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► **To cite this version:**

Lieven Verschaffel, Joke Torbeyns, Bert de Smedt. Young children's early mathematical competencies: Analysis and stimulation. CERME 10, Feb 2017, Dublin, Ireland. hal-01849595

HAL Id: hal-01849595

<https://hal.science/hal-01849595>

Submitted on 26 Jul 2018

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Young children's early mathematical competencies: Analysis and stimulation

Lieven Verschaffel, Joke Torbeyns, and Bert De Smedt

University of Leuven, Belgium, lieven.verschaffel@kuleuven.be

In this paper we take a critical look at the state-of-the-art in the research domain of early mathematical development and education. We start with a brief review of the influential and successful (neuro)cognitive research in this domain - which is heavily focused on the development and teaching of children's (non-symbolic and symbolic) magnitude representation and strongly dominated by the theory of an approximate number system (ANS). We confront and complement this (neuro)cognitive approach with various other lines of research that may help to provide a more comprehensive picture of the development and stimulation of children's early mathematical competence and how it relates to their later mathematical proficiency at school.

Keywords: Early mathematics, approximate number system, number concepts, mathematical patterns and structures, preschool education.

Introduction

The past 10-15 years have witnessed the emergence of a remarkably productive and highly influential line of research on children's early numerical magnitude representation, its development, its relation to school mathematics, and its assessment and stimulation (Torbeyns, Gilmore & Verschaffel, 2015).

The starting point of this line of research - which has its origins in cognitive (neuro)psychology -, is the idea that young children, like many other species, are equipped with some foundational innate core systems to process quantities. This "starter's kit" is thought to involve (a) an "object tracking system" that has a limit of three or four objects and is thought to underlie "subitizing" (= to immediate and accurate estimate of one to four objects without serial enumeration), and (b) an "analogue number system" – for the internal representation of numerical magnitudes as Gaussian distributions of activation on a "mental number line" with increasingly imprecise representations for increasing magnitudes (Dehaene, 2011) - allowing them to compare non-symbolic quantities that are too numerous to enumerate exactly or to perform some very basic approximate arithmetic on these quantities (Andrews & Sayers, 2015; Butterworth, 2015).

With these foundational core number sense systems, these magnitudes are represented non-verbally and non-symbolically, but, over development and through early (mathematics) education, verbal and symbolic representations are gradually mapped onto these foundational representations, to evolve into a more elaborated system for number sense (Torbeyns et al., 2015).

People's numerical magnitude representations are commonly assessed via magnitude comparison and/or number line estimation tasks, of which there exist both non-symbolic and symbolic versions (Butterworth, 2015; Andrews & Sayers, 2015; Torbeyns et al., 2015). Examples are shown in Figure 1.



Figure 1: Example of a non-symbolic magnitude comparison and a symbolic number line estimation task

During the past decade, several research teams have set up correlational, cross-sectional and longitudinal studies to determine the contribution of children's numerical magnitude understanding - sometimes in combination with other specific early numerical competencies (such as subitizing, counting or numeral recognition) - to their concurrent and/or later overall mathematical achievement or to specific parts of it such as mental arithmetic or algebra (see, e.g., Bailey, Geary, & Siegler, 2014; De Smedt, Verschaffel, & Ghesquière, 2009; Jordan, Glutting, & Ramineni, 2010; Nguyen, Watts, Duncan, Clements, et al., 2016; Reeve, Reynolds, Humberstone, & Butterworth, 2012). These studies have demonstrated that children's numerical magnitude understanding is positively related to their concurrent and future mathematics achievement in general or in particular subdomains of mathematics.

Two recent meta-analyses have yielded a good overview of the outcomes of this research on the association between various measures of children's numerical magnitude understanding and their concurrent and future mathematics achievement. Schneider, Beeres, Coban, Merz, et al. (2017) performed a meta-analysis on the research about the association between performance on the magnitude comparison task and measures of mathematical competence. Their literature search yielded 45 articles reporting 284 effect sizes found with 17,201 participants. The results support the view that magnitude processing is reliably associated with mathematical competence as measured at least up to the end of the elementary-school years and by a wide range of mathematical tasks, measures and subdomains. Furthermore, the effect size was significantly higher for the symbolic than for the non-symbolic magnitude comparison task and decreased very slightly with age. So - the authors conclude - symbolic magnitude processing might be a more eligible candidate than non-symbolic magnitude processing to be targeted by diagnostic screening instruments and interventions for school-aged children and for adults. The association was also higher for mathematical competences that rely more heavily on the processing of magnitudes (i.e., early mathematical abilities and mental arithmetic) than for others (i.e., more general curriculum-based tests).

Schneider, Merz, Stricke, De Smedt, et al. (submitted) performed a similar meta-analysis for the association between people's score on the other main task to assess numerical magnitude processing skills, namely the number line estimation task, and mathematical competence. Using exactly the

same analytic procedure, and working with a set of 37 studies, they found that the correlations with mathematic competence - both in general and for particular parts of the curriculum - were significantly higher for number line estimation than for symbolic magnitude comparison or for non-symbolic magnitude comparison. Whereas the correlations did not substantially increase with age for comparison, an increase with age was found for number line estimation, which suggests that different underlying cognitive systems and processes are involved in magnitude comparison vs. number line estimation.

Furthermore, researchers working within this research tradition have tried to stimulate children's mathematical skills with (game-based) intervention programs that were (primarily or exclusively) aimed at enhancing their numerical magnitude understanding before or at the beginning of formal instruction in number and arithmetic in elementary school. While some intervention studies have resulted in positive effects (e.g., Kucian, Grond, Rotzer, Henzi, et al., 2011; Ramani & Siegler, 2011; Wilson, Dehaene, Dubois, & Fayol, 2009), the overall results are mixed (Torbeys et al., 2015).

Being well aware of the prominence of this line of research in the international research of early mathematics education, the IPC of the 23rd ICMI study on "Whole number arithmetic" invited one of the leading scholars in that line of research, namely Brian Butterworth, as a plenary speaker of the conference, which took place in June 2015 in Macau, China. In his plenary lecture Butterworth (2015) presented a very informative overview of this cognitive (neuro)scientific line research, and strongly defended this research in the working groups and panels wherein he participated. However, at that conference, it also became clear that the dominant picture of early mathematical competences and education in current mainstream (neuro)cognitive research is dangerously narrow. In the present paper, we will try to broaden that picture in multiple ways. In doing so, we will partly rely on recent and current work done in our own research group, but also on the work of many colleagues who have been active in the field of early mathematics education during the past decade(s).

The ordinal and measurement aspect of number

A first important feature of the line of research summarized above is its focus on the cardinal aspect of number, or, to state it differently, its neglect of other constituent aspects of number, particularly its (1) ordinal and (2) measurement aspect. Hereafter we discuss these two neglected aspects.

The distinction between the *ordinal* and cardinal aspect of number knowledge is well known. Whereas cardinality refers to the capacity to link number symbols to collections, e.g., to know that four or 4 is the correct representation to denote a group of four objects, ordinality refers to the capacity to place number words and numerals in sequence; for example, to know that 4 comes before 5 and after 3 in the sequence of natural numbers. Given the wide recognition of the importance of ordinality for the constitution of number since Piaget (1952) developed his theory of children's concept of number, it is remarkable that, until recently, the ordinality aspect of number seems largely neglected in the above mainstream cognitive neuroscientific conceptualization, assessment and instruction of early numerical abilities.

Interestingly, recent neuroscientific evidence shows that accessing ordinal information from numerical symbols (e.g., decide whether three numbers are in order of size) relies on a different

network of brain regions and shows qualitatively different behavioral patterns when compared to the cardinal processing of magnitudes or numerical symbols or to the ordinal processing of perceptual magnitudes (Lyons & Beilock, 2011, 2013). And, how well a child is able to reason about ordinal relations between number symbols has been found to be one of the strongest predictors of mathematical skill such as mental arithmetic (Lyons, Price, Vaessen, Blomert, & Ansari, 2014) – much stronger, by the end of the first grade of elementary school, than non-symbolic or symbolic cardinal processing as measured by the numerical magnitude comparison task. So, the idea that emerges from this recent neuroscientific research is that children’s sense of ordinality of number symbols may be distinct from their sense of cardinality and, in terms of developing skills needed for success in mathematics, that ordinality may even be the more significant one (Sinclair & Coles, 2015, see also Vogel, Remark, & Ansari, 2015).

This line of research pointing to the importance of ordinality also led math educators to criticize the mainstream neuroscientific view on how children’s early number sense may be stimulated. The latter view suggests that working on linking symbols to sets of objects may reinforce the very way of thinking that young children need to overcome to become successful in school mathematics. But this is the current practice in many countries, where the emphasis in early mathematics education is firmly on linking number symbols to collections of objects - whether this is done through subitizing or counting. Based on the above theoretical and empirical arguments, Sinclair and Coles (2015, p. 253) asserted that this emphasis on cardinal awareness in learning number is misplaced and argued that what young children above all need is “support to work with symbols in their relationship to other symbols”. This plea for paying more attention to the importance of ordinality has led these authors to the design of an innovative iPad app, *TouchCounts* (Sinclair & Jackiw, 2011) wherein the way numerosities are built, labeled and manipulated does not primarily require sense of cardinality but rather ordinality.

The cardinal emphasis on number knowledge has also been attacked from another, more radical, perspective. In his plenary address at the ICMI23 conference, Bass (2015) described an approach to developing concepts of number using the notion of quantity *measurement*. This approach is not new, of course, and is quite well-known among mathematics educators (see e.g., Brousseau, Brousseau, & Warfield, 2004), even though it has, to the best of our knowledge, hardly led to actual and wide-scale implementation in national curricula.

It has been articulated most prominently by Davydov (1990), a Soviet psychologist and educator, who developed, together with his colleagues, in the 1960s and 1970s, a curriculum for number and arithmetic based on this measurement approach. This curriculum delayed the introduction of number instruction until late in the first grade. Early lessons rather concentrated on “pre-numerical” material: properties of objects such as color, shape, and size, and then quantities such as length, volume, area, mass, and amount of discrete objects, but without yet using number to enumerate “how many”. So, in this approach number is not intrinsically attached to a quantity; rather it arises from measuring one quantity by another, taken to be the “unit:” How “much” (or many) of the unit is needed to constitute the given quantity?

The discrete (counting) context in which whole numbers are typically developed in most approaches to early and elementary mathematics education is characterized by the use of the single-object set as

the unit, so that the very concept of the unit, and its possible variability, is rarely subject to conscious consideration. According to Bass (2015, p. 11), “this choice is so natural, and often taken for granted, that the concept of a chosen unit of measurement need not enter explicit discussion. If number is first developed exclusively in this discrete context, then fractions, introduced later, might appear to be, conceptually, a new and more complex species of number quite separate from whole numbers. This might make it difficult to see how the two kinds of numbers eventually coherently inhabit the same real number line. Indeed, this integration entails seeing the placement of whole numbers on the number line from the point of view (not of discrete counting, but) of continuous linear measure.” (see also Behr, Harel, Post, & Lesh, 1992, for a similar argument coming from the research literature on rational numbers).

According to Bass (2015), this measurement approach has a lot of advantages over the counting based approach, especially if one takes a broader long-time mathematics educational perspective. First, it is a way of providing coherent connections in the development of rational numbers. A second advantage is that it makes the geometric number line continuum present from the start of the school curriculum as a useful mathematical object and concept. Third, the approach provides opportunities for some early algebraic thinking.

The above analysis suggests that it is important to balance cardinal, ordinal and measurement aspects of number in early mathematics education. This requires some serious reflection on the ingrained ways in which cardinality is now privileged in early mathematics education as well as further creative explorations of how the two other elements of ordinality and measurement can be mobilized to promote the development of a broad and balanced number concept.

Arithmetic reasoning skills

It is apparent that the mainstream analysis of early mathematics-related competences has capitalized on measures that emphasize children’s *numerical* competences, i.e., their subitizing skills, counting skills, the ability to compare numerical magnitudes, and the ability to position numerical magnitudes on an empty number line. While such measures provided empirical evidence for the multi-componential nature and importance of young children’s early numerical competences for future mathematical development, they reflect also in another way a restricted view on children’s early mathematical competences.

Starting from Piaget’s (1952) logical operations framework, there is a recent renewed research attention to children’s early arithmetic reasoning skills, such as their understanding of the additive composition of number or their additive and multiplicative reasoning skills, as well as to their importance for later mathematical learning at school (e.g., Clements & Sarama, 2011; Nunes, Bryant, Barros, & Sylva, 2012; Robinson, 2016).

As documented in her extensive review of this research, Robinson (2016) points out that the research on children’s conceptual understanding of these arithmetic concepts is heavily focused on additive concepts, that is, concepts involving the operations of addition and/or subtraction. Various principles including the additive composition of number but also the arithmetical properties such as the commutativity, the associativity, the addition-subtraction inverse, and the addition-subtraction complement principle have been intensively studied, sometimes also in relation to children’s actual

use of these principles in their mental arithmetic (Baroody, Torbeyns, & Verschaffel, 2009; Verschaffel, Bryant, & Torbeyns, 2012). Quite a number of these studies already involve young children at or even before the age of 6-7 years old.

Similar multiplication and division principles have also been investigated, however, to a much lesser extent and with a more restricted developmental range, from late middle childhood to adulthood (Larsson, 2016; Robinson, 2016), which is not surprising given that, for most children, these operations are typically not yet formally introduced in the first grades of elementary school.

Only a few of these studies have explicitly addressed the question of how young children's emergent understanding of these additive and multiplicative principles is predictively related to their (later) achievement in school mathematics, in similar ways as has been done for the numerical aspects of early mathematical competence reviewed in the previous section. The limited available evidence from these few studies suggests that early mathematical reasoning of this sort makes a separate and specific contribution to achievement in school mathematics, even up to several years later (Nunes, Bryant, Evans, Bell, et al., 2007; Nunes et al., 2012).

As an illustration, we refer to the study of Nunes et al. (2012), which used data collected in the context of the Avon Longitudinal Study of Parents and Children (ALSPAC) involving about 4000 pupils, to assess whether arithmetic reasoning makes an independent contribution, besides calculation skills, to the longitudinal prediction of mathematical achievement over five years. Arithmetic reasoning was assessed at the start of children's elementary education (i.e., at 7 years) using a test that included three types of items: additive reasoning about quantities, additive reasoning about relations, and multiplicative reasoning items (see examples in Figure 2).

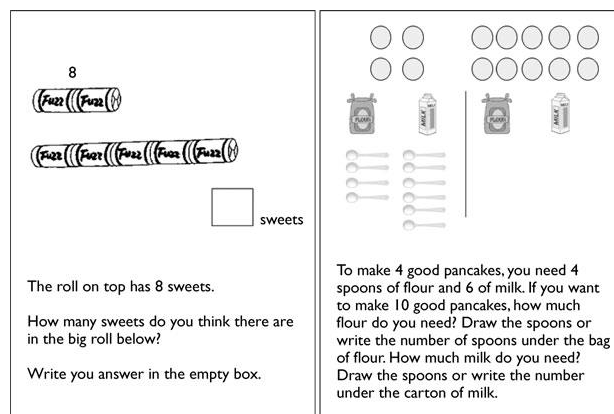


Figure 2: Examples of items from Nunes et al.'s (2012) arithmetic reasoning test

The outcome measures of mathematical achievement were standardized assessments designed to measure school standards by the end of elementary school. Hierarchical regression analyses were used to assess the independence and specificity of the contribution of arithmetic reasoning vs. arithmetic skill to the prediction of achievement in mathematics, science, and English at the end of elementary school, using age, intelligence, and working memory as controls in these analyses. Arithmetic reasoning and skill made independent contributions to the prediction of mathematical achievement, but arithmetic reasoning was by far the stronger predictor of the two. These

predictions were also specific, in so far that these measures were more strongly related to mathematics than to science or English.

In sum, according to Nunes et al. (2012), their findings provide a clear justification for making a distinction between arithmetic reasoning and numerical, counting and calculation skills. The implication for diagnosis and intervention in early mathematics education is that arithmetic reasoning should receive a greater emphasis from the early years in primary school on.

Understanding patterns and structures

In another attempt to identify and explain common underlying early bases of mathematical development and its stimulation, other researchers have looked at mathematical patterns and structures (Lüken, 2012; Mulligan & Mitchelmore, 2009; Mulligan, Mitchelmore, & Stephanou, 2015; Rittle-Johnson, Fyfe, Loehr, & Miller, 2015)

In what can be considered as one of the most enduring, systematic and influential research programs in this respect, based on a series of related studies with diverse samples of 4- to 8-year-olds, Mulligan and colleagues have identified and described a new construct, Awareness of Mathematical Pattern and Structure (AMPS) (Mulligan & Mitchelmore, 2009; Mulligan et al. 2015), that has been shown to be related to children's later mathematics achievement in school.

Mathematical pattern involves any predictable regularity involving number, space, or measure such as number sequences and geometrical patterns, whereas structure refers to the way in which the various elements are organized and related, such as iterating a single 'unit of repeat' (Mulligan & Mitchelmore, 2009). AMPS involves structural thinking based on recognizing similarities, differences and relationships, and also a deep awareness of how relationships and structures are connected.

An interview-based assessment instrument was developed and validated, the Pattern and Structure Assessment - Early Mathematics (PASA) (Mulligan et al., 2015). The PASA yields an overall AMPS score as well as scores on five individual structures: sequences, shape and alignment, equal spacing, structured counting, and partitioning. Some examples of tasks are sequences that have to be extended (e.g., a sequence of colored pearls on a string or a series of triangular dot configurations of increasing size) or structured counting tasks (e.g., counting by two's, counting the number of cells in a partly covered rectangular pattern). Based on the child's response, which may include drawn representations and verbal explanations of patterns and relationships, five broad levels of structural development were identified and described: pre-structural, emergent, partial, structural, and advanced structural (Mulligan & Mitchelmore, 2015). Validation studies indicated that high levels of AMPS were correlated with high performance on standardized achievement tests in mathematics with young students (Mulligan et al., 2015).

In alignment with the assessment of AMPS, an innovative, challenging alternative learning program, the Pattern and Structure Mathematics Awareness Program (PASMAMP) was developed and evaluated longitudinally in the kindergarten (= the first year of formal schooling in Australia). This study first showed that kindergartners are capable of representing, symbolizing and generalizing mathematical patterns and relationships, albeit at an emergent level (Mulligan,

Mitchelmore, English, & Crevensten, 2013). The study also tracked and described children's individual profiles of mathematical development and these analyses showed that core underlying mathematical concepts are based on AMPS, and that some children develop these more readily and in more complex ways than others. Finally, this study also involved an attempt to provide an empirical evaluation part involving 316 kindergartners from two schools with and two schools without the PASMMap program. Highly significant differences on PASA scores were found for PASMMap children in comparison to children from the control schools, also for those children labeled as low ability, both at the posttest and the retention test, when children had already moved to Grade 1. On the other hand, there was no significant impact of PASMMap on improving children's mathematical achievement as measured by a general mathematics achievement test.

Other researchers have also performed analyses of (1) elementary school children's perceptions and understandings of patterns and structures, providing nice descriptions and accounts of young children's abilities and difficulties with respect to various mathematical patterns and structures tasks, (2) the predictive value of their mastery of pattern and structure for their later mathematical, i.e. algebraic proficiency, and (3) how instruction on patterns and structures can not only transfer to similar and other patterns and structures, but also to other mathematical domains such as ratios, and mathematics achievement in general (for an overview, see Rittle-Johnson, Fyfe, Loehr, & Miller, 2015).

Of course, the idea that patterns and structures play an important role in the learning of mathematics, and should play an important role in its teaching, is not new (Orton, 1999). After all, is the definition of "mathematics as a science of patterns" (Müller, Selter, & Wittmann, 2012) not one of our favorite definitions of what mathematics is all about? The critically new element in the research of the work of Mulligan and associates is that they give it such a prominent role in their diagnostic and teaching materials for early mathematics. In doing so, they contribute to broadening the picture of what (early) mathematics is all about – a picture that is largely undervalued in current early and elementary school mathematics with its strong focus on learning about numbers and arithmetic facts and procedures.

Spontaneous focusing tendencies

The studies and views on the early development of children's mathematical competence reviewed so far typically take a purely "ability" perspective. In doing so, they neglect other aspects of young children's early mathematical competence, such as their attention to or feeling for, numerical magnitudes, mathematical relations, or mathematical patterns and structures. During the past decade, researchers have started to explore children's spontaneous tendency to focus on numerosity (SFON), its development, its cultivation, and its predictive relation to children's later mathematical achievement (Hannula & Lehtinen, 2005). To a lesser extent, similar attempts have been done for quantitative relations (SFOR) and, even much less, mathematical patterns and structures (SFOPS).

These SFON, SFOR or SFOPS tendencies are not about what children think and do *when they are guided to* the mathematical elements, relations or patterns in the situation, but *what they spontaneously focus on* in informal everyday situations. SFON assessment instruments must therefore capture whether children spontaneously use their available number recognition or

quantitative or mathematical reasoning and patterning skills in situations where they are not explicitly guided or instructed to do so. So, the instruments used to assess these spontaneous focusing tendencies must meet several strict methodological criteria (Hannula & Lehtinen, 2005).

As far as SFON is concerned, the most frequently used task so far is the Elsi Bird Imitation task, wherein the child is instructed to imitate the experimenter's play behavior with toys, i.e., feeding berries into the beak of a toy parrot. A SFON score is given on an item as soon as the child is observed doing or saying something that shows that he or she has spontaneously attended to the quantitative aspect of the situation. Meanwhile several other SFON tasks have been developed, such as the Picture Description task, with cartoon pictures displaying both non-numerical and numerical information and the request to tell what is in the picture. If the child spontaneously refers to the exact numerosities - correct or not – in his or her verbal descriptions of the pictures, (s)he gets a SFON score (for an overview and critical discussion of SFON measures, see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016).

Observations of children's activities in SFON assessment indicated that already at the age of 3-4 years children can be spontaneously engaged in mathematically relevant practices in their everyday environments (Hannula & Lehtinen, 2005). This research also revealed great inter-individual differences in children's tendency to spontaneously focus on number. It further showed that children's SFON at the age of 5 or 6 is a unique and strong predictor of later development of mathematical skills even up to the end of elementary school. The hypothetical explanation for these findings is that children who spontaneously focus on the numerical aspects of their environment in everyday situations get much more practice of magnitude recognition, number comparison, combining of numbers, etc. than children who only do this when explicitly instructed by parents or teachers. SFON may support the development of numerical skills and more elaborated numerical skills may further strengthen the SFON tendency. However, convincing direct empirical evidence for this explanatory mechanism is still scarce (Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016).

In many everyday activities exact numerosity is not the only mathematically relevant aspect that can be focused on. In young children's daily life there are many opportunities to focus on more complex quantitative aspects, such as quantitative relations. Children can also recognize and use mathematical or quantitative relations without explicit guidance to do so. Based on a series of studies, McMullen, Hannula, and Lehtinen (2013, 2014) proposed that there is a similar tendency to focus on quantitative relations as SFON, which indicates that instead of mere numerosity children and school pupils can also focus spontaneously on quantitative relations (SFOR). McMullen and colleagues (McMullen, Hannula-Sormunen, Laakkonen & Lehtinen, 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2013; McMullen, Hannula-Sormunen, & Lehtinen, 2014). designed the Teleportation Task to measure SFOR. This task involves a cover story telling that a set of supplies containing three sets of objects was sent from earth through space with a teleportation machine. However, when doing so, the objects are transformed. Children are asked, first, to describe the transformation in their own words in as many ways as possible, and, second, to draw what they expect to happen with a different numerosity of the same objects. When describing or drawing the transformation, learners can pay attention to the various non-mathematical changes (e.g., in terms of

the colors or shapes of the objects), but also to the quantitative relation between the original and final numerosity of the three sets. The results of the longitudinal study of McMullen, Hannula-Sormunen, Laakkonen, and Lehtinen (2016) showed that there were substantial individual differences in students' SFOR tendencies. It also revealed that SFOR tendency had a unique predictive relationship with rational number conceptual development in late primary school students during the 2-year follow-up period.

Interestingly, in their conceptualization of AMPS, Mulligan and Mitchelmore (2009) also tend to go beyond the pure ability aspect of early mathematical competence, by stating that AMPS may consist of “two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive, i.e., “spontaneous” (a tendency to seek and analyze patterns)” (p. 39). According to these authors, both are likely to be general features of how children perceive and react to their environment. However, neither in their assessment nor in their intervention materials, they have already tried to specifically and explicitly address this spontaneous focusing aspect.

Early mathematics and executive functions

In the previous sections, we have discussed various kinds of domain-specific competences that all have been claimed, and in many cases been shown, to be predictively related to general mathematical competence or to knowledge and/or skill in specific subdomains of the mathematics curriculum. However, it is a well-established research finding that formal mathematics achievement is also influenced by domain-general processes, such as sustained attention, inhibitory control, cognitive flexibility, working memory capacity, and - even more generally - intelligence (Bull & Scerif, 2001; De Smedt, Janssen, et al., 2009; Friso-van den Bos et al., 2013; LeFevre et al., 2010; Peng, Namkung, Barnes, & Sun, 2016).). While most of that research evidence comes from research with older participants, there is increasing evidence on the importance of executive functions in early mathematical thinking and learning too.

In one line of research, authors have analyzed the relative importance of general executive skills as compared to the role of domain-specific early numerical competences in predicting concurrent and later mathematical development. For instance, in a longitudinal study wherein we followed children during the first grades of elementary school, we were able to show that working memory at the start of primary education was predictively related to individual differences in mathematics achievement six months later in Grade 1 and one year later in Grade 2 (De Smedt, Janssen, et al., 2009). Interestingly, overviewing the research, Bailey et al. (2014), concluded that the contribution of domain-specific factors, such as children's early numerical competences to their later mathematical development is relatively small compared to these more stable domain-general factors, such as intelligence and working memory.

The relation between these executive functions and mathematical performance may also be more specific in nature. Research has revealed specific relations between certain executive functions, such as inhibition or working memory, on the one hand, and specific mathematical competences, such as mental arithmetic or word problem solving, on the other hand. Robinson and Dubé (2013), for instance, investigated the role of inhibition in children's use of the inversion and associativity shortcuts on mental addition and subtraction (e.g., $6 + 23 - 23 = ?$). Children who demonstrated the

highest use of conceptually-based shortcuts also scored highest on the Stop-Signal task, a standard measure of inhibitory abilities. This finding suggests that these children were able to inhibit their tendency to routinely solve problems from left-to-right and thereby process all of the presented numbers before executing the clever shortcut strategy.

So far, we have discussed the role of executive functions in children's performance on relatively complex mathematical tasks. However, to make the picture about the role of executive functions even more complicated, these executive functions are also assumed to play a critical function in the early mathematical tasks, such as the magnitude comparison task, the SFON tasks, the mathematical reasoning tasks, and the patterns and structures tasks discussed above. Take, for instance, the non-symbolic magnitude comparison task used to assess the approximate number system (ANS) and which lies at the basis of this whole line of research that has led to the pivotal role of the precision of children's early ANS representations in early mathematics diagnosis and intervention (see Section 1). In this task it is important to ensure that participants are basing their judgements on the numerosity of the visual arrays, rather than possible visual cues such as the size of the dots, or the area that the dot arrays cover. As Gilmore, et al. (2013) have argued, in an attempt to control for this possible confound, researchers introduce an inhibitory control aspect to the task, as for half of the items with which the child is confronted inconsistent visual cues must be inhibited to indicate the correct set. But if the non-symbolic comparison task is, in part at least, a measure of inhibitory control, then it is perhaps unsurprising that it is predictive of school-level mathematics achievement, but for other reasons than claimed by the advocates of this task.

Starting from the above research documenting in various ways the involvement of executive functions in mathematical thinking and learning, researchers have also asked the question about the possibility and efficacy of enhancing mathematical thinking and learning through training of these executive skills. At least for working memory, a recent meta-analysis by Schwaighofer, Fischer, and Bühner (2015) led to the general conclusion that attempts to improve working memory only improved performance on working memory tests but failed to improve mathematics achievement.

So, while there is increasing research evidence that, from a very young age on, an association between mathematics and executive functions exists, this complex and multi-aspectual association and its implications for early mathematics education and assessment is not well understood yet. Numerous questions remain (Robinson, 2016; Van Dooren & Inglis, 2015). As (early) mathematics educators we are traditionally not so much interested in these general executive functions. However, for various reasons related to theory, diagnosis and intervention, it may be unwise to neglect them.

The role of parents and early caregivers during the preschool years

As amply shown in the previous sections, before the start of formal mathematics education - typically at the age of 5-6, children already begin their initial explorations into everyday mathematics at home, progressively developing and refining their mathematical knowledge and skills as well as their mathematics-related orientations, beliefs, and affects. However, there is wide variation - linked in part to socio-economic status (SES) and culture - in the kinds of early mathematical learning experiences children have at home and the ways in which they are stimulated and helped by their parents. Further, in many cultures, the majority of young children spend

significant time in non-parental care, including family childcare and organized preschool education (DREME, 2016). Arguably, the quantity and quality of mathematics learning stimulation in these various settings also vary enormously, impacting children’s mathematical development. For evident reasons, mainstream cognitive (neuro)psychological research on early numerical competences has paid little or no attention to these informal mathematical learning environments. But also within the mathematics education research community this topic is “under-studied”. Indeed, we know relatively little about the role of parents and early caregivers during the preschool years when compared, on the one hand, to the development and stimulation of children’s emergent literacy, and, on the other hand, to mathematics education in the higher educational levels. Fortunately, the last few years have witnessed an increased research interest.

First, several researchers have aimed for an understanding of children’s preschool experiences at home and of how these experiences affect their early mathematical development. For instance, starting from the well-documented finding that children’s early numerical competence before the start of formal schooling is highly predictive of their acquisition of mathematics in (the first grades of) elementary school, several authors have pleaded for a better understanding of children’s preschool experiences at home. In a well-known study by Lefevre, Skwarchuk, et al. (2009), the mathematical skills of \pm 150 Canadian children in Kindergarten, Grade 1, and Grade 2 were correlated with the frequency with which parents reported informal activities that have quantitative components such as board and card games, shopping, or cooking on a questionnaire. The results support claims about the importance of home experiences in children’s acquisition of mathematics, given that effect sizes were consistent with those obtained in research relating home literacy experiences to children’s vocabulary skills. In a more recent and more sophisticated study, Susperreguy and Davis-Kean (2016) analyzed the relation between the amount of mathematical input that preschool children hear from their mothers in their homes and their early mathematics ability one year later. Forty mother–child dyads recorded their naturalistic exchanges in their homes using an enhanced audio-recording device. Results from a sample of naturalistic interactions during mealtimes indicated that all mothers involved their children in a variety of mathematics exchanges, although there were differences in the amount of input children received. Moreover, being exposed to more instances of mathematics talk was positively related to children’s early mathematical ability one year after the recordings, even after control for maternal education, self-regulation, and recorded minutes. Finally, starting from the well-documented finding that early numerical competences amongst children vary widely and from the belief that a better understanding of the sources of this variation may help to reduce SES-related differences in mathematics skills, Ramani, Rowe, Eason, and Leech (2015) examined two sources of this variation in low SES families: (1) caregiver reports of number-related experiences at home, and (2) caregivers’ and children’s talk related to math during a dyadic interaction elicited by the researchers. Frequency of engaging in number-related activities at home predicted children’s foundational number skills, while caregivers’ talk during the interaction about more advanced number concepts for preschoolers, such as cardinality and ordinal relations, predicted children’s advanced number skills that build on these foundational concepts. So, these findings suggest that the quantity and quality of number-related experiences that occur at home contribute to the variability found in low-income preschoolers’ numerical knowledge.

Complementary to these ascertaining studies, several intervention studies reported positive effects on children's early numerical and later mathematics performance at school. Again, we can give only a few examples. In a series of high-impact studies with children from low-income backgrounds, who were found to lag behind their peers from middle-income backgrounds already before the children enter school, Siegler and Ramani (2008) found that playing a research-based designed numerical board game for only a couple of hours already eliminated the differences in the two commonly used measures of understanding of numerical magnitudes, namely numerical magnitude comparison and number line estimation. Moreover, in a subsequent study (Siegler & Ramani, 2009), children who had played the number board game also performed better in a subsequent training on arithmetic problems. Thus, playing number board games was found to increase not only preschoolers' numerical knowledge but also to help them learn their school arithmetic. Van den Heuvel-Panhuizen, Elia, and Robitzsch (2016) report on a very recent field experiment with a pretest–posttest control group design, which investigated the potential of reading picture books to kindergarten children for supporting their mathematical understanding. During three months, the children from nine experimental classes were read picture books. Data analysis revealed that, when controlled for relevant covariates, the picture book reading programme had a positive effect on kindergartners' mathematics performance as measured by a test containing items on number, measurement and geometry. Finally, we refer to one of the best known research-based early mathematics programs, namely the Building Blocks (BB) program of Clements and Sarama (2011). This program, which is organized into five major strands: (numeric, geometric, measuring, patterning, and classifying and data analyzing), consists of daily lessons where children are encouraged to extend and mathematize their daily experiences through sequenced activities, games, and the use of technology. The daily lessons are organized in whole group activities, small group activities, free-choice learning centers, and reflection time. The program is complemented with a parallel in-service teacher training program. Studies on the effectiveness of the BB intervention program (Clements & Sarama, 2007, 2011) demonstrated that 3- and 4-year-olds who received the BB intervention program developed stronger mathematical abilities than children in the control group, with effects lasting up to the end of first grade. Bojorque (2017) recently successfully implemented the BB program in the Ecuadorian context, with significant effects on the quality of the kindergarten teachers' pedagogical actions as well as on children's progression both on a standard mathematics achievement test based on the national K3 curriculum and on their SFON.

The findings emerging from all these observational, correlational, and intervention studies are very informative for the design of educational environments and activities aimed at increasing young children's mathematics learning - far beyond the rather narrowly oriented (computer) games aimed at stimulating children numerical magnitude representations that have been derived from the cognitive neuroscientific line of research. But still a lot of work needs to be done to further advance knowledge on effective ways to increase parents' and professionals' engagement in preschoolers' mathematics learning, particularly in children growing up in poverty and/or in contexts of unfavorable immigration.

Preschool to elementary school transition

As explained in the previous sections, a large number of factors in the young child and in its home and caretaking environment have a strong impact on the ease with which (s)he will take the step to formal mathematics education at the age of 5-6 (depending on the country or culture) and profit from the elementary school mathematics curriculum. However, the child's mathematical development and achievement will evidently also be significantly affected by the quality of the transition from preschool to elementary school (see also Gueudet, Bosch, diSessa, Kwon, & Verschaffel, 2016).

Interestingly, researchers working on this theme typically take a much broader theoretical stance than the cognitive (neuro)scientific researchers who look for the elements in children's domain-specific and domain-general competences that are predictively related with success in school mathematics. Their inspiration comes from socio-cultural, sociological, anthropological, and critical mathematical theories (Dockett, Petriwskyj, & Perry, 2014; Perry, McDonald, & Gervasoni, 2015).

The transition from prior-to-school to school mathematics is primarily conceived by these researchers as a set of processes whereby individuals "cross borders" or undergo a "rite of passage" from one cultural c.q. educational context or community to another and, in doing so, also change their role in these contexts or communities. Dockett et al. (2014, p. 3) provide the following summation of this approach: "While there is no universally accepted definition of transition, there is acceptance that transition is a multifaceted phenomenon involving a range of interactions and processes over time, experienced in different ways by different people in different contexts. In very general terms, the outcome of a positive transition is a sense of belonging in the new setting." There is growing research evidence that developing practices that promote effective transitions, and that strive for giving agency of all involved and rely on the "Funds of Knowledge" available in children's home and local environments, results in positive effects - although most of this research is more qualitative and descriptive in nature and thus not primarily interested in following strict experimental designs and providing "hard" statistical data. A nice overview of this broader transition perspective is provided by Perry et al. (2015).

In an interesting newly funded project, Andrews and Sayers have begun to examine how two systems, England and Sweden, facilitate the early mathematical competences, and more specifically their foundational number sense (FoNS) (Andrews & Sayers, 2015), of children starting in Grade 1. Currently the project team is comparing the FoNS opportunities found in commonly used textbooks in the two countries (Löwenhielm, Marschall, Sayers, & Andrews, 2017a). Simultaneously the team has been interviewing first grade teachers in the two countries about their role as well as their perceptions of their pupils' parents' roles in the development of children's FoNS-related competence. Initial analyses (Löwenhielm, Marschall, Sayers, & Andrews, 2017b) have identified both similarities and considerable differences in the relationship between the school and home environment between the two countries.

It is a general complaint among stakeholders of early mathematics education that mathematics learning in preschool is often disconnected from the first grades of elementary school. This disconnect, which is particularly relevant for lower SES and immigrant children, can lead to

children experiencing uneven instructional practices, which can compromise their mathematical development in elementary school. So, policy makers, curriculum developers, teacher trainers, etc. should work toward creating greater alignment of and coherence between preschool and elementary school mathematics education, using research-based insights and recommendations. Unfortunately, there is still limited research on the impact of these policies and practices on the learning experiences and learning outcomes of children moving from preschool through the early elementary grades.

Professional development of caregivers and teachers

In the previous section, we emphasized the importance of a high-quality mathematical learning environment in the preschool years, the first years of elementary school, and the transition between the two. Evidently, this requires highly professional (mathematics) teachers, i.e., “teachers who know the content, who understand children’s thinking, who know how to engage in pedagogical practices that support learning, and who see themselves as capable math teachers” (DREME, 2016, p. 4).

At the same time, many teachers and caregivers in the early care and education field may not be adequately equipped to provide appropriate math-related experiences and instruction to these young children. Research suggests that many practitioners working with preschool, kindergarten and early grade children (1) are themselves not competent in mathematics, (2) have important shortcomings in the pedagogical content knowledge, particularly with respect to the components of the early math curriculum beyond counting, number, and simple addition and subtraction, and/or (3) do not see themselves as competent in mathematics (see e.g., Lee, 2010). And, even if practitioners are mathematically capable and do view themselves as such, they may still hold pedagogical reservations against teaching mathematics to young children, believing that early childhood programs should focus primarily on social emotional and literacy goals (Platas, 2008).

While these problems have shown to be partly due to these professionals’ restricted mathematical talents and negative earning histories in elementary and secondary education, research also indicates that the nature of the pre-service and in-service training they received does not greatly help to overcome these problems. As DREME (2016, p. 4) argues: “Professional teacher preparation programs rarely address how to identify the wide range of informal mathematical understandings that young children bring with them to the classroom, or how to translate these into intentional, individualized math experiences for children with diverse backgrounds and needs.” Indeed, surveys of early childhood education degree programs (e.g., Maxwell, Lim, & Early, 2006) reveal that early education practitioners are exposed rarely to high-quality pre-service or in-service courses that address children’s mathematical development, or the pedagogical content knowledge necessary for supporting it.

We emphasize that the above analysis is largely based on critical reflections upon the situation in the US. So, the situation may be better in other places in the world, although there are good reasons to restrain from being too optimistic, because the above observations about early math teachers’ professional knowledge and beliefs and previous educational histories seem to hold, at least to some extent, for many other countries too.

To support the training of prospective and practicing early childhood teachers, there is a need of creating and implementing research-based modules for professional development that can be used in a variety of pre-service and in-service settings (DREME, 2016). The way forward for research is to attempt to figure out what are the key levers of professional development that might effect significant change in the quality of early math education and its learning outcomes. Given the above-mentioned depiction of the complex and multi-sided nature of caregivers' and early math teacher's professional knowledge base, it seems reasonable to expect the greatest effect from modules that do not focus on one single aspect of professionalism but work on the development of early math related knowledge, skills and beliefs, and that convey the idea that early mathematics is more than teaching young children some basic number knowledge and counting skills.

Conclusion

Inspired by developments in the field of neuroscience (e.g., Butterworth, 2015), the past two decades have witnessed the emergence of a very productive and highly influential line of (neuro)cognitive research on children's early number sense, its development, and its relation to school mathematics. Cross-sectional and longitudinal studies have demonstrated that various core elements of children's early mathematical ability - especially their numerical magnitude understanding, their subitizing and counting skills, and their ability to transcode a number from one representation to another - are positively related to concurrent and future mathematics achievement (Torbeys et al., 2015).

However, other research, most of which is situated in other scientific circles and relying on other theoretical and methodological perspectives, has yielded increasing evidence for uniquely significant relations of mathematical achievement also with (1) young children's understanding of ordinal and measurement aspects of number, (2) their abilities related to mathematical relations, patterns and structures, and (3) their tendency to spontaneously attend to numerosities and to mathematical relations, patterns, and structures in their environment, and has confirmed the important role of domain general executive functions.

Moreover, researchers have started to explore and analyze the rich variety of early mathematical learning environments at home, in preschool and kindergarten settings, as well as the coherence between these informal learning settings and the first years of elementary school mathematics, with special attention to the professional quality of the early caregivers and teachers. Also, they started to set up various kinds of intervention studies aimed at the improvement of the quality of these environments and of the professionals operating in these environments. These studies have yielded evidence on the short- and long-term benefits of such attempts to provide high-quality early mathematics education in preschool settings and in the transition from preschool to elementary school.

While the small-scale, short-term and focused experimental intervention programs derived from the (neuro)cognitive research on early numeracy have their value in enhancing our theoretical insight into numerical cognition and learning, practitioners active in the field of early mathematics education may profit more from the studies describing the design, implementation, and evaluation of large-scale and more broadly conceived intervention programs that combine and balance several

of the elements that have been found to be foundational for future mathematics learning (see Sections 2-6) and that also integrate aspects of teacher development, working with parents, and community building (see Sections 7-9), with the *Building Blocks* program of Clements and Sarama (2007) and the *Pattern and Structure Mathematics Awareness Program* of Mulligan et al. (2013) as the most visible and successful examples. Still, as math educators, we should continue to follow, with an open but critical mind, the cognitive neuroscientific research on mathematical cognition and, equally important, also try to have an impact on their research agenda (De Smedt et al., 2011).

As a result of all this research, there is a lot of practically useful new knowledge, techniques and resources to promote young children's math learning. Still there remains much to learn about how to optimally enhance math learning at home and at school in the preschool years and about how to help teachers to be well prepared for delivering high-quality instruction to those young children, particularly the weaker ones. In this respect, we should applaud - and may-be also strive for an European counterpart – of the recent initiative called the DREME Network in the US, which is aimed at developing new researchers and enticing current elementary math education, child development, and policy researchers to expand their work to include young children's mathematical learning.

Acknowledgment

The authors want to thank Paul Andrews, Mariolina Bussi, and Jeremy Hodgen for their comments on earlier versions of this paper.

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