The use of integrals in Mechanics of Materials textbooks for engineering students: the case of the first moment of an area

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Research has reported on the difficulties engineering students face in relating the content of their mathematics courses to what is taught in their professional courses. One way to address these difficulties is by better understanding how mathematical notions are used in professional engineering courses. This paper analyses how the notion of first moment of an area – which is defined as an integral – is used in civil engineering courses. Basing our analysis on elements from the anthropological theory of the didactic, we are currently analysing a classic Mechanics of Materials book. Our findings indicate that although first moments are introduced as an integral, the textbook's tasks do not require students to use techniques typically introduced in a traditional calculus course.

Keywords: Mathematics for engineers, teaching and learning of analysis and calculus, textbooks, anthropological theory of the didactic, first moment of an area.

INTRODUCTION

Engineering courses are usually organized into two groups: basic science courses (which are taught in the first two years, including foundational skills in mathematics and physics), and technical courses (which appear later in the programme and are more specific to each field of engineering). However, research in engineering education and mathematics education indicates that engineering students encounter many difficulties in their mathematics courses in the first years of study, which can lead to high failure rates, and in many cases, result in students dropping out of engineering programmes (Ellis, Kelton, & Rasmussen, 2014). In this sense, "poor mathematics skills are a major obstacle to completing […] engineering programs" (Fadali, Johnson, Mortensen, & McGough, 2000, p. S2D-19).

Researchers have identified some negative situations for students who pass these mathematics courses. One situation is that these students often find it difficult to relate the learned mathematical content to the content of the professional courses. For Flegg, Mallet, and Lupton (2011, p. 718) "without the explicit connection between theory and practice, the mathematical content of engineering programs may not be seen by students as relevant". Another situation is that in spite of having passed the mathematics courses (with a rather rigid structure and rare concrete applications relevant to engineering), students must apply mathematics in their engineering courses, where many new mathematical notions appear without having been encountered in the previous mathematics courses (Hochmuth, Biehler, &

Schreiber, 2014, p. 694). Faced with these problems, the mathematics and engineering education communities have been engaged in research and discussion on "how to improve engineering students' mathematics learning, and hence their service teaching" (Bingolbali, Monaghan, & Roper, 2007, p. 764).

Our current research program investigates how calculus notions are used in engineering courses, aiming at identifying possible ruptures between how notions are first introduced and used in calculus, and how they are later used in professional courses. First, we analyse how engineering textbooks present these notions, working under the principle that most tertiary instructors organise their teaching using textbooks as an important resource (e.g., Mesa & Griffiths, 2012). The manner in which mathematics notions are used in professional courses has not been the subject of much research. However, we believe this type of research could help bridge the gap between two communities. On the one hand, mathematics lecturers in engineering programs could benefit from knowing how their course content is used in professional courses; on the other hand, professional course instructors could benefit from a critical analysis of their use of mathematics, to help their students make connections between the content of mathematics and professional courses. For example, our analysis of the way integrals are used to define bending moments for beams in strength of materials textbooks for civil engineering reveals different uses of "the same" object (González-Martín & Hernandes Gomes, 2017a). Although bending moments are defined as an integral, the tasks, techniques, and justifications used in calculus courses are very different from the ones presented in professional engineering courses; this may result in students not recognising "the same" object in two different courses, and they may question the relevance of integration techniques that are not used in tasks concerning bending moments. In this paper we develop the content of González-Martín & Hernandes-Gomes (2017b) as we explore the use of integrals to introduce another engineering notion: first moment of an area. We aim to address two questions: how is the content related to integrals used in engineering to work with first moments of an area, and how does this use relate to the content in calculus courses?

Defining first moment of an area

Moments of areas are topics commonly taught in engineering courses that cover strength of materials. Due to space limitations, in this paper we focus on the first moment of an area. In civil engineering, for example, to solve bending problems one must take into account some specific geometrical characteristics of cross-sections of a bar, which is the general term for structures that include beams (Feodosyev, 1973). In this situation, the notion of first moment of an area is used to calculate the centroid of an area and the shearing stresses in transverse bending. The centroid of an area A is its geometrical barycentre and is the point C of coordinates $\bar{x}$ and $\bar{y}$ such that the following relationships hold true: $\int_A x\,dA = A\bar{x}$ and $\int_A y\,dA = A\bar{y}$.

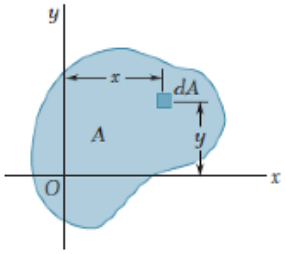

Figure 1: General area *A* with infinitesimal area *dA* in the *xy* plane (Beer et al., 2012, p. A2).

Let *A* be an area situated in the *xy* plane (Figure 1), using *x* and *y* as the coordinates of an element of area *dA*. According to Beer, Johnston, DeWolf, and Mazurek (2012, p. A2), the first moment of an area *A* with respect to the *x* axis (resp. *y* axis) is mathematically defined as the integral $Q_x = \int_A y\,dA$ (resp. $Q_y = \int_A x\,dA$). In both integrals, the index *A* in the integral sign indicates that the integral is calculated over the whole cross-sectional area. Both integrals characterize the sum of the products of each element of area *dA* and its distance to the respective axis (*x* or *y*) and are measured in cubic units (Beer et al., 2012). When an area possesses an axis of symmetry, the first moment with respect to that axis is zero, since every element of area *dA* of abscissa *x* (resp. ordinate *y*) corresponds to an element of area *dA'* of abscissa *–x* (resp. ordinate *–y*). This implies that when an area possesses an axis of symmetry, its centroid is located on that axis. For instance, in a rectangular cross-section (two axes of symmetry), its centroid *C* coincides with its geometric centre. Determining the position of the centroid is important, since several forces in a bar pass through its centroid.

To illustrate these definitions and their calculation with an example, let us consider the case of a bar with a rectangular cross-section (Figure 2). If we consider the expression above, $Q_x = \int_A y\,dA$, we can take *dA* as the area of the grey rectangle, whose dimensions are *b* and *dy*. Substituting *dA* in the integral, we have that $Q_x = \int_A y\,dA = \int_A y\,b\,dy$. Calculating this integral throughout all the vertical extension of the rectangular cross section, we obtain: $Q_x = \int_0^h y\,b\,dy$. Calculating the integral, we obtain: $Q_x = b\frac{y^2}{2}\Big|_0^h = b\frac{h^2}{2} - b\frac{0^2}{2}$, therefore $Q_x = b\frac{h^2}{2}$.

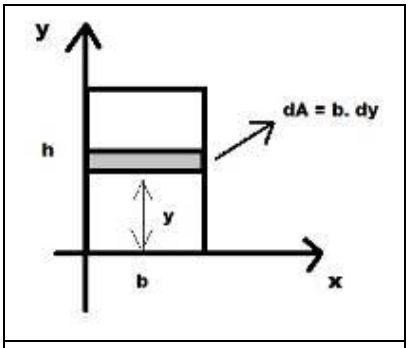

Figure 2: Determination of the first moment with respect to the *x*-axis of an area with rectangular cross-section.

THEORETICAL FRAMEWORK

As stated above, we are interested in analysing how calculus notions are used in professional engineering courses, aiming at identifying possible breaks from the content in calculus courses. For our research, we use tools from the anthropological theory of the didactic (ATD – Chevallard, 1999) because it considers human activities as institutionally situated. In this sense, knowledge about these activities and their raison d'être is also institutionally situated (Castela, 2016, p. 420). In particular, ATD offers a general epistemological model of mathematical knowledge,

where mathematics is seen as a human activity through which various types of problems are studied (Barbé, Bosch, Espinoza, & Gascón, 2005, p. 236).

The key element we use in our analysis is the notion of praxeology (or, in our case, mathematical organisation or mathematical praxeology – MO hereinafter), which is formed by a quadruplet [$T/\tau/\theta/\Theta$] consisting of a type of task T to perform, a technique τ which allows the task to be completed, a discourse (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. The first two elements [T/τ] are the practical block (or know-how), whereas the knowledge block [θ/Θ] describes, explains, and justifies what is done. These two blocks are important elements of the ATD model of mathematical activity that can be used to describe mathematical knowledge. Furthermore, ATD distinguishes different types of MO: punctual, which are associated with a specific type of task; local, which integrate multiple punctual MOs that can be explained using the same technological discourse; and regional, which integrate local MOs that accept the same theoretical discourse (Barbé et al., 2005, pp. 237-238).

Praxeologies, like knowledge in general, may move from the institution where they emerge to other institutions that find them useful (Castela & Romo Vázquez, 2011). This is the case, for instance, of mathematical notions that are used to solve engineering problems. In this process, there are *transposition* effects on the concerned praxeologies (Castela & Romo Vázquez, 2011; Chevallard, 1999). We consider the work of Castela (2016), who identified that "when a fragment of social knowledge, produced within a given institution I, moves to another one I_U in order to be used, the ATD's epistemological hypothesis states that such boundary crossing most likely results in some transformations of knowledge, called transpositive effects" (p. 420). In this boundary-crossing process, some (or all) elements of the original praxeology may evolve. Therefore, it is important to analyse the types of tasks and techniques as well as the discourses and theories employed. To that end, our research identifies specific local MOs present in professional courses; we analyse how calculus notions are used (practical block) and whether this use relates to the way the notions are usually presented in calculus courses (knowledge block).

METHODOLOGY

It is worth noting that, in order to understand how calculus notions are used in engineering courses, we have had several exchanges with an engineering teacher holding bachelor's and master's degrees in civil engineering, with more than 28 years of experience teaching a variety of professional engineering courses at Brazilian universities. This teacher has explained notions related to his field and has helped us identify course content in which first-year calculus notions are used.

At this teacher's university, first moments of area are introduced during the third semester of the programme (second year), in the Strength of Materials for Civil Engineering course (students take calculus in their first two semesters). The course's

reference book is Beer et al. (2012). First moments are initially cited in chapter 4 (4.2. *Stresses and deformations in the elastic range*). We proceeded in two stages:

- First, we analysed the general structure of the content related to integrals in the calculus courses. We identified the main tasks proposed to students, grouping them according to the technological elements needed, identifying therefore the main local MOs that structure this content.
- Second, we started our analyses of the reference book for the Strength of Materials course. We identified all instances where first moments appear in the book (using key words to search in an electronic version of this book). For each occurrence of this notion, we are currently analysing the tasks presented in the book where first moments are used. For each task, we are analysing the techniques and discourses (technologies) the textbook uses. As the notion of first moment is used in different chapters of the book, where different professional notions are introduced and explained, the technological discourses are quite varied, giving place to various MOs. The next section provides specific details of our analysis.

DATA ANALYSIS AND DISCUSSION

Calculus is taught in the first year of the program over two semesters in two courses: Calculus I and Calculus II. Up until a few years ago, integrals appeared only in Calculus II, but some content was moved to Calculus I because Physics II (a course in the second semester) requires a knowledge of integrals. Integrals appear towards the end of the first course and are the main topic in the second course (the second author of this paper has taught Calculus I for 15 years and Calculus II for two years). The content covers indefinite integral (antiderivative of *f*), Riemann sum and definite integral, applications for the calculation of areas, integration by substitution, volumes (Calculus I), and integration techniques, improper integrals, and arc length (Calculus II). The main source for the calculus courses is the classic book by Stewart (2012). The content concerning integrals is basically structured using two local MOs. The first, MO_{M1}, introduces techniques for calculating indefinite integrals (immediate integration to begin with, followed by various integration techniques); however, theoretical elements justifying the different integration methods are mostly absent and those present are explained without a proof. The second MO, MO_{M2}, introduces Riemann sums to formally define integrals and interpret them as areas, and leads to the Fundamental Theorem of Calculus and the calculation of definite integrals using Barrow's rule; this leads to some applications of the integral (area, volume …). Many of the techniques used in MO_{M2} are derived from MO_{M1}.

We are currently analysing the use of first moments and centroids in the engineering textbook (Beer et al., 2012), which numbers more than 800 pages. So far, our first analyses indicate that although this content is introduced as an integral, the

techniques employed do not call for integration. Our ongoing analyses of the use of first moments in the textbook are summarised in Figure 3.

Description of use	Terms used	Chapter – Sections
The term appears in a theoretical explanation. It appears with an expression using the integral sign, but no calculation is required.	First moment; First moment of an area; *Q*; centroid	4.2 (p. 245); 4.2 (p. 245); 4.4 (p. 262); 4.6 (p. 274)
		6.1 (p. 421); 6.1 (p. 421); 6.3 (p. 437)
The term appears in a theoretical explanation. It appears without an expression using the integral sign.	First moment; First moment of an area; *Q*; centroid	4.3 (p. 262)
		6.1C (p. 424); 6.1C (p. 424); 6.4 (p. 440); 6.6 (p. 454; pp. 459-460); Review (p. 467)
		8.1 (p. 559); Review (p. 591)
		9.5A (p. 651); 9.5A (p. 651); 9.5A (p. 654); 9.5A (p. 654); 9.6B (p. 666)
Concept application: It is involved in some calculations, but no calculation of integrals is required.	First moment; First moment of an area; *Q*; centroid	4.2 (p. 247; p. 248)
		6.1 (p. 422); 6.1 (p. 422); 6.3 (p. 438); 6.6 (pp. 456-457)
		8.3 (pp. 577-578)
		9.5A (p. 652); 9.5A (p. 653); 9.5A (p. 655); 9.5A (p. 656); 9.6B (p. 667); 9.6C (p. 669)
Sample problem: It is involved in some calculations, but no calculation of integrals is required	First moment; *Q*; centroid	4.3 (p. 251); 4.5 (p. 265); 4.10 (p. 326)
		6.2 (p. 429); 6.5 (pp. 443-446); 6.6 (p. 462)
		8.3 (p. 583)

Figure 3: Synthesis of uses of first moments in Beer et al. (2012).

Here, due to space limitations, we describe our analysis of two MOs present in the textbook at points where first moments come into play. It is worth noting the book advises students that they should already have completed a course in statics, that the properties of moments and centroids are explained in Appendix A, and that this material can be used to reinforce the discussion of the determination of normal and shearing stresses in beams in chapters 4, 5, and 6 (Preface, p. x).

First case: MO_{E1}

The initial use of first moments, MO_{E1}, concerns stresses and deformations in the elastic range (section 4.2 of the book). Its aim is to calculate the maximum stress that beams can resist, resulting in some recommendations about the size and shape of beams. Using some formulae, the book arrives at $\int ydA = 0$ and concludes: "This equation shows that the first moment of the cross section about its neutral axis must be zero. Thus, for a member subjected to pure bending and *as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section*" (p. 245, italics in the original). This is the first apparition of first moments in the book; however, they are not explained and the authors refer readers to Appendix A. In Appendix A, first moments and their link with the centroid are introduced in a similar manner as in this paper, using implicitly theoretical elements from MO_{M2} (namely, the interpretation of an integral). However, the book makes the connection with the centroid and deduces many integrals using geometric considerations (and the properties of the centroid), and adds "Centroids of common geometric shapes are given in a table inside the back cover" (p. A3). Therefore, this content is justified vaguely through some basic integral content (present in MO_{M2}), but mostly by using geometric considerations. The tasks in MO_{E1} calculate stresses and bending moments

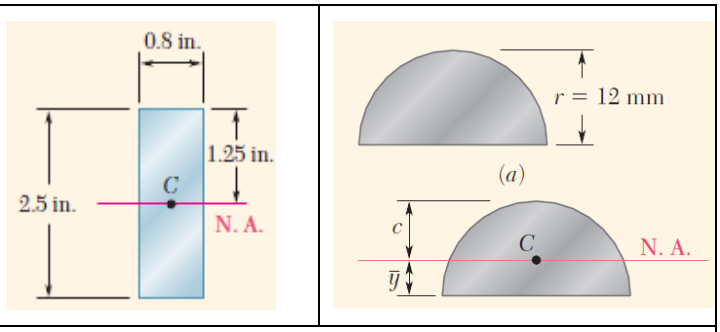

Figure 4: Left: The centroid is placed calculating the half measure of each side of the rectangle (Beer et al., 2012, p. 247). Right: The centroid is placed using geometric formulae (p. 248).

in known geometrical shapes. In the case of a rectangle (Figure 4-left), the coordinates of the centroid are deduced using geometry (and not techniques derived from MO_{M1} or MO_{M2}); the same approach is used in the case of a semicircular cross-section (Figure 4-right).

Thus, although the notions of first moment and centroid are necessary to solve tasks in MO_{E1}, the techniques employed are not based on elements derived from MO_{M1} or MO_{M2}. Students can solve the tasks present in this MO_{E1} without using any of the techniques learned in MO_{M1} or MO_{M2}, or hardly any of the technological elements present in them.

Second case: MO_{E2}

First moments and centroids are also used in chapter 6. In section 6.1A, *Shear on the horizontal face of a beam element*, MO_{E2} seeks to determine the horizontal shear per unit length (or shear flow) on a beam. Defining Δx as the length of a section of the beam, V as the shear force, ΔH as the horizontal shearing force exerted on the lower face of the element, Q as the first moment, and I as the centroidal moment of inertia, and using techniques and technological elements covered in this and previous chapters, the horizontal shear per unit length (q) is deduced as: $q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I}$. It is worth noting that the techniques used to arrive at this expression involve integrals, but they are referred to in terms of notions belonging to MO_{E2}. The above expression is used to solve tasks such as the one in Figure 5.

A beam is made of three planks, 20 by 100 mm in cross-section, and nailed together. Knowing that the spacing between nails is 25 mm and the vertical shear in the beam is V = 500N, determine the shearing force in each nail.

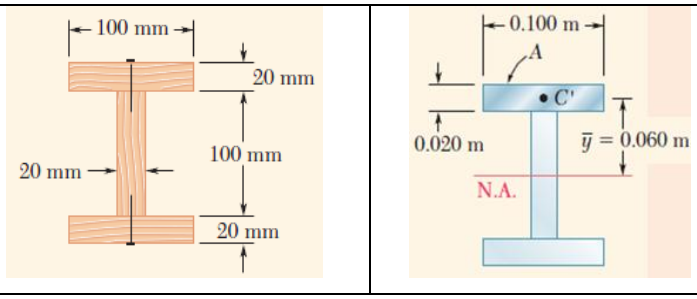

Figure 5: Task and diagrams used concerning horizontal shear (Beer et al., 2012, p. 422).

The resolution of the task is based on the determination, via different expressions, of Q and I (since V = 500N is provided) to find the horizontal force exerted on the lower face of the upper plank. For the first moment, Q, the following technique is presented: "Recalling that the first element of an area with respect to a given axis is equal to the product of the area and of the

distance from its centroid to the axis, $Q = A\bar{y}$" (p. 422). The area of the cross-section of the upper plank is calculated as 0.020m × 0.1m, and the coordinate of the centroid of this horizontal plank with respect to the axis of symmetry of the cross-section is 0.05m + 0.01m (that is, half the measure of the central plank, plus half the measure of the horizontal plank). Q is thus obtained as: $Q = A\bar{y}$ = (0.020m × 0.100m)(0.060m) = $120 \times 10^{-6}m^3$. We see that, once more, the tasks to solve in this MO_{E2} involve cross-sections with geometrical shapes that make use of geometrical considerations, thus avoiding techniques belonging to MO_{M1} or MO_{M2}.

Although the technological elements of MO_{E2} refer to elements that imply the use of integrals, tasks are presented in such a way that previously deduced formulae can be used and magnitudes can be deduced using these formulae and geometrical considerations. The book later provides a table with values (Figure 6). Therefore, it is possible for students to simply memorise the formulae or use the tables to solve the given tasks without actually using any technical or technological element derived from MO_{M1} or MO_{M2}.

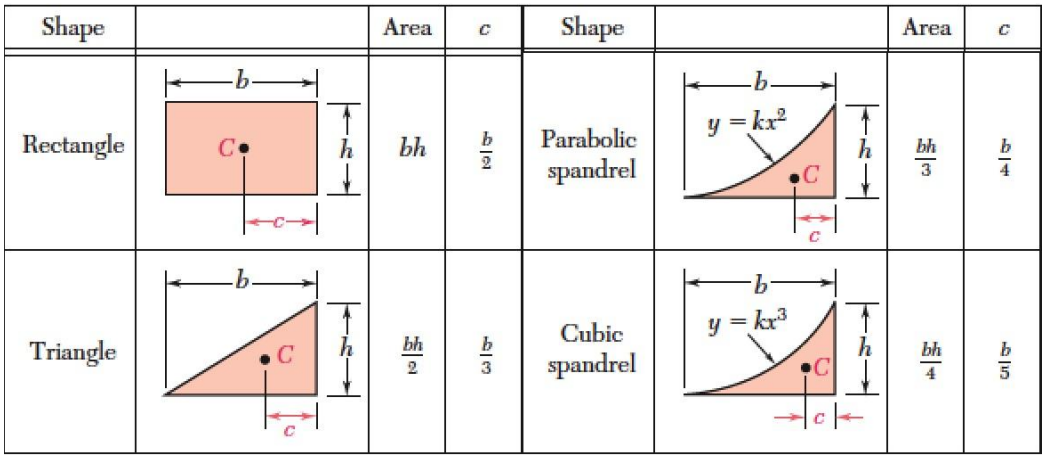

Figure 6: Areas and centroids of common shapes (Beer et al., 2012, p. 654).

FINAL CONSIDERATIONS

The data presented here, together with the data from González-Martín & Hernandes Gomes (2017a), indicate that two notions used in civil engineering (bending moment and first moment) are defined as integrals. This may often be used to justify the fact that "engineers need to learn integrals". However, our analyses show that the types of tasks and the techniques developed are not actually derived from praxeologies explored in a calculus course. In the two cases presented in this paper, both MO_{E1} and MO_{E2} have their own set of tasks and techniques, and both develop their own technological discourse, which uses the notion of integral to define their own notions and deduce properties. As Figure 3 shows, this seems to be the general situation throughout the textbook.

As Castela (2016) states, when a fragment of knowledge (in this case, the notion of integral) produced within a given institution moves to and is used by another

institution, this process results in a transformation of knowledge. In the case analysed here, it is clear that all the technological discourse proper to a calculus (or even an analysis) course pertaining to the notion of integral is transformed when it is used to define first moments (and centroid) in a professional course, where explanations mostly rely on basic geometric considerations. In this case, it seems that the transpositive effects cause the notion of integral to be used very differently in both courses. The techniques presented in the Strength of Materials course make use of given formulae and geometric considerations, rendering the techniques introduced in the calculus course irrelevant for the use of first moments in MO_{E1} and MO_{E2}. This may result in students not recognising the object "integral" when they move from MO_{M1} and MO_{M2} to MO_{E1} and MO_{E2}. Students may encounter many difficulties in learning MO_{M1} and MO_{M2}, but this knowledge is not necessary to solve tasks in engineering courses, so students may question the need to learn these MOs.

It is therefore important that mathematics lecturers in engineering programs become aware of how the notions they teach are used in professional courses. Once they develop a better understanding of the techniques and tasks used in professional courses, mathematics instructors may be prompted to reflect on the mathematical praxeologies developed in their own courses and make stronger connections with the techniques used in professional courses. This could help students transition from mathematics courses to professional courses, enabling them to relate mathematical content to the content of their professional courses and better understand its relevance (Flegg et al., 2011).

We plan to continue analysing the use of integrals in professional courses in engineering. This will be the source of future papers.

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