



HAL
open science

From advanced mathematical thinking to university mathematics education: A story of emancipation and enrichment

Elena Nardi

► **To cite this version:**

Elena Nardi. From advanced mathematical thinking to university mathematics education: A story of emancipation and enrichment. CERME 10, Feb 2017, Dublin, Ireland. hal-01849593

HAL Id: hal-01849593

<https://hal.science/hal-01849593>

Submitted on 26 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

From advanced mathematical thinking to university mathematics education: A story of emancipation and enrichment

Elena Nardi

University of East Anglia, UK, e.nardi@uea.ac.uk

Between CERME1 and CERME9 there have been approximately two hundred and fifty papers with their focus directly, or a little less so, on the teaching and learning of mathematics at university level, starting from about a dozen in CERME1 and rising to several dozens in CERME9. ERME recognised the increasing significance of this emerging field with the launch of Working Group 14 (Advanced Mathematical Thinking) in CERME4 in 2005 which evolved into Thematic Working Group 14 (University Mathematics Education) in CERME7 in 2011. In this lecture, I draw on my experience as researcher in this field, and as participant in both groups (and inaugural leader of the latter), to identify epistemological – theoretical, substantive and methodological – trends in the transition from the one to the other. I aim that the story I tell is one of gradual emancipation from a relatively limited initial focus on cognitive aspects of the student learning experience in university mathematics to the grander vista of issues – also inclusive of pedagogical, institutional, affective and social issues – that studies presented at CERME nowadays address. I also aim that the story I tell is one of enrichment as the depth and diversity of said vista has been accomplished also through thoughtful appropriation of results from those earlier studies.

Keywords: University mathematics education, developmental / cognitive and sociocultural approaches to the teaching and learning of mathematics.

Introduction

In tandem with ERME, the area of research that is the focus of this plenary, *University Mathematics Education* research, has also been evolving rapidly in the last twenty years or so. Here I focus on some of the milestones of this evolutionary journey, with the particular emphasis that I promised in the above title and abstract. Before proceeding to these though, here is a bit of a pre-ample: Figure 1 presents a still from a scene in the film *A Serious Man* (2009) directed by Ethan and Joel Coen.

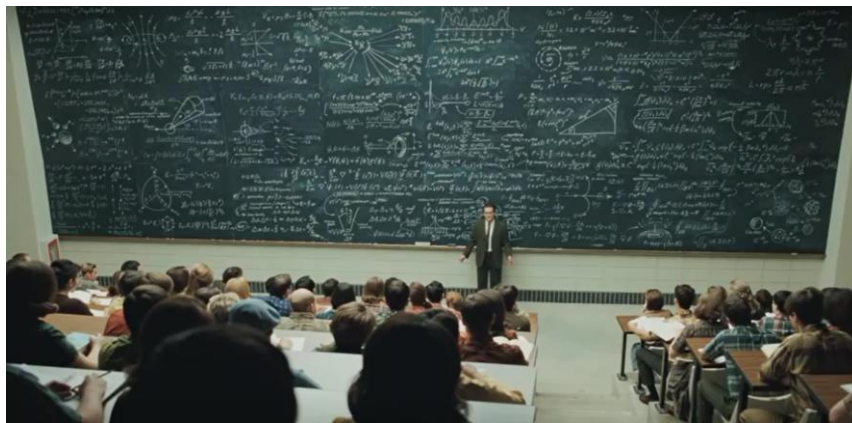


Fig.1. Still taken from *A Serious Man* (2009): <https://www.youtube.com/watch?v=7iggyFPls4w>

This is a typical imagining in popular culture of how mathematics teaching looks like at university. I will not go much further with a discourse analysis of what the still (or the scene, or the film itself) may convey. In what I see as some contrast, Figure 2 presents a sequence of images, taken from the publicity materials of my own institution's department of mathematics.

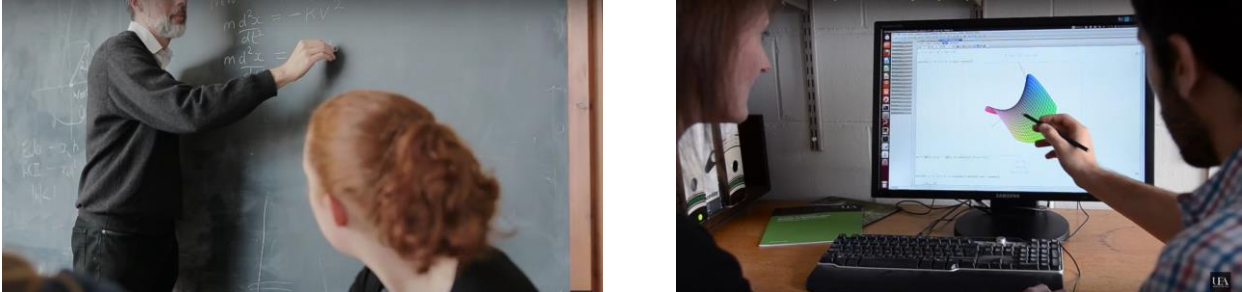


Fig. 2. Still taken from UEA promotional video: <https://www.youtube.com/watch?v=gRzVX8c1be4>

The students and the lecturer in these images work together, they are not physically too far from each other and there is a range of resources – from chalk to digital – present. The sequence illustrates how institutions may wish to present the kind of learning experience that potential incomers into a department of mathematics are likely to be offered.

To me, there is a clear contrast between the movie still from *A Serious Man* and these two images from the UEA promotional video. It is a contrast between a widespread perception of university mathematics lectures as the ultimate form of transmissive pedagogies – with all the repercussions of alienation and distancing these pedagogies may entail – and the aspiration (institutional but not only) for a more approachable, more inclusive and more engaging learning experience in university mathematics that is tailored to individual student needs.

As university lecturers today – in mathematics and in other disciplines – we lecture. But we also do much more: we coordinate seminars, we conduct individual or small group tutorials, we run workshops and drop-in clinics, we supervise dissertations, we advise students on academic and on pastoral matters and we assess students in a variety of ways (all the way from closed-book examinations to mini-projects and oral presentations). Our professional worlds are far from monotonous. In fact, they require us to be quite versatile.

I see as of little surprise, and rather pleasing, that the versatility of our jobs is being reflected in the diversity of University Mathematics Education research that is now presented at CERME. This diversity of focus – but also theoretical perspective and methodology – is to me a sign of richness. In fact, here I have taken the liberty of endorsing a metaphor, which originates in currently dominant theories of evolution and conservation (Figure 3). These theories equate species diversity with resilience. The story I tell here relies somewhat on whether this is a convincing metaphor.



Fig. 3. Image from: <https://conservationbytes.com/2014/01/08/more-species-more-resilience/>

I tell this story in five parts: *The “early years”, CERME 1, 2, 3; The AMT years, CERME 4, 5, 6; The UME years, CERME 7, 8, 9; CERME10, the split¹ ...; and, Taking stock / What next / Coming soon...* Before starting, I need to post a health warning though: that a lecture of this kind errs on the side of being impressionistic – and of course quite personal² too. I thank you in advance for your tolerance.

My own trajectory in CERME – and outside – mirrors some of the milestones and trends that this plenary aims to map out. I was present in 1999 at CERME1, in Osnabrueck, assisting with the coordination of Group 5, *Mathematical thinking and learning as cognitive processes*. To those more familiar with the increasingly sociocultural and discursive take that my work has been taking over the years, this commitment to Group 5 may sound a little surprising. It is not. I start Part I with an anecdote on exactly this.

Part I: The “early years”, CERME 1, 2, 3; UME research evidenced in several TWG groups

My 1996 doctorate’s title (Nardi, 1996) is *The novice mathematician’s encounter with mathematical abstraction: Tensions in concept image construction and formalization*. The statement of intentions in this doctorate are clear:

Mathematics is defined as an abstract way of thinking. Abstraction ranks among the least accessible mental activities. In [the UK educational context where the study took place], the encounter with mathematical abstraction is the crucial step of the transition from informal school mathematics to the formalism of university mathematics. This transition is characterised by cognitive tensions. This study aimed at the *identification and exploration of the tensions in the novice mathematician's encounter with mathematical abstraction*. (Nardi, 1996: Abstract)

However, the study’s stated theoretical perspective is a little more perplexing. It is declared as “consisting of cognitive and sociocultural theories on learning”. And, the two key parts of findings in the final chapter promise an account of the novice mathematician's encounter with mathematical abstraction “as a personal meaning-construction process and as an enculturation process” (ibid.).

It is quite easy, in hindsight, to be skeptical about the risky eclecticism of the approach – some may see this as standing on a fence, or, even, as pick-and-mix nonsense. But, I keep reminding myself that the study started in 1992 and was completed in 1996. It was therefore conducted at a time when

¹ Continuing with the biology inspired metaphors, I use the word “split” deliberately. Cell splitting is the process of subdividing a congested cell into smaller cells. Cell splitting or division is associated with reproduction and the creation of an entire new organism. This process is typically seen as increasing many of the capacities of a cellular system. In fact, in Parts III and IV, I aim to show the inevitability of cell splitting, emanating from the substantive, theoretical, and methodological diversity of UME research presented in CERME these days. It is in these parts that the main point of this lecture, signposted in the abstract by the words **emancipation** and **enrichment**, will, I hope, come through.

² I also need to thank at this juncture two overlapping groups of colleagues: my CERME 7, 8 and 9 TWG14 co-leaders and my co-authors of the 20-year anniversary ERME book in which UME research has been allocated a chapter (Winsløw et al., in press). Since 2010, when the UME TWG group was formed – for its first appearance in CERME7, in 2011 – these colleagues, have become what I like to call my academic family of friends.

the various shades of constructivism that form its theoretical backbones were then taking shape themselves. To signpost this a little more emphatically, allow me the gentle reminder that the seminal paper *Constructivist, emergent and sociocultural perspectives in the context of developmental research* (Cobb & Yackel, 1996) – a paper and a programme more broadly that impacted upon our debate around the co-determinants of mathematical learning in immense ways – appeared in *Educational Psychologist* in 1996, the year that my doctorate was completed. I often use this excuse when the slightly embarrassing thought comes to me that my study wanted to have its cake and eat it too!

So, here are some recollections from the early years, and, to start with, CERME1, that I see as pertinent for today: UME papers can be found in several groups but mostly in TWG1 (*Nature and content of mathematics and its relation to teaching and learning*) and TWG5 (*Mathematical thinking and learning as cognitive processes*). There is a pronounced epistemological focus on several papers – Grenier and Payan (1999) is one example – and there is a strong tendency in the few papers present to give a prominent position to the mathematical context and content of, for example, proposed course designs. Belousova and Byelyavtseva's (1999) paper on course design in Numerical Methods comes to mind; as do the Cabri designs for Linear Algebra put forward by Tommy Dreyfus, Joel Hillel and Anna Sierpinska (1999). There is also a tendency to consider this mathematical content regardless of whether this is present in school or university mathematics: there are, for example, propositions in this first CERME about using CAS (Computer Algebra Systems) for teaching functions; or, courseware for the teaching of Geometry from across school to university, and all the way to Differential Geometry.

There are two contributions to CERME1 though which, for me, stand out even more than those I sampled in my last comments. Both pre-empt the publication of two volumes that proved influential in the following years, in different, yet distinct ways. One is Leone Burton's (1999) preliminary analyses of interviewed mathematicians' epistemological perspectives which culminated in her monograph (Burton, 2004), *Mathematicians as Enquirers*. The other is Jean-Luc Dorier's paper (with Aline Robert, Jacqueline Robinet and Marc Rogalski, 1999) that sets the scene for the volume *On the teaching of linear algebra* (Dorier et al., 2000).

Both papers foreshadow – and I daresay contributed towards shaping – trends in UME research that became prominent in the years that followed. Burton's work signals a broadening of the UME church to include in its focus the university teacher (most other work at the time concerns the student or the mathematics alone). Dorier's work, and that of his colleagues, signals the still then not so imminent end of what I see as a shortcoming of UME research that is still present today, albeit to a lesser extent: the perception of research into university mathematics teaching practice as an a-theoretical aside of well-intended practitioners who are unaware of the epistemological and methodological underpinnings of mathematics education as an academic discipline. This work is distinct for its robust theoretical grounds and for its keen eye for intervention design, trial and evaluation – in a nutshell, for its systematic character. In this sense, of scope and ambition, it shares some common ground with another, powerful at the time – and still today – programme: that of APOS which originated in the USA and which was at the time also pushing the boundaries of work in UME beyond elementary Calculus and into Abstract Algebra.

Continuing with my observing trends that were to become influential in later years, within TWG5 (*Mathematical thinking and learning as cognitive processes*), which I mentioned earlier and which I assisted coordinating under the leadership of Inge Schwank, there are two themes that made an appearance – timidly and managing to occupy a small portion of the discussions only: the role of motivation in cognition (I see here inklings of evidence on the burgeoning importance of research on affect) and the emerging importance of theories of situated cognition. An observation that stands out from these discussions was made in the paper by Pier Luigi Ferrari (1999): in advanced mathematical thinking, wrote Pier Luigi at the time, some learner behaviours cannot be accounted for simply in terms of semantics. His paper presented an argument that brings the role of language – ordinary and mathematical – and of communicational structures to the fore.

CERME2 and CERME3 are the two CERMEs that I missed. Nonetheless, returning to the proceedings after all these years, there are several papers presented in CERME2 and a couple of dozen papers in CERME3 that can be found across several Working Groups and contain implicit references to advanced mathematics, often as extensions of what is typically found in the school syllabus that each paper revolves around.

In CERME2 these papers are mostly found in Working Group 5 (*Mathematical thinking and learning as cognitive processes*) and Working Group 1 (*Creating experience for structural thinking*). Mathematical thinking (including a growing focus on proof and proving) is at the heart of these papers which are only implicitly and only occasionally concerned with the institutional, curricular and pedagogical context of university level Mathematics Education. There is concern in these papers with internal mental structures. Nad'a Stehlíková and Darina Jirotková's paper (2001) is a good example: it focuses explicitly on processes of building an inner mathematical structure, which the authors abbreviate as IMS and which they acknowledge as hard to observe. They then resort to introspective, self-reporting accounts of mathematical thinking. John Mason's (1998) "researching from the inside" features largely as a theoretical influence on the paper. Nad'a Stehlíková will carry on in this strand of work also in CERME3.

These works concern the learning of mathematics often at the cusp of the transition from school to (what is in many places) university mathematics. One example of this trend is Bettina Pedemonte's (2001) study of cognitive unity, or break, in the context of constructing mathematical arguments and proofs. Another is the paper by Baruch Schwarz, Rina Hershkowitz, and Tommy Dreyfus (2001) which presents a perspective on abstraction as always occurring in context and which focuses on three epistemic actions (Recognising, Building-With and Constructing, RBC). Its theoretical close relatives are an eclectic mix and include elements of Activity Theory (Alexei Nikolaevich Leontiev) and the construct of situated abstraction per Richard Noss and Celia Hoyles (1996).

In tandem with abstraction, there are two studies of mathematical intuition that I would like to close my reference to CERME2 with. One (Tsamir, 2001) regards infinite sets and another (Chartier, 2001) regards geometrical intuition as a stepping stone to the study of Linear Algebra. Both refer extensively – and in some sense stand on the solid shoulders of – the essential work on mathematical intuition by Ephraim Fischbein. The analysis in (Chartier, 2001) is also embedded in curricular and pedagogical aspects of the experiences of the post-graduate students who are its focus and draws out of the students' responses the kinds of geometrical intuition – helpful and less helpful

– they bring into their practice of Linear Algebra. Those links between mathematical encounters of the students in earlier and later phases of their studies will be a focus for Ghislaine Gueudet (then Chartier) also in CERME3.

Transitions, for example from Algebra to Analysis – as in the work also in CERME2 by Michela Maschietto (2001), even though technically concerning secondary school – is a theme that features strongly in CERME ever after. I note though that both Gueudet and Maschietto had their CERME2 work presented in Working Group 7 (*Metaphors and Images*) and that Maschietto's paper has an explicit focus on the concept of limit. This is a mathematical topic which, to this day, is a flagship topic for much UME research. In CERME3, for example, there are five papers with this focus, with three of the studies carried out in a computational environment. Again, UME research can be found interspersed in five (on my count) Working Groups: 1. Metaphors and images (including embodied cognition); 3. Building structures in mathematical knowledge; 4. Argumentation and proof; 6. Algebraic thinking; 7. Geometrical thinking. Colleagues such as Uri Leron, Ted Eisenberg, Cécile Ouvrier-Bufferet contribute investigations that can be seen as closely relevant to those of us doing research in a university mathematics education context. However, these are works pitched beyond the context of the investigations at their heart. Participants are often called “subjects” and it is sometimes several pages into the papers that the reader learns whether these participants are school pupils, university undergraduates or pre-service teachers. This is a particularly evident tendency in the more explicitly psychologically-oriented works in Working Group 3 (*Building structures in mathematical knowledge*) and a little less pronounced in those in the rapidly growing Working Group 4 (*Argumentation and proof*) which had more than a dozen papers in it.

A clear exception to this rule is a paper that was not presented in any of the working groups I listed above: it was presented and discussed in Thematic Group 8 (*Social interactions in mathematical learning situations*) and, to me, it has an incredibly modern, up to date feel to it. It embodies several of the characteristics that were to become more salient in much later CERMEs. The paper is by Andreas Andersson (2003, later Ryve) and it involves observations of engineering students as they interact during mathematical activity. It also deploys the then just-emerging tools from the work of Anna Sfard and her colleagues (e.g. 2002). The tools are used to record patterns in participants' communication (*preoccupational analysis* for social aspects of the communication and *focal analysis* for patterns in the mathematical content of the communication). Both the explicit focus on a group of university students (and actually non-mathematics specialists) and the discursive tools deployed in the data analysis render the paper – retrospectively – a solid foreshadower of things to come, in CERME and elsewhere.

Part II: The AMT years, CERME 4, 5, 6

The quality and quantity of work I sampled so far from the first three CERMEs resulted in the recognition by ERME of the increasing significance of research in this area. Group 14 (*Advanced Mathematical Thinking*) was launched in CERME4 in 2005 with Joanna Mamona-Downs, Maria Meehan and John Monaghan as its inaugural leaders and attracted twelve papers.

There is a clear trend emerging from the bulk of these twelve papers: many of these works focus squarely on the students and their habits or preferences in mathematical thinking. The perspective is

largely developmental and dualist. Several papers explore perceived differences between the intuitive and the abstract, the procedural and the conceptual, processes and objects. The prevailing theoretical constructs are Richard Skemp's *instrumental and relational understanding* (1976), Shlomo Vinner and David Tall's *concept image – concept definition* (1981), Eddie Gray and David Tall's *procepts* (1994), APOS theory (Dubinsky, 1991) and Anna Sfard's *theory of reification and process – object duality* (1991).

These dualities prevail in the analysis in many of the papers – especially in studies that concern the mathematical topics of Calculus and Analysis, and proof and proving. Matthew Inglis and Adrian Simpson (2005) capture this well in their paper about dual process theory: intuition, formalism/abstraction. Students in these analyses – which have a strong developmental / cognitive flavour – appear frequently not at ease with the latter (formalism) and uncertain about the validity of the former (intuition). But, we are now well into the 2000s and the broader field is moving briskly towards what Steve Lerman (2000) had labelled a “social turn”. (A note here: I find myself agreeing more though with the later labelling, by Eva Jablonka and Christer Bergsten (2010), of “social brand”, and Lerman's own acknowledgment in the same volume that plurality is not a problem per se in mathematics education.) While attending CERME4, I was also preparing a review (Nardi, 2005) of Carolyn Kieran's, Ellice Forman's and Anna Sfard's 2002 volume *Learning Discourse: discursive approaches to research in mathematics education*. There was a palpable sense in the CERME4 sessions that this extended and accentuated tendency to use developmental/cognitive frameworks, rather than exploring connections between students' learning behaviours and the institutional, pedagogical and curricular context in which these behaviours manifest themselves, was leaving much more to desire from the presented analyses.

The paper by Erhan Bingolbali and John Monaghan (2005) on the impact of departmental settings for engineering and mathematics undergraduates' engagement with the notion of derivative, expressed this desire very well. The paper had a good go at exploring the dialectic between departmental setting, lecturers' teaching and student 'positioning'. Even better was the 2008 ESM paper by these authors, poignantly entitled *Concept image revisited*.

The paper that Paola Iannone and I presented at CERME4 (2005) also expresses, in a rudimentary form, this desire for more substantial exploration of the dialectic relationship between lecturers' and students' ways of communicating mathematically in writing and in speaking. We used the term “genre speech” (Bakhtin, 1986). The paper draws on the larger data pool that three years later became *Amongst Mathematicians* (Nardi, 2008) and has – a little over-ambitiously I admit – a multiple purpose. To explore the “genre speeches” of university mathematics is one. The other one is to bring to the fore an example of a “co-learning partnership” between university mathematics lecturers and mathematics education researchers. I note that “co-learning partnership” is a term that I had become familiar with from the work of my doctoral supervisor and research collaborator Barbara Jaworski (2003), who is also to be credited for introducing me to CERME in the first place! The rapprochement between the communities of university mathematicians and mathematics education researchers became a staple theme in much of the work that I became involved with in the years that followed – and it is one of the defining characteristics of the work that the UME group has showcased and also nurtured. More on this follows later.

Joanna Mamona-Downs continued to lead the AMT group in CERME5 too and the group grew bigger – about 50% bigger! But was it also healthier? I recall vividly the vibrancy of the sessions and also the fact that substantial findings were shared. Two strands made an impression on me at the time: the emerging strand of studies on students' generation of examples, non-examples and counter examples – for example by Maria Meehan (2007) – also emerging out of the then freshly published work in this area by Anne Watson and John Mason (2005). I also recall an emerging focus on studies that explore the easing of the transition from school to university – for example, in terms of the mathematical reasoning required. Matthew Inglis and Adrian Simpson (2007) at the time brought to our attention differences between 'vernacular logic' and 'mathematical logic' and belief biases in reasoning.

Closer to the focus that my work was gearing towards at the time, I also recall Winsløw and Møller Madsen's (2007) adaptation of ATD, the anthropological theory of the didactic, and their examination of the relationship between mathematicians' research activities and their teaching practices. Paola Iannone and I (2007) continued to report analyses from our interview study with university mathematicians: this time we chose to report a slice of our data that concerned the interplay between syntactic and semantic knowledge in proof production (Weber & Alcock, 2004).

With Lara Alcock, and also Matthew Inglis and Rina Zazkis, I was delighted to act as helper to Joanna Mamona-Downs and to observe the many elements of continuity from CERME4 – but also the elements of what I, to this day, see as evidence of healthy controversy. Mamona-Downs (2007), in her synopsis of the group's work captures this well. Here she lists the pertinent questions we were asked to engage with:

- (1) Is the perceived discontinuity between secondary and tertiary mathematics due to institutional and pedagogical practices, or is it caused by factors concerning the character of University Mathematics that demand new habits of behavior in reasoning?
- (2) What ways are there to ease the transition?
- (3) If AMT is taken as thinking skills needed for Advanced Mathematics, how are they beyond those required at school?
- (4) What commonalities or differences in mental processes are there in the two levels? (p.2228)

She then notes that our group discussion was:

“rather diffused and mostly sidestepped the questions despite their fundamental significance. It was dominated by the view of some that the research field of AMT has largely changed its main focus from cognitive-based studies starting in the early nineteen eighties, to the tendencies found nowadays based more on societal and affect factors that make the long established work 'obsolete'. Others countered strongly this position on the basis of the existence of different scientific 'paradigms', in the sense of Kuhn, and on much of the actual output of recent educational research. Opinions were often put in a partisan spirit. [...] A discussion was raised concerning the possibility that some tasks accessible to school students might pose the same kinds of problems in their resolution for undergraduates, and so it could be claimed that these tasks might be considered within the scope of AMT.” (p.2228)

No consensus was found possible in the group at CERME5 as this quotation from Mamona-Downs suggests:

“Several participants declared that the two interpretations are complementary and that there was no compelling reason not to retain the traditional name 'Advanced Mathematical Thinking' as an umbrella term [while there were] a few participants who felt that the themes stated in the program were mostly steered towards cognitive factors.” (p.2228-9)

And, I recall, for example, the paper from Corine Castela (2007) offering evidence and taking a clear stance that this persistent focus on cognitive approaches may not be the most inclusive – or fertile – way forward for the group.

This tendency to question whether UME research was appropriately congregating under the AMT umbrella continued in CERME6. The AMT group maintained its size and also, as the group leaders (Roza Leikin, Claire Cazes, Joanna Mamona-Dawns, Paul Vanderlind) observe in their notes on the proceedings (2009), attracted papers firmly focused on the latter of the two ways of interpreting AMT (advanced thinking in mathematics, A-MT or thinking about advanced mathematics, AM-T). As I was reporting a study about prospective and practising teachers' perspectives on proof, I attended the proof group on that occasion. So I missed the wealth of findings in the CERME6 AMT papers on conceptual attainment, approaches to proof and proving, problem solving, instructional approaches and processes of abstraction. It is fair to say though that UME research was gaining even more critical mass with about twenty five papers across six groups!

One of these is Barbara Jaworski's (2009) paper which proposes the exploration of university mathematics teaching practice through a sociocultural perspective that embroiders elements of *Activity Theory* and the *Communities of Practice Theory*. There will be a stream of papers thereafter in CERME with a focus on the practices and perspectives of the university mathematics teacher.

My own work in this period, a part of it also with Barbara Jaworski, illustrates this focus rather emphatically. In a nutshell, I would describe my research programme dating from 1990s to the mid-2000s as as shifting from studies of university **mathematics students' learning** of particular mathematical topics (as outlined earlier: Nardi, 1996; 2000) to a progressively growing focus on **university mathematics teachers' perspectives/practices** in mathematics and mathematics teaching (Nardi, Jaworski & Hegedus 2005; Nardi, 2008). These two sets of work illustrate the shift of my focus progressively towards university mathematics teachers' pedagogical and epistemological perspectives. UMTTP (University Mathematics Teaching Project) resulted in the 4-level Spectrum of Pedagogical Awareness (Nardi et al., 2005). *Amongst Mathematicians: Teaching and learning mathematics at University Level* (Nardi, 2008) was published in 2008, following a gestation period of several years that had started also in CERME with the presentations, with Paola Iannone, that I mentioned earlier.

Amongst Mathematicians (Nardi, 2008) tells the story of a co-learning partnership that illustrated research between mathematics educators and mathematicians with these five key characteristics: ***collaborative, mathematically focussed, context-specific, non-prescriptive and non-deficit*** as possible. In addition to reporting university mathematicians' pedagogical and epistemological perspectives, the book served a broader purpose too. It is written in the rather unconventional format of a dialogue between two fictional, yet data grounded characters – M, mathematician, and RME, researcher in mathematics education – and is intended as reflection on the perceived benefits,

obstacles and desires of the relationship between the two. Such conversations were of course not new. For example, Anna Sfard (1998) reported her discussion with Shimshon A. Amitsur, in the form of a dialogue and a range of authors from a variety of national and institutional contexts, including Michèle Artigue and Gerry Goldin, were writing at the time about this relationship. A common observation in these accounts was about its fragility. Research which consolidates and propels the rapprochement between the communities of mathematicians and mathematics educators remains a focus of my work today (e.g. Nardi, 2016) and it is fair to say that CERME, in the mid-2000s provided one of the first fora for kickstarting this work.

Let me conclude my reflections on what I labelled as “the AMT” years with a brief reference to a set of works that somehow foreshadow developments within the UME community in CERME: in the Modelling TWG, Berta Barquero, Marianna Bosch and Josep Gascón (2009) offered an ATD account of the institutional constraints hampering the teaching of mathematical modelling at university level. They coin the term “applicationism”, an epistemological perspective which proposes a strict separation between mathematics and other disciplines (especially the natural sciences) and sees mathematical tools as built to be applied to solve problems in other disciplines – with this application not causing any change in the discipline of mathematics or for the discipline in which the application is made. As UME research is rapidly growing in the area of teaching mathematics to non-mathematicians, works such as this, in CERME6 and earlier, now acquire added significance.

Part III: The UME years, *CERME 7, 8, 9*

The proposal to the ERME board for the launch of *TWG14: University Mathematics Education* was born out of two main sources. First was my reading and writing at the time: While writing *Amongst Mathematicians*, my search across the literature was broad. In fact, as Michèle Artigue (2016) has noted in her INDRUM2016 plenary, there is a synthesis feel to the book. A more explicit, deliberate synthesis of hitherto developments in research into the teaching and learning of university mathematics that was the chapter that Artigue (Artigue, Batanero & Kent, 2007) co-authored with Carmen Batanero and Philip Kent for the second NCTM Handbook. Secondly, at PME, in Morelia (Nardi & Iannone, 2008) and in Thessaloniki (Nardi et al., 2009), two Working Sessions / Discussion Groups that I had co-ordinated with colleagues many of whom ended up co-leading the UME TWG in CERME, had attracted many colleagues and had generated vital, urgent discussions.

I recall that this sensation of vibrancy and urgency was not universally shared outside the bubble of researchers in this area. I recall that when we proposed the launch of the group, we were gently reminded by members of the board that we would need to attract at least eight papers to make the new group viable! I recall that we – the inaugural co-leaders of TWG14 – were nudging each other that, if each one of us submitted a paper, we would only need to find three more to be able to launch the group! We were of course wrong.

I need to make two brief notes at this juncture: first, that the account of the group’s work since 2011 borrows heavily from the collectively authored texts in the CERME7, 8 and 9 proceedings (Nardi et al., 2011; 2013; 2015); second, that, given the volume of work presented at these conferences, I will

from now on stay largely away from extensive exemplification from specific papers. I will instead focus on the themes that mark the “emancipation” and “enrichment” themes promised in the title.

Our rationale for a UME TWG ((Nardi et al., 2011) was in a nutshell as follows.

Research on university level mathematics education is a relatively young field, which embraces an increasingly wider range of theoretical approaches (e.g. cognitive/developmental, socio-cultural, anthropological and discursive) and methods/methodologies (e.g. quantitative, qualitative and narrative). Variation also characterises research in this area with regard to at least two further issues:

- the *role of the participants*, students and university teachers, in the research – from ‘just’ subjects of the research to fully-fledged co-researchers; and,
- the *degree of intervention* involved in the research – from external, non-interventionist research, to developmental/action research in which researchers identify problems and devise, implement and evaluate reforms of practice (Artigue et al, 2007).

2011 marked the 20th anniversary of the publication of *Advanced Mathematical Thinking* edited by David Tall (1991). This is a volume that is often heralded as a first signal of the emergence of this new area of research. A few years later, a second signal was given by the 1998 ICMI study that resulted in *The teaching and learning of mathematics at university level*, edited by Derek Holton (2001). In the meantime, Advanced Mathematical Thinking (AMT) groups ran both in previous CERME and PME conferences; sessions exclusively on university mathematics education have been part of the EMF (*Espace Mathématique Francophone*) conferences since 2006; the RUME, UMT and Delta conferences emerged in the USA, the UK and South Africa respectively; the International Conferences on the Teaching of Mathematics at University Level were launched in 1998; etc. The UME TWG emerged out of the above developments and out of the realisation that this is a distinct area of mathematics education research.

The distinctiveness of UME research can be attributed to several characteristics.

Firstly, the classic distinction between ‘teacher’ and ‘researcher’ does not always apply in UME as researchers in mathematics education in this area are often university-level teachers of mathematics themselves. In particular, there is a growing group of mathematicians specializing in research on mathematics education at university level, where expertise and experience in advanced mathematics is really an asset (if not a necessity). Secondly, mathematics education theories and research methods find new uses, and adaptations, at the university level. These adaptations are often quite radical as the post-compulsory educational context is different in many ways – including the voluntary presence of students, the important role of mathematics as a service subject, the predominance of lecturing to large numbers of students, the absence of national programmes for university education, the required shift to the distinctly different practices of university mathematics, to mention but a few. In this sense, UME is a distinct area of mathematics education research, not merely a mirror of mathematics education research at a more advanced educational level. Finally, in recent years, research in this area has been growing in different parts of the world. TWG14 is one forum where evidence of this growing research activity from Europe and beyond has been accumulating.

Across CERME7, 8 and 9, the WG14 Calls for Papers invited contributions from as wide a range of research topics as possible. Here is, for example, the list from CERME9: the teaching and learning of advanced topics; mathematical reasoning and proof; transition issues “at the entrance” to university mathematics, or beyond; challenges for, and novel approaches to, teaching (including the teaching of students in non-mathematics degrees); the role of ICT tools (e.g. CAS) and other resources (e.g. textbooks, books and other materials); assessment; the preparation and training of university mathematics teachers; collaborative research between university mathematics teachers and researchers in mathematics education; and, theoretical approaches to UME research.

We opted for widening participation as much as possible, both in terms of the substantive, methodological and theoretical takes of the proposed papers but also in terms of the disciplinary background and experience of the proposers. The 21, 29 and 45 (31 long 14 short) papers accepted for publication in the respective proceedings met those terms.

Across the WG14 discussions, certain themes and questions emerged as crucial. These included: exploring whether UME needs to generate new theories or adapt already existing ones; attending to issues of both theory and practice; acknowledging that research on teaching and learning in higher education develops also outside mathematics education, and benefiting from these developments; working towards the generation of new theories while valuing already accumulated knowledge in the field; etc. One oft-repeated observation was that, beyond staple references to classic constructs from the AMT years, several works presented in TWG14 employ (often in tandem with the above) approaches such as the *Anthropological Theory of the Didactic* (Chevallard, 1999) and discursive approaches, such as Anna Sfard’s (2008) theory of *commognition*.

In CERME7 (Nardi, et al., 2011), we noted that an area of growth has certainly been studies that examine the different role of mathematics in courses towards a mathematics degree, courses for pre-service teachers, as a ‘service’ subject (physics, biology, economics etc.). While a substantial number of papers remains in the increasingly well-trodden area of students’ perceptions of specific mathematical concepts (again calculus prevails in these), a focus on university teachers and teaching is also emerging, if often a little timidly, and diplomatically, resulting in descriptive, openly non-judgemental studies. In conjunction with those studies, a genre of collaborative studies, with mathematicians engaged as co-researchers, also seems to be on the rise. We signal the emerging trends in the CERME7 papers as: *Transitions; Affect; Teacher practices; Mathematical topics*.

In CERME8 (Nardi et al., 2013), we noted the appearance of new mathematical topics: infinite series and abstract algebra. We also noted that some of these papers are written by research mathematicians, using a mathematical, epistemological, or historical analysis, and drawing on their teaching experience. Others present research that makes use of different theoretical frameworks, and methodological tools, to analyse students’ difficulties with these specific topics, to better understand the teaching of a specific topic and the consequences of this teaching, or to formulate propositions for the design of teaching to overcome these difficulties. The range of approaches vary from developmental ones (such as *concept image – concept definition*), to models for abstraction (such as

the *RBC model*), to analysis of discourse (*theory of commognition*) and the consideration of institutional matters (*anthropological theory of the didactic*)³.

After CERME8, the team – in collaboration with TWG14 participants and others – worked towards a *Research in Mathematics Education Special Issue on Institutional, sociocultural and discursive approaches to research in university mathematics education* which focused on research that is conducted in the spirit of the following theoretical frameworks: *Anthropological Theory of the Didactic*, *Theory of Didactic Situations*, *Instrumental and Documentational Approaches*, *Communities of Practice and Inquiry* and *Theory of Commognition*. As we noted in the Editorial of the RME Special Issue (Nardi et al., 2014), there is a clear surge of sociocultural and discursive approaches – and the number of papers using ATD and TDS is also remarkable. An emerging focus seems to be also on systematic investigations of innovative course design and implementation and there is certainly a rise in the number of studies that examine the teaching and learning of mathematics in the context of disciplines other than mathematics, such as engineering and economics. Furthermore, this time we welcomed more colleagues from outside Europe and also noted the rise in the number of papers on assessment and examination⁴.

In CERME9 (Nardi et al., 2015), there was a notable shift in terms of numbers of papers (two to one) in favour of the second of our two umbrella themes: *Teaching and learning of specific topics in university mathematics; Teachers' and students' practices at university level*. The breadth of topics covered especially in the latter is also telling: curriculum and assessment; innovative course design in UME; student approaches to study; relating research mathematicians' practices to student practices; views and practices of mathematics lecturers; and, methodological and theoretical contributions to UME research.

In CERME9 we also observed the further strengthening, maturity and increasingly more robust theorizing of studies into teaching practices. And, we also noticed in several papers the establishing of promising liaisons across different theoretical perspectives such as a discursive take on *mathematical knowledge for teaching* or an anthropological take on *documentational approaches*.

The critical – and growing – mass and quality of the work presented at CERME9 TWG14 led to the launch of an ERME Topic Conference, [INDRUM2016](#), a conference of the newly established *International Network for Didactic Research in University Mathematics* (Montpellier, March 31 – April 2, 2016)⁵. The conference attracted more than 80 submissions and more than 100 participants. [INDRUM2018](#) is currently in preparation.

³ By the way, we closed our CERME8 text for the proceedings with a *Concluding note on rigour and quality* of UME research. While there is no space here to elaborate, I invite the reader to what I see as pertinent observations from the TWG14 team about these issues in CERME at large.

⁴ In CERME10 there is a new TWG on assessment that spans across educational levels led by former TWG14 co-leader Paola Iannone.

⁵ I chaired this conference with the tireless Carl Winsløw. Its launch and its 2016 success (Montpellier, France) relied heavily on the sterling work of ERME president Viviane Durand-Guerrier and the commitment of Thomas Hausberger.

Part IV: CERME10, the split...

There were 47 UME papers and 16 UME posters accepted for presentation and discussion in CERME10. Their presentation and discussion was in two isomorphic groups: TWG14A and TWG14B. From CERME11, it is expected that papers may be invited for two, also thematically distinct, groups – and the debate on possible configurations for this dominated some of the discussions at the conference. One way forward that I personally favour is for a grouping by the following distinction: studies that concern the transition to university studies of mathematics and the transition from university studies into the (various forms of) workplace; and, studies that concern the teaching and learning of mathematics while at university. The challenge of debating the numerous configurations of how the (new) group(s) can be (re)defined is certainly non-negligible. Isn't this a most wonderful place to find ourselves though, having to manage the now critical mass and quality of UME research present in CERME?

Part V: Taking stock / What next / Coming soon...

As I am drawing to a close, I would like to ask the question: what did we want to achieve with the establishment of TWG14? Have we achieved these objectives? Are we going to? For example: did we manage to encourage fledgling topics in UME research? Have we planted the seed for new ones?

In the sprawling vista of works that I aimed to sample in this lecture – and I am fully aware of the wafer thin way in which I have done so – I have aimed to identify trends in UME research (overall, in CERME, in my own work) that signify the benefits (the **richness!**) of opening up, of widening our substantive, theoretical and methodological horizons (the what, the how and the why of our research). Most of my examples have aimed to illustrate the benefits that **emancipation** from an individualistic, narrowly psychological, cognitive perspective has brought to UME research.

There are still though foci that have not yet merited our sufficient attention. One such research focus that seems to me to be not within the radar of current works is UME research is on more advanced topics in mathematics – and by that, I mean mathematics that is typically taught beyond the first two years of university studies.

On a less deficit tone, I am generally satisfied that we have come a long way but I also acknowledge that there is an even longer way to go. It is fair to say that, within the various UME communities around the world, we have gone (or are still going) through what I would like to label as a dismissive phase: that all so-called traditional pedagogies are “bad”, lecturing in particular. I am observing – but I am also asking that we do so even better – that we become more nuanced and embracing of possibility. We are starting, for example, to recognise that lecturing can serve some purposes rather well; that it can be complemented by formats more tailored to the serving of students' individual needs; that there are interactive lecture formats that give participants the buzz of community belonging and building and prepare students for the less cocooned, less protected world of work where interaction, team work and communication are key. We are finding out that not all interaction and all the time is good per se and that there are particular types of communal engagement with mathematics that work better than others. TWG14 papers have been offering the evidence base for these claims, steadily and cumulatively. In a way, I find the choice made by the

mathematics department in my institution (see earlier snip in Figure 2) to include in its promotional materials images of lectures and also to close its promotional video (<https://www.youtube.com/watch?v=gRzVX8c1be4>) with a close-up of white chalk on a blackboard (Figure 4) somewhat refreshing. We are perhaps starting after all to embrace diversity in the ways that the students need to experience mathematics!

I believe the answers to the questions with which I started this section are reservedly optimistic and affirmative. In Part II, I showed an outline of my own research programme over the years and I am pleased to be able to say that most of the items there – and what followed these – have emerged out of collaborations with colleagues in CERME, including research plans for the immediate future.

CERME has indeed been a platform where I am trialling new topics for research. My CERME8 paper (Nardi, 2013) offers analyses of the challenges of teaching a graduate course on mathematics education to students with a variety of backgrounds, including bachelor degrees in pure mathematics, and native languages other than the language of instruction. The paper also outlines key didactic techniques and principles to cope with these challenges. It finally morphed into the more substantial analyses present in a paper included in the inaugural issue of IJRUME (Nardi, 2015) which examined ways to facilitating paradigm shifts in the supervision of mathematics graduates upon entry into mathematics education.

CERME has also been a platform where I have trialled new approaches to analysing data. In fact, I credit CERME for allowing me the creative space to have a go – and converse about – discursive, particularly commognitive, approaches to the analyses of my data. My CERME7 paper (Nardi, 2011) outlined interviewed mathematicians' perspectives on their newly arriving students' verbalisation skills; and, observed that discourse on verbalisation in mathematics tends to be risk-averse and not as explicit in teaching as necessary. At CERME9, Bill Barton and I (Nardi & Barton, 2015) presented a *commognitive* analysis of a “low lecture” episode (student-led inquiry oriented discussion on open-ended problems) to illustrate crucial steps of student enculturation into mathematical ways of acting and communicating, including a shift away from the lecturer's ‘ultimate substantiator’ role. Finally, both the papers I am involved in as co-author in CERME10 (Virman & Nardi, 2017; Thoma & Nardi, 2017) present commognitive analyses in contexts that said analyses are now just about starting to appear (teaching mathematics to non-mathematicians; analyses of closed-book examination tasks and student/lecturers' assessment discourses).

Returning to the anecdote that I started with, a somewhat self-deprecating recollection of the theoretical ambivalence of my doctoral work, I see my own research programme as an illustration of the **richness emanating from the emancipation**, from what I now see as a narrow, individualistic perspective in my earlier work. To me there is nothing vacantly rhetorical about the three Cs in the CERME spirit: COMMUNICATION, COOPERATION, COLLABORATION. The growth of my research programme through each one of these is to me unshakeable evidence of the pragmatic strength of these three words. In TWG14 these words have taken shape as specific actions. Here are two: (1) Certainly, we have assisted with the arrival of several new researchers in this field, some of whom are currently co-leaders; many have used the reviewing process as a stepping stone for their writing (from poster to conference paper then to completing theses and journal papers). (2) We have engaged practitioners of university mathematics teaching who now see themselves also as UME

researchers. To do so, we deploy the reviewing process and the discussions at the conference to convey the rigour that is required for UME research (in terms of engaging with theory, prior research and methodology) and to bridge the epistemological differences between the academic disciplines of mathematics and education.

I invite the reader to the collections of papers published in the TWG14 sections of the Proceedings, the 2014 *Research in Mathematics Education* Special Issue that followed CERME8, the proceedings of the 2016 *INDRUM* conference and the imminent (publication expected in 2018) *International Journal for Research in Undergraduate Mathematics Education* Special Issue that is following INDRUM2016 as testimonials of the growth and diversity I have tried to map here. And there is more to come: INDRUM2018 will be hosted by MatRIC at the University of Agder (Kristiansand, Norway) in April 2018 and its Scientific Committee aims to follow it up with a state-of-the-art volume soon after. And, of course, there is the UME chapter (Winsløw et al., in press) in the ERME 20th Anniversary Book that we aim to celebrate in CERME11, in 2019. The promise of UME research on the global scene is further corroborated by the healthy growth of the [RUME](#) and [DELTA](#) conferences, and the respective group within [EMF](#). In closing, I return to the words of Michèle Artigue whose thoughtful INDRUM2016 plenary (Artigue, 2016) triggered the focus of the synthesis and analysis presented here:

“The emergence of the [UME] field was [...] characterized by the domination of cognitive and constructivist perspectives. I consider as a strength of our field the fact that we have succeeded in emancipating ourselves from these perspectives, whose limitations are evident, but also the fact evidenced by the consideration of most research publications, that this emancipation went along a reconstruction of their main outcomes, thus making possible some form of incorporation of these outcomes in the new paradigms.”

Michèle Artigue, from *Mathematics education research at university level: Achievements and challenges*, INDRUM2016 plenary lecture (p.19)

References

- Andersson, A. (2003). The discourse of engineering students constructing concept maps in Linear Algebra. In M. A. Mariotti, (Ed.) *European Research in Mathematics Education III: Proceedings of the Third Conference of the European Society for Research in Mathematics Education*. Bellaria, Italy: University of Pisa and ERME. Available at: <http://www.mathematik.uni-dortmund.de/~erme/CERME3/>.
- Artigue, M. (2016). Mathematics education research at university level: Achievements and challenges. In E. Nardi, C. Winsløw & T. Hausberger (Eds.) *Proceedings of the 1st INDRUM (International Network for Didactic Research in University Mathematics) Conference: an ERME Topic Conference* (pp. 11-27). Montpellier, France.
- Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. K. Lester (Ed.), *The Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1011-1049). Charlotte: IAP.
- Bakhtin, M. M. (1986). *Genre Speeches and Other Late Essays*. USA: University of Texas Press.

- Barquero, B., Bosch, M. & Gascón, J. (2009). The ‘ecology’ of mathematical modelling: constraints to its teaching at university level. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) *Proceedings of the Sixth Congress of the European Mathematical Society for Research in Mathematics Education*, (pp. 2146-2155). Lyon, France: Institut National de Recherche Pédagogique and ERME.
- Belousova, L. & Byelyavtseva T. (1999). Training explorations on numerical methods course using technology. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education*. Vol. 1, (pp. 201-208). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.
- Burton, L. (1999). Mathematics and their epistemologies – and the learning of mathematics. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education*. Vol. 1, (pp. 87-102). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.
- Burton, L. (2004). *Mathematicians as Enquirers*. New York: Springer.
- Castela, C. (2007). Analysis of the autonomy required from mathematics students in the French Lycée. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME5* (pp. 2260-2269). Larnaca, Cyprus: University of Cyprus and ERME.
- Chartier, G. (2001). «Geometrical intuition» to learn Linear Algebra. In J. Novotná (Ed.), *European Research in Mathematics Education II: Proceedings of the Second Conference of the European Society for Research in Mathematics Education*. (pp. 533-542). Mariánské Lázně, Czech Republic: Charles University, Faculty of Education and ERME.
- Chevallard, Y. (1999). L’analyse des pratiques enseignantes en théorie anthropologique du didactique [The analysis of teaching practice in the anthropological theory of the didactic]. *Recherches en Didactique des Mathématiques*, 19, 221–266.
- Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist* 31, 175-90.
- Dorier, J-L., Robert, A., Robinet, J. & Rogalski, M. (1999). Teaching and learning Linear Algebra in first year of French science university. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education*. Vol. 1, (pp. 103-112). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.
- Dorier, J-L., Robert, A., Robinet, J. & Rogalski, M. (2000). *On the teaching of linear algebra*. New York: Springer.
- Dreyfus, T., Hillel, J. & Sierpinska, A. (1999). Cabri-based Linear Algebra Transformations. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education*. Vol. 1, (pp. 209-221). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.

- Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht: Kluwer Academic Publishers.
- Ferrari, P.L. (1999). Co-operative principles and linguistic obstacles in advanced mathematics learning. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education*. Vol. 2, (pp. 24-37). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.
- Grenier, D. & Payan, C. (1999). Discrete mathematics in relation to learning and teaching proof and modelling. In I. Schwank (Ed.), *European Research in Mathematics Education I: Proceedings of the First Conference of the European Society for Research in Mathematics Education*. Vol. 1, (pp. 140-152). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.
- Holton, D. A. (Ed.). (2001). *The teaching and learning of mathematics at university level: An ICMI study*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Iannone, P. & Nardi, E. (2005). To appear and to be: acquiring the ‘genre speech’ of university mathematics. In M. Bosch (Ed.), *European Research in Mathematics Education IV: Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education*. (pp. 1800-1810). Sant Feliu de Guíxols, Spain: FUNDEMI IQS – Universitat Ramon Llull and ERME.
- Iannone, P. & Nardi, E. (2007). The interplay between syntactic and semantic knowledge in proof production: mathematicians’ perspectives. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME5* (pp. 2300-2310). Larnaca, Cyprus: University of Cyprus and ERME.
- Inglis, M. & Simpson, A. (2005). Characterising mathematical reasoning: Studies with the Wason selection task. In M. Bosch (Ed.), *European Research in Mathematics Education IV: Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education*. (pp. 1768-1777). Sant Feliu de Guíxols, Spain: FUNDEMI IQS – Universitat Ramon Llull and ERME.
- Inglis, M. & Simpson, A. (2007). Belief bias and the study of mathematics. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME5* (pp. 2310-2319). Larnaca, Cyprus: University of Cyprus and ERME.
- Jablonka, E., & Bergsten, C. (2010). Commentary on theories of mathematics education—Is plurality a problem? In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 11–117). New York, NY: Springer.
- Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54(2/3), 249 - 282.
- Jaworski, B. (2009). The practice of (university) mathematics teaching: mediational inquiry in a community of practice or an activity system. In V. Durand-Guerrier, S. Soury-Lavergne & F.

- Arzarello (Eds.) *Proceedings of the Sixth Congress of the European Mathematical Society for Research in Mathematics Education*, (pp. 1585-1594). Lyon, France: Institut National de Recherche Pédagogique and ERME.
- Leikin, R., Cazes, C., Mamona-Downs, J. & Vanderlind, P. (2009). Advanced mathematical thinking. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) *Proceedings of the Sixth Congress of the European Mathematical Society for Research in Mathematics Education*, (pp. 2238-2245). Lyon, France: Institut National de Recherche Pédagogique and ERME.
- Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Westport, CT: Ablex.
- Mamona-Downs, J. (2007). Advanced mathematical thinking. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME5* (pp. 2221-2230). Larnaca, Cyprus: University of Cyprus and ERME.
- Maschietto, M. (2001). The transition from algebra to analysis: The use of metaphors in a graphic calculator environment. In J. Novotná (Ed.), *European Research in Mathematics Education II: Proceedings of the Second Conference of the European Society for Research in Mathematics Education*. (pp. 542-553). Mariánské Lázně, Czech Republic: Charles University, Faculty of Education and ERME.
- Mason, J. (1998). Researching from the inside in mathematics education. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (pp. 357 - 377). Dordrecht / Boston / London: Kluwer Academic Publishers.
- Meehan, M. (2007) Student generated examples and the transition to advanced mathematical thinking. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME5* (pp. 2349-2358). Larnaca, Cyprus: University of Cyprus and ERME.
- Nardi, E. (1996). *The novice mathematician's encounter with mathematical abstraction: Tensions in concept-image construction and formalisation* Unpublished doctoral thesis, University of Oxford, UK. Available at <http://www.uea.ac.uk/~m011>.
- Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in Group Theory. *Educational Studies in Mathematics* [43\(2\), 169-189](#).
- Nardi, E. (2005). 'Beautiful minds' in rich discourses: On the employment of discursive approaches to research in mathematics education. *European Educational Research Journal* [4\(2\), 145-154](#).
- Nardi, E. (2011). 'Driving noticing' yet 'risking precision': University mathematicians' pedagogical perspectives on verbalisation in mathematics. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2053-2062). Rzeszow, Poland.
- Nardi, E. (2013). Shifts in language, culture and paradigm: the supervision and teaching of graduate students in mathematics education. In B. Ubuz, C. Haser, & M.A. Mariotti (Eds.), *Proceedings of*

the Eighth Congress of the European Mathematical Society for Research in Mathematics Education (pp. 2396-2405). Ankara, Turkey: Middle East Technical University and ERME.

- Nardi, E. (2015). Not like a big gap, something we could handle: Facilitating shifts in paradigm in the supervision of mathematics graduates upon entry into mathematics education. *International Journal for Research in Undergraduate Mathematics Education* [1\(1\), 135–156](#).
- Nardi, E. (2016). Where form and substance meet: Using the narrative approach of *re-storying* to generate research findings and community rapprochement in (university) mathematics education. *Educational Studies in Mathematics* [92\(3\), 361–377](#).
- Nardi, E. & Barton, B. (2015). Challenging the mathematician's 'ultimate substantiator' role in a low lecture innovation. In K. Krainer, & N. Vondrova (Eds.), *Proceedings of the Ninth Congress of the European Mathematical Society for Research in Mathematics Education* (pp. 2207-2213). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.
- Nardi, E., Biza, I., González-Martin, A., Gueudet, G., Iannone, P., Viirman, O. & Winsløw, C. (2015). University Mathematics Education. In K. Krainer, & N. Vondrova (Eds.), *Proceedings of the Ninth Congress of the European Mathematical Society for Research in Mathematics Education* (pp. 2048-2051). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.
- Nardi, E., Biza, I., González-Martin, A., Gueudet, G., & Winsløw, C. (2013). University Mathematics Education. In B. Ubuz, C. Haser, & M.A. Mariotti (Eds), *Proceedings of the Eighth Congress of the European Mathematical Society for Research in Mathematics Education* (pp. 2270-2275). Ankara, Turkey: Middle East Technical University and ERME.
- Nardi, E., Biza, I., González-Martin, A., Gueudet, G., & Winsløw, C. (2014). Institutional, sociocultural and discursive approaches to research in university mathematics education. *Research in Mathematics Education* [16\(2\), 91-94](#).
- Nardi, E., González-Martin, A., Gueudet, G., Iannone, P. & Winsløw, C. (2011). University Mathematics Education. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Mathematical Society for Research in Mathematics Education* (pp. 1923-1927). Rzeszów, Poland: University of Rzeszów and ERME.
- Nardi, E., Jaworski, B., & Hegedus, S. (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From 'tricks' to 'techniques'. *Journal for Research in Mathematics Education* [36\(4\), 284-316](#).
- Nardi, E. & Iannone, P. (2008). Shifts in generating pedagogical theory in university-level mathematics education research. Working Session. In O. Figueras, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the XX North American Chapter* Vol. 1, pp. 200. Morelia, Michoacán, México: PME.
- Nardi E., Iannone P., Biza I., Giraldo V., González-Martín A. & Pinto, M. (2009). The role of theory in university-level mathematics education research. In M. Tzekaki, M. Kaldrimidou, & C.

- Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 298. Thessaloniki, Greece: PME.
- Noss, R. & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer Academic Publishers.
- Pedemonte, B. (2001). Relation between argumentation and proof in mathematics: cognitive unity or break? In J. Novotná (Ed.), *European Research in Mathematics Education II: Proceedings of the Second Conference of the European Society for Research in Mathematics Education*. (pp. 70-80). Mariánské Lázně, Czech Republic: Charles University, Faculty of Education and ERME.
- Schwarz, B., Hershkowitz, R. & Dreyfus, T. (2001). Emerging knowledge structures in and with algebra. In J. Novotná (Ed.), *European Research in Mathematics Education II: Proceedings of the Second Conference of the European Society for Research in Mathematics Education*. (pp. 81-91). Mariánské Lázně, Czech Republic: Charles University, Faculty of Education and ERME.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Sfard, A. (1998). A mathematician's view of research in mathematics education: An interview with Shimshon A. Amitsur. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (pp. 445 - 458). Dordrecht / Boston / London: Kluwer Academic Publishers.
- Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. In C. Kieran, E. Forman & A. Sfard (Eds.), *Learning Discourse: Discursive approaches to research in mathematics education*. Dordrecht/Boston/London: Kluwer Academic Publishers.
- Sfard, A. (2008). *Thinking as communicating: Human development, development of discourses, and mathematizing*. New York, NY: Cambridge University Press.
- Sierpiska, A., & Kilpatrick, J. (Eds.). (1998). *Mathematics Education as a research domain: A search for identity*. Dordrecht / Boston / London: Kluwer Academic Publishers.
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20-26.
- Stehlíková, N. & Jirotková, D. (2001). Building a finite algebraic structure. In J. Novotná (Ed.), *European Research in Mathematics Education II: Proceedings of the Second Conference of the European Society for Research in Mathematics Education*. (pp. 101-111). Mariánské Lázně, Czech Republic: Charles University, Faculty of Education and ERME.
- Tall, D. O. (Ed.). (1991). *Advanced mathematical thinking*. Dordrecht: Kluwer.
- Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Thoma, A. & Nardi, E. (2017, in press). Discursive shifts from school to university mathematics and lecturer assessment practices: Commognitive conflict regarding variables. In T. Dooley

& G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Mathematical Society for Research in Mathematics Education*. Dublin, Ireland: DCU and ERME.

Tsamir, P. (2001). Intuitive structures: The case of comparisons of infinite sets. In J. Novotná (Ed.), *European Research in Mathematics Education II: Proceedings of the Second Conference of the European Society for Research in Mathematics Education*. (pp. 112-122). Mariánské Lázně, Czech Republic: Charles University, Faculty of Education and ERME.

Vuurman, O. & Nardi, E. (2017, in press). From ritual to exploration: The evolution of Biology students' mathematical discourse through Mathematical Modelling activities. In T. Dooley & G. Gueudet (Eds.) *Proceedings of the Tenth Congress of the European Mathematical Society for Research in Mathematics Education*. Dublin, Ireland: DCU and ERME.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. London: Lawrence Erlbaum Associates, Inc.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, 56, 209 - 234.

Winsløw, C. Gueudet, G., Hochmuth, R. & Nardi, E. (2019, in preparation). [university mathematics education]. In Dreyfus, T., M. Artigue, D. Potari, D., S. Prediger, & K. Ruthven (Eds.), *Developments in European Research in Mathematics Education - Twenty Years of Communication, Cooperation and Collaboration*. New York, NY: Routledge.

Winsløw, C. & Møller Madsen, L. (2007). Interplay between research and teaching from the perspective of mathematicians. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME5* (pp. 2379-2388). Larnaca, Cyprus: University of Cyprus and ERME.

Young children's early mathematical competencies: Analysis and stimulation

Lieven Verschaffel, Joke Torbeyns, and Bert De Smedt
University of Leuven, Belgium, lieven.verschaffel@kuleuven.be

In this paper we take a critical look at the state-of-the-art in the research domain of early mathematical development and education. We start with a brief review of the influential and successful (neuro)cognitive research in this domain - which is heavily focused on the development and teaching of children's (non-symbolic and symbolic) magnitude representation and strongly dominated by the theory of an approximate number system (ANS). We confront and complement this (neuro)cognitive approach with various other lines of research that may help to provide a more comprehensive picture of the development and stimulation of children's early mathematical competence and how it relates to their later mathematical proficiency at school.

Keywords: Early mathematics, approximate number system, number concepts, mathematical patterns and structures, preschool education.

Introduction

The past 10-15 years have witnessed the emergence of a remarkably productive and highly influential line of research on children's early numerical magnitude representation, its development, its relation to school mathematics, and its assessment and stimulation (Torbeyns, Gilmore & Verschaffel, 2015).

The starting point of this line of research - which has its origins in cognitive (neuro)psychology -, is the idea that young children, like many other species, are equipped with some foundational innate core systems to process quantities. This "starter's kit" is thought to involve (a) an "object tracking system" that has a limit of three or four objects and is thought to underlie "subitizing" (= to immediate and accurate estimate of one to four objects without serial enumeration), and (b) an "analogue number system" – for the internal representation of numerical magnitudes as Gaussian distributions of activation on a "mental number line" with increasingly imprecise representations for increasing magnitudes (Dehaene, 2011) - allowing them to compare non-symbolic quantities that are too numerous to enumerate exactly or to perform some very basic approximate arithmetic on these quantities (Andrews & Sayers, 2015; Butterworth, 2015).

With these foundational core number sense systems, these magnitudes are represented non-verbally and non-symbolically, but, over development and through early (mathematics) education, verbal and symbolic representations are gradually mapped onto these foundational representations, to evolve into a more elaborated system for number sense (Torbeyns et al., 2015).

People's numerical magnitude representations are commonly assessed via magnitude comparison and/or number line estimation tasks, of which there exist both non-symbolic and symbolic versions (Butterworth, 2015; Andrews & Sayers, 2015; Torbeyns et al., 2015). Examples are shown in Figure 1.