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An empirical study of the understanding of formal propositions about sequences, with a focus on infinite limits

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In this paper, we analyze the answers of one group of high-school students and two groups of first-year University students to a questionnaire designed to test their level of recognition and understanding of the formal definition of the concept of infinite limit. Although this empirical study is ancillary to a larger project centered on didactic engineering, its analysis sheds light on the key issue of the logical prerequisites for the learning of the fundamental concepts of analysis. It also provides a new tool to investigate students' concept-image of limits, and assess the impact of teaching contexts and teaching paths.

Keywords: Teaching and learning of analysis and calculus, teaching and learning of logic, reasoning and proof, definitions, limits.

CONTEXT AND RATIONALE

At the INDRUM 2016 conference, Cécile Ouvrier-Buffet and Renaud Chorlay presented a poster outlining a medium-scale project on definitions in analysis (Chorlay & Ouvrier-Buffet, 2016), with a focus on the formal definition of the limit of a numerical sequence. This topic lied at the intersection of the research interests of the two researchers: Cécile Ouvrier-Buffet is a maths-education researcher with a strong epistemological background, whose work bears mainly on *definitions*, their use, and the conditions for their genesis in teaching-contexts (Ouvrier-Buffet 2011). Since most of her former work bore on discrete mathematics, she wanted to investigate the extent to which the theoretical tools she had developed in this context had to be adapted to deal with a teaching context with very different mathematical (continuous vs discrete) and didactical (transition from *calculus* to *analysis*) features. Renaud Chorlay is a historian of mathematics and teacher educator with a long-standing interest in the history (Chorlay, 2011) and didactics of analysis.

We selected the topic of limits because we felt many years of didactical investigations had made it a *mature* topic; a topic about which knowledge has accumulated to form a sound and coherent body of knowledge. Indeed, we know a lot about limits in terms of conceptions and misconceptions (Robert, 1982); also in terms of obstacles (Sierpiska, 1985). As far as the genesis or rediscovery of the (or *a*) definition is concerned, many attempts have been made and reported upon in details, whether in the framework of didactic engineering¹ (Robert, 1983) (Bloch & Gibel, 2011) or with other research tool-boxes (Mamona-Downs, 2001) (Przenioslo, 2005) (Swinyard, 2011) (Lecorre, 2016)

(Roh & Lee, 2017). The tricky logical aspects were studied, in particular, in (Arsac & Durand-Guerrier, 2005).

On this solid basis, our work on the genesis and use of definitions has so far been engaged along three different lines of investigation; we will distinguish between *ex-ante* studies – *before* students' first encounter with formal definitions of limits – and *ex-post* studies.

- *Ex-ante* 1: For year 12 (final year of secondary education), the French curriculum requires that students majoring in mathematics and the sciences study a definition of limits (finite or infinite) of numerical sequences. Students are not really expected to use this definition on their own; rather, the teacher is expected to use these definitions on a few occasions, to show that some properties of limits can actually be proved mathematically (in particular: any unbounded and increasing sequence tends to $+\infty$). The underlying idea is that early encounter with a few rigorous definitions and proofs should ease the transition between high-school *calculus* – with its combination of algorithmic procedures and graphical intuition – and university *analysis*. This classroom work on the formal definition of limits is connected to another requirement of the current curriculum, namely that throughout high-school, the basic notions and the standard notations of mathematical logic be gradually made explicit. In this context, the discovery of a definition for limit, with its specific sequence of nested quantifiers, is supposed to be the culmination of this gradual process. In 2016, one of us (Chorlay) designed a teaching-session in the spirit of didactic engineering, for students to gradually formulate a formal definition of the infinite limit. We will report on this in detail in another context.
- *Ex-post* 1: in 2015-2016 we studied how – if at all – prospective maths-teachers made use of the definition of limits in order to identify and analyze vague, informal or erroneous statements regarding limits. We reported on this in a poster presented at the INDRUM 2016 conference.
- *Ex-post* 2: in 2016-2017 we designed a questionnaire in order to assess the level of recognition and understanding of the formal definition of the infinite limit. This questionnaire, and the answers collected with three groups of students are the topic of this paper.

QUESTIONNAIRE - DATA COLLECTION

The questionnaire was of the True/False type, divided in two parts. We give below an English translation, along with indications on the correct answers.

Part I. For each one of the implications below, circle either “True” or “False”. If you circle “False”, justify your answer.

#1	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$
T - F	Justification (if “False”):
#2	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \forall n \in \mathbf{N} u_n \geq A$
T - F	
#3	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R} \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$
T - F	
#4	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R} \forall n \in \mathbf{N}, u_n \geq A$
T - F	
#5	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \exists n_A \in \mathbf{N}$ such that for any integer n greater than n_A $u_n \geq A$
T - F	

Correct answers:

#1 True: Here the consequent means “not bounded above”.

#2 False: Here the consequent is a property which never holds; hence the implication is always invalid.

#3 True: Here the consequent is always valid, hence the implication is always valid.

#4 True: Here the consequent means “bounded below”.

#5 True: Here the consequent is the definition, worded semi-formally.

Part II. The four implications below are taken from part I. For each one of them, first state its converse, then circle “True” or “False” regarding the converse. Justify if “False”.

#1	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$ Converse :
T - F	Justification (if “False”):
#3	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R} \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$ Converse :
T - F	
#4	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R} \forall n \in \mathbf{N}, u_n \geq A$ Converse :
T - F	
#5	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \exists n_A \in \mathbf{N}$ such that for any integer n greater than n_A $u_n \geq A$ Converse:
T - F	

Conv. of #1 False: standard counter-examples are $(-2)^n, (-1)^n \times n \dots$

C of #3 False: the antecedent being always true while the consequent can be false, the implication is invalid.

C of #4 False: being bounded below does not imply $\lim = +\infty$.

C of #5 True: definition.

The specific form of the questionnaire derives from its original intended use. It was first designed to assess the didactic engineering, which focused on the formal definition of the infinite limit. Other forms of assessment of the ability to recognize, and of the level of understanding of the formal definition were ruled out, in particular interviews (as in (Robert, 1982)) or proof-writing (as in (Roh & Lee, 2017)). We felt this questionnaire would give us feedback regarding two key features of the engineering, namely (1) the role of logic, hence the flood of formulae with nested quantifiers in this questionnaire; (2) the fact that “not bounded above” is a necessary condition for $\lim u_n = +\infty$ but not a sufficient condition, hence the importance of item #1 and its converse.

We did not ask for justifications when the item was deemed “True” by the students, mainly to save time and keep the questionnaire feasible in about 20 minutes. In addition, the justificatory task for True statements could vary a lot across teaching-contexts and would not easily lend itself to comparison. For instance, considering item #4 (if $\lim u_n = +\infty$ then the sequence is bounded below): in some contexts citing a theorem studied in class would suffice whereas in other contexts students would have to devise and write a non trivial proof. We also chose to drop the converse of item #2, since the fact that an implication whose antecedent is False is considered valid is a purely logical matter.

In the spring of 2017, the questionnaire was administered to three groups of students: Group 1 is one of the two French Year-12 classes which had experienced the engineering; Group 2 and 3 are first-year university students in Mons University (Belgium), with high-achieving maths majors in Group 2 and medium-achieving² computer science majors in Group 3. In all three cases, the questionnaire was given several months after the course on limits had been taught, and students had not been asked to revise anything in particular. They were told the questionnaire was given for research purposes, and would not be graded. They were given between 20 and 30 minutes. The number of students was: 31 (group 1), 50 (group 2), and 17 (group 3).

We originally hoped a comparison between the three groups would enable us to study the effects of three teaching units: our engineering (group 1), a “standard” maths-lecturer course (group 2), and Robert’s engineering (group 3, as reported upon in (Bridoux, 2016)). Unfortunately, we were not able to do that, since other factors seemed to have had a more significant impact.

FINDINGS

Result #1

A first result is that this questionnaire is *not* unfeasible. In group 2, 14 of the 50 questionnaires were answered perfectly correctly, with relevant counter-examples for the False statements. Some of these counter-examples had been

studied in class (such as $(-1)^n \times n$ for the converse of #1); in these cases, students managed to interpret “ $\forall A \in \mathbf{R} \quad \exists n_A \in \mathbf{N} \quad u_{n_A} \geq A$ ” as “not bounded above” and selected a relevant counter-example in a memorized repertoire. In other cases, counter-examples had not been studied in the course on limits – because they had nothing to do with limits – and students crafted *ad-hoc* counter-examples, displaying some command of logic (for instance, to prove that the negation of “ $\forall A \in \mathbf{R} \quad \forall n_A \in \mathbf{N} \quad u_{n_A} \geq A$ ” always holds).

Result #2

A second set of results sheds light on the role of an explicit teaching of logic. When we collected the data we first engaged in quantitative analysis, and were pretty unhappy about the following result: in group 1 (our engineering), only 26% of the students considered #4 to be “True”, compared to 86% in group 2 and 71% in group 3. A closer look at the answers showed that in group 1, a significant number of students had actually engaged in another task than the prescribed task. In Fig. 1 and 2 we translated extracts of answer-sheets from group 1:

#2	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \quad \forall n \in \mathbf{N} \quad u_n \geq A$
T (F)	$\exists n \in \mathbf{N}$ such that $u(n) \geq A$
#3	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R} \quad \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$
T (F)	$\forall A \in \mathbf{R}$
#4	If $\lim u_n = +\infty$ then $\exists A \in \mathbf{R} \quad \forall n \in \mathbf{N}, \quad u_n \geq A$
T (F)	It's beyond some rank n

Figure 1. Student 29 of group 1

#1	If $\lim u_n = +\infty$ then $\forall A \in \mathbf{R} \quad \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$
True	Justification (if “False”): One forgot to specify $\forall n \geq n_A$ and $\exists A$
False	This implication proves that there exists a term greater than $\forall A \in \mathbf{R}$

Figure 2. Student 3 of group 1

In these answer-sheets, the students did not engage in an assessment of the logical implications but in a comparison between the formal statements given as consequents (in part I) and the definition of $\lim u_n = +\infty$. In these examples the comparison can be clumsy (as for #2 for student 29, or the “and $\exists A$ ” for student 3). Nevertheless, it rests on the fact that the definition is known (correct answers for #5 and its converse), and is seen as the relevant template against which other quantified formulae ought to be contrasted. Moreover, the comparison is not purely syntactical: in her assessment of #4, student 29 did not only spot that “ \exists

$A \in \mathbf{R} \quad \forall n \in \mathbf{N}, \quad u_n \geq A$ ” is *not* the definition, but also elicited in her own words *why* it could not be, namely “ $\forall n \in \mathbf{N} \quad u_n \geq A$ ” does not capture “beyond a certain rank”, which is a key element of the definition. The reinterpretation of the prescribed task is typical of at least one third of the questionnaires from group 1.

By contrast, only one of the 67 students from Mons University reinterpreted the implication-assessment task as a comparison-with-the-definition task. A key difference between group 1, on the one hand, and groups 2 and 3, on the other hand, is that at Mons University students had studied logic in the first term, whereas the French high-school students had only occasionally been exposed to logic. The French students were familiar with the notion of converse, and had some knowledge of the meaning of quantifiers \forall and \exists , but were not familiar with sequences of quantifiers; much less with the negation of such sequences. These formal aspects were not problematic for a large majority of the Mons students. This does not mean that all the logical aspects were mastered by the Mons students. In particular, when it came to proving that some formal statement was valid, many answer-sheets showed misconceptions regarding the use of \forall and \exists .

This sheds some light on the standard but thorny issue of prerequisites: since the formal definition of limits involves a sequence of nested quantifiers, how much logic should be taught (either beforehand or along the way) for students to be able to *do* anything with it? Our results suggest that the answer depends on how “do” something with a definition is construed. *Using* the formal definition to design and write proofs probably requires some know-how regarding the interpretation of hitherto unknown sequences of quantifiers, and the negation of such sequences; for a significant proportion of the French student, their occasional and in-context encounters with logical notations did not allow them to acquire such know-how. However, if “do” is taken to mean “remember the definition” and even “understand the definition”, then for a large majority of the French students, their command of logic was adequate. For instance, we consider the work of student 29 of group 1 (fig.1) to display some degree of conceptual understanding of definition, namely some understanding of the specific role of each of the three quantifiers. Student 3 is clearly able to interpret “ $\exists A \in \mathbf{R} \quad \forall n \in \mathbf{N} \quad u_n \geq A$ ”. This understanding does not rest on a general ability to make sense of and formally manipulate logical formulae, but is limited to the context of the definition of limits. Since it relies on the specific connections between the concept-image and concept-definition of “limit” targeted (and, apparently, stabilized) in the didactically engineered teaching-session, this understanding is probably not only context-dependent but also path-dependent.

Result #3.

In the *a priori* analysis for the engineering, we studied the relations between three mathematical properties of numerical sequences:

- (1) $\lim u_n = +\infty$;
- (2) (u_n) is not bounded above;
- (3) (u_n) is strictly increasing, at least from a certain rank onward.

Our hypothesis was that properties (2) and (3) were part of the concept-image of (1) for most students; of a concept-image² in which all three properties are considered to “go together”, without any specific and explicit logical connections being part of the cognitive structure. This hypothesis was based on the didactical literature (Robert 1982) (Mamona-Downs 2001) (Swinyard 2011), and was perfectly confirmed during the two implementations of the engineering. For this reason, our design aimed for conceptual differentiation, to be achieved first through the study a few well-chosen sequences, and then through the formal explicitation of the logical connections between (1), (2), and (3). Consequently, we wanted our post-experiment questionnaire to help us assess to what extent students knew that $(1) \Rightarrow (2)$ is valid, while its converse is not.

Due to the significant level of reinterpretation of the prescribed task in group 1, the data gathered do not easily lend themselves to quantitative comparison. However, the fact that “not bounded above” (2) is a key component of the concept image of $\lim u_n = +\infty$ (1) is again confirmed beyond doubt. Let us first compare groups 2 and 3. In group 3 – the medium-achieving computer science majors – all 17 students deemed the converse of #1 to be True. Leaving out 3 students whose answer-sheets show an inadequate command of the logical aspects, it seems that Aline Robert’s engineering (which targeted the definition of finite limits) had no impact on the belief that if a sequence (u_n) takes on arbitrarily large values, then $\lim u_n = +\infty$. In group 2, that of high-achieving maths majors, the results were not as striking; they were telling just as well. Among the 50 answer-sheets, let us focus on the subpopulation of those for which all the answers to part I were correct (including relevant counter-examples for #2), and all the converses were stated correctly. Among these 33 students, 17 deemed the converse of #1 to be False – which is the correct answer – and all but one provided a relevant counter-example (usually $(-1)^n \times n$ which – as the lecturer confirmed – had been studied in detail). Student 25 even wrote: “ $\forall A \in \mathbf{R} \quad \exists n_A \in \mathbf{N}$ such that $u_{n_A} \geq A$ means that the sequence is not bounded above, but it doesn’t mean it tends to $+\infty$, it may oscillate. Let’s consider $(-1)^n \cdot n$ (...)”. However, the other 16 students ticked “True” for the converse of #1. The resistance of this belief, even among students with a reasonable command of logic, who know the definition of $\lim u_n = +\infty$ (item #5 and its converse), and who had been exposed to a teaching which explicitly tackled this issue suggest that the conflation of (1) and (2) is an epistemological

obstacle (Chorlay & de Hosson, 2016). It is probably not independent from the belief that all sequences are monotonous, at least after a certain rank (Robert 1982), but our questionnaire offers no new insight as to this.

This confirms – in hindsight – that we were justified to take (2) into account when designing a teaching-session on the formal definition of (1). However, it does not tell us whether targeting the formulation of the definition of (1) through a process fostering the conceptual differentiation between (1) and (2) was didactically relevant – as standard constructivist tenets suggest – or just foolhardy.

The results of group 1 allow us to be cautiously optimistic. From a purely quantitative viewpoint, 58% of the students of group 1 deemed the converse of #1 to be “False” – which is the correct answer – but no conclusions can be drawn from this fact beyond that this 58% stands in sharp contrast with the 0% of “False” on the subpopulation of OK-answer-sheets of group 3. In group 1, for instance, the third of the students who clearly reinterpreted the task as “compare with the definition” ticked “False”, but this does not indicate that they are aware of the connections between properties (1) and (2), or that they were able to reformulate “ $\forall A \in \mathbf{R} \quad \exists n_A \in \mathbf{N} \quad \text{such that} \quad u_{n_A} \geq A$ ” as “not bounded above”. Answer-sheet 30 of group 1 shows, again, that some conceptual understanding can be achieved in a formal context in spite of a poor level of command of symbolic logic. This student systematically stated $\neg B \Rightarrow A$ as converse of $A \Rightarrow B$; hence one has to study her assessment of the converse of #4 – instead of #1 – to see if she mistakes (2) for (1); which she does *not*, actually. Of the 31 students of group 1, only two interpreted the task correctly *and* provided relevant correct answers for the converse of #1, either with a formulaic counter-example $(-5)^n$ or with a graphical counter-example (of the $y = x \cdot \sin x$ type). However, about one fourth of the students deemed the converse of #1 to be false, interpreted the task as “assess the implications” and provided arguments which we could be indicative of some conceptual understanding. In these cases, they justified their assessment not by displaying a counter-example, but by explaining why the antecedent was not strong enough to warrant the consequent: under the hypothesis “ $\forall A \in \mathbf{R} \quad \exists n_A \in \mathbf{N} \quad \text{such that} \quad u_{n_A} \geq A$ ”, the sequence can oscillate; or: the antecedent does not imply that the sequence is increasing. Our empirical data does not enable us to tell which of the following is the case: either, students argue on the basis of the fact that if a sequence is increasing and not-bounded above then it tends toward $+\infty$ (a theorem they are familiar with); or, students conflate (1) and (3).

CONCLUSIONS AND RESEARCH PERSPECTIVES

While the questionnaire studied in this paper was originally designed to compare the effectiveness of three teaching-modules on the definition of limits of sequences, it turned out that they could not serve that purpose due to the

decisive impact of another factor, namely the level of familiarity with predicate calculus – both in terms of syntactic command, and in terms of ability to make sense of logical formulae involving nested quantifiers. Nevertheless, we claim that meaningful conclusions or insights can be gained from the analysis of our empirical results.

For students with some command of logic – a command which cannot be gained through an occasional and in-context use of logical formalism – this questionnaire does provide insight into the connections between concept-image and concept-definition for limits, thus providing a new investigative tool to study this issue; a tool which does not involve conducting interviews or studying students ability to use the definition in proofs. As far as students are concerned, the comparison between group 2 and group 3 suggest that not all teachings on limits are equivalent in this respect; the case of group 2 shows that – under circumstances which call for further investigation – first-year university students can display a reasonable command of the concept of limit.

As far as group 1 is concerned, the result show that the prerequisites in logic may not be as high as one might expect, if what is targeted is the ability to memorize the formal definition, and the ability to display understanding of some key features of the concept. As far as our didactic engineering is concerned, these results show that (1) it was not a complete failure, (2) some of its guiding principles – such as the importance of the conceptual differentiation between infinite-limit and not-bounded-above, or the use of logical formalism – seem relevant. However, in this context, this questionnaire is probably not the best tool for a fine-grained assessment of what the actual impact of this engineering is.

1. For introduction to didactic engineering as task-design oriented research method, see (Bosch & Barquero 2015).

2. This assessment of the overall level of the groups is that of the team of maths lecturers at Mons University, as communicated to us by Stéphanie Bridoux, who is both a member of that team and a mathematics education researcher (LDAR). Many thanks to her for her collaboration on this project.

3. D. Tall and S. Vinner introduced the distinction between the image and the definition of a concept to stress the difference between mathematics as a mental activity and as a formal system. “We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. (...) it needs not be coherent (...)” (Quoted in (Tall 1991, 7)).

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