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Laura Branchetti, Viviane Durand-Guerrier

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Laura Branchetti[1] and Viviane Durand-Guerrier[2]

[1] Department of Mathematical, Physical and Computer sciences, Univ. of Parma, Italy, laura.branchetti@unipr.it

[2] Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier, France

In this paper we study cognitive conflicts on the issue of number sets being dense, ordered and denumerable. We first provide historical-epistemological background related to these notions. Then we consider the cognitive conflicts under the lenses of concept image and concept definition, which we use to analyse empirical data collected in order to understand better the didactical and cognitive issues at stake.

Keywords: density, ordered set, denumerable set, concept image versus definition.

INTRODUCTION

Our interest in the title question comes from our teaching experiences in the first year of the Master degree in Mathematics course in Italy for the first author, and in first-year university courses in France for the second author. In both cases, the focus was on the distinction between density and continuity for an ordered set of numbers. Both authors were surprised by the following students' questions concerning the denumerable sets N and Q.

> Q1. How is it possible to find an order in Q if Q is dense, i.e. when the consecutive number of a rational does not exist?
>
> Q2. How is it possible that there is a bijection between N and Q, but N is discrete and Q is dense?

The students' questions highlighted potential conflicts which emerge when making explicit the properties of density[1] and denumerability of the set of rational numbers at the same time. This motivates a research investigation into the didactical transposition of the objects that underlie the two properties, namely order on a set, properties of both discrete[2] and denumerable sets, bijection, ordered isomorphism, difference between cardinal and ordinal numbers, and enumeration[3]. Our general research question is: how should we deal with these questions in classroom activities in order to help students overcome the apparent contradictions? In this paper, we will focus on a less ambitious sub-question concerning the first question posed by students (Q1).

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[1] From the point of view of order, an order dense set is a linearly ordered set (X,<) with the property that if x<y then there exists z∈X with x<z<y (Jech, 2003). Here with the term "density" we refer to order density.

[2] A set S is discrete in a topological space X if every point x∈X has a neighbourhood U such that $S \cap U = \{x\}$ (points are said to be isolated) (Krantz, 1999, p.63).

[3] An enumeration is a complete, ordered listing of all the items of a set. An enumeration for an infinite set is a one-to-one correspondence between this set and the set of positive integers.

> RQ: What are university students' and teachers' concept images and concept definitions of *dense*, *ordered* and *denumerable* set? How do they connect them?

We analysed students' and teachers' answers from two perspectives: i. a historical-epistemological analysis of the topics which emerged in the students' questions: ii. *concept image* and *concept definition* (Tall & Vinner, 1981). The first point is addressed in the first section: we refer to historical works in which infinite sets were studied from the point of view of cardinality, ordering and enumeration (Galilei, 1638; Lolli, 2013; Peano, 1889). The second point is addressed in the second section; we use our historical-epistemological analysis, together with results in mathematics education (Tirosh & Tsamir, 1996; Bergé, 2010; Durand-Guerrier, 2016; Branchetti, 2016), as resources to identify *a priori* possible students' concept images that could conflict with each other. In the third section, we describe the contexts and methodology of data collection and analysis, carried out in parallel in France and Italy which involved university teachers and Master degree students. Finally, we provide a brief overview of the way the concepts are introduced in scholastic stages prior to university studies in both countries, as a relevant background for our conclusions and starting point for further developments.

HISTORICAL EPISTEMOLOGICAL ISSUES

Galileo's view on numbers and their squares: an issue about cardinality

In one of the most famous books by Galileo Galilei (1564 - 1642), *Dialogues concerning Two New Sciences* (1638), the Italian physician, mathematician and philosopher introduced the one-to-one correspondence between natural numbers and their squares. We report here a brief summary of the main ideas. The dialogue concerns the difficulty that appears when trying to compare the number of points contained in two segments, one being longer than the other. Salviati, the voice of Galilei, states; "This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited" (English translation, 1914, p.31). He then moves to numbers and develops an argument on the impossibility of comparing the totality of all numbers with the numbers of squares since they are both infinite:

> "neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and, finally, the attributes "equal," "greater," and "less" are not applicable to infinite, but only to finite, quantities." (p. 32-33)

Density of *Q*, cardinal and ordinal numbers: Cantor's contribution

Cantor (1845 - 1918), working on trigonometric series and their convergence, moved on to the creation of a new theory of transfinite numbers, and the perspectives of Number Theory and Set Theory. Cantor came firstly to the definitions of a derived set – the set of limit points – and of a dense set and then of a dense-in-itself set, like the rational numbers set (Lolli, 2013). Studying infinite sets, he introduced the diagonal argument to prove that not only do the squares have the same cardinality as natural numbers, but so also does the Cartesian product of the set of natural numbers by itself, and hence the set of rational numbers, thanks to the existence of a surjection from the Cartesian product to this latter set. In other words, he had to find a way to enumerate the ordered pairs of natural numbers, being sure to consider every pair once and once only, following an ordering principle. To do this, he moved from the usual linear image of order to a 2-dimensional image. A crucial distinction, introduced by Cantor when he was facing such problems, is between cardinal numbers and ordinal numbers (Lolli, 2013). While in the set of natural numbers with its standard structure (formalized by Peano in 1889) the relation of order is strictly connected to the problem of ordering and with the induction principle, this is no longer the case in Q. Indeed, the standard order on Q (i.e. that consistent with measurement of magnitudes onto the line) is not consistent with Cantor's diagonal ordering principle, i.e. in the resulting order, $\frac{3}{4}$ is listed before $\frac{5}{4}$, but after $\frac{3}{2}$.

Peano's formalization of Arithmetic and the issue of order in natural numbers

In Peano's Arithmetic, the ideas developed by Cantor were formalized and used as principles to grasp the "essence" of natural numbers: the injective function that establishes Cantor's "first generation principle" (Lolli, 2013) for the consecutive element of a natural number is strictly linked to the operation of addition (the consecutive of n, being $S(n)$, is equal to $n+1$) and to the comparison between natural numbers, if we consider the standard order. In this structure, the consecutive element is always greater than its precedent in respect to the standard order.

RESEARCH FRAMEWORK

According to Tall & Vinner (1981) every concept, from a cognitive point of view, is associated to different *concept images*. A relevant cognitive feature of conceptualization concerns the introduction of formal definitions: it often happens that the *concept definition* is not introduced appropriately by teachers in relation to the concept images. According to the authors, "a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure" (p. 3). Some students' concept images may be recognized as conflicting and inconsistent from an expert point of view, but they can coexist in their mind until a conflict is shown evoking them together simultaneously (p. 2). Such *cognitive conflicts* are occasions for learning and advancing in the process of conceptualization, but if not recognized and suitably overcome, they can become obstacles in the learning processes. We hypothesised that the students who asked our two questions (Q1 and Q2) were facing cognitive conflicts and trying to manage the following apparent conflicting images of

Q: i. a dense set; ii. an ordered set; iii. a set with the same cardinality as a discrete set, like *N*.

A priori identification of concept images

The concept definitions relevant to our topic are the following: *linear ordering*; *enumeration*; *dense* set; *discrete* set; *denumerable* set; *cardinality*; *bijection*; *isomorphism*. Relying on literature review and historical-epistemological analysis, we identified possible features of *concept images* that may cause cognitive conflicts:

CI1) *Consecutive element is greater*: generalizing an association that is typical of N, reinforced by the spatial image of the oriented line (a greater element is on the right as the consecutive element). Students may think that a consecutive element in a list must be greater than the previous one (considering the standard order).

CI2) *A dense set cannot be enumerated*: Students might have a concept image of ordered sets as sets in which the elements are "one after the other" on the line: an enumeration must move from left to right consistently with the standard order.

CI3) *Dense not discrete*: *Q* might be said to be dense as opposed to *N*, which is not. The difference is "shown" either on the line or with numerical examples as an absolute difference (to have elements in "between" or not), independent of the particular order. Density may thus be considered an absolute property of a set and the "visual contrary" of discrete.

CI4) *Linear or bidimensional representation*: the cardinalities of *Q* and *N* are shown to be equal, "re-ordering" *Q* and constructing a bijection between the two sets. The ordering procedure is usually represented in two dimensions (the "dovetail" counting method) while the standard order is represented using the line.

CI5) *Bijection is identification*: students may associate the term "bijection" with a total identification between the structures (A=B), not just in terms of cardinality.

CI6) *Finite versus infinite*: we represent indeed finite quantities of corresponding integer and rational numbers. This may lead the students to implicitly compare images of finite subsets of *N* and *Q* (maybe in the same graphic representation of intervals), concluding that *N* and *Q* cannot be composed by the same quantity of elements, since "rationals are more than integers" (see Galilei, 1638; finite reasoning applied to infinite sets, Tirosh & Tsamir, 1996).

METHODOLOGY AND PRELIMINARY DATA ANALYSIS

To answer our research question, we collected the following data: i. the Italian student's explanation of her doubt in written form; ii. answers to a similar questionnaire by an Italian university teacher and Master degree students in France preparing the selection procedure exam to become mathematics secondary teachers. The questionnaire was written in English and then translated into Italian and French; it is based on the historical-epistemological analysis and on the hypothesized student concept images. We created different versions of the same questionnaire, modifying them according to the two different countries' curricula for high school and university syllabi, and to the target of the questionnaire (university teachers or Master students).

At the beginning of both questionnaires, we present a realistic didactical situation, and we pose appropriate questions (see below).

We analyse one excerpt from a written interview to the Italian student, then we comment on excerpts of the answers from an Italian university teacher. Finally, we provide a brief summary of the answers to a version of the questionnaire submitted to Master students in France. To carry out the analysis of the students' interviews, consistently with our research question, we searched for their concept images and concept definitions before comparing the two source groups for connections and potential conflicts: in the case of the students, we consider personal concept images, while in the case of the teacher we look for examples that can reveal the hypothesized student concept images and concept definitions.

Analysis of the first student's comment (Italy)

The student who asked the first question explained her doubt as follows:

> "The doubt arose when [1] looking at the schemata that is used to [2] find a bijection between *N* and *Q*. [image] *N* is [3] ordered by definition, because, starting from 0, every element has a consecutive element, while [4] *Q* is not, since it is dense. But what prevents me from saying that 3/2 is the consecutive of 4/1? According to Peano's axiomatization, *N* is an [5] abstract structure that we can apply to natural numbers but also to *Q*, thanks to the bijection. I could thus say: [6] there are infinite rational numbers between 1/3 and ½, so there are infinite elements between the natural numbers associated with 1/3 and ½, using the [7] bijection in the reverse way. Reflecting more deeply, I realize that the problem is caused by the fact that [8] the bijection between *N* and *Q* is not "ordered" like that between *N* and *P* (pair numbers): 2 < 3, 2/1 > ½. I still do not understand why *Q* is dense and *N* is not, since we said that Peano's axioms can be used for several models and not just the natural numbers we already knew, but [9] if I enumerate *Q* with the natural numbers, it no longer has any sense to say < or > in *Q*."

The student was reasoning according to images (1,2) rather than definition and in mentioning the definition of order (3) she said: "every element has a consecutive element", revealing how she is not really using a formal definition but an image of ordering where each one is set after the other [CI2]; she was surprised that the consecutive could not be greater [CI1], so much so as to claim that in *Q* the meaning of < and > disappears. Also, she uses a representation of the bijection that she herself mentions as an identification (5, 7), reasoning on the schemata; indeed, she said that we can use it in both directions, identifying completely a couple of elements, one in the first and one in the second set [CI5]. She also used the image of "infinite elements in between" to say that *N* must be dense since we can image infinite elements between the natural numbers associated to a couple of rational numbers [CI3]. The student tried to connect different concept images and conflicts which emerged. The conflict that is expressed in the question, as she said, is caused by the use of images according to which [CI2] natural numbers and rational numbers are completely identified by means of a bijection, resulting in contradictions. It is impossible for the following to be simultaneously true: 1. Q is dense and N is not dense and N and Q are "the same structure"; 2. in the identification, Q loses the property that the consecutive element is greater, while N conserves this same property.

Analysis of one university teacher's questionnaire (Italy)

For this instance of the questionnaire, we provided the following realistic situation: during the lesson, a teacher is interrupted by a student asking the first question. The university teachers were explained what students are expected to have been taught before on *density*, *infinite cardinality* and the *problem of "consecutive numbers" in* Q. We asked the teachers to interpret the students' doubts and to propose how to deal with said situation in the classroom. We report and comment some excerpts from the answers to the questions (1 refers to Q1, and 2 refers to Q2):

1a. How would you answer the questions? How would you explain it to the whole class?

I. "The [1] order is not linked to the state of consecutive-ness: when she speaks of consecutive numbers we are in the domain of the [2] induction principle, which is only valid in N. In N we have much more: it is [3] well-ordered so there are no lower-unlimited subsets. Once the properties of N had been observed, i.e. the [4] "smallness" of the set that must satisfy all these features, I would move to the [5] differences between N and Q. Finally, I would observe that everything results from the fact that we are able to say that [6] one number is greater than another and this definition is also valid in N, as N is a subset of Q. I would say that [7] we cannot compare two properties linked to different definitions. Also, I would say that [8] in N we can give a definition of order that is linked to the induction principle but NOT generalized to bigger sets, but I would avoid going indepth into this issue, so as not to create confusion for the weaker students. Here, [9] the problem is that the sets are infinite".

II. "To be [10] dense is very different from being denumerable, I would [11] remind them of the definitions."

2. *Do you find a possible connection between the two questions, whereby these chance episodes could be used to help deal with some important topics in mathematics? What further examples or explanations would you propose to the student or what activities would you design in order to deal with such a question?*

"I would [12] show that {1/n} becomes more and more dense as it approaches 0. Then I would point out that the student should not be surprised when one proves that there is a [13] bijection between pair numbers and N, as well as with odds, the multiples of 3, the prime numbers, the negative integers. This is to [14] see (and prove, showing the application) examples of sets that are in a bijection with N."

3. Would you take the opportunity to explain something to students that could help them in trying to answer these kinds of questions on their own in the future?

"What we learn from these questions is that science proceeds with [15] analogies and differences, that we [16] must always consider the definitions, and that it's important that these are very precise."

On several occasions the teacher mentions definitions (6, 7, 8, 10, 11, 16), with different goals: we can't compare properties linked to different definitions; to be dense and to be denumerable are not linked because they concern different definitions (6, 7, 8, 10); we must use the definitions (11, 16).

When the teacher proposes examples, though, he uses words like show and see, and he uses images that may cause conflicts: he refers to "greater" and "smaller" related to infinite sets [CI5] and uses cardinality to compare *N* and *Q*. He mentions the bijection between *N* and its subsets and *Q* by showing the application, identifying it thus without stressing the issue of ordering [CI5]. In one case [CI2], the ordering is consistent with the linear order and in the other not, but it is not stressed. He shows that {1/n} becomes increasingly denser near 0, encouraging the use of images of density [CI3, CI5]. If we think about the student's comments, these answers would not have clarified the point she was "struggling" with: what she thought of as identification did not identify *N* and *Q* exactly. He mentions the definitions but, in the examples, he uses images and never seems to connect the definition to the images.

Preliminary analysis of the data collected in France

In France, an adapted version of the questionnaire was submitted to 30 first-year Master's students on October 12^{th}, 2017, in the first half-hour of a teaching session on didactic and epistemology of mathematics. No epistemological or didactic work on this topic had been done before with these students. The answers were then discussed later in the fall as a starting point in a session devoted to epistemological and didactical aspects of numbers construction. For both Q1 and Q2, we asked students (in French): 1. *Have you ever asked yourself this question? If so, in what context and how did you answer this question yourself? 2. Imagine that a student of a lyceum or of a preparatory class for the "grandes écoles" is asking you this question. How would you answer?* Finally, the last question was: *Do you find a possible connection between the two questions, whereby these chance episodes could be used to help deal with some important topics in mathematics?* The questionnaire was anonymous. The students were asked to indicate their previous university studies.

To the first question, 6 students answered "yes" and commented on their answer; 8 answered "no" and commented on their answers; 16 answered "no" without comments. Some Master's students claimed that the notion of density was still unclear for them. In answering the second question, some of these students proposed an incorrect explanation for hypothetical younger students, which relied on *concept images* of successor without any reference to definition:

> M1-18 “the successor of an integer exists, so in *Q* there are elements having successors. Thus, we can say that the successor of a rational does not ‘always’ exist rather than ‘does not exist’”.

Several students explained that thanks to the definition of density, it is possible to define an order, while in order to define density-in-itself, it is necessary to already have an order, as in the example below:

> M1-15 - in first year university when the set theory was introduced - I told myself that in a dense set like *Q*, the notion of successor as may be imagined on *N* [1] does not exist, but thanks to the definition of density [2] of a set, it was possible to define an ordering [3] of this set and consequently order this set [4]

The student begins with reference to *an image* [1], then refers to *the definition* [2], and concludes with the possibility *to define an ordering* [3], which is an inversion of the definitions between density and order. It is noticeable that the student makes a distinction between “ordering” and “order”, while there is no reference to the already-known order of *Q*. Among the 30 answers, only one student relies on the existence of the standard order in *Q* to justify that it is possible to define an order on *Q*. This brief summary of the students’ answers accounts for the weakness of their knowledge of the concept of density, and of its link with the concept of order.

Insights into the Italian Curricula and traditional didactical practices

In the Italian high school curricula from grade 9 to 13, order and density are never mentioned explicitly; teachers are, however, advised to introduce the concept of infinite, showing the connection between mathematics and philosophy, in grades 11 or 12 while introducing transcendental numbers. Natural, integer and rational numbers are mentioned, but attention is focused on computation techniques, representations of numbers (fractions, decimal numbers, points of a line) and approximation. In the curriculum for primary school, both “sequential” and “cardinal” sense of numbers appears. In middle and high school, students are taught that between two rational numbers you can always find a rational number, and that it follows that a rational number has no consecutive element. What is generally not made explicit by high school teachers is that this is not an absolute property of *Q*, but depends on the order chosen in *Q*. Also, numbers are usually represented in Italy on a number line, so the discrete and the dense are distinguished using more visual than theoretical considerations (the existence or not of “something in between”). Discrete is often counter-posed to continuous. At the end of high school and/or in the first year of university, the concept of accumulation point is introduced for dealing with limits and discontinuity of real functions with real variables. The existence of an infinite quantity of real numbers “between” two real numbers is said to be due to the density property of *R*. This kind of practice is likely to reinforce the concept images we identified and to reinforce the habit of reasoning in absolute terms while referring implicitly to the standard order, without stressing the dependence of the properties on the choice of order relation.

The French context

In France, the situation is not very different. Durand-Guerrier (2016, p.341) presented it briefly, and provides evidence of the weakness of fresh university students' knowledge about numbers, which could be related to the curriculum. Briefly, high school students deal with approximations, mainly with the use of calculators. In grade 12, they learn the *mean value theorem for derivatives* without a proof, and without discussion about the fact that this theorem holds in the set of real numbers but no longer applies in the sets of decimal or rational numbers. Consequently, students beginning university generally have no idea of the differences and interplay between finite decimal numbers, rational numbers and non-terminating decimal expansions, and thus are not prepared for what they will be taught at university. Indeed, in many French universities, in first-year mathematical courses, an axiomatic definition of the set of real numbers is given, most often via "the supremum property", without any explicit construction. In some cases, the representation of real numbers as non-terminating decimal expansions and the corresponding characterization of the type of numbers are introduced, and improper expansions such as $0.\underline{9}$ are discussed with students (Durand-Guerrier, 2016, p.341). A topological course is generally offered, but it is mainly theoretical, and students have very few opportunities to connect the theoretical concepts with their interpretation in the ordered field of real numbers.

CONCLUSIONS AND DEVELOPMENTS

A first relevant result is that the framework is suitable to interpret our data. Our epistemological investigation and empirical data analysis do indeed help to formulate interpretations of the conflicts which appeared in the first question, as confirmed by the Italian student's interview analysis. Also, we observed a total identification between structures due to the constructions of bijections between elements of the sets, which is implicitly present in the high school practices. University teachers mentioned merely definitions, but, as Tall & Vinner (1981) showed in the case of discontinuity of functions, concept definition may be largely inactive in the cognitive structure and concept images may be used instead of the definition in order to grasp better its meaning. In this case, in the first question (that we analysed in depth here), the conflict emerged at the level of concept images, so definitions would not have been sufficient to solve the students' doubts. For the French Master students who answered our questionnaire, this lack of awareness of the links between the concepts of density-in-itself and order of Q might prevent them from designing appropriate learning situations, once they pass the selection procedure exam and become teachers. We hypothesise that, even if such questions emerge in university courses, the reasoning and consequent conflicts can be due to the lack of explicit reference to the dependence of Q properties on the order relation which still exists in the high school, a lack which calls for epistemological and didactic clarification in teacher training. As developments, we consider it crucial to identify the curricula issues where non-recognition could generate such conflicts, and to look for suitable teaching strategies in high school and university to deal appropriately with these concepts. “When the teacher is aware of the possible concept images, it may be possible to bring incorrect images to the surface and, by discussion, rationalise the problem” (Tall & Vinner, 1981, p. 17).

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