterization. While the selection of the examples to be presented was somewhat eclectic and partly determined by EC members who were willing to write about a topic, the EC as a whole discussed and approved the proposed topics; the EC also

revised every draft several times. As result a sequence of brief articles has been published presenting a rather representative selection including solid findings about cognition and about affect, about teaching and about learning, about elementary school and about university, about specific mathematical contents and about cross domain issues such as the use of technology, and maybe most importantly about theoretical and about empirical results. Most of the issues of the EMS newsletter from Issue 82 (December 2011) to Issue 95 (March 2015) present such a solid finding.

Here, I briefly present two of these, one reason for my choice again being personal preference and the other representativeness, at least in the empirical – theoretical dimension.

Do theorems admit exceptions?

Empirical studies on students' conceptions of proof have found that many students provide examples when asked to prove a universal statement. Universality refers to the fact that a mathematical claim is considered true only if it is true in all admissible cases without exception. A student who seeks to prove a universal claim by showing that it holds in some cases is said to have an *empirical proof scheme*. The same student is also likely to expect that a statement, even if it has been 'proved', may still admit exceptions. There is considerable evidence that many mathematics students, and some mathematics teachers, rely on validation by means of one or several examples to support general statements. The majority of students who begin studying mathematics in high school have empirical proof schemes, and many students continue to act according to empirical proof schemes for many years, sometimes into their college years.

While the issue of empirical proof schemes has already been mentioned by Polya (1945), Bell (1976) may have been the first to report an empirical study about students' proof schemes. Following Fischbein's (1982) seminal investigation on universality, the issue has been re-examined many times, usually with similar results. For example, findings by Sowder and Harel (2003) indicate the appearance of empirical proof schemes among university mathematics graduates.

The phenomenon of empirical proof schemes is general in the sense that it has been found in many cultures, countries, school systems, and age groups. It is persistent in the sense that many students continue to do so even after explicit instruction about the nature of mathematical proof. However, it is also complex. For example, the London proof study (Healy and Hoyles, 2000) showed that even for relatively simple and familiar questions, 14-15 years old high-attaining students' most popular approach was empirical verification but that many students correctly incorporated some deductive reasoning into their proofs and most valued general and explanatory arguments.

How can this pervasive phenomenon be explained? The notion of a "universally valid statement" is not as obvious as it might seem to mathematicians. Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences, as well as everyday experience. In everyday life, the "exception that confirms the rule" is pertinent. Students, in particular young children, have little experience with mathematics as a wonderful world with its own objects and rules. According to Fischbein (1982), the concept of formal proof is completely outside mainstream thinking, and we require students to acquire a new, non-natural basis of belief

when we ask them to prove. These explanations contribute to the trustworthiness of the findings on empirical proof schemes.

In summary, the studies on empirical proof schemes, only a few of which have been referred to here, firmly establish the solidity of the phenomenon of empirical proof schemes. (For a more detailed exposition, see Education Committee of the European Mathematical Society, 2011b.)

Concept images in students' mathematical reasoning

Vinner and Hershkowitz (1980) were the first ones to point out that students' geometrical thinking is frequently based on prototypes rather than on definitions. They have shown, for example, that junior high school students tend to think that the altitude has to reach the base (rather than its extension). Hence, they draw the altitude inside the triangle, even in a triangle with an obtuse base angle. Students' prototype altitude is one that is inside the triangle. This is so, even if the students know and can recite the (general) definition of altitude in a triangle.

Authors from many countries have reported, over the past 35 years, analogous patterns in students' reasoning in other areas of mathematics, even among talented students in elementary school, high school and college. For example, and in spite of 'knowing' the appropriate definitions, students tend to act according to rules such as multiplication makes bigger, inflection points have horizontal tangents, definite integrals must be positive, and sequences are monotonous.

A commonality in these and parallel studies is that students do not base their reasoning on the definition of the concepts under consideration (even though they are often aware of these definitions and can recite and explain them) but rather on what Tall and Vinner (1981) have called their *concept image*: "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). A student's concept image need not be globally coherent and may have aspects which are at variance with the formal concept definition.

The notion of a student's concept image is complex since it is influenced by all of this student's experiences associated with the concept. These include examples, problems the student has solved, prototypes the student may have met substantially more often than non-prototypical examples, and different representations of the concept including visual, algebraic and numerical ones. Images may deeply influence concept formation. As a consequence, the concept image is personal and continuously changing through the student's mathematical experiences.

How can this pervasive phenomenon be explained? While it is not possible to introduce a concept without giving examples, particular instances of the concept never suffice to fully determine the concept. As a consequence, specific elements of the examples, even if not pertinent to the mathematical definition of the concept, become for the student key elements characterizing the concept. And even if at the stage a concept is introduced a teacher might make an effort to present a rather varied set of examples, as the concept is being used over the coming months or years, some of these properties tend to be reinforced because they appear much more frequently than others that may recede. Examples abound, and the height of a triangle being vertical in the sense explained above is a typical one. Students may see many triangles in which the altitude is inside the triangle, and few in which it is not. They might consider these few cases as exceptions (Lakatos might say

monsters). This explanation contributes to the trustworthiness of the findings on empirical proof schemes.

In summary, a solid finding of mathematics education research, supported by dozens of studies in many different contexts, is that students' mathematical reasoning is frequently based on their concept images rather than on a mathematical concept definition. A more detailed exposition of this solid finding has been published elsewhere (Dreyfus, on behalf of the EC of the EMS, 2014).

Conclusion

The list of solid findings presented by the EC of the EMS is, of course, not exhaustive but limited by the time of service of the committee and the people who served on it. I would like to encourage CERME members (and other researchers) to write and publish articles about solid findings they are aware of and consider important. This might have the desirable effect of producing a type of article lacking almost completely from our literature – review articles. Let me make just one suggestion: Work to raise the awareness of issues and of research on teaching and learning among university lecturers and tutors is necessary; it usually improves students' attitudes but effects on learning are limited. Research in at least four countries (USA, Germany, England, Finland) has shown that work with students has more potential for large scale effects. It seems to me that a suitable review article might not only inform many mathematics educators of an important line of research but might have a considerable effect on university teaching centres, an effect that a single study report could (and should) never have.

In conclusion, the researchers and teams referred to above have shown that mathematics education has, over the past 50 years, produced theoretical and empirical results that are solid in the sense that they have explanatory and predictive power, that they can be applied in contexts beyond those involved in previous studies, and that they are recognised as important contributions that have significantly influenced the research field, for example by providing a theoretical lens that allows to see an observed phenomenon differently from how it was seen before.

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