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Abstract

We consider a generalization of (pro)algebraic loops defined on general categories of algebras and the dual notion of a coloop bialgebra suitable to represent them as functors. We prove that the natural loop of formal diffeomorphisms with associative coefficients is proalgebraic, and we give the closed formulas of the codivisions on its coloop bialgebra. This result provides a generalization of the Lagrange inversion formula to series with non-commutative coefficients, and a loop-theoretic explanation to the existence of the non-commutative Faà di Bruno Hopf algebra.

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1 Introduction

1.1 Presentation and overview of the results

An affine proalgebraic group G is a representable functor in groups defined on the category of commutative associative algebras over a field $\mathbb{F}$. The algebra representing G is the commutative Hopf algebra $\mathbb{F}[G]$ of regular functions. In this paper we consider two generalizations of proalgebraic groups, on one side to representable functors *on categories of non-commutative algebras*, on the other side to functors taking values *in non-associative groups with divisions*, that is, *loops*.

Our main motivation comes from two proalgebraic groups of formal series appearing in renormalization in quantum field theory: the group of invertible series with constant term equal to 1, represented by the Hopf algebra of symmetric functions, and that of formal diffeomorphisms tangent to the identity, represented by the Faà di Bruno Hopf algebra. Details on the role played by these series in quantum field theory are given in a separate section below.

Both types of series make sense with non-commutative coefficients, and both representative Hopf algebras admit a non-commutative version [8]. We are interested in the relationship

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between the non-commutative algebras and the sets of series. For this, we first consider generalizations of proalgebraic groups to categories of non-commutative algebras.

Functors in groups on general categories have been studied by algebraic topologists in the late 50's. D. Kan considered them on the category of groups [23], and B. Eckmann and P. Hilton [15] introduced them on general categories. Their representative Hopf-type object is called a *cogroup*. In a category with coproduct $\amalg$ and initial object, a cogroup is an object H endowed with a comultiplication, a counit and an antipode satisfying the usual properties of Hopf algebras, where the comultiplication takes values in $H \amalg H$ instead of $H \otimes H$ (which is not necessarily defined). Cogroups are then generalizations of commutative Hopf algebras which, unlike quantum groups in the case of associative algebras, preserve the functorial properties and the adjoint constructions. They have proved to be very fruitful in homotopy theory, where they appear as special H-spaces [24], as shown by I. Berstein [3]. A comprehensive study of cogroups in many varieties of algebras can be found in G. Bergman and A. Hausknecht's book [2].

Not all proalgebraic groups admit an extention to non-commutative algebras. For instance, while the group of invertible formal series naturally extends as a proalgebraic group to the category of associative algebras, the group of formal diffeomorphism does not. We show, on this example, that the extention of the functor is sometimes possible if we regard the original group as a *loop*.

Loops are multiplicative sets with unit and with a left and a right division instead of two-sided inverses. They first appeared, with some extra properties, in the work of R. Moufang [36] on alternative rings, that is, rings where the associator $(a, b, c) = (ab)c - a(bc)$ is skew-symmetric. For an excellent historical review on loops, see [42]. Associative loops are groups. Similarly to Lie groups, the tangent space of a smooth loop carries a particular algebraic structure called a Sabinin algebra [44, 33], which reduces to a Mal'cev algebra [32] for smooth Moufang loops. The notion of universal enveloping algebra has been extended to Sabinin algebras by I. Shestakov, U. U. Umirbaev [46] and J. Mostovoy, J. M. Pérez-Izquierdo [34].

In this paper we consider functors in loops on a general category $\mathbf{C}$ with coproduct and initial object and call their representative objects *coloops in* $\mathbf{C}$. We specialise $\mathbf{C}$ to be a variety of algebras over a field $\mathbb{F}$ to have a reasonable notion of generalized (pro)algebraic loop. The first simple example is the extention of the functors of invertible elements in a unital algebra and that of unitary elements in a unital involutive algebra. As expected, the largest category on which these functors are representable as loops turn out to be respectively that of alternative and of alternative involutive algebras (Prop. 3.1.4 and Prop. 3.2.4). We also show that the loop of unitary elements in the Cayley-Dickson extention of an involutive algebra is not representable on non-commutative algebras (Prop. 3.3.1), even if examples of such loops exist. Then we turn to loops of formal series with coefficients in a non-commutative algebra. First we consider the set of invertible series (with constant term equal to 1). The algebra of series with coefficients in an alternative algebra is alternative. Surprisingly, in contrast to the previous results, we find that the set of invertible series is a proalgebraic loop on all algebras, not necessarily alternative (Thm. 4.2.3). Finally, our main result concerns the natural loop of formal diffeomorphisms (tangent to the identity) with associative coefficients. We show that it is proalgebraic, and give the closed formulas of the codivisions on its representative Faà di Bruno coloop bialgebra (Def. 5.2.6 and Thm. 5.4.10). For this, we express the co-operations in terms of some recursive operators defined on any positively graded algebra (Thm. 5.3.14), which extend the natural pre-Lie product of the Witt Lie algebra (cf. [9, 16]) but not as a multibrace product (cf. [27]), and which turn out to be very rich in combinatorial properties. The coefficients appearing in the divisions show up sequences of integer numbers typical of the Lagrange inversion formula (as Catalan numbers) and some new ones, that we call *(labeled) Lagrange coefficients* (Def. 5.2.1 and 5.2.3). This result is a generalization of the Lagrange inversion formula to series with non-commutative coefficients, and gives a loop-theoretic explanation to the existence of the

non-commutative Faà di Bruno Hopf algebra [8].

1.2 Motivation: formal series in quantum field theory

The main object of study in perturbative quantum field theory are the correlation functions of the fields describing some elementary particles, from which one can compute the probability amplitude of any event involving the particles. These quantities are asymptotic series in the powers of a measurable parameter λ, such as the electric charge, called the coupling constant. For instance, for a self-interacting field ϕ with coupling λ and mass m, the k-point correlation function is a series

$$G^{(k)}(x_1,...,x_k) = \langle \phi(x_1)\cdots\phi(x_k)\rangle = \sum_{n=0}^{\infty} G_n^{(k)}(x_1,...,x_k;m,\hbar)\ \lambda^n$$

where the nth coefficient is a finite sum of amplitudes of suitable Feynman graphs with k fixed external legs, which depend on the mass m and on the Plank constant's $\hbar$, and n is related to the number of internal vertices of the graph.

The computation of the correlation functions gives rise to some divergent integrals, or equivalently to the ill-defined product of singular distributions. Giving a meaning to such terms requires a renormalization procedure, which globally amounts to suitably multiplying and composing the correlation functions with some others series, called renormalization factors, obtained by assembling the counterterms needed to cure each divergence [14], [21]. Given an ambient algebra A, typically $\mathbb{C}$ or the algebra $\mathbb{C}((\varepsilon))$ of Laurent series in a regularization parameter ε, in renomalization theory there appear two groups of formal series in the variable λ and coefficients in A:

- the set $\mathrm{Inv}(A) = \Big\{a(\lambda) = \sum_{n\geqslant 0} a_n\ \lambda^n \mid a_0 = 1, a_n \in A\Big\}$ of invertible series, endowed with the pointwise multiplication $(ab)(\lambda) = a(\lambda)\ b(\lambda)$ and the unit $1(\lambda) = 1$, which represent the Green's functions (up to an invertible factor) and the renormalization factors;
- the set $\mathrm{Diff}(A) = \Big\{a(\lambda) = \sum_{n\geqslant 0} a_n\ \lambda^{n+1} \mid a_0 = 1, a_n \in A\Big\}$ of formal diffeomorphisms, endowed with the composition law $(a\circ b)(\lambda) = a\big(b(\lambda)\big)$ and the unit $e(\lambda) = \lambda$, which represent the bare coupling constants (i.e. the coupling constants before the renormalization is performed).

Dyson's renormalization formulas [14] are modeled by the semi-direct product $\mathrm{Diff}(A) \ltimes \mathrm{Inv}(A)$, endowed with the law

$$(a_1,b_1)\cdot(a_2,b_2) = \big(a_1\circ a_2, (b_1\circ a_2)\ b_2\big),$$

where $a_1, a_2 \in \mathrm{Diff}(A)$ and $b_1, b_2 \in \mathrm{Inv}(A)$, which is well defined because formal diffeomorphisms act on invertible series from the right, by composition.

These groups are proalgebraic on commutative algebras, so they are perfectly described by their re*resentative Hopf algebra. Physically, this means that the overall renormalization procedure (except the scheme which says how to compute the counterterms) is independent of the chosen field theory, whenever the latter leads to commutative amplitudes. The recent results on the Renormalization Hopf algebras, initiated by A. Connes and D. Kreimer [11, 12], show even a stronger result: co-operations dual to the multiplication and the composition of series exist even on Hopf algebras generated by Feynman graphs, which contain the coordinate ring of the usual groups of power series. In other words, there exist proalgebraic groups of series expanded over Feynman graphs, or over various types of trees, which project onto the groups

Inv qnd Diff and which turn out to be extremely efficient in handling the combinatorial content of renormalization procedures [6, 7, 8, 47, 43].

The toy model ϕ^3 theory used by Connes-Kreimer is a scalar field theory and leads to the commutative algebra $A = \mathbb{C}$ of amplitudes. However, interesting physical situations involve non-commutative algebras. In fact, Feynman amplitudes are complex numbers for single scalar fields, the coupling constants and the renormalization factors, but they are 4×4 complex matrices for the fermionic or bosonic fields, and may be represented by higher order matrices for theories involving several interacting fields. In this case, forcing the final counterterms to be scalar, as imposed by the fact that the renormalization factors act on the (scalar) Lagrangian, prevents us from describing the renormalization in a functorial way, as shown by the results in [48], where the Hopf algebra does not represent a functorial group on $A = M_4(\mathbb{C})$. In order to preserve this functoriality, there is a need to understand Dyson's formulas for sets of series $\mathrm{Inv}(A)$ and $\mathrm{Diff}(A)$ also when A is not a commutative algebra. This is the motivation for the present work.

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Contents

2 Loops and coloops

2.1 Loops and functors in loops

A **loop** is a non-empty set Q endowed with a **multiplication** $Q \times Q \longrightarrow Q$, $(a, b) \longmapsto a \cdot b$, a (two-sided) **unit** $1 \in Q$, a **left division** $\backslash : Q \times Q \longrightarrow Q$ and a **right division** $/ : Q \times Q \longrightarrow Q$ satisfying the **cancellation properties**

$$a \cdot (a\backslash b) = b, \qquad a\backslash(a \cdot b) = b, \tag{2.1.1}$$

$$(a/b) \cdot b = a, \qquad (a \cdot b)/b = a. \tag{2.1.2}$$

Given two loops Q and Q', a **homomorphism of loops** $f : Q \longrightarrow Q'$ is of course a map which preserves the multiplication, and therefore the unit and the divisions.

The multiplication in a loop Q is not necessarily associative, if it is associative then the loop is a group. Any element a in a loop Q has a **right inverse** $1/a$ and a **left inverse** $a\backslash 1$, which do not necessarily coincide and do not necessarily determine the divisions, in the sense that they do not satisfy the identities

$$(a\backslash 1) \cdot b = a\backslash b \qquad \text{and} \qquad b \cdot (1/a) = b/a \tag{2.1.3}$$

for any $a, b \in Q$, which hold in any group.

Examples of loops which are not groups are known since a long time, see for instance [10] or [41]. For finite loops, the multiplication table is a Latin square, and the number of non isomorphic loops is known up to order 11 (cf. [37]). For instance, the subset $\{\pm 1, \pm i, \pm j, \pm k\}$ of the hyperbolic quaternions [30] (where $i^2 = j^2 = k^2 = 1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$) forms a finite loop. Among infinite loops, two well known examples are the set of invertible octonions and that of unitary octonions, which is homeomorphic to $\mathbb{S}^7$.

Denote by **Loop** the category of loops and let $F : \mathbf{Loop} \longrightarrow \mathbf{Set}_*$ be the forgetful functor to the category of pointed sets. As for functors in groups, a functor $Q : \mathbf{C} \longrightarrow \mathbf{Loop}$ on a given category $\mathbf{C}$ is said to be **representable** if the composite functor FQ is representable, cf. [31]. This means that Q is naturally isomorphic to a hom-set functor $A \mapsto \mathrm{Hom}_{\mathbf{C}}(H, A)$ for a given object H in $\mathbf{C}$, and implies that the loop operations on any loop $Q(A)$ are determined by dual co-operations on H, by convolution. Following an established terminology on cogroups, the representative object H can then be called a **coloop in C**. A reasonable notion of **(pro)algebraic loop** is obtained for $\mathbf{C}$ being a variety of algebras over a field $\mathbb{F}$, its representative coloop then being a sort of bialgebra.

In this section we describe coloops in an axiomatic way. In the next sections we give some easy examples of algebraic and non-algebraic loops on associative and non-associative algebras, and then study extensively the loop of invertible series and that of formal diffeomorphisms.

2.2 Coloops in general categories

Given a category $\mathbf{C}$, Yoneda Lemma says that the category of representable functors from $\mathbf{C}$ to **Set**, with natural transformations, is equivalent to $\mathbf{C}$. The equivalence is realized by the contravariant Yoneda functor Y from $\mathbf{C}$ to the functor category $\mathbf{Set}^{\mathbf{C}}$, defined on any object H in $\mathbf{C}$ by the functor $Y(H) = \mathrm{Hom}_{\mathbf{C}}(H, \ )$, and on any map $\phi : H' \to H$ by the natural transformation $Y(\phi) : Y(H) \to Y(H') : \alpha \mapsto \alpha\,\phi$ (cf. [31] for details). In this section we characterise the subcategory of $\mathbf{C}$ equivalent to representable functors from $\mathbf{C}$ to **Loop**.

The cartesian product of two functors $Y(H_1)$ and $Y(H_2)$ is known to be represented by the categorical coproduct $H_1 \amalg H_2$, i.e. $Y(H_1) \times Y(H_2) = Y(H_1 \amalg H_2)$, and the constant functor to the base point is known to be represented by an initial object I, i.e. it is of the form $Y(I)$. We recall the categorical notations about the coproduct, the initial object and some related categorical maps we need to define coloops.

The coproduct in a category $\mathbf{C}$ is a bifunctor $\amalg$ defined on two objects A and B as the unique object $A \amalg B$ together with two maps $i_1 : A \to A \amalg B$ and $i_2 : B \to A \amalg B$ satisfying the following universal property: for any maps $f : A \to C$ and $g : B \to C$, there exists a unique map $\langle f, g\rangle : A \amalg B \to C$ such that $\langle f, g\rangle i_1 = f$ and $\langle f, g\rangle i_2 = g$. On two maps $f : A \to A'$ and $g : B \to B'$, the bifunctor is defined as the map $f \amalg g = \langle i'_1 f, i'_2 g\rangle : A \amalg B \to A' \amalg B'$. The coproduct can be extended to several objects and maps with similar universal constructions, and turns out to be an associative bifunctor, in the sense that $(A \amalg B) \amalg C = A \amalg (B \amalg C) = A \amalg B \amalg C$ for any three objects and $(f \amalg g) \amalg h = f \amalg (g \amalg h) = f \amalg g \amalg h$ for any three maps in $\mathbf{C}$.

An initial object in $\mathbf{C}$ is an object I together with a unique map $u_A : I \to A$ on any object, which commutes with any map $f : A \to B$, that is, $f\, u_A = u_B$. Then, there are canonical isomorphisms $A \amalg I \cong A \cong I \amalg A$ given by the universal maps

$$\begin{aligned} &\varphi_1 := i_1 : A \longrightarrow A \amalg I \qquad \text{with inverse} \qquad \psi_1 = \langle \mathrm{id}_A, u_A\rangle : A \amalg I \longrightarrow A, \\ &\varphi_2 := i_2 : A \longrightarrow I \amalg A \qquad \text{with inverse} \qquad \psi_2 = \langle u_A, \mathrm{id}_A\rangle : I \amalg A \longrightarrow A. \end{aligned}$$

In particular, we have $I \amalg I \cong I$, $\langle u_A, u_A\rangle = u_A$ and therefore also $u_{A \amalg B} = u_A \amalg u_B$.

For any objects A and B, there is a canonical symmetry operator $\tau_{A,B} = \langle i_2, i_1\rangle : A \amalg B \to B \amalg A$ such that $\tau_{A,B}^{-1} = \tau_{B,A}$. Note that $B \amalg A = A \amalg B$ as objects in $\mathbf{C}$, but the maps i_1 and i_2 are inverted. The twist τ is precisely the map which identifies $A \amalg B$ and $B \amalg A$ as universal objects. To sum up, $(\mathbf{C}, \amalg, I, \tau)$ *is a strict symmetric monoidal category.*

Furthermore, for any A, there exists a canonical folding map $\mu_A = \langle \mathrm{id}_A, \mathrm{id}_A\rangle : A \amalg A \to A$ such that, for any maps $f, g : A \to B$, we have

$$\langle f, g\rangle = \mu_B\, (f \amalg g).$$

It follows that μ preserves the unit, i.e. $\mu_A(u_A \amalg u_A) = \langle u_A, u_A\rangle = u_A$, that it is associative, i.e. $\mu_A(\mu_A \amalg \mathrm{id}_A) = \mu_A(\mathrm{id}_A \amalg \mu_A)$, and that it is commutative, i.e. $\mu_A\, \tau_{A,A} = \mu_A$. It also follows that μ commutes with any map $f : A \to B$ in $\mathbf{C}$, i.e. $\mu_B\, (f \amalg f) = f\, \mu_A$. To sum up, we can say that *any object* (A, μ_A, u_A) *is a commutative monoid in* $\mathbf{C}$, with respect to the monoidal product $\amalg$, and that *any map* $f : A \to B$ *in* $\mathbf{C}$ *is a morphism of monoids*. Finally, one can prove that the folding map on $A \amalg B$ is given by $\mu_{A \amalg B} = (\mu_A \amalg \mu_B)\, (\mathrm{id}_A \amalg \tau_{B,A} \amalg \mathrm{id}_B)$.

Definition 2.2.1 Let us call **coloop in C** an object H endowed with the following maps in $\mathbf{C}$:

i) a **comultiplication** $\Delta : H \longrightarrow H \amalg H$;

ii) a **counit** $\varepsilon : H \longrightarrow I$ satisfying the **counitary** property

$$(\varepsilon \amalg \mathrm{id})\, \Delta = \varphi_2 \qquad \text{and} \qquad (\mathrm{id} \amalg \varepsilon)\, \Delta = \varphi_1, \tag{2.2.2}$$

where $\varphi_1 : H \to H \amalg I$ and $\varphi_2 : H \to I \amalg H$ are the canonical isomorphisms;

iii) a **right codivision** $\delta_r : H \longrightarrow H \amalg H$ satisfying the **right cocancellation** properties

$$(\mathrm{id} \amalg \mu)\, (\delta_r \amalg \mathrm{id})\, \Delta = i_1 \qquad \text{and} \qquad (\mathrm{id} \amalg \mu)\, (\Delta \amalg \mathrm{id})\, \delta_r = i_1, \tag{2.2.3}$$

where $i_1 : H \to H \amalg H$ can be factorized as $i_1 = (\mathrm{id} \amalg u)\, \varphi_1$;

iv) a **left codivision** $\delta_l : H \longrightarrow H \amalg H$ satisfying the **left cocancellation** properties

$$(\mu \amalg \mathrm{id})\,(\mathrm{id} \amalg \delta_l)\,\Delta = i_2 \qquad \text{and} \qquad (\mu \amalg \mathrm{id})\,(\mathrm{id} \amalg \Delta)\,\delta_l = i_2, \tag{2.2.4}$$

where $i_2 : H \to H \amalg H$ can be factorized as $i_2 = (u \amalg \mathrm{id})\,\varphi_2$.

If H and H' are two coloops in $\mathbf{C}$, we say that a map $f : H \longrightarrow H'$ is a **homomorphism of coloops** if it commutes with the coproducts, the counits and the codivisions.

Proposition 2.2.5 *Let H be a coloop in $\mathbf{C}$.*

1. *The codivisions verify the identities*

$$\mu\,\delta_r = u\,\varepsilon \qquad \textit{and} \qquad \mu\,\delta_l = u\,\varepsilon, \tag{2.2.6}$$

and the following **partial counitality properties**

$$(\mathrm{id} \amalg \varepsilon)\,\delta_r = \varphi_1 \qquad \textit{and} \qquad (\varepsilon \amalg \mathrm{id})\,\delta_l = \varphi_2. \tag{2.2.7}$$

2. *We can define a* **right antipode** $S_r : H \longrightarrow H$ *and a* **left antipode** $S_l : H \longrightarrow H$ *by setting*

$$S_r := \psi_2\,(\varepsilon \amalg \mathrm{id})\,\delta_r \qquad \textit{and} \qquad S_l := \psi_1\,(\mathrm{id} \amalg \varepsilon)\,\delta_l\ , \tag{2.2.8}$$

where $\psi_1 = \langle \mathrm{id}, u_H \rangle : H \amalg I \to H$ and $\psi_2 = \langle u_H, \mathrm{id} \rangle : I \amalg H \to H$ are isomorphisms. The antipodes satisfy the following **left** *and* **right 5-terms identities**

$$\mu\,(S_r \amalg \mathrm{id})\,\Delta = u\,\varepsilon \qquad \textit{and} \qquad \mu\,(\mathrm{id} \amalg S_l)\,\Delta = u\,\varepsilon. \tag{2.2.9}$$

These properties are easily verified. A proof using tangle diagrams is given in the Appendix.

Theorem 2.2.10 *Let $\mathbf{C}$ be a category with coproduct and initial object. Then the Yoneda functor is a contravariant equivalence of categories from the category of coloops in $\mathbf{C}$ to that of covariant representable functors $Q : \mathbf{C} \longrightarrow$ **Loop**.*

Proof. We follow the ideas of Eckmann-Hilton [15], who characterized the subcategories of $\mathbf{C}$ equivalent to the category of representable functors from $\mathbf{C}$ respectively to the category **Mag** of unital multiplicative sets, called unital magmas in [45, 5], and to the category **Grp** of groups[1].

Let us first prove that the Yoneda functor, applied to coloops in $\mathbf{C}$, gives rise to a functor in loops. On a given coloop H, let us call $Q = Y(H)$. We define the multiplication and the divisions on each set $Q(A) = \mathrm{Hom}_{\mathbf{C}}(H, A)$ as usual convolution with the coproduct and the codivisions in H, namely

$$\begin{aligned} \alpha \cdot \beta &= \langle \alpha, \beta \rangle\,\Delta = \mu_A\,(\alpha \amalg \beta)\,\Delta \\ \alpha/\beta &= \langle \alpha, \beta \rangle\,\delta_r = \mu_A\,(\alpha \amalg \beta)\,\delta_r \\ \alpha\backslash\beta &= \langle \alpha, \beta \rangle\,\delta_l = \mu_A\,(\alpha \amalg \beta)\,\delta_l, \end{aligned} \tag{2.2.11}$$

for any $\alpha, \beta \in Q(A)$. The unit in $Q(A)$ is given, as usual, by the map $1_A = u_A\,\varepsilon$, and the left and right inverses of α are then easily described as $\alpha\backslash 1 = \alpha\,S_l$ and $1/\alpha = \alpha\,S_r$. Then, using the cocancellation identities (2.2.3) and (2.2.4), and because μ_A is associative and commutes

[1] Eckmann-Hilton require $\mathbf{C}$ to have zero-maps, we replace them with an initial object.

with **C**-maps, it is easy to verify that the divisions given by (2.2.11) satisfy the cancellation properties (2.1.2) and (2.1.1).

Now fix a homomorphism of coloops $\phi : H' \longrightarrow H$, and call $Q = Y(H)$, $Q' = Y(H')$ and $\Phi = Y(\phi)$. Yoneda Lemma tells us already that Φ is given on an object A by the map

$$\begin{aligned} \Phi_A : Q(A) &\longrightarrow Q'(A) \\ \alpha &\longmapsto \Phi_A(\alpha) = \alpha\,\phi, \end{aligned}$$

and that, for any $f : A \to B$, Φ acts on the map $Q(f) : Q(A) \to Q(B)$ given by $Q(f)(\alpha) = f\,\alpha$ as a natural transformation, i.e.

$$\Phi_B\big(Q(f)(\alpha)\big) = f\,\alpha\,\phi = Q'(f)\big(\Phi_A(\alpha)\big).$$

It is then easy to verify that Φ_A is a homomorphism of loops, that is, for any $\alpha, \beta \in Q(A)$, we have

$$\Phi_A(\alpha \cdot \beta) = \Phi_A(\alpha) \cdot \Phi_A(\beta),$$

and similarly for the other co-operations.

Viceversa, let us describe how a functor in loops Q gives rise to a coloop structure on its representative object H. Suppose that the covariant functor Q is represented by an object H, i.e. $Q = Y(H)$, that the set $Q(A)$ is a loop for any A in **C**, and that for any map $f : A \to B$ the induced map $Q(f) : Q(A) \to Q(B)$ given by $\alpha \mapsto Q(f)(\alpha) = f\alpha$ is a loop homomorphism. We use repeatedly the fact that, given $\alpha, \beta \in Q(A)$, for the composite maps $f(\alpha \cdot \beta), f\alpha, f\beta \in Q(B)$ we have

$$f\,(\alpha \cdot \beta) = Q(f)(\alpha \cdot \beta) = Q(f)(\alpha) \cdot Q(f)(\beta) = (f\alpha) \cdot (f\beta) \tag{2.2.12}$$

and similarly for the operations / and \. Seeing $i_1, i_2 : H \to H \amalg H$ as elements of the loop $Q(H \amalg H)$, we define the comultiplication and the codivisions on H by

$$\Delta = i_1 \cdot i_2, \qquad \delta_r = i_1/i_2, \qquad \delta_l = i_1 \backslash i_2$$

and the counit ε as the unit 1_I in $Q(I)$. It follows that the antipodes are the inverses of the identity map, $S_r = 1_H/\mathrm{id}_H$ and $S_l = \mathrm{id}_H \backslash 1_H$.

Let us show that these maps give a coloop structure to H, and that the functor $Q \mapsto H$ is inverse to the Yoneda one, $H \mapsto Q = Y(H)$. For any $\alpha, \beta \in Q(A)$, we apply (2.2.12) to $A = H \amalg H$, $B = A$, $f = \langle \alpha, \beta \rangle : H \amalg H \to A$ and to the elements $\alpha = i_1$, $\beta = i_2$ of $Q(H \amalg H)$, and get

$$\begin{aligned} \langle \alpha, \beta \rangle\, \Delta &= \langle \alpha, \beta \rangle\, (i_1 \cdot i_2) \\ &= (\langle \alpha, \beta \rangle\, i_1) \cdot (\langle \alpha, \beta \rangle\, i_2) = \alpha \cdot \beta \end{aligned} \tag{2.2.13}$$

and similarly for the operations / and \. Now apply Q to a unit map $u_A : I \to A$. Since $Q(u_A) : Q(I) \to Q(A)$ is a homomorphism of loops, it preserves the units, and therefore, for $\varepsilon = 1_I \in Q(I)$, we have

$$Q(u_A)(\varepsilon) = u_A\,\varepsilon = 1_A.$$

In particular we have $u_H\,\varepsilon = 1_H$, and therefore, using (2.2.13), we have

$$\langle u_H \varepsilon, \mathrm{id}_H \rangle\, \Delta = 1_H \cdot \mathrm{id}_H = \mathrm{id}_H.$$

On the other side, we have

$$\begin{aligned} \langle u_H \varepsilon, \mathrm{id}_H \rangle\, \Delta &= \langle u_H, \mathrm{id}_H \rangle\, (\varepsilon \amalg \mathrm{id})\, \Delta \\ &= \psi_2\, (\varepsilon \amalg \mathrm{id})\, \Delta, \end{aligned}$$

and we obtain the equality $\psi_2\,(\varepsilon \amalg \mathrm{id})\,\Delta = \mathrm{id}$. Since ψ_2 is the inverse map to φ_2, we obtain $(\varepsilon \amalg \mathrm{id})\,\Delta = \varphi_2$, which proves (2.2.2). Let us show equalities (2.2.3). Firstly, we have trivially that

$$(\mathrm{id} \amalg u_H)\, i_1 = \langle i_1, i_2\, u_H\rangle\, i_1 = i_1.$$

Secondly, note that $\delta_r \cdot i_2 = (i_1/i_2)\cdot i_2 = i_1$, therefore

$$\begin{aligned}(\mathrm{id} \amalg \mu)\,(\delta_r \amalg \mathrm{id})\,\Delta &= \langle i_1, i_2\mu\rangle\langle i_1\delta_r, i_2\rangle\,(i_1 \cdot i_2)\\ &= \langle i_1, i_2\mu\rangle\,(\delta_r \cdot i_2)\\ &= \langle i_1, i_2\mu\rangle\, i_1 = i_1.\end{aligned}$$

Finally, since $\langle\alpha,\beta\rangle(i_1/i_2) = \alpha/\beta$ and $(i_1\cdot i_2)/i_2 = i_1$, we also have

$$\begin{aligned}(\mathrm{id} \amalg \mu)\,(\Delta \amalg \mathrm{id})\,\delta_r &= \langle i_1, i_2\mu\rangle\langle i_1\Delta, i_2\rangle\,(i_1/i_2)\\ &= \langle i_1, i_2\mu\rangle\,\big((i_1\cdot i_2)/i_2\big)\\ &= \langle i_1, i_2\mu\rangle\, i_1 = i_1.\end{aligned}$$

The same arguments apply to the left codivision. □

The relationship between coloops and cogroups is straightforward. As usual, a coloop H is **coassociative** if

$$(\Delta \amalg \mathrm{id})\,\Delta = (\mathrm{id} \amalg \Delta)\,\Delta, \tag{2.2.14}$$

and H is **cocommutative** if $\tau\,\Delta = \Delta$.

We say that H has the **left** and **right coinverse property** if the codivisions are determined by the antipodes, that is,

$$\delta_l = (S_l \amalg \mathrm{id})\,\Delta. \qquad \text{and} \qquad \delta_r = (\mathrm{id} \amalg S_r)\,\Delta \tag{2.2.15}$$

These identities correspond to the analogues (2.1.3) in the loop $Q = Y(H)$.

Furthermore, an **antipode** on H is a map $S : H \to H$ satisfying the **5-terms identity**

$$\mu\,(S \amalg \mathrm{id})\,\Delta = \mu\,(\mathrm{id} \amalg S)\,\Delta = u\,\varepsilon. \tag{2.2.16}$$

This can happen if and only if $S_r = S_l =: S$. Note that the unicity of the antipode satisfying (2.2.16) does not imply that the coinverse properties (2.2.15) are verified. A counterexample is given by the coloop of formal diffeomorphisms, cf. Section 5.

A cogroup in a category $\mathbf{C}$ is an object H endowed with a coassociative comultiplication Δ, a counit ε satisfying the counitary property (2.2.2) and an antipode S satisfying the 5-terms identity (2.2.16), cf. [3]. It follows from the dual statement on loops and groups (cf. [10]), that

Proposition 2.2.17 *If H is a coassociative coloop, then it is a cogroup.*

In the Appendix we prove with tangles that coassociativity implies that the left and the right antipodes coincide, and therefore H has an antipode satisfying the coinverse property.

2.3 (Pro)algebraic loops

Let $\mathbf{A}$ be a variety of unital algebras over a field $\mathbb{F}$, that is, the subcategory of vector spaces over $\mathbb{F}$ which collects all algebras of a certain *type*, given by a set of operations of various arities, including the unit of arity 0, defined by a set of identities (cf. [31] ch. V). For instance, $\mathbf{A}$ can be the category of $\mathcal{P}$-algebras, where $\mathcal{P}$ is an algebraic operad with $\mathcal{P}(0) = \{\,\text{unit map}\,\}$ (cf. [28]).

Then, $\mathbf{A}$ has a coproduct and an initial object (cf. [31] ch. IX), therefore we can apply to $\mathbf{A}$ the results of the previous section. More precisely, the initial object is given by the trivial unital algebra $\mathbb{F}$. Suppose that in $\mathbf{A}$ there are operations p of arity $n > 0$, and let $\bar{A}$ denote the subalgebra of an algebra A determined by such operations. Then, given two algebras A and B in $\mathbf{A}$, the coproduct $A \amalg B$ is the quotient of the free algebra $\mathbf{A}(\bar{A} \oplus \bar{B})$ (which always exists, cf. [31] ch. V) by the ideal generated by the identities

$$\begin{aligned} &p_{\mathbf{A}(A\oplus B)}(a_1, ..., a_n) = p_A(a_1, ..., a_n) \in \bar{A}, \qquad \text{for any } a_1, ..., a_n \in \bar{A} \\ &p_{\mathbf{A}(A\oplus B)}(b_1, ..., b_n) = p_B(b_1, ..., b_n) \in \bar{B}, \qquad \text{for any } b_1, ..., b_n \in \bar{B} \\ &1_A = 1_B = 1_{\mathbf{A}(A\oplus B)}, \end{aligned} \tag{2.3.1}$$

for all the operations p admitted in $\mathbf{A}$. The universal properties of $\amalg$ follow from the universal properties of the free algebra $\mathbf{A}(\bar{A} \oplus \bar{B})$.

Examples 2.3.2

1. In the category $\mathbf{Com}_{\mathbb{F}}$ of unital commutative and associative algebras over $\mathbb{F}$, the free algebra $\mathbf{Com}_{\mathbb{F}}(V)$ on a vector space V is the symmetric algebra $S(V)$, and the coproduct of two algebras A and B is the tensor product $A \otimes B$.

2. In the category $\mathbf{As}_{\mathbb{F}}$ of unital associative algebras over $\mathbb{F}$, the free algebra $\mathbf{As}_{\mathbb{F}}(V)$ is the tensor algebra $T(V)$, and the coproduct[2] $A \amalg B$ of two algebras A and B is the tensor algebra $T(\bar{A} \oplus \bar{B})$ modulo relations (2.3.1), which mean that $a \otimes a' = a\, a$ whenever a and a' are both in $\bar{A}$ or both in $\bar{B}$. As a vector space, we then have

$$A \amalg B = \mathbb{F} \oplus \bigoplus_{n \geqslant 1} \Big[\underbrace{\bar{A} \otimes \bar{B} \otimes \bar{A} \otimes \cdots}_{n} \oplus \underbrace{\bar{B} \otimes \bar{A} \otimes \bar{B} \otimes \cdots}_{n} \Big],$$

   and the multiplication in $A \amalg B$ is given by the concatenation modulo the above relations. For instance, if we denote the multiplication in $A \amalg B$ by $\bullet$, we have

$$\begin{aligned} (b \otimes a) \bullet (b' \otimes a' \otimes b'') &= b \otimes a \otimes b' \otimes a' \otimes b'' \\ (b \otimes a) \bullet (a' \otimes b' \otimes a'') &= b \otimes (a\, a') \otimes b' \otimes a''. \end{aligned}$$

3. Let $\mathbf{Alg}_{\mathbb{F}}$ be the category of unital algebras (not necessarily associative, also called magmatic) over $\mathbb{F}$. The free unital algebra on a vector space V is the tensor algebra with parenthesizing $T\{V\}$, and the coproduct $A \amalg B$ of two algebras A and B is the quotient of $T\{\bar{A} \oplus \bar{B}\}$ modulo relations (2.3.1), which again mean that $a \otimes a' = a\, a$ whenever a and a' are both in $\bar{A}$ or both in $\bar{B}$. The multiplication in $A \amalg B$ is the concatenation with parenthesis.

4. An **alternative algebra** is an algebra A such that the associator $(a, b, c) = (ab)c - a(bc)$ is skew-symmetric, that is

$$(b, a, c) = -(a, b, c) \qquad \text{and} \qquad (a, c, b) = -(a, b, c) \tag{2.3.3}$$

   for any $a, b, c \in A$. This is equivalent to requiring that

$$(ab)b = a(bb) \qquad \text{and} \qquad (aa)b = a(ab)$$

   for any $a, b \in A$. Unital alternative algebras over a field $\mathbb{F}$ form a subcategory of $\mathbf{Alg}_{\mathbb{F}}$, denoted by $\mathbf{Alt}_{\mathbb{F}}$, which is a variety with initial object $\mathbb{F}$. The coproduct $A \amalg B$ of two unital alternative algebras is the quotient of the coproduct in $\mathbf{Alg}_{\mathbb{F}}$ by the relations (2.3.3). For details see [50] or [25].

[2] In associative algebras, the coproduct is usually called *free product* and denoted by $\star$.

5. In a category $\mathbf{A}$, an (anti) **involution** is a unary linear operation $^* : A \longrightarrow A$ such that

$$(a^*)^* = a \qquad \text{and} \qquad (a_1\, a_2)^* = a_2^*\, a_1^*, \tag{2.3.4}$$

for any $a, a_1, a_2 \in A$. Each of the four previous categories of algebras can be considered with involution, and denote by $\mathbf{A}^*$. For such algebras, the initial object and the coproduct are the same as in $\mathbf{A}$, the involution on $A \amalg B$ is automatically defined from the involutions on A and B by properties (2.3.4). Note that, in $\mathbf{Alg}_{\mathbb{F}}$ and in $\mathbf{Alt}_{\mathbb{F}}$, the parenthesizing of a word $a_1 \otimes \cdots \otimes a_n$ is inverted from left to right by the involution, together with the single letters of the word.

Definition 2.3.5 A coloop H in a variety of unital algebras $\mathbf{A}$ is called a **coloop A-bialgebra**. Its associated functor in loops $Q = Y(H)$ is then called an **algebraic loop on A** if H is a finitely generated algebra, and a **proalgebraic loop on A** if H is not finitely generated. In this case, it is an inductive limit of finitely generated coloop $\mathbf{A}$-bialgebras.

There are not many known examples of algebraic loops on non-commutative algebras, but some of them are quite special. In section 3, we give two easy examples of algebraic groups on commutative algebras which can be extended as groups to associative algebras (the groups of invertible elements and that of unitary ones), and another one which can not be extended to a functor on associative algebras even as a loop (the Cayley-Dickson loop). Viceversa, in section 5 we give the example of a proalgebraic group which can be extended to associative algebras only as a proalgebraic loop (the loop of formal diffeomorphisms).

Finally, the two groups of invertible and unitary elements can be extended to alternative algebras if we regard them as algebraic loops, and in section 4 we also give an example of an algebraic group which can be extended to all associative algebras as a group, and to non-associative algebras as a loop (the loop of invertible series).

While the functoriality of the first examples is straightforward, for the two loops of formal series it is not. The group of formal diffeomorphisms is a local approximation of the most simple group of smooth diffeomorphisms on a manifold, and the existence of a proalgebraic version on associative algebras is a new step in the study of non-commutative geometry. In particular, its existence as a proalgebraic loop allows us to consider a physical "renormalization loop" to replace the standard group [21], which could be applied in any perturbative theory when the function rings have to be replaced by tensor algebras, as in [20].

Remark 2.3.6 All known examples of algebraic groups and loops on non-commutative algebras have free underlying algebra structure. The fact that this should hold in any category (under certain completeness hypothesis) has not been proved, but it was proved for cogroups in several categories: by D. Kan [23] in the category of groups, by I. Berstein [3] (and later reproved by J. Zhang in [51]) in the category of graded connected associative algebras, and by B. Fresse [17] in the category of complete algebras over any operad. For coloops, this result is proved by G. Bergman and A.O. Hausknecht [2] in the category of graded connected associative rings.

Before giving the examples, we mention two maps which allow us to compare coloop and cogroup bialgebras to usual Hopf algebras. A coloop $\mathbf{A}$-bialgebra has the operations $p : H^{\otimes n} \longrightarrow H$ from $\mathbf{A}$, and the categorical folding map $\mu : H \amalg H \to H$ needed to describe the coloop axioms, which can be iterated on n copies of H. In general, there is no relationship between these two types of operations, since $H^{\otimes n}$ need not be an algebra in $\mathbf{A}$.

Assume that $\mathbf{A}$ is a category of algebras such that, for any $\mathbf{A}$-algebras A and B, the tensor product $A \otimes B$ is again an $\mathbf{A}$-algebra with componentwise operations

$$p^{(n)}_{A\otimes B}(a_1 \otimes b_1, \cdots, a_n \otimes b_n) = p^{(n)}_A(a_1, \cdots, a_n) \otimes p^{(n)}_B(b_1, \cdots, b_n)$$

and unit $1_{A\otimes B} = 1_A \otimes 1_B$.

Definition 2.3.7 For any $n \geqslant 2$ and for any n algebras A_k, with $k = 1, ..., n$, we call **canonical projection** of $A_1 \amalg \cdots \amalg A_n$ onto $A_1 \otimes \cdots \otimes A_n$ the algebra homomorphism

$$\pi := \langle j_1, ..., j_n \rangle : A_1 \amalg \cdots \amalg A_n \longrightarrow A_1 \otimes \cdots \otimes A_n$$

induced by the injective algebra maps $j_k : A_k \to A_1 \otimes \cdots \otimes A_n$ given by

$$j_k(a_k) = 1_{A_1} \otimes \cdots \otimes a_k \otimes \cdots \otimes 1_{A_n}.$$

The map π reorders the elements of $A_1 \amalg \cdots \amalg A_n$ and then multiplies them within each A_k to get elements in $A_1 \otimes \cdots \otimes A_n$. For instance, if we denote by $a^{(k)}$ an element $a \in A_k$ seen in the coproduct $A_1 \amalg \cdots \amalg A_n$, we have

$$\pi\big(a^{(1)}b^{(2)}c^{(1)}d^{(2)}\big) = (ac) \otimes (bd).$$

This map is surjective, because a preimage of any $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A_1 \otimes \cdots \otimes A_n$ by π is given by $a^{(1)}_1 a^{(2)}_2 \cdots a^{(n)}_n \in A_1 \amalg \cdots \amalg A_n$.

Note that, when all A_k coincide and we are given an operation p of arity n, the map $p\,\pi : A^{\amalg n} \longrightarrow A$ is not, in general, an algebra homomorphism (because p is not), and therefore it surely differs from the folding map $\mu = \langle \mathrm{id}_A, ..., \mathrm{id}_A \rangle$. In fact, μ multiplies the elements of A in the order they appear in $A^{\amalg n}$ (it is a *concatenation*), while $p\,\pi$ first reorders the factors in $A^{\amalg n}$ with π, as explained above, then multiplies them (it is a *componentwise operation*).

Definition 2.3.8 On the other side, for any $n \geqslant 2$ and for any n algebras A_k, with $k = 1, ..., n$, there are categorical maps $i_k : A_k \to A_1 \amalg \cdots \amalg A_n$. For any operation p of arity n in $\mathbf{A}$, we call **canonical inclusion** of $A_1 \otimes \cdots \otimes A_n$ in $A_1 \amalg \cdots \amalg A_n$ the linear map $\iota_p : A_1 \otimes \cdots \otimes A_n \longrightarrow A_1 \amalg \cdots \amalg A_n$ defined by

$$\iota_p(a_1 \otimes \cdots \otimes a_n) := p_{\amalg}\big(i_1(a_1), ..., i_n(a_n)\big),$$

where $p_{\amalg} : (A_1 \amalg \cdots \amalg A_n)^{\otimes n} \to A_1 \amalg \cdots \amalg A_n$ denotes the operation p on the coproduct algebra $A_1 \amalg \cdots \amalg A_n$. It follows from the definition of $\amalg$ that this map is injective.

Note that ι_p is not, in general, an algebra homomorphism, because the operation p in $\mathbf{A}$ is not. However, when all A_k coincide (say, with A), the map ι_p allows us to recover the operation $p_A : A^{\otimes n} \to A$ from the folding map μ, in the sense that $\mu\,\iota_p = p_A$, because

$$\mu\; p_{\amalg}\big(i_1(a_1), ..., i_n(a_n)\big) = p_A(a_1, ..., a_n)$$

for any $a_1, ..., a_n \in A$.

Proposition 2.3.9 *When the map ι_p is well defined, we have $\pi\,\iota_p = \mathrm{id}_{A_1\otimes\cdots\otimes A_n}$.*

Proof. Denote by $p_\otimes$ the operation p on the tensor algebra $A_1 \otimes \cdots \otimes A_n$. Since π is an algebra homomorphism, for any $a_k \in A_k$, with $k = 1, ..., n$, we have

$$\begin{aligned}\pi\, \iota_p(a_1 \otimes \cdots \otimes a_n) &= \pi\, p_{\amalg}\big(i_1(a_1), ..., i_n(a_n)\big) = p_\otimes\Big(\pi\big(i_1(a_1)\big), ..., \pi\big(i_n(a_n)\big)\Big) \\ &= p_\otimes\Big(\langle j_1, ..., j_n\rangle\big(i_1(a)\big), ..., \langle j_1, ..., j_n\rangle\big(i_n(a_n)\big)\Big) \\ &= p_\otimes\Big(j_1(a_1), ..., j_n(a_n)\Big) \\ &= a_1 \otimes \cdots \otimes a_n.\end{aligned}$$

□

Remark 2.3.10 These maps allow us in particular to compare the coloop bialgebra representing some loop to other types of bialgebras related to it which appear in the literature. In particular, the universal enveloping algebra of the Sabinin algebra associated to the loop has been studied in [39, 34, 35]. Because of the axioms, it is clear that the graded dual of this universal enveloping algebra does not coincide with the bialgebra $H^\otimes$ induced by a coloop bialgebra H, nor in $\mathbf{Alg}_\mathbb{F}$ nor in $\mathbf{As}_\mathbb{F}$.

Finally, let us use these maps to compare associative coloop bialgebras and Hopf algebras. Let H be a coloop bialgebra in $\mathbf{As}_\mathbb{F}$. Denote by $H^\otimes$ the algebra H endowed with the usual co-operations

$$\Delta^\otimes = \pi\, \Delta,\ \delta_r^\otimes = \pi\, \delta_r,\ \delta_l^\otimes = \pi\, \delta_r : H^\otimes \longrightarrow H^\otimes \otimes H^\otimes,$$

the counit ε and the antipodes S_r, S_l, which are all still algebra homomorphisms on $H^\otimes$.

Proposition 2.3.11 *If Δ is coassociative, then $\Delta^\otimes$ is coassociative. Moreover, we have*

$$S_r = (\varepsilon \otimes \mathrm{id})\, \delta_r^\otimes \qquad \textit{and} \qquad S_l = (\mathrm{id} \otimes \varepsilon)\, \delta_l^\otimes.$$

Proof. If Δ is coassociative, the two terms

$$(\Delta^\otimes \otimes \mathrm{id})\, \Delta^\otimes = (\pi \otimes \mathrm{id})\, \pi_{(H \amalg H) \amalg H}\, (\Delta \amalg \mathrm{id})\, \Delta$$

and

$$(\mathrm{id} \otimes \Delta^\otimes)\, \Delta^\otimes = (\mathrm{id} \otimes \pi)\, \pi_{H \amalg (H \amalg H)}\, (\mathrm{id} \amalg \Delta)\, \Delta$$

coincide, because Δ is coassociative and because the two maps $(\pi \otimes \mathrm{id}) \otimes \pi_{(H \amalg H) \amalg H}$ and $(\mathrm{id} \otimes \pi) \otimes \pi_{H \amalg (H \amalg H)}$ coincide with the standard projection $\pi : H \amalg H \amalg H \longrightarrow H \otimes H \otimes H$. Therefore $\Delta^\otimes$ is coassociative.

For any $a \in H$, the term $\delta_r(a) \in H^{(1)} \amalg H^{(2)}$ is a finite sum of products of elements of $H^{(1)}$ and of $H^{(2)}$ in alternative order. The right antipode $S(a)_r = (\varepsilon \amalg \mathrm{id})\, \delta_r(a)$ turns all the factors belonging to $H^{(1)}$ into scalars, which can then be positioned on the lefthand side of all the remaining elements belonging to $H^{(2)}$. Therefore the result is the same that we obtain if we first reorder the factors in $H^{(1)}$ all at the leftmost position by applying $\delta_r(a)^\otimes$. Same with S_l by putting all the scalars on the rightmost position. □

Note however that S_r and S_l do not necessarily satisfy the left and right 5-terms identities for $\Delta^\otimes$ on $H^\otimes$, because

$$m\, (S_r \otimes \mathrm{id})\, \Delta^\otimes = m\, \pi\, (S_r \amalg \mathrm{id})\, i\, \Delta^\otimes = \mu\, \iota\, \pi\, (S_r \amalg \mathrm{id})\, \iota\, \pi\, \Delta$$

and $\iota\, \pi$ is not the identity map on $H \amalg H$. Therefore, even if H is a cogroup bialgebra, $H^\otimes$ is not necessarily a Hopf algebra.

3 Coloops of invertible and unitary elements

3.1 Loop of invertible elements

In this section we give an example of an abelian algebraic group which can be extended to associative algebras as a group, to alternative algebras as a loop, but not to non-associative algebras, even as a loop.

Let $\mathbb{F}$ be a field. For any unital commutative algebra A over $\mathbb{F}$, the set

$$I(A) = \{a \in A \mid a \text{ admits an inverse } a^{-1} \}$$

is the abelian group of invertible elements in A. The functor I is represented on $\mathbf{Com}_{\mathbb{F}}$ by the commutative (and cocommutative) Hopf algebra of Laurent polynomials $H_{\mathrm{I}} = \mathbb{F}[x, x^{-1}]$, with co-operations

$$\Delta(x) = x \otimes x, \qquad \varepsilon(x) = 1, \qquad S(x) = x^{-1}.$$

In fact, elements $a \in I(A)$ are in bijection with algebra homomorphisms $\alpha : H_{\mathrm{I}} \to A$ such that $\alpha(x) = a$ and $\alpha(x^{-1}) = a^{-1}$. Then, if α and β give respectively the elements a and b, their convolution product coincides with the product in A, because we have

$$\begin{aligned}(\alpha \cdot \beta)(x) &= \mu_A(\alpha \otimes \beta)\Delta(x) = \alpha(x)\beta(x) = ab,\\ \alpha^{-1}(x) &= \alpha S(x) = \alpha(x^{-1}) = a^{-1}.\end{aligned}$$

We show that the functor I admits an extention to associative algebras as a group, and that it admits an extention to non-associative algebras, as a loop, only on alternative algebras.

Definition 3.1.1 We call **invertible coloop bialgebra** on $\mathbb{F}$ the associative algebra $H_{\mathrm{I}}^{\amalg} = \mathbb{F}[x, x^{-1}]$ endowed with the following co-operations with values in the coproduct $H_{\mathrm{I}}^{\amalg} \amalg H_{\mathrm{I}}^{\amalg}$ of the category $\mathbf{As}_{\mathbb{F}}$:

$$\begin{aligned}&\Delta(x) = x^{(1)}\, x^{(2)} && \Delta(x^{-1}) = (x^{-1})^{(2)}\, (x^{-1})^{(1)},\\ &\epsilon(x) = 1 && \epsilon(x^{-1}) = 1,\\ &\delta_r(x) = x^{(1)}\, (x^{-1})^{(2)} && \delta_r(x^{-1}) = x^{(2)}\, (x^{-1})^{(1)},\\ &\delta_l(x) = (x^{-1})^{(1)}\, x^{(2)} && \delta_l(x^{-1}) = (x^{-1})^{(2)}\, x^{(1)},\end{aligned}$$

where $x^{(k)} = i_k(x)$ is the generator x seen in the kth copy of $H_{\mathrm{I}}^{\amalg}$ of the coproduct algebra $H_{\mathrm{I}}^{\amalg} \amalg H_{\mathrm{I}}^{\amalg}$, for $k = 1, 2$, and similarly for $(x^{-1})^{(k)} = i_k((x^{-1})^{(k)})$.

It follows that there is a two-sided antipode given by $S(x) = x^{-1}$ and $S(x^{-1}) = x$.

Proposition 3.1.2 *The algebra $H_{\mathrm{I}}^{\amalg}$ is a cogroup bialgebra in $\mathbf{As}_{\mathbb{F}}$ and represents, for any associative algebra A, the group*

$$I(A) = \mathrm{Hom}_{\mathbf{As}_{\mathbb{F}}}(H_{\mathrm{I}}^{\amalg}, A)$$

of invertible elements of A. Moreover, the group $I(A)$ is abelian if A is commutative.

Proof. The axioms of a coloop bialgebra for the codivisions are easily verified. For instance, the computations

$$\begin{aligned}(\mathrm{id}\amalg\mu)(\delta_r\amalg\mathrm{id})\Delta(x) &= (\mathrm{id}\amalg\mu)(\delta_r\amalg\mathrm{id})(x^{(1)}x^{(2)})\\ &= (\mathrm{id}\amalg\mu)(x^{(1)}(x^{-1})^{(2)}x^{(3)})\\ &= x^{(1)}(x^{-1})^{(2)}x^{(2)}\\ &= x^{(1)} = i_1(x)\\ (\mathrm{id}\amalg\mu)(\Delta\amalg\mathrm{id})\delta_r(x) &= (\mathrm{id}\amalg\mu)(\Delta\amalg\mathrm{id})(x^{(1)}(x^{-1})^{(2)})\\ &= (\mathrm{id}\amalg\mu)(x^{(1)}x^{(2)}(x^{-1})^{(3)})\\ &= x^{(1)}x^{(2)}(x^{-1})^{(2)}\\ &= x^{(1)} = i_1(x)\end{aligned}$$

and the analogue computations for x^{-1} prove the cocancellation (2.2.3) for δ_r. The first claim is then ensured by the fact that Δ is coassociative. In fact,

$$(\Delta\amalg\mathrm{id})\Delta(x) = x^{(1)}\,x^{(2)}\,x^{(3)} = (\mathrm{id}\amalg\Delta)\Delta(x),$$

and similarly for x^{-1}. Thus, $I(A)$ is a group by Theorem 2.2.10 and Proposition 2.2.17.

The fact that the group $I(A)$ is abelian if A is commutative is less evident because Δ is not cocommutative. In fact, we have

$$\tau\,\Delta(x) = \tau(x^{(1)}\,x^{(2)}) = x^{(2)}\,x^{(1)} \neq x^{(1)}\,x^{(2)} = \Delta(x).$$

It is however true because the generators x and x^{-1} are group-like, and therefore the commutativity of the convolution product only depends on that of the multiplication in A. □

Example 3.1.3 The group $I(A)$ is the simplest algebraic group at all: it describes invertible elements in an associative algebra A whatever is the nature of A, that is, without making use of any internal structure of A. The simplest non-trivial example is the group $I(M_n(\mathbb{F})) = GL_n(\mathbb{F})$, which is recovered as the set of $M_n(\mathbb{F})$-valued algebra homomorphisms on H_{I} without using the non-homogeneous relation $\det(A) \neq 0$ which defines invertible matrices (or, more precisely, the relation $\det(A) = t$ where t determines a new scalar invertible generator of the coordinate ring).

Proposition 3.1.4 *The algebraic group I can be extended as a loop to a variety of algebras $\mathbf{A} \subset \mathbf{Alg}_{\mathbb{F}}$ if $\mathbf{A}$ is a subcategory of alternative algebras $\mathbf{Alt}_{\mathbb{F}}$ admitting coproduct and initial object. In particular, it is an algebraic loop on $\mathbf{Alt}_{\mathbb{F}}$.*

Proof. If I could be extended as an algebraic loop to **Alg**, its representative coloop bialgebra should be the algebra $H_{\mathrm{I}}^{\amalg} = \mathbb{F}[x, x^{-1}]$ with co-operations defined on generators as in Def. 3.1.1 but taking values in the coproduct $H_{\mathrm{I}}^{\amalg}\amalg H_{\mathrm{I}}^{\amalg}$ of the category **Alg**. This algebra is not a coloop bialgebra in **Alg**, because the codivisions do not satisfy the cocancelation properties (2.2.4) and (2.2.3). In fact, the element

$$(\mathrm{id}\amalg\mu)\,(\delta_r\amalg\mathrm{id})\,\Delta(x) = \left(x^{(1)}\,(x^{-1})^{(2)}\right)x^{(2)}$$

can not coincide with $i_1(x) = x^{(1)}$ in $H_{\mathrm{I}}^{\amalg}\amalg H_{\mathrm{I}}^{\amalg}$. However, the conditions under which the cocancellation properties hold, all similar to the one above, are guaranteed in the category of alternative algebras, where $(ab)b^{-1} = a = b^{-1}(ba)$ for any a and any invertible b (cf. [50]). □

Example 3.1.5 The octonions $\mathbb{O}$ form an alternative algebra, therefore one can apply I to $\mathbb{O}$. The set $I(\mathbb{O})$ of invertible octonions is a well known Moufang loop (cf. [10]), that is, it is a loop satisfying the Moufang identities

$$\begin{aligned} a(b(ca)) &= ((ab)a)c & (ab)(ca) &= (a(bc))a \\ a(b(cb)) &= ((ab)c)b & (ab)(ca) &= a((bc)a \end{aligned}$$

for any elements a, b, c.

3.2 Loop of unitary elements

Consider now involutive algebras A, and the subgroup of $I(A)$ made of unitary elements in A, namely

$$U(A) = \{a \in A \mid a\, a^* = 1\},$$

when A is commutative. Exactly as for I, the functor U is represented on $\mathbf{Com}^*_{\mathbb{F}}$ by the commutative Hopf algebra $H_{\mathrm{U}} = \mathbb{F}[x, x^* \mid x\, x^* = 1]$, with co-operations

$$\Delta(x) = x \otimes x, \qquad \varepsilon(x) = 1, \qquad S(x) = x^*.$$

Definition 3.2.1 Let us call **unitary coloop bialgebra** on $\mathbb{F}$ the associative algebra $H_{\mathrm{U}}^{\mathrm{II}} = \mathbb{F}[x, x^* \mid x\, x^* = 1]$ endowed with the co-operations defined on generators exactly as those in Def. 3.1.1, where the generator x^{-1} is replaced by x^*.

As for the invertible coloop bialgebra, one can prove that

Proposition 3.2.2 *The algebra $H_{\mathrm{U}}^{\mathrm{II}}$ is a cogroup bialgebra in $\mathbf{As}^*_{\mathbb{F}}$ and represents, for any involutive associative algebra A, the group*

$$U(A) = \mathrm{Hom}_{\mathbf{As}^*_{\mathbb{F}}}(H_{\mathrm{U}}^{\mathrm{II}}, A)$$

of unitary elements of A. Moreover, the group $U(A)$ is abelian if A is commutative.

Examples 3.2.3 For $\mathbb{F} = \mathbb{R}$, this functor allows us to describe several groups of unitary matrices.

1. Applied to the algebra $M_n(\mathbb{R})$, if we take the transposition of matrices as involution, it gives $U(\mathbb{R}) = \{1, -1\}$, and the orthogonal group $U\big(M_n(\mathbb{R})\big) = O(n)$ for $n > 1$.

2. On $M_n(\mathbb{C})$, we take as involution the complex conjugate of the transposition. Then $U(\mathbb{C}) = U(1) \cong \mathbb{S}^1$ and $U\big(M_n(\mathbb{C})\big) = U(n)$ is the unitary group.

3. Let $\mathbb{H}$ be the algebra of quaternions, spanned over $\mathbb{R}$ by 1 and by three imaginary units i, j, k which anticommute with each other. The conjugate of a quaternion $q = a + b\,i + c\,j + d\,k$ is the quaternion $q^* = a - b\,i - c\,j - d\,k$. The conjugation is an involution, and the real number $\|q\| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2}$ defines a multiplicative norm on $\mathbb{H}$. Then, the functor U applied to $\mathbb{H}$ gives the subgroup $U(\mathbb{H}) \cong Sp(1) \cong SU(2) \cong \mathbb{S}^3$ of $I(\mathbb{H})$ consisting of unit norm quaternions.

   On the set of matrices $M_n(\mathbb{H})$, we take as involution the quaternionic conjugate of the transposition. Then $U\big(M_n(\mathbb{H})\big) \cong U(n, \mathbb{H}) \cong Sp(n)$ is the compact symplectic group, also called the hyperunitary group.

Again exactly as for the invertible coloop bialgebra, one can prove the next result.

Proposition 3.2.4 *The algebraic group U can be extended as a loop to a variety of algebras $\mathbf{A} \subset \mathbf{Alg}^*_{\mathbb{F}}$ if $\mathbf{A}$ is a subcategory of involutive alternative algebras $\mathbf{Alt}^*_{\mathbb{F}}$ admitting coproduct and initial object. In particular, U is an algebraic loop on $\mathbf{Alt}^*_{\mathbb{F}}$.*

Example 3.2.5 An alternative algebra of octonions $\mathbb{O} = \mathbb{O}(\alpha, \beta, \gamma)$ is spanned over a field $\mathbb{F}$ of characteristic not 2 by $e_0 = 1$ and by seven imaginary units e_i, for $i = 1, ..., 7$, with an involved table of multiplication (cf. [10], [50, Ch.2], [25]). Over the field $\mathbb{R}$ there are two non-isomorphic octonion algebras: the classical division Cayley octonions $\mathbb{O}$ and the split matrix Cayley-Dickson algebra Zorn($\mathbb{R}$) [10], also known as the Zorn vector-matrix algebra [38]. The last one may be defined over an arbitrary commutative ring. The conjugate of an octonion $q = \sum_{i=0}^{7} a_i e_i$ is the octonion $q^* = a_0 e_0 - \sum_{i=1}^{7} a_i e_i$. Again, the conjugation is an involution, and the scalar $n(q) = qq^* = q^*q$ defines a multiplicative norm on $\mathbb{O}$ (and an isotropic quadratic form on Zorn($\mathbb{R}$)).

Then, for the classical Cayley octonions $\mathbb{O}$, the set $U(\mathbb{O})$ is the Moufang subloop of the loop $I(\mathbb{O})$ consisting of unit norm octonions, which is homeomorphic to the sphere S^7, while for the matrix Cayley-Dickson algebra Zorn($\mathbb{R}$) the loop U(Zorn($\mathbb{R}$)) is not compact. The loops $U(\mathbb{O})$ and U(Zorn($\mathbb{R}$)) can be compared to the groups $U(\mathbb{H}) \cong \mathbb{S}^3$ and $U(\mathbb{P}) \cong SL_2(\mathbb{R})$ obtained respectively for division quaternions $\mathbb{H}$ and for split quaternions $\mathbb{P}$.

3.3 Unitary Cayley-Dickson loops

In this section we give an example of a loop which is not algebraic on associative algebras.

Let $\mathbb{F}$ be a field and j denote an imaginary unit. For any involutive commutative algebra A over $\mathbb{F}$, the set

$$U_{CD}(A) = \{a + b\,j \in A + A\,j \mid a\,a^* + b\,b^* = 1\}$$

gives the group of unitary elements in the Cayley-Dickson algebra $A + A\,j$ with multiplication

$$(a + b\,j)\,(c + dj) = (ac - d^*b) + (da + bc^*)\,j,$$

unit 1, and involution $(a + b\,j)^* = a^* - b\,j$.

The functor $A \longmapsto U_{CD}(A)$ is representable on $\mathbf{Com}^*_{\mathbb{F}}$, by the commutative Hopf algebra

$$H_{\mathrm{UCD}} = \mathbb{F}[x, x^*, y, y^* \mid x\,x^* + y\,y^* = 1]$$

with co-operations

$$\begin{aligned} &\Delta(x) = x \otimes x - y \otimes y^* && \Delta(y) = x \otimes y + y \otimes x^*, \\ &\varepsilon(x) = 1 && \varepsilon(x) = 0, \\ &S(x) = x^* && S(y) = -y. \end{aligned}$$

Proposition 3.3.1 *The algebraic group U_{CD} can not be extended as an algebraic loop to the category of involutive associative algebras.*

Proof. If U_{CD} could be extended to an algebraic loop to $\mathbf{As}^*_{\mathbb{F}}$, its representative coloop bialgebra $H^{\mathrm{II}}_{\mathrm{UCD}}$ should be an associative algebra generated by x, x^*, y and y^* submitted to conditions which give $x\,x^* + y\,y^* = 1$ if the variables commute. The co-operations should then be defined on

generators exactly as in the commutative case, but taking values in the coproduct $H^{\amalg}_{\mathrm{UCD}} \amalg H^{\amalg}_{\mathrm{UCD}}$ of the category $\mathbf{As}^*_{\mathbb{F}}$.

The conditions $x\,x^* = x^*x$ and $x\,x^* + y^*y = 1$ are enough to guarantee that the algebra $H^{\amalg}_{\mathrm{UCD}}$ has a well defined comultiplication, a counit and an antipode satisfying the 5-terms relations. However, the codivisions, defined according to the coinverse properties (2.2.15) as

$$\delta_r(x) = x^{(1)}\,(x^*)^{(2)} + (y^*)^{(2)}\,y^{(1)} \qquad \delta_r(y) = -y^{(2)}\,x^{(1)} + y^{(1)}\,x^{(2)}$$
$$\delta_l(x) = (x^*)^{(1)}\,x^{(2)} + (y^*)^{(2)}\,y^{(1)} \qquad \delta_l(y) = y^{(2)}\,(x^*)^{(1)} - y^{(1)}\,(x^*)^{(2)},$$

satisfy the cocancellation identities (2.2.3) and (2.2.4) if and only if

$$x^{(1)}\,(x^*x)^{(2)} = (x^*x)^{(2)}\,x^{(1)} \qquad \text{and} \qquad y^{(1)}\,(y^*y)^{(2)} = (y^*y)^{(2)}\,y^{(1)}$$

in $H^{\amalg}_{\mathrm{UCD}} \amalg H^{\amalg}_{\mathrm{UCD}}$. This could happen for two reasons. The first is that the map $n : H^{\amalg}_{\mathrm{UCD}} \longrightarrow H^{\amalg}_{\mathrm{UCD}}$ given by $n(a) = aa^* = a^*a$ has scalar values, i.e. its image is in $u(\mathbb{F}) \subset H^{\amalg}_{\mathrm{UCD}}$. This is the case if $H^{\amalg}_{\mathrm{UCD}}$ is a *composition algebra*, cf. [1]. But composition algebras do not have a categorical coproduct. The second possibility to verify these conditions is that the identity $a^{(1)}b^{(2)} = b^{(2)}a^{(1)}$ holds in $H^{\amalg}_{\mathrm{UCD}} \amalg H^{\amalg}_{\mathrm{UCD}}$ for any elements $a, b \in H^{\amalg}_{\mathrm{UCD}}$. This means that $\amalg = \otimes$ and therefore it is only possible in the category $\mathbf{Com}^*_{\mathbb{F}}$. □

Examples 3.3.2 In agreement with this result, namely that the construction U_{CD} is not functorial on associative algebras, there are few examples of loops arising as sets of unitary elements in the Cayley-Dickson algebra constructed on an associative algebra. For instance, we can consider the associative algebras of matrices $A = M_n(K)$ with entries in involutive algebras K over the field $\mathbb{F} = \mathbb{R}$, with involution given by the transposition of the matrices plus the involution of their matrix elements. The unitary elements in $A + Aj$ are preserved by divisions if A is a composition algebra, and matrix algebras, in general, are not. So, in general, $U_{CD}(A)$ is not a loop. There are few exceptions:

1. The set $U_{CD}(M_n(\mathbb{R}))$ is a loop for $n = 1, 2$. For $n = 1$ (when $A = \mathbb{R}$ is commutative) it is an abelian group $U_{CD}(\mathbb{R}) = U(\mathbb{C}) \cong \mathbb{S}^1$, and for $n = 2$ the loop $U_{CD}(M_2(\mathbb{R}))$ coincides with the loop $U(\mathrm{Zorn}(\mathbb{R}))$ from Example 3.2.5, since $M_2(\mathbb{R}) + M_2(\mathbb{R})j \cong \mathrm{Zorn}(\mathbb{R})$ is a matrix Cayley-Dickson algebra.

2. The set $U_{CD}(M_n(\mathbb{C}))$ is a loop for $n = 1, 2$. For $n = 1$ (when $A = \mathbb{C}$ is commutative) it is a group $U_{CD}(\mathbb{C}) = U(\mathbb{H}) \cong \mathbb{S}^3$, and for $n = 2$ the loop $U_{CD}(M_2(\mathbb{C}))$ coincides with the loop $U(\mathrm{Zorn}(\mathbb{C}))$ of unital elements in the split matrix Cayley-Dickson algebra over the complex numbers $\mathbb{C}$.

3. The set $U_{CD}(M_n(\mathbb{H}))$ is a Moufang loop only for $n = 1$, and we have $U_{CD}(\mathbb{H}) \cong U(\mathbb{O}) \cong \mathbb{S}^7$. For $n > 1$, the set $U_{CD}(M_n(\mathbb{H}))$ is not a loop because $M_n(\mathbb{H})$ is not a composition algebra.

4 Coloop of invertible series

The group of invertible series (with constant term equal to 1), is the set of formal series

$$\mathrm{Inv}(A) = \left\{ \; a(\lambda) = \sum_{n \geqslant 0} a_n\,\lambda^n \mid a_0 = 1,\; a_n \in A \; \right\}$$

with coefficients a_n taken in a commutative algebra A, endowed with the pointwise multiplication $(ab)(\lambda) = a(\lambda)\ b(\lambda)$, unit $1(\lambda) = 1$, and where the inverse of a series $a(\lambda)$ is found by recursion. It is an abelian proalgebraic group on **Com**, represented by the cocommutative Hopf algebra

$$H_{\mathrm{inv}} = \mathbb{F}\big[x_n,\ n \geqslant 1\big] \qquad (x_0 = 1)$$
$$\Delta_{\mathrm{inv}}(x_n) = \sum_{m=0}^{n} x_m \otimes x_{n-m}$$

known as Hopf algebra of symmetric functions [18].

The functor Inv admits an evident extention to associative algebras as a functor in groups (but not abelian), represented by the cogroup bialgebra [8]

$$H_{\mathrm{inv}} = \mathbb{F}\langle x_n,\ n \geqslant 1\rangle \qquad (x_0 = 1)$$
$$\Delta_{\mathrm{inv}}(x_n) = \sum_{m=0}^{n} x_m^{(1)}\ x_{n-m}^{(2)}$$

with antipode defined recursively. The projection of this bialgebra by the canonical map π given in Def. 2.3.7 coincides with the Hopf algebra of non-commutative symmetric functions (cf. [19]).

In this section we show that the functor Inv can be extended to non-associative algebras, as a proalgebraic loop.

4.1 Loop of invertible series

Definition 4.1.1 Let A be a unital algebra and let λ be a formal variable. We call invertible series in λ with coefficients in A the formal series in the set

$$\mathrm{Inv}(A) = \Big\{\ a = \sum_{n\geqslant 0} a_n\,\lambda^n \mid a_0 = 1,\ a_n \in A\ \Big\},$$

endowed with the multiplication

$$a \cdot b = \sum_{n\geqslant 0}\ \sum_{m=0}^{n}\ a_m\ b_{n-m}\,\lambda^n$$

and the unit e given by $e_0 = 1$ and $e_n = 0$ for all $n > 1$. For instance,

$$(a \cdot b)_1 = a_1 + b_1,$$
$$(a \cdot b)_2 = a_2 + a_1 b_1 + b_2,$$
$$(a \cdot b)_3 = a_3 + a_2 b_1 + a_1 b_2 + b_3.$$

Proposition 4.1.2 *For any unital algebra A, the set of invertible series* $\mathrm{Inv}(A)$ *is a loop.*

Proof. It is clear that the series e is a unit for the given multiplication, so we only have to show that there exist a left and a right divisions satisfying the cancellation properties (2.1.2) and (2.1.1). Since the multiplication is completely symmetric in the the two variables, the proof for the two divisions is exactly the same. We do it for the right division.

Given two series $a = \sum a_n\,\lambda^n$ and $b = \sum b_n\,\lambda^n$, we define the right division $a/b = \sum (a/b)_n\,\lambda^n$ so that $(a/b)\dot{b} = a$, that is

$$\sum_{m=0}^{n} (a/b)_m\ b_{n-m} = a_n \qquad \text{for any } n \geqslant 0.$$

These equations are solved recusively from $(a/b)_0 = 1$, and give the nth term

$$(a/b)_n = a_n - \sum_{m=0}^{n-1} (a/b)_m \; b_{n-m}.$$

Let us then prove by induction that $(a \cdot b)/b = a$, that is, $\big((a \cdot b)/b\big)_n = a_n$ for any $n \geqslant 0$. We have $\big((a \cdot b)/b\big)_0 = a_0 = 1$ and, for any $n \geqslant 1$,

$$\begin{aligned}\big((a \cdot b)/b\big)_n &= (a \cdot b)_n - \sum_{m=0}^{n-1} (a \cdot b)_m \, b_{n-m} \\ &= a_n + \sum_{m=0}^{n-1} \big(a_m - (a \cdot b)_m\big) \, b_{n-m},\end{aligned}$$

so if we suppose that $\big((a \cdot b)/b\big)_m = a_m$ for any $m \geqslant n-1$, we have $\big((a \cdot b)/b\big)_n = a_n$. □

For instance, for the right division we find

$$\begin{aligned}(a/b)_1 &= a_1 - b_1, \\ (a/b)_2 &= a_2 - a_1 b_1 - b_2 + b_1 b_1, \\ (a/b)_3 &= a_3 - (a_1 b_2 + a_2 b_1) + (a_1 b_1) b_1 - b_3 + (b_1 b_2 + b_2 b_1) - (b_1 b_1) b_1,\end{aligned}$$

and for the left division we find

$$\begin{aligned}(a\backslash b)_1 &= b_1 - a_1, \\ (a\backslash b)_2 &= b_2 - a_1 b_1 - a_2 + a_1 a_1, \\ (a\backslash b)_3 &= b_3 - (a_1 b_2 + a_2 b_1) + a_1 (a_1 b_1) - b_3 + (a_1 a_2 + a_2 a_1) - a_1 (a_1 a_1).\end{aligned}$$

4.2 Coloop bialgebra of invertible series

For any $n \geqslant 1$, let x_n be a graded variable of degree n. For $X = \mathrm{Span}_{\mathbb{F}}\{x_n, \; n \geqslant 1\}$, the tensor algebra $H = T(X)$ can be seen as the set of non-commutative polynomials in the variables $x_1, x_2, ...$, that we denote by $\mathbb{F}\langle x_n, \; n \geqslant 1\rangle$. It is then useful to denote the unit 1 of H by x_0.

The unital associative coproduct algebra $H \amalg H$ is then the tensor algebra $T(X^{(1)} \oplus X^{(2)})$ on two identical sets of variables, and similarly $H \amalg H \amalg H = T(X^{(1)} \oplus X^{(2)} \oplus X^{(3)})$. To simplify the notations, in this section we denote by $x_n = x_n^{(1)}$, $y_n = x_n^{(2)}$ and $z_n = x_n^{(3)}$ the generators taken in the different copies of X in a coproduct algebra.

For any integer $n \geqslant 1$ and any $1 \leqslant \ell \leqslant n$, let $\mathcal{C}_n^\ell$ denote the set of compositions of n of length ℓ, that is, the set of ordered sequences $\mathbf{n} = (n_1, ..., n_\ell)$ such that

$$n_1 + \cdots + n_\ell = n, \qquad \text{and} \qquad n_1, ..., n_\ell \geqslant 1. \tag{4.2.1}$$

For instance, for $\ell = 1, 2, 3$, we have

$$\begin{aligned}&\mathcal{C}_1^1 = \big\{(1)\big\}, \qquad \mathcal{C}_2^1 = \big\{(2)\big\}, \quad \mathcal{C}_2^2 = \big\{(1,1)\big\}, \\ &\mathcal{C}_3^1 = \big\{(3)\big\}, \quad \mathcal{C}_3^2 = \big\{(2,1),(1,2)\big\}, \quad \mathcal{C}_3^3 = \big\{(1,1,1)\big\}.\end{aligned}$$

Definition 4.2.2 Let us call **coloop bialgebra of invertible series** the free unital algebra

$$H_{\mathrm{inv}}^{\amalg} = T\{x_n \mid n \geqslant 1\}$$

with the following graded co-operations:

- comultiplication $\Delta^{\amalg}_{\mathrm{inv}} : H^{\amalg}_{\mathrm{inv}} \longrightarrow H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}}$ given by

$$\Delta^{\amalg}_{\mathrm{inv}}(x_n) = \sum_{m=0}^{n} x_m \, y_{n-m};$$

- counit $\varepsilon : H^{\amalg}_{\mathrm{inv}} \longrightarrow \mathbb{F}$ given by $\varepsilon(x_n) = \delta_{n,0}$;

- right codivision $\delta_r : H^{\amalg}_{\mathrm{inv}} \longrightarrow H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}}$ given by

$$\delta_r(x_n) = x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \Big(\big(((x_{n_1} - y_{n_1})\, y_{n_2})\, y_{n_3} \big) \cdots \Big) y_{n_{\ell+1}},$$

  where $\mathcal{C}_n^{\ell+1}$ is the set of compositions of n of length $\ell+1$, cf. (4.2.1);

- left codivision $\delta_l : H^{\amalg}_{\mathrm{inv}} \longrightarrow H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}}$ given by

$$\delta_l(x_n) = y_n - x_n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} x_{n_1} \Big(\cdots \big(x_{n_2} \, (x_{n_\ell}(y_{n_{\ell+1}} - x_{n_{\ell+1}})) \big) \Big).$$

Theorem 4.2.3 *The algebra $H^{\amalg}_{\mathrm{inv}}$ is a coloop bialgebra and represents the loop of invertible series as a functor* $\mathrm{Inv} : \mathbf{Alg} \longrightarrow \mathbf{Loop}$.

As a consequence, given an algebra A, a series $a = \sum_{n \geqslant 0} a_n \, \lambda^n \in \mathrm{Inv}(A)$ can be seen as an algebra homomorphism $a : H^{\amalg}_{\mathrm{inv}} \longrightarrow A$ defined on the generators of $H^{\amalg}_{\mathrm{inv}}$ by $a(x_n) = a_n$, and the right and left division a/b and $a\backslash b$ are given at any order n by the following closed formulas:

$$\begin{aligned}
(a/b)_n &= \mu_A \, (a \amalg b) \, \delta_r(x_n) \\
&= a_n - b_n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \Big(\big(((a_{n_1} - b_{n_1}) b_{n_2}) b_{n_3} \big) \cdots \Big) b_{n_{\ell+1}}, \\
(a\backslash b)_n &= \mu_A \, (a \amalg b) \, \delta_l(x_n) \\
&= b_n - a_n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} a_{n_1} \Big(\cdots \big(a_{n_2} \, (a_{n_\ell}(b_{n_{\ell+1}} - a_{n_{\ell+1}})) \big) \Big).
\end{aligned}$$

Proof. The algebra $H^{\amalg}_{\mathrm{inv}}$ clearly represents the functor Inv with values in sets, and the comultiplication $\Delta^{\amalg}_{\mathrm{inv}}$ represents the pointwise multiplication of series. The only thing which should be proved is that $H^{\amalg}_{\mathrm{inv}}$ is a coloop bialgebra with the given codivisions. The formulas for the left and for the right codivisions are perfectly symmetric, in the sense that $\delta_l = \tau \, \delta_r$, so it suffices to give the details for one codivision. Let us then show that the right codivision satisfies the two equations (2.2.3).

Concerning the first one, we have

$$(\delta_r \amalg \mathrm{id}) \Delta^{\amalg}_{\mathrm{inv}}(x_n) = \delta_r(x_n) + z_n + \sum_{m=1}^{n-1} \delta_r(x_m) \, z_{n-m},$$

which is an element of $H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}}$, and since $\mathrm{id} \amalg \mu : H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}} \longrightarrow H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}}$ multiplies the variables y and z (in the order they appear) and puts the result in the right-hand

side copy of $H^{\amalg}_{\mathrm{inv}}$ in $H^{\amalg}_{\mathrm{inv}} \amalg H^{\amalg}_{\mathrm{inv}}$, we have

$$\begin{aligned}(\mathrm{id} \amalg \mu)(\delta_r \amalg \mathrm{id})\Delta^{\amalg}_{\mathrm{inv}}(x_n) &= \delta_r(x_n) + y_n + \sum_{m=1}^{n-1} \delta_r(x_m)\, y_{n-m} \\ &= x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \Big(\big((u_{n_1}\, y_{n_2})\, y_{n_3} \big) \cdots \Big) y_{n_{\ell+1}} + y_n \\ &\quad + \sum_{m=1}^{n-1} u_m\, y_{n-m} + \sum_{m=1}^{n-1} \sum_{\lambda=1}^{m-1} (-1)^{\lambda} \sum_{\mathbf{m} \in \mathcal{C}_m^{\lambda+1}} \Big(\big((u_{m_1}\, y_{m_2}) \cdots \big) y_{m_{\lambda+1}} \Big)\, y_{n-m}\end{aligned}$$

where we set $u_n := x_n - y_n$ and therefore we have

$$\sum_{m=1}^{n-1} u_m\, y_{n-m} = \sum_{\mathbf{n} \in \mathcal{C}_n^2} u_{n_1}\, y_{n_2}.$$

Setting $\ell = \lambda + 1$ in the last sum, we have $2 \leqslant \ell \leqslant n-1$ and $\ell \leqslant m \leqslant n-1$ with

$$\bigcup_{m=\lambda}^{n-1} \mathcal{C}_m^{\ell} \times \mathcal{C}_{n-m}^{1} = \mathcal{C}_n^{\ell+1},$$

therefore

$$\sum_{m=1}^{n-1} \sum_{\lambda=1}^{m-1} (-1)^{\lambda} \sum_{\mathbf{m} \in \mathcal{C}_m^{\lambda+1}} \Big(\big((u_{m_1}\, y_{m_2}) \cdots \big) y_{m_{\lambda+1}} \Big)\, y_{n-m} = \sum_{\ell=2}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \Big(\big((u_{n_1}\, y_{n_2})\, y_{n_3} \big) \cdots \Big) y_{n_{\ell+1}}.$$

Thus, we finally obtain

$$(\delta_r \amalg \mathrm{id})\Delta^{\amalg}_{\mathrm{inv}}(x_n) = x_n.$$

For the second identity, we rewrite the comultiplication as

$$\Delta^{\amalg}_{\mathrm{inv}}(x_n) = x_n + y_n + \sum_{\mathbf{n} \in \mathcal{C}_n^2} x_{n_1}\, y_{n_2}$$

and using the fact that

$$\mathcal{C}_n^{\ell+1} = \bigcup_{m=1}^{n-1} \mathcal{C}_m^{1} \times \mathcal{C}_{n-m}^{\ell},$$

and setting $\lambda = \ell$, we rewrite the right codivision as

$$\delta_r(x_n) = u_n + \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m} (-1)^{\lambda} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^{\lambda}} \Big(\big((u_m\, y_{k_1})\, y_{k_2} \big) \cdots \Big) y_{k_\lambda}.$$

We then have

$$
\begin{aligned}
(\Delta^{\amalg}_{\mathrm{inv}} \amalg \mathrm{id})\delta_r(x_n) &= \Delta^{\amalg}_{\mathrm{inv}}(x_n) - z_n + \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\Big(\big((\Delta^{\amalg}_{\mathrm{inv}}(x_m)-z_m)\,z_{k_1}\big)\cdots\Big)z_{k_\lambda}\\
&= x_n + y_n + \sum_{\mathbf{n}\in\mathcal{C}^2_n} x_{n_1}\,y_{n_2} + \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\big((x_m\,z_{k_1})\cdots\big)z_{k_\lambda} - z_n\\
&\quad + \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\big((y_m\,z_{k_1})\cdots\big)z_{k_\lambda}\\
&\quad + \sum_{m=1}^{n-1}\sum_{\mathbf{m}\in\mathcal{C}^2_m}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\Big(\big((x_{m_1}y_{m_2})\,z_{k_1}\big)\cdots\Big)z_{k_\lambda}\\
&\quad - \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\big((z_m\,z_{k_1})\cdots\big)z_{k_\lambda}.
\end{aligned}
$$

When we then apply $\mathrm{id}\amalg\mu$, we identify $z_m = y_m$ and $z_{k_i} = y_{k_i}$ for $i = 1, ..., \lambda$, and therefore we have

$$
\begin{aligned}
(\mathrm{id}\amalg\mu)(\Delta^{\amalg}_{\mathrm{inv}}\amalg\mathrm{id})\delta_r(x_n) &= x_n + \sum_{\mathbf{n}\in\mathcal{C}^2_n} x_{n_1}\,y_{n_2} + \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\big((x_m\,y_{k_1})\cdots\big)y_{k_\lambda}\\
&\quad + \sum_{m=1}^{n-1}\sum_{\mathbf{m}\in\mathcal{C}^2_m}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\Big(\big((x_{m_1}y_{m_2})\,y_{k_1}\big)\cdots\Big)y_{k_\lambda}
\end{aligned}
$$

where

$$
\sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\big((x_m\,y_{k_1})\cdots\big)y_{k_\lambda} = \sum_{\ell=1}^{n-1}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}^{\ell+1}_n}\big((x_{n_1}\,y_{n_2})\cdots\big)y_{n_{\ell+1}},
$$

and

$$
\begin{aligned}
&\sum_{\mathbf{n}\in\mathcal{C}^2_n} x_{n_1}\,y_{n_2} + \sum_{m=1}^{n-1}\sum_{\mathbf{m}\in\mathcal{C}^2_m}\sum_{\lambda=1}^{n-m}(-1)^{\lambda}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}\Big(\big((x_{m_1}y_{m_2})\,y_{k_1}\big)\cdots\Big)y_{k_\lambda}\\
&\qquad = -\sum_{\mathbf{n}\in\mathcal{C}^{1+1}_n}(-1)^1 x_{n_1}\,y_{n_{1+1}} - \sum_{\ell=2}^{n-1}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}^{\ell+1}_n}\big((x_{n_1}\,y_{n_2})\cdots\big)y_{n_{\ell+1}}.
\end{aligned}
$$

This we finally have

$$
(\mathrm{id}\amalg\mu)(\Delta^{\amalg}_{\mathrm{inv}}\amalg\mathrm{id})\delta_r(x_n) = x_n.
$$

□

4.3 Properties of the loop of invertible series

Loops satisfying weak versions of associativity have many applications, for instance in Blaschke's Web Geometry through nets [4]. It is therefore interesting to ask what kind of identities are satisfied by the loops $\mathrm{Inv}(A)$ of invertible series.

Proposition 4.3.1 *Given an algebra A, the loop* $\mathrm{Inv}(A)$ *satisfies an identity*

$$(*) \qquad u(a,b,...,c) = v(a,b,...,c)$$

for any series $a,b,...,c \in \mathrm{Inv}(A)$ if and only if the identity $(*)$ is satisfied in A, that is, for any elements $a,b,...,c \in A$.

Proof. Roughly speaking, this result follows from the fact that the comultiplication $\Delta^{\amalg}_{\mathrm{inv}}$ is linear on both sides on generators. More precisely, if in the identity $(*)$ the operators u and v are multilinear, the implication "$(*)$ on $A \;\Rightarrow\; (*)$ on $\mathrm{Inv}(A)$" is proved by direct inspection, and the opposite implication is proved by considering series of the form $a = 1 + a_1\,\lambda$, $b = 1 + b_1\,\lambda$,..., $c = 1 + c_1\,\lambda$.

If in the identity $(*)$ the operators u and v are not multilinear, for instance the element a appears k times, it suffices to linearize them, by considering the sum $a = a^1 + \cdots + a^k$ of k different elements. □

In particular, this result implies that $\mathrm{Inv}(A)$ is a Moufang loop if and only if A is alternative, that $\mathrm{Inv}(A)$ is a group if and only if A is associative, and that $\mathrm{Inv}(A)$ is an abelian group if and only if A is commutative and associative.

We give below counterexamples to some interesting properties of loops which fail on the loops $\mathrm{Inv}(A)$ for associative algebras A, which can be deduced by the coloop bialgebra $H^{\amalg}_{\mathrm{inv}}$.

Example 4.3.2 The left and the right inverses of any $a \in \mathrm{Inv}(A)$ do not coincide, that is

$$a\backslash e \neq e/a.$$

In fact, the left and right inverses in $\mathrm{Inv}(A)$ coincide if and only if the left antipode S_l and the right antipode S_r of $H^{\amalg}_{\mathrm{inv}}$ coincide. Applying equations (2.2.8), we find

$$\begin{aligned} S_r(x_n) &:= (\varepsilon \amalg \mathrm{id})\,\delta_r(x_n) \\ &= -x_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \Big(\big((x_{n_1}\, x_{n_2})\, x_{n_3} \big) \cdots \Big) x_{n_{\ell+1}} \end{aligned}$$

and

$$\begin{aligned} S_l(x_n) &:= (\mathrm{id} \amalg \varepsilon)\,\delta_l(x_n) \\ &= -x_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} x_{n_1} \Big(\cdots \big(x_{n_2}\, (x_{n_\ell} x_{n_{\ell+1}}) \big) \Big), \end{aligned}$$

therefore the two antipodes do not coincide. For instance, for a series $a = 1 + a_1\,\lambda$, we have

$$\begin{aligned} e/a &:= a\,S_r = 1 - a_1\,\lambda + a_1 a_1\,\lambda^2 - (a_1 a_1) a_1\,\lambda^3 + \big((a_1 a_1) a_1\big) a_1\,\lambda^4 + \cdots \\ a\backslash e &:= a\,S_l = 1 - a_1\,\lambda + a_1 a_1\,\lambda^2 - a_1 (a_1 a_1)\,\lambda^3 + a_1 \big(a_1 (a_1 a_1)\big)\,\lambda^4 + \cdots \end{aligned}$$

To have a counterexample to the equality $e/a = a\backslash e$, take for A the algebra of 2×2 matrices over the sedenions, spanned by 1 and by the imaginary units e_i for $i = 1, ..., 15$. If $a = 1 + a_1\,\lambda$ is the series with coefficient

$$a_1 = \begin{pmatrix} e_1 + e_{10} & e_5 + e_{14} \\ 0 & 1 \end{pmatrix},$$

we have

$$e/a - a\backslash e = \begin{pmatrix} 0 & -2(e_5 + e_{14}) \\ 0 & 0 \end{pmatrix} \lambda^3 + O(\lambda^4).$$

Example 4.3.3 The left and right inversions in $\mathrm{Inv}(A)$ do not allow us to construct the divisions, that is,

$$a/b \neq a\,(e/b) \qquad \text{and} \qquad a\backslash b \neq (a\backslash e)\,b$$

for any $a, b \in \mathrm{Inv}(A)$.

In fact, to show that $a/b \neq a\,(e/b)$ and $a\backslash b \neq (a\backslash e)\,b$ in the loop $\mathrm{Inv}(A)$ is equivalent to show that

$$\delta_r \neq (\mathrm{id} \amalg S_r)\,\Delta^{\amalg}_{\mathrm{inv}} \qquad \text{and} \qquad \delta_l \neq (S_l \amalg \mathrm{id})\,\Delta^{\amalg}_{\mathrm{inv}}$$

in the coloop bialgebra $H^{\amalg}_{\mathrm{inv}}$. Let us show it for the right codivision. For any generator x_n, we have

$$\begin{aligned}(\mathrm{id} \amalg S_r)\,\Delta^{\amalg}_{\mathrm{inv}}(x_n) &= x_n + S_r(y_n) + \sum_{m=1}^{n-1} x_m\, S_r(y_{n-m}) \\ &= x_n - y_n - \sum_{\ell=1}^{n-1}(-1)^{\ell} \sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}} \big((y_{n_1}\, y_{n_2})\cdots\big) y_{n_{\ell+1}} \\ &\quad - \sum_{m=1}^{n-1} x_m\, y_{n-m} - \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m-1}(-1)^{\lambda} \sum_{\mathbf{k}\in\mathcal{C}_{n-m}^{\lambda+1}} x_m \Big(\big((y_{k_1}\, y_{k_2})\cdots\big) y_{k_{\lambda+1}}\Big).\end{aligned}$$

Writing the last two sums in terms of compositions of n yields

$$(\mathrm{id} \amalg S_r)\,\Delta^{\amalg}_{\mathrm{inv}}(x_n) = u_n + \sum_{\ell=1}^{n-1}(-1)^{\ell} \sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}} \big(x_{n_1}\,\big(\big((y_{n_2}\, y_{n_3})\cdots\big) y_{n_{\ell+1}}\big) - \big((y_{n_1}\, y_{n_2})\cdots\big) y_{n_{\ell+1}}\big)\,,$$

which is clearly different from the expression of $\delta_r(x_n)$.

Example 4.3.4 A loop Q is **left alternative** if $a(ab) = (aa)b$ for any $a, b \in Q$, and it is **right alternative** if $(ab)b = a(bb)$. The proalgebraic loop Inv on the category **Alg** is not left nor right alternative.

For this, it suffices to show that the coloop bialgebra $H^{\amalg}_{\mathrm{inv}}$ is not right coalternative, that is $(\mathrm{id} \amalg \mu)\,K \neq 0$, where $K = (\Delta^{\amalg}_{\mathrm{inv}} \amalg \mathrm{id})\,\Delta^{\amalg}_{\mathrm{inv}}(\mathrm{id} \amalg \Delta^{\amalg}_{\mathrm{inv}})\,\Delta^{\amalg}_{\mathrm{inv}}$ is the coassociator. The first deviation from right alternativity appears on the generator x_3, since we have

$$\begin{aligned}K(x_3) &= (x_1y_1)z_1 - x_1(y_1z_1) \\ (\mathrm{id} \amalg \mu)\,K(x_3) &= (x_1y_1)y_1 - x_1(y_1y_1) \neq 0.\end{aligned}$$

For instance, if A is the algebra of sedenions, the deviation from right alternativity can be seen comparing $(ab)b$ and $a(bb)$ for the two series

$$a = 1 + (e_1 + e_{10})\lambda \qquad \text{and} \qquad b = 1 + (e_5 + e_{14})\lambda$$

because $(e_1 + e_{10})(e_5 + e_{14}) = 0$ and therefore

$$(ab)b - a(bb) = -(e_1 + e_{10})(e_5 + e_{14})^2\,\lambda^3 = 2(e_1 + e_{10})\,\lambda^3.$$

Example 4.3.5 A loop Q is **power associative** if every element of the loop generates an abelian subgroup. The proalgebraic loop Inv on the category **Alg** is not power associative.

In particular, power associativity requires the associativity $(aa)a = a(aa)$ for any element. Therefore, it suffices to show that

$$\mu\,(\mathrm{id}\amalg\mu)\,K(x_3) = (x_1x_1)x_1 - x_1(x_1x_1) \neq 0.$$

For instance, if we take A to be the algebra of 2×2 matrices with coefficients in the sedenion algebra, for the series $a = 1 + a_1\lambda$ of Example 4.3.3 with

$$a_1 = \begin{pmatrix} e_1 + e_{10} & e_5 + e_{14} \\ 0 & 1 \end{pmatrix}$$

we have

$$a_1^2 a_1 = \begin{pmatrix} -2(e_1 + e_{10}) & -(e_5 + e_{14}) \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad a_1\, a_1^2 = \begin{pmatrix} -2(e_1 + e_{10}) & e_5 + e_{14} \\ 0 & 1 \end{pmatrix}$$

and therefore

$$(aa)a - a(aa) = \begin{pmatrix} 0 & -2(e_5 + e_{14}) \\ 0 & 0 \end{pmatrix} \lambda^3.$$

5 Coloop of formal diffeomorphisms

The group of formal diffeomorphisms (tangent to the identity) is the set of series

$$\mathrm{Diff}(A) = \Big\{\ a = \sum_{n\geqslant 0} a_n\,\lambda^{n+1}\ \mid\ a_0 = 1,\ a_n \in A\ \Big\}$$

with coefficients a_n taken in a commutative algebra A, endowed with the composition law $(a\circ b)(\lambda) = a\big(b(\lambda)\big)$, unit $e(\lambda) = \lambda$, and where the inverse of a series $a(\lambda)$ is given by the Lagrange inversion formula [26]. It is a proalgebraic group on **Com**, represented by the Faà di Bruno Hopf algebra [13], [22]

$$H_{\mathrm{FdB}} = \mathbb{F}[x_n,\ n\geqslant 1] \qquad (x_0 = 1)$$
$$\Delta_{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{(p)} \frac{(m+1)!}{p_0!p_1!\cdots p_n!}\ x_1^{p_1}\cdots x_n^{p_n}$$

where the sum is done over the set of tuples $(p_0, p_1, p_2, ..., p_n)$ of non-negative integers such that $p_0 + p_1 + p_2 + \cdots + p_n = m + 1$ and $p_1 + 2p_2 + \cdots + np_n = n - m$. In this section we show that this group can be extended as a proalgebraic loop to the category **As**.

5.1 Loop of formal diffeomorphisms

Definition 5.1.1 Let A be a unital associative algebra, non necessarily commutative, and let λ be a formal variable. We call formal diffeomorphisms in λ with coefficients in A the formal series in the set

$$\mathrm{Diff}(A) = \Big\{\ a = \sum_{n\geqslant 0} a_n\,\lambda^{n+1}\ \mid\ a_0 = 1,\ a_n \in A\ \Big\},$$

endowed with the composition law

$$a\circ b = \sum_{n\geqslant 0}\ \sum_{m=0}^{n}\ \sum_{\substack{k_0+\cdots+k_m = n-m \\ k_0,\ldots,k_m\geqslant 0}} a_m\ b_{k_0}\cdots b_{k_m}\,\lambda^{n+1}$$

$$= \sum_{n\geqslant 0}\left(a_n + b_n + \sum_{m=1}^{n-1} a_m\ \sum_{l=1}^{n}\binom{m+1}{l}\sum_{\substack{k_1+\cdots+k_m = n-m \\ k_1,\ldots,k_m\geqslant 1}} b_{k_1}\cdots b_{k_m}\right)\lambda^{n+1}$$

and the unit e given by $e_0 = 1$ and $e_n = 0$ for all $n > 1$. For instance,

$$\begin{aligned}(a \circ b)_1 &= a_1 + b_1,\\ (a \circ b)_2 &= a_2 + 2a_1b_1 + b_2,\\ (a \circ b)_3 &= a_3 + 3a_2b_1 + a_1(2b_2 + b_1^2) + b_3.\end{aligned}$$

The indeterminate λ is not necessary to define the loop law, but helps to keep track of the degree of the terms in the sum.

Proposition 5.1.2 *For any unital associative algebra A, the set* $\mathrm{Diff}(A)$ *is a loop.*

Proof. It is clear that the composition is a well-defined operation, and that e is a unit. Let us show that the left and right divisions exist.

i) Let us prove that there exists a right division / satisfying the two equations (2.1.2). Given two series $a = \sum a_n \lambda^{n+1}$ and $b = \sum b_n \lambda^{n+1}$, let us define the series $a/b = \sum (a/b)_n \lambda^{n+1}$ so that $(a/b) \circ b = a$, that is

$$\sum_{m=0}^{n} \sum_{\substack{k_0+\cdots+k_m=n-m\\ k_0,\ldots,k_m\geqslant 0}} (a/b)_m \; b_{k_0} \cdots b_{k_m} = a_n \qquad \text{for any } n \geqslant 0.$$

From now on, in the sum over the integers $k_0, ..., k_p$ we omit to write that all integers can be zero. These equations are solved recursively, starting from $(a/b)_0 = a_0 = 1$. The nth term is given by

$$(a/b)_n = a_n - \sum_{m=0}^{n-1} \sum_{k_0+\cdots+k_m=n-m} (a/b)_m \; b_{k_0} \cdots b_{k_m}.$$

To prove that $(a \circ b)/b = a$, i.e. that $\Big((a \circ b)/b\Big)_n = a_n$ for any $n \geqslant 0$, we proceed by induction. We have $\Big((a \circ b)/b\Big)_0 = (a \circ b)_0 = 1$, therefore

$$\Big((a \circ b)/b\Big)_1 = (a \circ b)_1 - \Big((a \circ b)/b\Big)_0 b_1 = a_1 + b_1 - b_1 = a_1$$

and

$$\begin{aligned}\Big((a \circ b)/b\Big)_n &= (a \circ b)_n - \sum_{m=0}^{n-1} \sum_{k_0+\cdots+k_m=n-m} \Big((a \circ b)/b\Big)_m \; b_{k_0} \cdots b_{k_m}\\ &= \sum_{m=0}^{n} \sum_{k_0+\cdots+k_m=n-m} a_p \; b_{k_0} \cdots b_{k_m} - \sum_{m=0}^{n-1} \sum_{k_0+\cdots+k_m=n-m} \Big((a \circ b)/b\Big)_m \; b_{k_0} \cdots b_{k_m}\\ &= a_n + \sum_{m=0}^{n-1} \sum_{k_0+\cdots+k_m=n-m} \Big(a_m - \Big((a \circ b)/b\Big)_m\Big) \; b_{k_0} \cdots b_{k_m},\end{aligned}$$

so, if we suppose that $\Big((a \circ b)/b\Big)_m = a_m$ for any $m \leqslant n-1$, we have $\Big((a \circ b)/b\Big)_n = a_n$.

ii) To prove the existence of the left division we proceed in the same way: the series $a\backslash b$ that satisfies the identity $a \circ (a\backslash b) = b$ of equations (2.1.1), that is,

$$\sum_{m=0}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \ (a\backslash b)_{k_0} \cdots (a\backslash b)_{k_m} = b_n \qquad \text{for any } n \geqslant 0,$$

is given recursively by $(a\backslash b)_0 = 1$ and

$$(a\backslash b)_n = b_n - \sum_{m=1}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \ (a\backslash b)_{k_0} \cdots (a\backslash b)_{k_m}.$$

The identity $a\backslash(a \circ b) = b$ means that, for any $n \geqslant 0$, we have $\Big(a\backslash(a \circ b)\Big)_n = b_n$. This is proved by induction. We have $\Big(a\backslash(a \circ b)\Big)_0 = 1$, therefore

$$\Big(a\backslash(a \circ b)\Big)_1 = (a \circ b)_1 - a_1 \Big(a\backslash(a \circ b)\Big)_0 \Big(a\backslash(a \circ b)\Big)_0 = a_1 + b_1 - a_1 = b_1$$

and

$$\begin{aligned}
\Big(a\backslash(a \circ b)\Big)_n &= (a \circ b)_n - \sum_{m=1}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \Big(a\backslash(a \circ b)\Big)_{k_0} \cdots \Big(a\backslash(a \circ b)\Big)_{k_m} \\
&= \sum_{m=0}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \ b_{k_0} \cdots b_{k_m} - \sum_{m=1}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \Big(a\backslash(a \circ b)\Big)_{k_0} \cdots \Big(a\backslash(a \circ b)\Big)_{k_m} \\
&= a_0 b_n + \sum_{m=1}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \ \left(b_{k_0} \cdots b_{k_m} - \Big(a\backslash(a \circ b)\Big)_{k_0} \cdots \Big(a\backslash(a \circ b)\Big)_{k_m} \right) \\
&= b_n + \sum_{m=1}^{n-1} \sum_{k_0+\cdots+k_m=n-m} a_m \ \left(b_{k_0} \cdots b_{k_m} - \Big(a\backslash(a \circ b)\Big)_{k_0} \cdots \Big(a\backslash(a \circ b)\Big)_{k_m} \right) \\
&\qquad + a_n \ \left(b_0 \cdots b_0 - \Big(a\backslash(a \circ b)\Big)_0 \cdots \Big(a\backslash(a \circ b)\Big)_0 \right) \\
&= b_n + \sum_{m=1}^{n-1} \sum_{k_0+\cdots+k_m=n-m} a_m \ \left(b_{k_0} \cdots b_{k_m} - \Big(a\backslash(a \circ b)\Big)_{k_0} \cdots \Big(a\backslash(a \circ b)\Big)_{k_m} \right),
\end{aligned}$$

so, if we suppose that $\Big(a\backslash(a \circ b)\Big)_m = b_m$ for any $m \leqslant n-1$, we have $\Big(a\backslash(a \circ b)\Big)_n = b_n$. □

For instance, the first terms of the right division are

$$\begin{aligned}
(a/b)_1 &= a_1 - b_1, \\
(a/b)_2 &= a_2 - \Big[b_2 + 2(a/b)_1 b_1\Big] = a_2 - 2a_1 b_1 - (b_2 - 2b_1^2), \\
(a/b)_3 &= a_3 - \Big[b_3 + (a/b)_1(2b_2 + b_1^2) + 3(a/b)_2 b_1\Big] \\
&= a_3 - \big(2a_1 b_2 + 3a_2 b_1\big) + 5a_1 b_1^2 - \big[b_3 - (2b_1 b_2 + 3b_2 b_1) + 5b_1^3\big],
\end{aligned}$$

and the first terms of the left division are

$$\begin{aligned}
(a\backslash b)_1 &= b_1 - a_1, \\
(a\backslash b)_2 &= b_2 - \Big[2a_1(a\backslash b)_1 + a_2\Big] = b_2 - 2a_1 b_1 - (a_2 - 2a_1^2), \\
(a\backslash b)_3 &= b_3 - \Big[a_1(2(a\backslash b)_2 + (a\backslash b)_1^2) + a_2(3(a\backslash b)_1) + a_3\Big] \\
&= b_3 - \big(2a_1 b_2 + 3a_2 b_1\big) + \big(5a_1^2 b_1 + a_1 b_1 a_1 - a_1 b_1^2\big) - \big[a_3 - (2a_1 a_2 + 3a_2 a_1) + 5a_1^3\big].
\end{aligned}$$

We now prove that the loop of formal diffeomorphism is proalgebraic over associative algebras, and give its representative coloop bialgebra.

5.2 Faà di Bruno coloop bialgebra

As in Section 4.2, let $X = \mathrm{Span}_{\mathbb{F}}\{x_n,\ n \geqslant 1\}$ be the set of graded variables x_n of degree n, and identify the tensor algebra $T(X) = \mathbb{F}\langle x_n,\ n \geqslant 1\rangle$ with the set of non-commutative polynomials $\mathbb{F}\langle x_n,\ n \geqslant 1\rangle$. We endow this algebra with the structure of a coloop biagebra which represents the loop Diff.

As before, to simplify the notations, we denote by $x_n = x_n^{(1)}$ and $y_n = x_n^{(2)}$ the generators taken in the two copies of X in $T(X) \amalg T(X) \cong T(X^{(1)} \oplus X^{(2)})$.

To describe the codivisions we need to introduce some sets of sequences and two types of related integer coefficients.

Definition 5.2.1 For any $\ell \geqslant 1$, let $\mathcal{M}_\ell$ denote the set of sequences $\mathbf{m} = (m_1, ..., m_\ell)$ such that

$$m_1 + \cdots + m_\ell = \ell, \qquad \text{and} \qquad m_1 + \cdots + m_j \geqslant j \quad \text{for} \quad j = 1, ..., \ell - 1. \tag{5.2.2}$$

For instance, for $\ell = 1, 2, 3$, we have

$$\mathcal{M}_1 = \big\{(1)\big\}, \quad \mathcal{M}_2 = \big\{(2,0),(1,1)\big\}, \quad \mathcal{M}_3 = \big\{(3,0,0),(2,1,0),(2,0,1),(1,2,0),(1,1,1)\big\}.$$

For any $\ell \geqslant 1$ and any sequence $(n_1, ..., n_{\ell+1})$ of positive integers, we call **Lagrange coefficient**[3] the number

$$d_\ell(n_1, ..., n_\ell) = \sum_{\mathbf{m} \in \mathcal{M}_\ell} \binom{n_1+1}{m_1} \cdots \binom{n_\ell+1}{m_\ell}.$$

For $\ell = 0$, $\mathcal{M}_0$ is empty and we set $d_0 = 1$.

For instance, for $\ell = 1, 2, 3$, we have

$$\begin{aligned}
d_1(n_1) &= \binom{n_1+1}{1}, \\
d_2(n_1, n_2) &= \binom{n_1+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}, \\
d_3(n_1, n_2, n_3) &= \binom{n_1+1}{3} + \binom{n_1+1}{2}\binom{n_2+1}{1} + \binom{n_1+1}{2}\binom{n_3+1}{1} \\
&\quad + \binom{n_1+1}{1}\binom{n_2+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}\binom{n_3+1}{1}.
\end{aligned}$$

Definition 5.2.3 For any $\ell \geqslant 1$, let $\mathcal{E}_\ell$ be the set of sequences $\mathbf{e} = (e_1, ..., e_\ell)$ of bits $e_i \in \{1, 2\}$. For any $\mathbf{e} \in \mathcal{E}_\ell$, let $\mathcal{M}_\ell^{\mathbf{e}}$ be the set of sequences $\mathbf{m} = (m_1, ..., m_\ell) \in \mathcal{M}_\ell$ such that

$$m_i = 0 \quad \text{if} \quad e_i = 2, \quad \text{for} \quad i = 2, ..., \ell.$$

The bits e_i will be used in Eq. (5.2.7) to label the generators x_i of the coloop bialgebra in order to determine to which copy of the coproduct algebra the variables $x_i^{(e_i)}$ belong. To simplify the final formulas for the codivisions, we chose for e_i the bits 1 and 2, even if, for the present discussion, the bits 1 and 0 would be more appropriate.

In particular, if $\mathbf{e} = (1, 1, ..., 1)$ then $\mathcal{M}_\ell^{\mathbf{e}} = \mathcal{M}_\ell$. If $\mathbf{e}$ starts with the bit $e_1 = 2$, then $\mathcal{M}_\ell^{\mathbf{e}}$ is empty, because the condition (5.2.2) implies that $m_1 \geqslant 1$. If $\mathbf{e}$ starts with the bit $e_1 = 1$ and contains at least a bit value 2, then the set $\mathcal{M}_\ell^{\mathbf{e}}$ is a proper subset of $\mathcal{M}_\ell$ obtained by keeping

[3] These coefficients appear in the Lagrange inversion formula [26], cf. [8].

only those sequences $\mathbf{m}$ which have the value 0 in all the positions where the bit value of $\mathbf{e}$ is 2. For instance, $\mathcal{M}_1^{(1)} = \mathcal{M}_1 = \{(1)\}$ and for $\ell = 2$ we have

$$\begin{aligned} \mathcal{M}_2^{(1,1)} &= \mathcal{M}_2 = \{(2,0),(1,1)\}, \\ \mathcal{M}_2^{(1,2)} &= \{(2,0)\}. \end{aligned} \tag{5.2.4}$$

For $\ell = 3$, we have

$$\begin{aligned} \mathcal{M}_3^{(1,1,1)} &= \mathcal{M}_3 = \big\{(3,0,0),(2,1,0),(2,0,1),(1,2,0),(1,1,1)\big\} \\ \mathcal{M}_3^{(1,1,2)} &= \{(3,0,0),(2,1,0),(1,2,0)\} \\ \mathcal{M}_3^{(1,2,1)} &= \{(3,0,0),(2,0,1)\} \\ \mathcal{M}_3^{(1,2,2)} &= \{(3,0,0)\}. \end{aligned} \tag{5.2.5}$$

For any $\ell \geqslant 1$, any sequence $\mathbf{e} \in \mathcal{E}_\ell$ and any sequence $(n_1, ..., n_{\ell+1})$ of positive integers, we call **labeled Lagrange coefficient** the number

$$d_\ell^{\mathbf{e}}(n_1, ..., n_\ell) = \sum_{\mathbf{m} \in \mathcal{M}_\ell^{\mathbf{e}}} \binom{n_1+1}{m_1} \cdots \binom{n_\ell+1}{m_\ell}.$$

For $\ell = 0$, $\mathcal{M}_0$ and $\mathcal{E}_0$ are empty and we set $d_0^{\mathbf{e}} = d_0 1$. Of course, if $\mathbf{e} = (1,1,...,1)$ then $d_\ell^{\mathbf{e}}(n_1, ..., n_\ell) = d_\ell(n_1, ..., n_\ell)$, if $\mathbf{e}$ starts by 2 then $d_\ell^{\mathbf{e}} = 0$, and if $\mathbf{e}$ starts by 1 and contains some bit values equal to 2, then $d_\ell^{\mathbf{e}}(n_1, ..., n_\ell) < d_\ell(n_1, ..., n_\ell)$.

Here are the values of $d_\ell^{\mathbf{e}}$ for $\ell = 1, 2, 3$, obtained by summing the patterns according to the labeled sequences given in (5.2.4) and (5.2.5):

$$\begin{aligned} d_1^{(1)}(n_1) &= d_1(n_1) = \binom{n_1+1}{1}, \\ d_2^{(1,1)}(n_1,n_2) &= d_2(n_1,n_2) = \binom{n_1+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}, \\ d_2^{(1,2)}(n_1,n_2) &= \binom{n_1+1}{2}, \end{aligned}$$

where the term $\binom{n_1+1}{1}\binom{n_2+1}{1}$ disappears because it corresponds to the sequence $\mathbf{m} = (1,1)$ which does not have the value 0 in the same position as the bit 2 in $\mathbf{e} = (1,2)$,

$$\begin{aligned} d_3^{(1,1,1)}(n_1,n_2,n_3) &= d_3(n_1,n_2,n_3) \\ &= \binom{n_1+1}{3} + \binom{n_1+1}{2}\binom{n_2+1}{1} + \binom{n_1+1}{2}\binom{n_3+1}{1} \\ &\quad + \binom{n_1+1}{1}\binom{n_2+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}\binom{n_3+1}{1}, \\ d_3^{(1,1,2)}(n_1,n_2,n_3) &= \binom{n_1+1}{3} + \binom{n_1+1}{2}\binom{n_2+1}{1} + \binom{n_1+1}{1}\binom{n_2+1}{2}, \\ d_3^{(1,2,1)}(n_1,n_2,n_3) &= \binom{n_1+1}{3} + \binom{n_1+1}{2}\binom{n_3+1}{1}, \\ d_3^{(1,2,2)}(n_1,n_2,n_3) &= \binom{n_1+1}{3}. \end{aligned}$$

Definition 5.2.6 We call **Faà di Bruno coloop bialgebra** the free unital associative algebra

$$H_{\mathrm{FdB}}^{\amalg} = \mathbb{F}\langle x_n,\ n \geqslant 1\rangle, \qquad x_0 = 1$$

of non-commutative polynomials in the graded variables x_n, with the following graded cooperations:

- comultiplication $\Delta_{\mathrm{FdB}}^{\amalg} : H_{\mathrm{FdB}}^{\amalg} \longrightarrow H_{\mathrm{FdB}}^{\amalg} \amalg H_{\mathrm{FdB}}^{\amalg}$ given by

$$\Delta_{\mathrm{FdB}}^{\amalg}(x_n) = x_n + y_n + \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in\, \mathcal{C}_n^{\ell+1}} \binom{n_1+1}{\ell}\, x_{n_1}\ y_{n_2} \cdots y_{n_{\ell+1}},$$

  where $\mathcal{C}_n^{\ell+1}$ is the set of compositions of n of length $\ell+1$, cf. (4.2.1);

- counit $\varepsilon : H_{\mathrm{FdB}}^{\amalg} \longrightarrow \mathbb{F}$ given by $\varepsilon(x_n) = \delta_{n,0}$;

- right codivision $\delta_r : H_{\mathrm{FdB}}^{\amalg} \longrightarrow H_{\mathrm{FdB}}^{\amalg} \amalg H_{\mathrm{FdB}}^{\amalg}$ given by

$$\delta_r(x_n) = \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in\, \mathcal{C}_n^{\ell+1}} d_\ell(n_1, ..., n_\ell)\ (x_{n_1} - y_{n_1})\, y_{n_2} \cdots y_{n_{\ell+1}},$$

  where the Lagrange coefficients d_ℓ are given in Def. 5.2.1;

- left codivision $\delta_l : H_{\mathrm{FdB}}^{\amalg} \longrightarrow H_{\mathrm{FdB}}^{\amalg} \amalg H_{\mathrm{FdB}}^{\amalg}$ given by

$$\delta_l(x_n) = \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in\, \mathcal{C}_n^{\ell+1}} \sum_{\mathbf{e} \in\, \mathcal{E}_\ell} (-1)^{\mathbf{e}}\ d_\ell^{\mathbf{e}}(n_1, ..., n_\ell)\ x_{n_1}^{(e_1)} x_{n_2}^{(e_2)} \cdots x_{n_\ell}^{(e_\ell)}\, (y_{n_{\ell+1}} - x_{n_{\ell+1}})$$

  where the set of sequences $\mathcal{E}_\ell$ and the labeled Lagrange coefficients $d_\ell^{\mathbf{e}}$ are given in Def. 5.2.3, and where we set

$$(-1)^{\mathbf{e}} = (-1)^{e_1 + \cdots + e_\ell - \ell}$$

  and, according to the previous convention, we set

$$x_n^{(e_i)} = \begin{cases} x_n & \text{if } e_i = 1 \\ y_n & \text{if } e_i = 2 \end{cases}. \tag{5.2.7}$$

  In particular, since $d_\ell^{\mathbf{e}} = 0$ if $e_1 = 2$, the first variable is always $x_{n_1}^{(e_1)} = x_{n_1}$.

For instance, on the first five generators, the comultiplication is

$$\begin{aligned}
\Delta_{\mathrm{FdB}}^{\amalg}(x_1) &= x_1 + y_1 \\
\Delta_{\mathrm{FdB}}^{\amalg}(x_2) &= x_2 + y_2 + 2x_1y_1 \\
\Delta_{\mathrm{FdB}}^{\amalg}(x_3) &= x_3 + y_3 + \big(2x_1y_2 + 3x_2y_1\big) + x_1y_1^2 \\
\Delta_{\mathrm{FdB}}^{\amalg}(x_4) &= x_4 + y_4 + \big(2x_1y_3 + 3x_2y_2 + 4x_3y_1\big) + \big(x_1(y_1y_2 + y_2y_1) + 3x_2y_1^2\big) \\
\Delta_{\mathrm{FdB}}^{\amalg}(x_5) &= x_5 + y_5 + \big(2x_1y_4 + 3x_2y_3 + 4x_3y_2 + 5x_4y_1\big) \\
&\qquad + \big(x_1(y_1y_3 + y_2^2 + y_3y_1) + 3x_2(y_1y_2 + y_2y_1) + 6x_3y_1^2\big) + x_2y_1^3
\end{aligned}$$

the right codivision, with $u_n = x_n - y_n$, is

$$\begin{aligned}
\delta_r(x_1) &= u_1\\
\delta_r(x_2) &= u_2 - 2u_1y_1\\
\delta_r(x_3) &= u_3 - \big(2u_1y_2 + 3u_2y_1\big) + 5u_1y_1^2\\
\delta_r(x_4) &= u_4 - \big(2u_1y_3 + 3u_2y_2 + 4u_3y_1\big) + \big(5u_1y_1y_2 + 7u_1y_2y_1 + 9u_2y_1^2\big) - 14u_1y_1^3\\
\delta_r(x_5) &= u_5 - \big(2u_1y_4 + 3u_2y_3 + 4u_3y_2 + 5u_4y_1\big)\\
&\qquad + \big(5u_1y_1y_3 + 7u_1y_2^2 + 9u_1y_3y_1 + 9u_2y_1y_2 + 12u_2y_2y_1 + 14u_3y_1^2\big)\\
&\qquad - \big(14u_1y_1^2y_2 + 19u_1y_1y_2y_1 + 23u_1y_2y_1^2 + 28u_2y_1^3\big) + 42u_1y_1^4
\end{aligned}$$

and the left codivision has additional terms which contain both variables x and y in alternative order beside the first position which is always x, and last position which is always $v_n = y_n - x_n$:

$$\begin{aligned}
\delta_l(x_1) &= v_1\\
\delta_l(x_2) &= v_2 - 2x_1v_1\\
\delta_l(x_3) &= v_3 - \big(2x_1v_2 + 3x_2v_1\big) + 5x_1^2v_1 - x_1y_1v_1\\
\delta_l(x_4) &= v_4 - \big(2x_1v_3 + 3x_2v_2 + 4x_3v_1\big) + \big(5x_1^2v_2 + 7x_1x_2v_1 + 9x_2x_1v_1\big) - 14x_1^3v_1\\
&\qquad - \big(x_1y_1v_2 + x_1y_2v_1 + 3x_2y_1v_1\big) + \big(4x_1^2y_1v_1 + 2x_1y_1x_1v_1\big)\\
\delta_l(x_5) &= v_5 - \big(2x_1v_4 + 3x_2v_3 + 4x_3v_2 + 5x_4v_1\big)\\
&\qquad\quad + \big(5x_1^2v_3 + 7x_1x_2v_2 + 9x_1x_3v_1 + 9x_2x_1v_2 + 12x_2^2v_1 + 14x_3x_1v_1\big)\\
&\qquad\quad - \big(14x_1^3v_2 + 19x_1^2x_2v_1 + 23x_1x_2x_1v_1 + 28x_2x_1^2v_1\big) + 42x_1^4v_1\\
&\qquad - \big(x_1y_1v_3 + x_1y_2v_2 + x_1y_3v_1 + 3x_2y_1v_2 + 3x_2y_2v_1 + 6x_3y_1v_1\big)\\
&\qquad\quad + \big(4x_1^2y_1v_2 + 4x_1^2y_2v_1 + 9x_1x_2y_1v_1 + 10x_2x_1y_1v_1\\
&\qquad\qquad + 2x_1y_1x_1v_2 + 3x_1y_1x_2v_1 + 2x_1y_2x_1v_1 + 7x_2y_1x_1v_1 - x_2y_1^2v_1\big)\\
&\qquad\quad - \big(14x_1^3y_1v_1 + 9x_1^2y_1x_1v_1 + 5x_1y_1x_1^2v_1 - x_1^2y_1^2v_1 - x_1y_1x_1y_1v_1\big).
\end{aligned}$$

We now want to prove that the algebra given above is indeed a coloop bialgebra. The only difficulty is to prove that the codivisions satisfy the cocancellation properties (2.2.4) and (2.2.3), which are equivalent to some recurrence relations on the Lagrange coefficients d_ℓ and $d^{\mathbf{e}}_\ell$.

We prove in fact a stronger result, namely, that there exist some operators R_ℓ and $R^{\mathbf{e}}_\ell$ defined on the tensor space $T(A)$ over any positively graded algebra A, which produce the Lagrange coefficients and which satisfy the wished recurrence relations. These operators provide an alternative definition of the Faà di Bruno codivisions when applied to the non-unital associative coproduct algebra $A = \overline{H^{\amalg}_{\mathrm{FdB}}} \amalg \overline{H^{\amalg}_{\mathrm{FdB}}} = \overline{T}(X^{(1)} \oplus X^{(2)})$.

5.3 Faà di Bruno co-operations in terms of recursive operators

Let $A = \oplus_{n\geqslant 1} A_n$ be a positively graded associative algebra over a field $\mathbb{F}$, and let us denote by $|a|$ the degree of an element $a \in A$, that is, the integer n such that $a \in A_n$. The tensor algebra $T(A) = \bigoplus_{\ell\geqslant 0} A^{\otimes \ell}$ is then bigraded, on one side by the tensor power ℓ, that we call **length**, and on the other side by the grading induced by that of A, that we call **degree**,

$$|a_1 \otimes a_2 \otimes \cdots \otimes a_\ell| = \sum_{i=1}^{\ell} |a_i|.$$

A multi-monomial is a homogeneous element of $T(A)$ with respect to the length, that is, an element of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_\ell$ for some $\ell \geqslant 1$. Then $T(A)$ can be decomposed into the following direct sum with respect to the degree[4]:

$$T(A) = \mathbb{F} \oplus \bigoplus_{n \geqslant 1} \left(\bigoplus_{\ell=1}^{n} \bigoplus_{\mathbf{n} \in \mathcal{C}_n^\ell} A_{n_1} \otimes \cdots \otimes A_{n_\ell} \right),$$

where the compositions $\mathbf{n} \in \mathcal{C}_n^\ell$ are defined by eq. (4.2.1).

Definition 5.3.1 Let us define a graded linear operation

$$\rhd : T(A) \otimes T(A) \longrightarrow \mathbb{F} \oplus A$$

by setting

$$\begin{aligned} a \rhd (b_1 \otimes \cdots \otimes b_\ell) &= \binom{|a|+1}{\ell} a\, b_1 \cdots b_\ell \\ (a_1 \otimes \cdots \otimes a_{\ell'}) \rhd (b_1 \otimes \cdots \otimes b_{\ell''}) &= a_1 \rhd (a_2 \otimes \cdots \otimes a_{\ell'} \otimes b_1 \otimes \cdots \otimes b_{\ell''}) \\ &= \binom{|a_1|+1}{\ell'+\ell''-1} a_1 \cdots a_{\ell'}\, b_1 \cdots b_{\ell''} \end{aligned}$$

where the expressions on the right-hand side mean the product in the algebra A.

In particular, if we apply these rules to $1 \in \mathbb{F} = A^{\otimes 0}$, we have

$$\begin{aligned} 1 \rhd 1 &= 1 \\ 1 \rhd b &= b \\ 1 \rhd (b_1 \otimes \cdots \otimes b_\ell) &= 0 \qquad \text{if } \ell > 1 \\ a \rhd 1 &= a \\ (a_1 \otimes \cdots \otimes a_\ell) \rhd 1 &= \binom{|a_1|+1}{\ell-1} a_1 \cdots a_\ell. \end{aligned}$$

Remark 5.3.2 The restriction $\rhd : A \otimes T(A) \longrightarrow A$ is a brace product on A which is symmetric if A is commutative and generalises the natural pre-Lie product of the Lie subalgebra of strictly positive generators in the Witt algebra (cf. [9, 16]). Note however that $\rhd$ on $T(A) \otimes T(A)$ is not a multibrace product (cf. [27]), even excluding the scalar component, because the first non-trivial multibrace identity $M_{21}(a \otimes b + b \otimes a; c) + M_{11}(M_{11}(a; b); c) = M_{11}(a; M_{11}(b; c)) + M_{12}(a; b \otimes c + c \otimes b)$ is not satisfied. Moreover, a unit for $\rhd$ can not exist, because of length arguments, and $\rhd$ is not associative, since for any $a, b, c \in A$ we have

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (|a|+1)|a|\, abc \neq 0.$$

The algebraic structure described by the operator $\rhd$ in terms of generators and relations is an open question.

[4]Note that if A had a null degree component A_0, then $T(A)$ would contain an infinite sum of terms in each degree, namely $T(A)_0 = \bigoplus_{p \geqslant 0} A_0^{\otimes p}$ and $T(A)_n = \bigoplus_{\ell=1}^{n} \bigoplus_{\mathbf{n} \in \mathcal{C}_n^\ell} \bigoplus_{p \geqslant 0} \binom{\ell+p}{\ell} A_0^{\otimes p} \otimes A_{n_1} \otimes \cdots \otimes A_{n_\ell}$ for $n \geqslant 1$.

Definition 5.3.3 We call **left recursive operator** $L : T(A) \longrightarrow T(A)$ the collection $L = \{L_\ell,\ \ell \geqslant 0\}$ of (non homogeneous) linear operators $L_0 = \mathrm{id} : \mathbb{F} \longrightarrow \mathbb{F}$ and

$$L_\ell : A^{\otimes \ell} \longrightarrow \bigoplus_{\lambda=1}^{\ell} A^{\otimes \lambda}, \qquad \ell \geqslant 1$$

defined recursively by

$$L_\ell(a_1, ..., a_\ell) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \Big(L_i(a_1, ..., a_i) \rhd a_{i+1} \Big) \otimes a_{i+2} \otimes \cdots \otimes a_\ell,$$

where we denote $L_\ell(a_1, ..., a_\ell) := L_\ell(a_1 \otimes \cdots \otimes a_\ell)$ and L_0 is understood as acting on 1.

The first left operators give

$$\begin{aligned}
L_1(a) &= L_0(1) \rhd a = 1 \rhd a = a, \\
L_2(a,b) &= -(L_0(1) \rhd a) \otimes b + L_1(a) \rhd b = -a \otimes b + a \rhd b \\
&= -a \otimes b + \binom{|a|+1}{1} ab, \\
L_3(a,b,c) &= (L_0(1) \rhd a) \otimes b \otimes c - (L_1(a) \rhd b) \otimes c + L_2(a,b) \rhd c \\
&= a \otimes b \otimes c - (a \rhd b) \otimes c - (a \otimes b) \rhd c + (a \rhd b) \rhd c \\
&= a \otimes b \otimes c - \binom{|a|+1}{1} ab \otimes c + \Big(\binom{|a|+1}{1} \binom{|a|+|b|+1}{1} - \binom{|a|+1}{2} \Big) abc.
\end{aligned}$$

The left operators L_ℓ can be easily described in a closed way.

Lemma 5.3.4 *For any $\ell \geqslant 2$ and any $a_1, ..., a_\ell \in A$ we have*

$$L_\ell(a_1, ..., a_\ell) = L_{\ell-1}(a_1, ..., a_{\ell-1}) \rhd a_\ell - L_{\ell-1}(a_1, ..., a_{\ell-1}) \otimes a_\ell.$$

As a consequence, $L_\ell(a_1, ..., a_\ell)$ is the sum of the $2^{\ell-1}$ possible multi-monomials obtained by combining the operations $\rhd$ and $\otimes$ with fixed parenthesizing on the left, namely

$$L_\ell(a_1, ..., a_\ell) = \sum_{\sigma_1, ..., \sigma_{\ell-1} \in \{0,1\}} (-1)^{\sigma_1 + \cdots + \sigma_{\ell-1}} \big(\cdots \big((a_1 *_{\sigma_1} a_2) *_{\sigma_2} a_3 \big) *_{\sigma_3} \cdots *_{\sigma_{\ell-2}} a_{\ell-1} \big) *_{\sigma_{\ell-1}} a_l$$

where we set

$$*_\sigma = \begin{cases} \rhd & \text{if } \sigma = 0 \\ \otimes & \text{if } \sigma = 1 \end{cases}.$$

Proof. By induction on ℓ. For $\ell = 2$, we have

$$L_1(a_1) \rhd a_2 - L_1(a_1) \otimes a_2 = a_1 \rhd a_2 - a_1 \otimes a_2 = L_2(a_1, a_2).$$

Now suppose that for any $i = 2, ..., \ell - 1$ we have

$$L_i(a_1, ..., a_i) = L_{i-1}(a_1, ..., a_{i-1}) \rhd a_i - L_{i-1}(a_1, ..., a_{i-1}) \otimes a_i.$$

Let us expand the sum defining $L_\ell(a_1,...,a_\ell)$. At each step, we separate the first two terms of the sum over $i=0,...,\ell-1$:

$$\begin{aligned}
L_\ell(a_1,...,a_\ell) &= \sum_{i=0}^{\ell-1}(-1)^{\ell-1-i}\Big(L_i(a_1,...,a_i)\triangleright a_{i+1}\Big)\otimes a_{i+2}\otimes\cdots\otimes a_\ell \\
&= (-1)^{\ell-1}a_1\otimes a_2\otimes\cdots\otimes a_\ell + (-1)^{\ell-2}(a_1\triangleright a_2)\otimes\cdots\otimes a_\ell \\
&\qquad + \sum_{i=2}^{\ell-1}(-1)^{\ell-1-i}\Big(L_i(a_1,...,a_i)\triangleright a_{i+1}\Big)\otimes a_{i+2}\otimes\cdots\otimes a_\ell \\
&= (-1)^{\ell-2}L_2(a_1,a_2)\otimes a_3\otimes\cdots\otimes a_\ell + (-1)^{\ell-3}\Big(L_2(a_1,a_2)\triangleright a_3\Big)\otimes\cdots\otimes a_\ell \\
&\qquad + \sum_{i=3}^{\ell-1}(-1)^{\ell-1-i}\Big(L_i(a_1,...,a_i)\triangleright a_{i+1}\Big)\otimes a_{i+2}\otimes\cdots\otimes a_\ell = \dots
\end{aligned}$$

Iterating this expansion we obtain

$$\begin{aligned}
L_\ell(a_1,...,a_\ell) &= (-1)^{\ell-(\ell-1)}L_{\ell-1}(a_1,...,a_{\ell-1})\otimes a_\ell + (-1)^{\ell-\ell}L_{\ell-1}(a_1,...,a_{\ell-1})\triangleright a_\ell \\
&= L_{\ell-1}(a_1,...,a_{\ell-1})\triangleright a_\ell - L_{\ell-1}(a_1,...,a_{\ell-1})\otimes a_\ell.
\end{aligned}$$

□

Definition 5.3.5 We call **right recursive operator** $R: T(A)\longrightarrow T(A)$ the collection $R=\{R_\ell,\ \ell\geqslant 0\}$ of (non homogeneous) linear operators $R_0=\mathrm{id}:\mathbb{F}\longrightarrow\mathbb{F}$ and

$$R_\ell : A^{\otimes\ell}\longrightarrow \bigoplus_{\lambda=1}^{\ell} A^{\otimes\lambda},\qquad \ell\geqslant 1$$

defined recursively by

$$\begin{aligned}
R_\ell(a_1,...,a_\ell) = \sum_{j=1}^{\ell}\sum_{\mathbf{p}\in\mathcal{C}_\ell^j} &\big(a_1\triangleright R_{p_1-1}(a_2,...,a_{p_1})\big) \\
&\otimes\big(a_{p_1+1}\triangleright R_{p_2-1}(a_{p_1+2},...,a_{p_1+p_2})\big)\otimes\cdots \\
&\otimes\big(a_{p_1+\cdots+p_{j-1}+1}\triangleright R_{p_2-1}(a_{p_1+\cdots+p_{j-1}+2},...,a_{p_1+\cdots+p_j})\big),
\end{aligned} \tag{5.3.6}$$

where we denote $R_\ell(a_1,...,a_\ell):=R_\ell(a_1\otimes\cdots\otimes a_\ell)$ and where R_0 is understood as acting on 1.

For instance, the first right operators are

$$\begin{aligned}
R_1(a) &= a\triangleright R_0(1) = a, \\
R_2(a,b) &= a\triangleright R_1(b) + (a\triangleright R_0(1))\otimes(b\triangleright R_0(1)) \\
&= a\triangleright b + a\otimes b = \binom{|a|+1}{1}ab + a\otimes b, \\
R_3(a,b,c) &= a\triangleright R_2(b,c) + (a\triangleright R_1(b))\otimes c + a\otimes(b\triangleright R_1(c)) + a\otimes b\otimes c \\
&= a\triangleright(b\triangleright c) + a\triangleright(b\otimes c) + a\otimes(b\triangleright c) + (a\triangleright b)\otimes c + a\otimes b\otimes c \\
&= \left(\binom{|a|+1}{1}\binom{|b|+1}{1} + \binom{|a|+1}{2}\right)abc + \binom{|b|+1}{1}a\otimes bc \\
&\qquad + \binom{|a|+1}{1}ab\otimes c + a\otimes b\otimes c.
\end{aligned}$$

Note that the right recursive operator is not just a flip of the left recursive one, basically because the recursion defining the two operators takes place on the left and on the right-hand side of $\triangleright$, which is not a symmetric operation. The precise relationship between R_ℓ and L_ℓ is given in Cor. 5.4.3, after some preliminary results.

The right operators R_ℓ can also be described in a closed way.

Definition 5.3.7 Let $\mathcal{M}_\ell$ be the set of sequences satisfying (5.2.2). For any $\mathbf{m} \in \mathcal{M}_\ell$, we define a length-homogeneous linear operator $R^\ell_{\mathbf{m}} : A^{\otimes \ell} \longrightarrow T(A)$ which nests the operation $\triangleright$ in a multi-monomial $a_1 \otimes \cdots \otimes a_\ell$ according to the sequence $\mathbf{m} = (m_1, ..., m_\ell) \in \mathcal{M}_\ell$. The idea is the following:

- The multi-monomial $R^\ell_{\mathbf{m}}(a_1, ..., a_\ell)$ is constructed by nesting tensor monomials of the form $a_i \triangleright Q_{i+1}(a_{i+1}, a_{i+2}, ...)$ one into the other one, where $Q_{i+1}(a_{i+1}, a_{i+2}, ...)$ is a multi-monomial whose tensor factors can be single variables or monomials of the same form $a_j \triangleright Q'_{j+1}(a_{j+1}, a_{j+2}, ...)$.
- Every tensor monomial $a_i \triangleright Q_{i+1}(a_{i+1}, a_{i+2}, ...)$ is determined by the length of the multi-monomial $Q_{i+1}(a_{i+1}, a_{i+2}, ...)$ and that of the nested monomials of the same form. The sequence $\mathbf{m}$ fixes the lengths of all the nested multi-monomials:
  - The coefficient m_1 is the overall length of the multi-monomial $R^\ell_{\mathbf{m}}(a_1, ..., a_\ell)$ in the tensor algebra $T(A)$, that is, we have $R^\ell_{\mathbf{m}}(a_1, ..., a_\ell) \in A^{\otimes m_1}$.
  - For $i = 2, ..., \ell-1$, the coefficient m_i is the length of the multi-monomial $Q_i(a_i, a_{i+1}, ...)$ on which a_{i-1} acts by $\triangleright$: if $m_i = 0$, then a_{i-1} appears as an insolated tensor factor, if $m_i \neq 0$, then a_{i-1} acts by $\triangleright$ on a multimonomial of length m_i, which is determined by the values m_j for $j > i$.

And now we give the algorithm to construct $R^\ell_{\mathbf{m}}(a_1, ..., a_\ell)$:

(1) The coefficient m_1 tells us how many tensor factors we have to construct.

(2) Start with a_1 and read the coefficient m_2: if $m_2 = 0$ write $a_1 \otimes a_2 \cdots$, if $m_2 \neq 0$ write $a_1 \triangleright (a_2 \cdots)$ and expect to close the parenthesis after a multi-monomial of length m_2.

(3) Then read the next coefficient of $\mathbf{m}$ and repeat the procedure of (2). For any $i = 2, ..., \ell$, if $m_i = 0$ write $a_{i-1} \otimes a_i \cdots$, if $m_i \neq 0$ write $a_{i-1} \triangleright (a_i \cdots)$ and expect to close this parenthesis after a multi-monomial of length m_i.

(4) The procedure stops with the coefficient m_ℓ which, by definition of $\mathbf{m}$, can be only 0 or 1, and tells if the last pattern is $a_{\ell-1} \otimes a_\ell$ or $a_{\ell-1} \triangleright a_\ell$.

Example 5.3.8 Let us give some examples of this algorithm, for $\ell = 5$. Fix $a_1, ..., a_5 \in A$ and set $n_i = |a_i|$ for $i = 1, ..., 5$. For $\mathbf{m} = (2, 1, 0, 2, 0)$, the multi-monomial $R^5_{(2,1,0,2,0)}(a_1, ..., a_5)$ is composed of two tensor factors (because $m_1 = 2$). The variable a_1 acts by $\triangleright$ on a multi-monomial of length 1 (because $m_2 = 1$) which starts necessarily by a_2, and since a_2 does not act by $\triangleright$ (because $m_3 = 0$), the first tensor factor is necessarily of the form $a_1 \triangleright a_2$. Then the second tensor factor starts with a_3 acting by $\triangleright$ on a multi-monomial of length 2 (because $m_4 = 2$), which starts necessarily by a_4. Since a_4 does not act by $\triangleright$ (because $m_5 = 0$), the second tensor factor is necessarily of the form $a_3 \triangleright (a_4 \otimes a_5)$. Therefore we finally have

$$\begin{aligned} R^5_{(2,1,0,2,0)}(a_1, ..., a_5) &= (a_1 \triangleright a_2) \otimes \big(a_3 \triangleright (a_4 \otimes a_5)\big) \\ &= \binom{n_1 + 1}{1} \binom{n_3 + 1}{2} a_1 a_2 \otimes a_3 a_4 a_5. \end{aligned}$$

For $\mathbf{m} = (2,1,2,0,0)$, the variable a_1 still acts by $\triangleright$ on a multi-monomial of length 1 which starts necessarily by a_2, but this time a_2 itself acts by $\triangleright$ on a multi-monomial of length 2, and this exhausts the possible $\triangleright$ operations. Finally, this time we have

$$\begin{aligned} R^5_{(2,1,2,0,0)}(a_1,...,a_5) &= [a_1 \triangleright (a_2 \triangleright (a_3 \otimes a_4)]] \otimes a_5 \\ &= \binom{n_1+1}{1}\binom{n_2+1}{2} a_1a_2a_3a_4 \otimes a_5. \end{aligned}$$

Note that the binomial coefficients given by a sequence $\mathbf{m} \in \mathcal{M}_\ell$ can be determined directly from the last $\ell - 1$ digits, plus an extra null value. For $(2,1,0,2,0)$ we have exactly

$$\binom{n_1+1}{1}\binom{n_2+1}{0}\binom{n_3+1}{2}\binom{n_4+1}{0}\binom{n_5+1}{0}.$$

and for $(2,1,2,0,0)$ we have

$$\binom{n_1+1}{1}\binom{n_2+1}{2}\binom{n_3+1}{0}\binom{n_4+1}{0}\binom{n_5+1}{0}.$$

Two more examples of the algorithm: for $\mathbf{m} = (3,0,2,0,0)$,

$$\begin{aligned} R^5_{(3,0,2,0,0)}(a_1,...,a_5) &= a_1 \otimes \big(a_2 \triangleright (a_3 \otimes a_4)\big) \otimes a_5 \\ &= \binom{n_2+1}{2} a_1 \otimes a_2a_3a_4 \otimes a_5, \end{aligned}$$

and for $\mathbf{m} = (4,0,1,0,0)$,

$$\begin{aligned} R^5_{(4,0,1,0,0)}(a_1,...,a_5) &= a_1 \otimes (a_2 \triangleright a_3) \otimes a_4 \otimes a_5 \\ &= \binom{n_2+1}{1} a_1 \otimes a_2a_3 \otimes a_4 \otimes a_5. \end{aligned}$$

Lemma 5.3.9 *For any $\ell \geqslant 1$ and any $(a_1,...,a_\ell)$, we have*

$$R_\ell(a_1,...,a_\ell) = \sum_{\mathbf{m} \in \mathcal{M}_\ell} R^\ell_{\mathbf{m}}(a_1,...,a_\ell).$$

For instance, for $\ell = 1,2,3$ we have

$$\begin{aligned} R^1_{(1)}(a) &= a, \\ R^2_{(1,1)}(a,b) &= a \triangleright b = \binom{n_1+1}{1} ab \\ R^2_{(2,0)}(a,b) &= a \otimes b \\ R^3_{(1,1,1)}(a,b,c) &= a \triangleright (b \triangleright c) = \binom{n_1+1}{1}\binom{n_2+1}{1} abc. \\ R^3_{(1,2,0)}(a,b,c) &= a \triangleright (b \otimes c) = \binom{n_1+1}{2} abc \\ R^3_{(2,0,1)}(a,b,c) &= a \otimes (b \triangleright c) = \binom{n_2+1}{1} a \otimes bc \\ R^3_{(2,1,0)}(a,b,c) &= (a \triangleright b) \otimes c = \binom{n_1+1}{1} ab \otimes c \\ R^3_{(3,0,0)}(a,b,c) &= a \otimes b \otimes c \end{aligned}$$

Comparing with the value of R_1, R_2 and R_3 given above, the assertion is easily verified.

Proof. Let us call $\tilde{R}_\ell(a_1, ..., a_\ell)$ the sum over $\mathbf{m} \in \mathcal{M}_\ell$ of Lemma 5.3.9, and prove that it solves equation (5.3.6) by induction on ℓ.

For $\ell = 1, 2, 3$ the assertion was proved in the examples. For any $\ell \geqslant 1$, we then suppose that on the right-hand side of eq. (5.3.6) we have $R_{p_i-1} = \tilde{R}_{p_i-1}$ for any $1 \leqslant i \leqslant j$, and we set

$$P_i = p_1 + p_2 + \cdots + p_i,$$

so that

$$R_\ell(a_1, ..., a_\ell) = \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_\ell^j} \sum_{i=1}^{j} \sum_{\mathbf{q}^{(i)} \in \mathcal{M}_{p_i-1}} \big(a_1 \triangleright \tilde{R}^{p_1-1}_{\mathbf{q}^{(1)}}(a_2, ..., a_{p_1})\big) \otimes$$
$$\otimes \big(a_{p_1+1} \triangleright \tilde{R}^{p_2-1}_{\mathbf{q}^{(2)}}(a_{p_1+2}, ..., a_{P_2})\big) \otimes \cdots \otimes \big(a_{P_{j-1}+1} \triangleright \tilde{R}^{p_j-1}_{\mathbf{q}^{(j)}}(a_{P_{j-1}+2}, ..., a_{P_j})\big).$$

In this sum, we can note the following things:

- The running value j gives the length of the corresponding multi-monomial.
- In the first tensor factor, the value $q_1^{(1)}$ represents the length of $\tilde{R}^{p_1-1}_{\mathbf{q}^{(1)}}(a_2, ..., a_{p_1})$, that is, a sequence number associated to a_1, and more generally $\mathbf{q}^{(1)}$ rules the nested operations up to the variable a_{p_1-1}. The last variable a_{p_1} does not act on further variables and so it should be associated to a missing value 0. Therefore, the nested operations in the whole first tensor factor are ruled by the sequence $(\mathbf{q}^{(1)}, 0)$.
- Similarly, for any $i \leqslant j$, the nested operations in the ith tensor factor are ruled by the sequence $(\mathbf{q}^{(i)}, 0)$.

Let us then associate to this expression the sequence

$$\mathbf{m} = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, ..., \mathbf{q}^{(j)}),$$

that is,

$$\begin{aligned}
&m_1 = j \\
&m_2 = q_1^{(1)}, \ \dots \ , \ m_{p_1} = q_{p_1-1}^{(1)} \ , \ m_{p_1+1} = 0 \ , \\
&m_{P_{i-1}+2} = q_1^{(i)}, \ \dots \ , \ m_{P_i} = q_{p_i-1}^{(i)} \ , \ m_{P_i+1} = 0 \ , \quad \text{for } 1 \leqslant i \leqslant j-1 \\
&m_{P_{j-1}+2} = q_1^{(j)}, \ \dots \ , \ m_{P_j} = q_{p_j-1}^{(j)},
\end{aligned}$$

which has precisely length

$$1 + (p_1 - 1) + 1 + (p_2 - 1) + \cdots + 1 + (p_j - 1) = p_1 + \cdots + p_j = P_j = \ell.$$

Note that in the sum over the sequences $\mathbf{p} = (p_1, ..., p_j) \in \mathcal{C}_\ell^j$, where $p_i \geqslant 1$ for $i = 1, ..., j$, there occur the terms with $p_i - 1 = 0$. In this case the multipolynomial $\tilde{R}_{p_i-1} = R_0 = 1$ has no variables, and the set $\mathcal{M}_{p_1-1} = \mathcal{M}_0$ is empty. The corresponding sequence $\mathbf{q}^{(i)}$ is then absent in $\mathbf{m}$, but its associated null value must be present, for any $i = 1, ..., j-1$, to preserve the total length ℓ. Following the rules of the algorithm given in Def. 5.3.7, we can therefore write

$$\big(a_1 \triangleright \tilde{R}^{p_1-1}_{\mathbf{q}^{(1)}}(a_2, ..., a_{p_1})\big) \otimes \big(a_{p_1+1} \triangleright \tilde{R}^{p_2-1}_{\mathbf{q}^{(2)}}(a_{p_1+2}, ..., a_{P_2})\big) \otimes \cdots$$
$$\cdots \otimes \big(a_{P_{j-1}+1} \triangleright \tilde{R}^{p_j-1}_{\mathbf{q}^{(j)}}(a_{P_{j-1}+2}, ..., a_{P_j})\big) \quad = \quad R_{\mathbf{m}}(a_1, ..., a_\ell).$$

Let us call

$$\mathcal{N}_\ell = \{\mathbf{m} = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, ..., \mathbf{q}^{(j)}) \mid 1 \leqslant j \leqslant \ell,\ \mathbf{p} \in \mathcal{C}_\ell^j,\ \mathbf{q}^{(i)} \in \mathcal{M}_{p_i - 1} \text{ for } 1 \leqslant i \leqslant j\}$$

the set of sequences obtained in this way. Then the equality

$$R_\ell(a_1, a_2, ..., a_\ell) = \sum_{\mathbf{m} \in \mathcal{N}_\ell} R_{\mathbf{m}}^\ell(a_1, ..., a_\ell) = \tilde{R}_\ell(a_1, a_2, ..., a_\ell)$$

holds if we show that $\mathcal{N}_\ell = \mathcal{M}_\ell$.

Let us first show that $\mathcal{N}_\ell \subset \mathcal{M}_\ell$. For fixed j, $\mathbf{p}$ and $\mathbf{q}^{(1)}, ..., \mathbf{q}^{(j)}$, we have

$$\begin{aligned} m_1 + \cdots + m_\ell &= j + \sum_{i=1}^{j} \left(q_1^{(i)} + \cdots + q_{p_i - 1}^{(i)}\right) \\ &= j + (p_1 - 1) + (p_2 - 1) + \cdots + (p_j - 1) \\ &= j + p_1 + \cdots + p_j - j = \ell. \end{aligned}$$

For any $h = 1, ..., \ell$, suppose that h belongs to the rth block, for some $r \leqslant j$, that is,

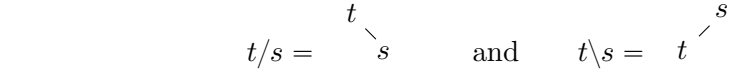
$$h = P_{r-1} + 1 + k = p_1 + \cdots + p_{r-1} + 1 + k,$$

with $1 \leqslant k \leqslant p_r - 1$. Then we have

$$\begin{aligned} m_1 + \cdots + m_h &= j + \sum_{i=1}^{r-1} \left(q_1^{(i)} + \cdots + q_{p_i - 1}^{(i)}\right) + \left(q_1^{(r)} + \cdots + q_k^{(r)}\right) \\ &= j + (p_1 - 1) + (p_2 - 1) + \cdots + (p_{r-1} - 1) + \left(q_1^{(r)} + \cdots + q_k^{(r)}\right) \\ &= (p_1 + \cdots + p_{r-1}) + \left(q_1^{(r)} + \cdots + q_k^{(r)}\right) + (j - r) + 1 \\ &\geqslant P_{r-1} + 1 + k = h \end{aligned}$$

because $q_1^{(r)} + \cdots + q_k^{(r)} \geqslant k$ and $j - r \geqslant 0$.

Finally, let us show that there is a bijection between $\mathcal{N}_\ell$ and $\mathcal{M}_\ell$. The set $\mathcal{M}_\ell$ is well known to be in bijection with the set $PBT_{\ell+1}$ of planar binary trees with $\ell + 1$ leaves (and a root). An explicit bijection $\Phi : \mathcal{M}_\ell \longrightarrow PBT_{\ell+1}$ is described in [8], Definition 2.16, using the *over* and *under* grafting operations on trees, namely



The first values of Φ, for the empty sequence in $\mathcal{M}_0$ and for $(1) \in \mathcal{M}_1$ and $(2,0), (1,1) \in \mathcal{M}_2$, are

$$\Phi(\ ) = \mid, \qquad \Phi(1) = \curlyvee, \qquad \Phi(2,0) = \curlyvee\!\curlyvee, \qquad \Phi(1,1) = \curlyvee\!\curlyvee.$$

So, for our purpose, it is enough to show that $\mathcal{N}_\ell$ is in bijection with $PBT_{\ell+1}$. For this, since $\mathcal{N}_\ell \subset \mathcal{M}_\ell$, consider the map Φ restricted to $\mathcal{N}_\ell$ and let us show that the image $\Phi(\mathcal{N}_\ell)$ coincides with $PBT_{\ell+1}$.

For a given sequence $\mathbf{m} = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, ..., \mathbf{q}^{(j)}) \in \mathcal{N}_\ell$, we have:

- The sequence $\mathbf{m}$ is decomposable as $\mathbf{m} = (\mathbf{m}', \mathbf{q}^{(j)})$ into the two well-defined sequences $\mathbf{m}' = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, ..., \mathbf{q}^{(j-1)}, 0) \in \mathcal{M}_{P_{j-1}+1}$ and $\mathbf{q}^{(j)} \in \mathcal{M}_{p_j - 1}$. In fact, if we set $\ell' = P_{j-1} + 1$, we have

$$m'_1 + \cdots m'_{\ell'} = j + (p_1 - 1) + \cdots + (p_{j-1} - 1) = j + P_{j-1} - (j - 1) = \ell',$$

and for any $h = 1, ..., \ell'$ one can see that $m'_1 + \cdots m'_h \geqslant h$ with a computation similar to that used to show that $\mathbf{m} \in \mathcal{M}_\ell$.

According to the definition of Φ, we then have $\Phi(\mathbf{m}) = \Phi(\mathbf{m}')\backslash\Phi(\mathbf{q}^{(j)})$. Graphically, if we denote the trees by $t = \Phi(\mathbf{m})$, $t' = \Phi(\mathbf{m}')$ and $t_j = \Phi(\mathbf{q}^{(j)})$, this means that

$t =$.

- The sequence $\mathbf{m}'$ is surely not decomposable because it is of the form

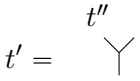
$$\mathbf{m}' = (m''_1 + 1, m''_2, ..., m''_{\ell''}, 0)$$

with

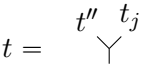
$$\mathbf{m}'' = (j-1, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, ..., \mathbf{q}^{(j-1)}) \in \mathcal{M}_{\ell''}, \qquad \ell'' = \ell' - 1.$$

The sequence $\mathbf{m}''$ indeed belongs to $\mathcal{M}_{\ell''}$ for the same reason used to show that $\mathbf{m} \in \mathcal{M}_\ell$. Then, the sequence $\mathbf{m}'$ is not decomposable in position 1 because $m'_1 = m''_1 + 1 \geqslant 2$, and it is not decomposable in any position $h = 2, ..., \ell'$ because $\mathbf{m}'' \in \mathcal{M}_{\ell''}$ implies that $m'_1 + \cdots + m'_h = m''_1 + 1 + m''_2 + \cdots + m''_h \geqslant h + 1$, and therefore surely $m'_1 + \cdots + m'_h \neq h$.

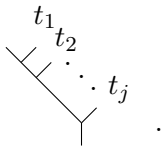
Finally, according to the definition of Φ, we then have $\Phi(\mathbf{m}') = \Phi(\mathbf{m}'')/\curlyvee$. If we set $t'' = \Phi(\mathbf{m}'')$, this means that

$t' =$ 

and therefore

$t =$  .

- The same arguments can be applied to the sequence $\mathbf{m}''$ and its new components, until we reach a full description of the tree $t = \Phi(\mathbf{m})$ in terms of the trees $t_i = \Phi(\mathbf{q}^{(i)})$, for $i = 1, ..., j$, namely

$t =$  .

Let us denote this tree by $G^j(t_1, ..., t_j)$.

In conclusion, if we let j run from 1 to ℓ, we consider all possible sequences $\mathbf{p} \in \mathcal{C}^j_\ell$ and for any $i = 1, ..., j$ all trees $t_i \in \Phi(\mathcal{M}_{p_i - 1}) = PBT_{p_i}$, the result $G^j(t_1, ..., t_j)$ is any possible tree with number of leaves given by

$$\big|G^j(t_1, ..., t_j)\big| = 1 + |t_1| + \cdots + |t_j| = 1 + p_1 + \cdots + p_j = \ell + 1.$$

In other words, we have

$$\Phi(\mathcal{N}_\ell) = \big\{t = G^j(t_1, ..., t_j) \mid j = 1, ..., \ell,\ \mathbf{p} \in \mathcal{C}^j_\ell,\ t_i \in PBT_{p_i},\ 1 \leqslant i \leqslant j\big\} = PBT_{\ell+1}.$$

□

Corollary 5.3.10 *For any $\ell \geqslant 0$ and any $a_1, ..., a_{\ell+1} \in A$, set $n_i = |a_i| \geqslant 1$ for $i = 1, ..., \ell+1$. Then, for any sequence $\mathbf{m} \in \mathcal{M}_\ell$, we have*

$$a_1 \triangleright R_{\mathbf{m}}(a_2, ..., a_{\ell+1}) = \binom{n_1+1}{m_1} \cdots \binom{n_\ell+1}{m_\ell} \binom{n_{\ell+1}+1}{0} \; a_1 a_2 \cdots a_{\ell+1}.$$

Therefore

$$a_1 \triangleright R_\ell(a_2, ..., a_{\ell+1}) = d(n_1, ..., n_{\ell+1}) \; a_1 a_2 \cdots a_{\ell+1},$$

where the Lagrange coefficients $d(n_1, ..., n_{\ell+1})$ are given in Def. 5.2.1.

To describe the left codivision we introduce a last set of operators corresponding to the labeled Lagrange coefficients.

Definition 5.3.11 For any $\ell \geqslant 1$, let $\mathcal{E}_\ell$ be the set of sequences of bits $e_i \in \{1, 2\}$, as in Def. 5.2.3. We call **labeled right recursive operator** $R^{\mathbf{e}} : T(A) \longrightarrow T(A)$ the collection $R^{\mathbf{e}} = \{R^{\mathbf{e}}_\ell, \; \ell \geqslant 0, \; \mathbf{e} \in \mathcal{E}_\ell\}$ of (non homogeneous) linear operators $R^{\mathbf{e}}_0 = \mathrm{id} : \mathbb{F} \longrightarrow \mathbb{F}$ and

$$R^{\mathbf{e}}_\ell : A^{\otimes \ell} \longrightarrow \bigoplus_{\lambda=1}^{\ell} A^{\otimes \lambda}, \qquad \ell \geqslant 1$$

defined recursively by

$$R^{(1)}_0(a) = a \qquad \text{and} \qquad R^{(2)}_0(a) = 0,$$

and, for $\ell \geqslant 2$ and for any $\mathbf{e} = (e_1, ..., e_\ell) \in \mathcal{E}_\ell$, by

$$\begin{aligned} R^{\mathbf{e}}_\ell(a_1, ..., a_\ell) = \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}^j_\ell} & \big(R^{(e_1)}_1(a_1) \triangleright R^{(e_2, ..., e_{p_1})}_{p_1 - 1}(a_2, ..., a_{p_1}) \big) \\ & \otimes \big(a_{p_1+1} \triangleright R^{(e_{p_1+2}, ..., e_{p_1+p_2})}_{p_2-1}(a_{p_1+2}, ..., a_{p_1+p_2}) \big) \otimes \cdots \\ & \cdots \otimes \big(a_{p_1+\cdots+p_{j-1}+1} \triangleright R^{(e_{p_1+\cdots+p_{j-1}+2}, ..., e_\ell)}_{p_j-1}(a_{p_1+\cdots+p_{j-1}+2}, ..., a_\ell) \big). \end{aligned} \tag{5.3.12}$$

It turns out that $R^{\mathbf{e}}_\ell = R_\ell$ if $\mathbf{e} = (1, 1, ..., 1)$. If $\mathbf{e}$ starts by 2, then $R^{\mathbf{e}}_\ell = 0$. If $\mathbf{e}$ starts by 1 and contains a bit value $e_i = 2$ (in position i), then $R^{\mathbf{e}}_\ell$ is obtained from R_ℓ by removing the term which contains the factor $a_{i-1} \triangleright Q_i$.

For instance, for $\ell = 2$, we have $\mathcal{E}_1 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and therefore

$$\begin{aligned} R^{(1,1)}_2(a, b) &= R^{(1)}_1(a) \triangleright R^{(1)}_1(b) + R^{(1)}_1(a) \otimes (b \triangleright R_0(1)) \\ &= a \triangleright b + a \otimes b \\ R^{(1,2)}_2(a, b) &= R^{(1)}_1(a) \triangleright R^{(2)}_1(b) + R^{(1)}_1(a) \otimes (b \triangleright R_0(1)) \\ &= a \otimes b \\ R^{(2,1)}_2(a, b) &= R^{(2)}_1(a) \triangleright R^{(1)}_1(b) + R^{(2)}_1(a) \otimes (b \triangleright R_0(1)) = 0 \\ R^{(2,2)}_2(a, b) &= R^{(2)}_1(a) \triangleright R^{(2)}_1(b) + R^{(2)}_1(a) \otimes (b \triangleright R_0(1)) = 0. \end{aligned}$$

For $\ell = 3$, the set $\mathcal{E}_2$ contains 8 sequences, which give

$$\begin{aligned} R_3^{(1,1,1)}(a,b,c) &= a \rhd R_2^{(1,1)}(b,c) + \big(a \rhd R_1^{(1)}(b)\big) \otimes c + a \otimes \big(b \rhd R_1^{(1)}(c)\big) + a \otimes b \otimes c \\ &= a \rhd (b \rhd c) + a \rhd (b \otimes c) + (a \rhd b) \otimes c + a \otimes (b \rhd c) + a \otimes b \otimes c \end{aligned}$$

$$\begin{aligned} R_3^{(1,1,2)}(a,b,c) &= a \rhd R_2^{(1,2)}(b,c) + \big(a \rhd R_1^{(1)}(b)\big) \otimes c + a \otimes \big(b \rhd R_1^{(2)}(c)\big) + a \otimes b \otimes c \\ &= a \rhd (b \otimes c) + (a \rhd b) \otimes c + a \otimes b \otimes c \end{aligned}$$

$$\begin{aligned} R_3^{(1,2,1)}(a,b,c) &= a \rhd R_2^{(2,1)}(b,c) + \big(a \rhd R_1^{(2)}(b)\big) \otimes c + a \otimes \big(b \rhd R_1^{(1)}(c)\big) + a \otimes b \otimes c \\ &= a \otimes (b \rhd c) + a \otimes b \otimes c \end{aligned}$$

$$\begin{aligned} R_3^{(1,2,2)}(a,b,c) &= a \rhd R_2^{(2,2)}(b,c) + \big(a \rhd R_1^{(2)}(b)\big) \otimes c + a \otimes \big(b \rhd R_1^{(2)}(c)\big) + a \otimes b \otimes c \\ &= a \otimes b \otimes c \end{aligned}$$

and finally $R_3^{(2,1,1)} = R_3^{(2,1,2)} = R_3^{(2,2,1)} = R_3^{(2,2,2)} = 0$.

The labeled right operations can also be given by a closed formula.

Lemma 5.3.13 *For any $\ell \geqslant 2$, for any sequence $\mathbf{e} \in \mathcal{E}_\ell$ and for any $a_1, ..., a_\ell \in A$, we have*

$$R_\ell^{\mathbf{e}}(a_1, ..., a_\ell) = \sum_{\mathbf{m} \in \mathcal{M}_\ell^{\mathbf{e}}} R_{\mathbf{m}}^{\ell}(a_1, ..., a_\ell).$$

As a consequence, if for $a_1, ..., a_{\ell+1} \in A$ we denote $n_i = |a_i|$ for $i = 1, ..., \ell + 1$, we then have

$$a_1 \rhd R_\ell^{\mathbf{e}}(a_2, ..., a_{\ell+1}) = d_\ell^{\mathbf{e}}(n_1, ..., n_\ell)\ a_1 \cdots a_{\ell+1} \in A,$$

where the labeled Lagrange coefficients $d_\ell^{\mathbf{e}}(n_1, ..., n_\ell)$ are given in Def. 5.2.3.

Proof. If $\mathbf{e} = (1, 1, ..., 1)$, and if $\mathbf{e}$ starts by 2, there is nothing to prove. Otherwise, for any value $e_i = 2$ in $\mathbf{e}$, we obtain $R_\ell^{\mathbf{e}}(a_1, ..., a_\ell)$ from $R_\ell(a_1, ..., a_\ell)$ by removing the term containing the factor $a_{i-1} \rhd Q_i$. By Lemma 5.3.9, such a term is associated to a sequence $\mathbf{m} \in \mathcal{M}_\ell$, and by Def. 5.3.7 the factor $a_{i-1} \rhd Q_i$ corresponds to a non-zero value m_i. Therefore, in order to remove such terms, it suffices to consider sequences $\mathbf{m}$ with $m_i = 0$ whenever $e_i = 2$. □

Theorem 5.3.14 *The co-operations of the Faà di Bruno coloop bialgebra can be equivalently defined in terms of the recursive operators as follows:*

$$\begin{aligned} \Delta_{\mathrm{FdB}}^{\amalg}(x_n) &= x_n + y_n + \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} x_{n_1} \rhd \big(y_{n_2} \otimes \cdots \otimes y_{n_{\ell+1}}\big), \\ \delta_r(x_n) &= u_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} L_\ell(u_{n_1}, y_{n_2}, ..., y_{n_\ell}) \rhd y_{n_{\ell+1}} \\ &= u_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} u_{n_1} \rhd R_\ell(y_{n_2}, ..., y_{n_{\ell+1}}), \\ \delta_l(x_n) &= v_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \sum_{\mathbf{e} \in \mathcal{E}_\ell} (-1)^{\mathbf{e}}\ x_{n_1}^{(e_1)} \rhd R_\ell^{\mathbf{e}}(x_{n_2}^{(e_2)}, ..., x_{n_\ell}^{(e_\ell)}, v_{n_{\ell+1}}) \end{aligned} \tag{5.3.15}$$

where we recall that $u_n = x_n - y_n$, $v_n = y_n - x_n$, and also that $(-1)^{\mathbf{e}} = (-1)^{e_1 + \cdots + e_\ell - \ell}$ and that the bit value in $x_n^{(e)}$ tells us in which copy of $H_{\mathrm{dif}}^{\amalg} \amalg H_{\mathrm{dif}}^{\amalg}$ falls the generator x_n, cf. Def. 5.2.6.

Proof. It follows from the definition of $\triangleright$ given in Def. 5.3.1, the expression of R_ℓ given in Cor. 5.3.10, and that of $R_\ell^{\mathbf{e}}$ given in Lemma 5.3.13. The equivalence of the presentations of the right codivision in terms of R_ℓ and L_ℓ is proved in Cor. 5.4.3 in next section. □

Note that in the term $x_{n_1}^{(e_1)} \triangleright R_\ell^{\mathbf{e}}(x_{n_2}^{(e_2)}, ..., x_{n_\ell}^{(e_\ell)}, v_{n_{\ell+1}})$ of the left codivision (5.3.15), the labeled operator $R_\ell^{\mathbf{e}} = R_\ell^{(e_1,...,e_\ell)}$ is applied to variables which are also labeled, but only by the last $\ell - 1$ bits of $\mathbf{e}$. For instance, no labels affect the variables in

$$\begin{aligned}\delta_l(x_2) &= v_2 - x_1^{(1)} \triangleright R_1^{(1)}(v_1) + x_1^{(2)} \triangleright R_1^{(2)}(v_1)\\ &= v_2 - x_1 \triangleright v_1 + y_1 \triangleright 0\\ &= v_2 - 2x_1v_1\end{aligned}$$

but labels do affect the variables in

$$\begin{aligned}\delta_l(x_3) &= v_3 - \big(x_1^{(1)} \triangleright R_1^{(1)}(v_2) + x_2^{(1)} \triangleright R_1^{(1)}(v_1)\big) - \big(-x_1^{(2)} \triangleright R_1^{(2)}(v_2) - x_2^{(1)} \triangleright R_1^{(2)}(v_1)\big)\\ &\quad + x_1^{(1)} \triangleright R_2^{(1,1)}(x_1^{(1)}, v_1) - x_1^{(1)} \triangleright R_2^{(1,2)}(x_1^{(2)}, v_1) - x_1^{(2)} \triangleright R_2^{(2,1)}(x_1^{(1)}, v_1)\\ &\quad + x_1^{(2)} \triangleright R_2^{(2,2)}(x_1^{(2)}, v_1)\\ &= v_3 - (x_1 \triangleright v_2 + x_2 \triangleright v_1) + x_1 \triangleright (x_1 \triangleright v_1) + x_1 \triangleright (x_1 \otimes v_1) - x_1 \triangleright (y_1 \otimes v_1)\\ &= v_3 - \left(\binom{2}{1}x_1v_2 + \binom{3}{1}x_2v_1\right) + \left(\binom{2}{1}\binom{2}{1} + \binom{2}{2}\right)x_1^2v_1 - \binom{2}{2}x_1y_1v_1\\ &= v_3 - (2x_1v_2 + 3x_2v_1) + 5x_1^2v_1 - x_1y_1v_1.\end{aligned}$$

5.4 Functoriality of the diffeomorphisms loop

To prove the main theorem of this section we need some preliminary recurrence relations for the recursive operators, and consequently for the Lagrange coefficients.

Corollary 5.4.1 *For any $\ell \geqslant 1$ and any sequence $(n_1, ..., n_{\ell+1})$ of positive integers, the coefficients $d_{\ell+1}(a_1, ..., a_\ell)$ satisfy the following recursive equation:*

$$\begin{aligned}d_\ell(n_1, ..., n_\ell) = \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_\ell^j} \binom{n_1+1}{j}\, d_{p_1-1}(n_2, ..., n_{p_1})\; d_{p_2-1}(n_{p_1+2}, ..., n_{p_1+p_2}) \cdots\\ \cdots d_{p_j-1}(n_{p_1+\cdots+p_{j-1}+2}, ..., n_\ell).\end{aligned}$$

Proof. Applying $a_1 \triangleright (\ )$ to the recursive expression (5.3.6) of $R_\ell(a_2, a_3, ..., a_{\ell+1})$, and using Cor. 5.3.10, immediately gives the result. □

Lemma 5.4.2 *For any $\ell \geqslant 0$ and any $a_1, ..., a_{\ell+1} \in A$, the following recursive equation holds:*

$$a_1 \triangleright R_\ell(a_2, ..., a_{\ell+1}) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \big(a_1 \triangleright R_i(a_2, ..., a_{i+1})\big) \triangleright (a_{i+2} \otimes \cdots \otimes a_{\ell+1}).$$

Modulo the factor $a_1a_2 \cdots a_{\ell+1}$, this means that

$$d_\ell(n_1, ..., n_\ell) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \binom{n_1 + \cdots + n_{i+1} + 1}{\ell - i}\, d_i(n_1, ..., n_i).$$

Proof. The two assertions are equivalent, and the second one appears as a recursion for the coefficients in the non-commutative Lagrange inversion formula. It is essentially based on the Chu-Vandermonde identity and can be proved[5] using the hypergeometric function ${}_2F_1$ or using some trick as in [8], Lemma 2.15. □

Corollary 5.4.3 *For any $\ell \geqslant 0$ and any $a_1, ...a_{\ell+1} \in A$ we have*

$$a_1 \rhd R_\ell(a_2, ..., a_{\ell+1}) = L_\ell(a_1, ..., a_\ell) \rhd a_{\ell+1}$$

Proof. By induction on ℓ. For $\ell = 0, 1$ the identity is easily verified, because

$$\begin{aligned} a_1 \rhd 1 = a_1 = 1 \rhd a_1 \\ a_1 \rhd R_1(a_2) = a_1 \rhd a_2 = L_1(a_1) \rhd a_2. \end{aligned}$$

Now suppose that for $i = 1, ..., \ell - 1$ we have $a_1 \rhd R_i(a_2, ..., a_{i+1}) = L_i(a_1, ..., a_i) \rhd a_{i+1}$. Then by Lemma 5.4.2 and Def. 5.3.3 we have

$$\begin{aligned} a_1 \rhd R_\ell(a_2, ..., a_{\ell+1}) &= \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \big(a_1 \rhd R_i(a_2, ..., a_{i+1})\big) \rhd (a_{i+2} \otimes \cdots \otimes a_{\ell+1}) \\ &= \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \big(L_i(a_1, ..., a_i) \rhd a_{i+1}\big) \rhd (a_{i+2} \otimes \cdots \otimes a_{\ell+1}) \\ &= \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \Big(\big(L_i(a_1, ..., a_i) \rhd a_{i+1}\big) \otimes a_{i+2} \otimes \cdots \otimes a_\ell\Big) \rhd a_{\ell+1} \\ &= L_\ell(a_1, ..., a_\ell) \rhd a_{\ell+1}. \end{aligned}$$

□

Remark 5.4.4 In the case $\mathbf{e} \neq (1, 1, ..., 1)$, whether there exists an operator $L_\ell^{\mathbf{e}} : A^{\otimes \ell} \longrightarrow T(A)$ such that

$$a_1 \rhd R_\ell^{\mathbf{e}}(a_2, ..., a_{\ell+1}) = L_\ell^{\mathbf{e}}(a_1, ..., a_\ell) \rhd a_{\ell+1}$$

is an open question.

Lemma 5.4.5 *For any $\ell \geqslant 1$ and any $a_1, ..., a_{\ell+1} \in A$, the following recursive equation holds:*

$$a_1 \rhd R_\ell(a_2, ..., a_{\ell+1}) = \sum_{i=1}^{\ell} (-1)^{i-1} \big(a_1 \rhd (a_2 \otimes \cdots \otimes a_{i+1})\big) \rhd R_{\ell-i}(a_{i+2} \otimes \cdots \otimes a_{\ell+1}).$$

Modulo the factor $a_1 a_2 \cdots a_{\ell+1}$, and if we call $n_i = |a_i|$ for $i = 1, ..., \ell + 1$, this means that

$$d_\ell(n_1, ..., n_\ell) = \sum_{i=1}^{\ell} (-1)^{i-1} \binom{n_1 + 1}{i} \, d_{\ell-i}(n_1 + \cdots + n_{i+1}, n_{i+2}, ..., n_\ell). \tag{5.4.6}$$

[5] We warmly thank Jiang Zeng for pointing out this method to us.

Proof. The two assertions are equivalent. Let us prove the second one by induction on ℓ. Let us call $\tilde{d}_\ell(n_1, ..., n_\ell)$ the right-hand side of equation (5.4.6). For $\ell = 1$, the sum in $\tilde{d}_1(n)$ has only one term for $i = 1$, which gives

$$\tilde{d}_1(n) = (-1)^{1-1}\binom{n+1}{1}\, d_0 = d_1(n).$$

Now suppose that eq. (5.4.6) holds for any $1 \leqslant k \leqslant \ell - 1$, that is, we have

$$d_k(n_1, ..., n_k) = \sum_{i=1}^{k}(-1)^{i-1}\binom{n_1+1}{i}\, d_{k-i}(n_1+\cdots+n_{i+1}, n_{i+2}, ..., n_k),$$

and prove it for ℓ. For this, we write $d_\ell(n_1, ..., n_\ell)$ using the recursion given in Lemma 5.4.2 as a sum over $0 \leqslant k \leqslant \ell - 1$, and separate the term $k = 0$ to which we can not apply the inductive hypothesis. Then we expand the factor $d_k(n_1, ..., n_k)$ using the inductive hypothesis and exchange the sums over k and i. We finally obtain

$$\begin{aligned} d_\ell(n_1, ..., n_\ell) &= \sum_{k=1}^{\ell-1}(-1)^{\ell-1-k}\binom{n_1+\cdots+n_{k+1}+1}{\ell-k}\, d_k(n_1, ..., n_k) + (-1)^{\ell-1}\binom{n_1+1}{\ell} \\ &= \sum_{i=1}^{\ell-1}(-1)^{i-1}\binom{n_1+1}{i} \sum_{k=i}^{\ell-1}(-1)^{\ell-1-k}\binom{n_1+\cdots+n_{k+1}+1}{\ell-k}\, d_{k-i}(n_1+\cdots+n_{i+1}, n_{i+2}, ..., n_k) \\ &\qquad + (-1)^{\ell-1}\binom{n_1+1}{\ell}. \end{aligned}$$

Then, $d_\ell(n_1, ..., n_\ell)$ is equal to

$$\begin{aligned} \tilde{d}_\ell(n_1, ..., n_\ell) = \sum_{i=1}^{\ell-1}(-1)^{i-1}\binom{n_1+1}{i}\, d_{\ell-i}(n_1+\cdots+n_{i+1}, n_{i+2}, ..., n_\ell) \\ + (-1)^{\ell-1}\binom{n_1+1}{\ell}, \end{aligned}$$

if and only if, for any $1 \leqslant i \leqslant \ell - 1$, we have

$$\begin{aligned} \sum_{k=i}^{\ell-1}(-1)^{\ell-1-k}\binom{n_1+\cdots+n_{k+1}+1}{\ell-k}\, d_{k-i}(n_1+\cdots+n_{i+1}, n_{i+2}, ..., n_k) \\ = d_{\ell-i}(n_1+\cdots+n_{i+1}, n_{i+2}, ..., n_\ell). \end{aligned}$$

This identity is easily verifyed by setting $j = k - i$, $p_1 = n_1 + \cdots + n_{i+1}$ and $p_j = n_{i+j}$ for $2 \leqslant j \leqslant \ell - 1 - i$, since it gives

$$\sum_{j=0}^{\ell-i-1}(-1)^{\ell-i-1-j}\binom{p_1+\cdots+p_{j+1}+1}{\ell-i-j}\, d_j(p_1, p_2, ..., p_j) = d_{\ell-i}(p_1, p_2, ..., p_{\ell-i})$$

which holds again by Lemma 5.4.2. □

Corollary 5.4.7 *For any $\ell \geqslant 1$ and any $a_1, ..., a_\ell \in A$, the following recursive equation holds:*

$$L_\ell(a_1, ..., a_\ell) = \sum_{i=1}^{\ell-1}(-1)^{i-1} L_{\ell-i}\big(a_1 \triangleright (a_2 \otimes \cdots \otimes a_{i+1}), a_{i+2}, ..., a_\ell\big) + (-1)^{\ell-1} a_1 \otimes \cdots \otimes a_\ell.$$

Proof. It suffices to write

$$a_1 \rhd R_\ell(a_2,...,a_{\ell+1}) = \sum_{i=1}^{\ell-1}(-1)^{i-1}\big(a_1 \rhd (a_2 \otimes \cdots \otimes a_{i+1})\big) \rhd R_{\ell-i}(a_{i+2} \otimes \cdots \otimes a_{\ell+1}) + (-1)^\ell a_1 \rhd (a_2 \otimes \cdots \otimes a_{\ell+1})$$

after Lemma 5.4.5, and to apply the equality $b_1 \rhd R_j(b_2,...,b_{j+1}) = L_j(b_1,...,b_j) \rhd b_{j+1}$ everywhere. □

Lemma 5.4.8 *For any $\ell \geqslant 2$, any $\mathbf{e} \in \mathcal{E}_\ell$ and any $a_1,...,a_\ell \in A$, we have*

$$R^{\mathbf{e}}_\ell(a_1,...,a_\ell) = R_1^{(e_1)}(a_1) \rhd R_{\ell-1}^{(e_2,...,e_\ell)}(a_2,...,a_\ell) + \sum_{i=1}^{\ell-1} R_i^{(e_1,...,e_i)}(a_1,...,a_i) \otimes \Big(a_{i+1} \rhd R_{\ell-i-1}^{(e_{i+2},...,e_\ell)}(a_{i+2},...,a_\ell)\Big).$$

Proof. The term $j = 1$ in the defining recursion (5.3.12) gives exactly

$$R_1^{(e_1)}(a_1) \rhd R_{\ell-1}^{(e_2,...,e_{\ell-1})}(a_2,...,a_\ell),$$

se it remains to prove that

$$\begin{aligned}\sum_{j=2}^{\ell} \sum_{\mathbf{p}\in\mathcal{C}_\ell^j} &\big(R_1^{(e_1)}(a_1) \rhd R_{p_1-1}^{(e_2,...,e_{p_1})}(a_2,...,a_{p_1})\big) \\ &\otimes \big(a_{p_1+1} \rhd R_{p_2-1}^{(e_{p_1+2},...,e_{p_1+p_2})}(a_{p_1+2},...,a_{p_1+p_2})\big) \otimes \cdots \\ &\cdots \otimes \big(a_{p_1+\cdots+p_{j-1}+1} \rhd R_{p_j-1}^{(e_{p_1+\cdots+p_{j-1}+2},...,e_\ell)}(a_{p_1+\cdots+p_{j-1}+2},...,a_\ell)\big) = \\ &= \sum_{i=1}^{\ell-1} R_i^{(e_1,...,e_i)}(a_1,...,a_i) \otimes \Big(a_{i+1} \rhd R_{\ell-i-1}^{(e_{i+2},...,e_\ell)}(a_{i+2},...,a_\ell)\Big).\end{aligned} \tag{5.4.9}$$

Let us prove this identity by induction. For $\ell = 2$ and 3, it is easy to verify on the above examples that

$$R_2^{\mathbf{e}}(a,b) = R_1^{(e_1)}(a) \rhd R_1^{(e_2)}(b) + R_1^{(e_1)}(a) \otimes \big(b \rhd R_0(1)\big),$$

$$R_3^{\mathbf{e}}(a,b,c) = R_1^{(e_1)}(a) \rhd R_2^{(e_2,e_3)}(b,c) + R_1^{(e_1)}(a) \otimes \big(b \rhd R_1^{(e_3)}(c)\big) + R_2^{(e_1,e_2)}(a,b) \otimes (c \rhd R_0(1)).$$

Now suppose it holds up to order $\ell - 1$, and let us prove it at order ℓ.

Consider the left-hand side of eq. (5.4.9). Since $j \geqslant 2$, we can write

$$\mathcal{C}_\ell^j = \bigcup_{i=j-1}^{\ell-1} \mathcal{C}_i^{j-1} \times \mathcal{C}_{\ell-i}^1$$

and decompose $\mathbf{p} \in \mathcal{C}_\ell^j$ into $(p_1,...,p_{j-1}) \in \mathcal{C}_i^{j-1}$ and $(p_j) \in \mathcal{C}_{\ell-i}^1$ for any value $i = j-1,...,\ell-1$. We then have $p_1 + \cdots + p_{j-1} = i$ and $p_j = \ell - (p_1 + \cdots + p_{j-1}) = \ell - i$. Therefore the left-hand

side can be written as

$$\sum_{j=2}^{\ell}\sum_{i=j-1}^{\ell-1}\sum_{(p_1,\dots,p_{j-1})\in\mathcal{C}_i^{j-1}}\delta_{e_1,1}\,\big(a_1\rhd R_{p_1-1}^{(e_2,\dots,e_{p_1})}(a_2,\dots,a_{p_1})\big)\otimes\cdots$$
$$\cdots\otimes\big(a_{P_{j-2}+1}\rhd R_{p_{j-1}-1}^{(e_{p_{j-2}+2},\dots,e_{p_{j-1}})}(a_{P_{j-2}+2},\dots,a_i)\big)\otimes$$
$$\otimes\big(a_{i+1}\rhd R_{\ell-i-1}^{(e_{i+2},\dots,e_\ell)}(a_{i+2},\dots,a_\ell)\big)=$$

$$=\sum_{i=1}^{\ell-1}\left(\sum_{k=1}^{i}\sum_{(p_1,\dots,p_k)\in\mathcal{C}_i^k}\delta_{e_1,1}\,\big(a_1\rhd R_{p_1-1}^{(e_2,\dots,e_{p_1-1})}(a_2,\dots,a_{p_1})\big)\otimes\cdots\right.$$
$$\left.\cdots\otimes\big(a_{P_{k-1}+1}\rhd R_{p_k-1}^{(e_{P_{k-1}+2},\dots,e_{P_k})}(a_{P_{k-1}+2},\dots,a_i)\big)\right)\otimes$$
$$\otimes\big(a_{i+1}\rhd R_{\ell-i-1}^{(e_{i+2},\dots,e_\ell)}(a_{i+2},\dots,a_\ell)\big),$$

where $P_k=p_1+\cdots+p_k$. Applying the inductive hypothesis to the sum over k leads to the result. □

Theorem 5.4.10 *The associative algebra $H_{\mathrm{FdB}}^{\amalg}$ is indeed a coloop bialgebra and represents the loop of formal diffeomorphisms* Diff *as a functor* Diff : **As** $\longrightarrow$ **Loop**.

As a consequence, given an associative algebra A, a series $a=\sum_{n\geqslant 0}a_n\,\lambda^{n+1}\in\mathrm{Diff}(A)$ can be seen as an algebra homomorphism $a: H_{\mathrm{FdB}}^{\amalg}\longrightarrow A$ defined on the generators of $H_{\mathrm{FdB}}^{\amalg}$ by $a(x_n)=a_n$, and the right and left division a/b and $a\backslash b$ are given at any order n by the following closed formulas:

$$\begin{aligned}(a/b)_n&=\mu_A\,(a\amalg b)\,\delta_r(x_n)\\&=a_n-b_n+\sum_{\ell=1}^{n-1}(-1)^\ell\sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}}d_\ell(n_1,\dots,n_\ell)\;(a_{n_1}-b_{n_1})\,b_{n_2}\cdots b_{n_{\ell+1}},\\(a\backslash b)_n&=\mu_A\,(a\amalg b)\,\delta_l(x_n)\\&=b_n-a_n+\sum_{\ell=2}^{n-1}(-1)^\ell\sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}}\sum_{\mathbf{e}\in\mathcal{E}_\ell}(-1)^{\mathbf{e}}\;d_\ell^{\mathbf{e}}(n_1,\dots,n_\ell)\;a_{n_1}c_{n_2}^{(e_1)}\cdots c_{n_\ell}^{(e_{\ell-1})}\,(b_{n_{\ell+1}}-a_{n_{\ell+1}}),\end{aligned}$$

where $c_{n_i}^{(e_i)}=a_{n_i}$ if $e_i=1$ and $c_{n_i}^{(e_i)}=b_{n_i}$ if $e_i=2$.

Proof. The free associative algebra $H_{\mathrm{FdB}}^{\amalg}$ clearly represents the sets $\mathrm{Diff}(A)$ over associative algebras A, and the comultiplication $\Delta_{\mathrm{FdB}}^{\amalg}$ is just the Faà di Bruno comultiplication Δ_{FdB} seen with values in $H_{\mathrm{FdB}}\amalg H_{\mathrm{FdB}}$ instead of $H_{\mathrm{FdB}}\otimes H_{\mathrm{FdB}}$, therefore it clearly represents the loop law given in Definition 5.1.1. Thus, the theorem is proved if we show that $H_{\mathrm{FdB}}^{\amalg}$ is indeed a coloop bialgebra.

The comultiplication $\Delta_{\mathrm{FdB}}^{\amalg}$ satisfies the compatibility relation with the standard counit, because Δ_{FdB} does, and coassociativity is not required. So it remains to check that the codivisions δ_r and δ_l given in Def. 5.1.1 satisfy the identities (2.2.3) and (2.2.4). Since these maps are algebra morphisms, it suffices to verify these identities on the generators x_n, for any $n\geqslant 1$.

i) Let us start with the right codivision and show that it satisfies the first identity (2.2.3), namely

$$(\mathrm{id}\amalg\mu)\,(\delta_r\amalg\mathrm{id})\,\Delta_{\mathrm{FdB}}^{\amalg}(x_n)=x_n,$$

which explicitely gives the recurrence (with $u_n = x_n - y_n$)

$$\delta_r(x_n) = u_n - \sum_{m=1}^{n-1} \sum_{\ell=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^{\ell}} \delta_r(x_m) \rhd \big(y_{k_1} \otimes \cdots \otimes y_{k_\ell}\big). \tag{5.4.11}$$

Expanding $\delta_r(x_n)$ in terms of the left recursive opeators, this equation becomes

$$\begin{aligned}
&u_n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} L_\ell(u_{n_1}, y_{n_2}, ..., y_{n_\ell}) \rhd y_{n_{\ell+1}} \\
&\quad = u_n - \sum_{m=1}^{n-1} \sum_{j=1}^{n-m} \sum_{\mathbf{q} \in \mathcal{C}_{n-m}^{j}} \sum_{i=0}^{m-1} (-1)^{i} \sum_{\mathbf{p} \in \mathcal{C}_m^{i+1}} \Big(L_i(u_{p_1}, y_{p_2}, ..., y_{p_i}) \rhd y_{p_{i+1}} \Big) \rhd \big(y_{q_1} \otimes \cdots \otimes y_{q_j}\big) \\
&\quad = u_n + \sum_{\ell=i+j=1}^{n-1} \sum_{i=0}^{\ell-1} (-1)^{i+1} \sum_{m=i+1}^{n-1} \sum_{\substack{\mathbf{p} \in \mathcal{C}_m^{i+1} \\ \mathbf{q} \in \mathcal{C}_{n-m}^{\ell-i}}} \Big(L_i(u_{p_1}, y_{p_2}, ..., y_{p_i}) \rhd y_{p_{i+1}} \Big) \rhd \big(y_{q_1} \otimes \cdots \otimes y_{q_{\ell-i}}\big).
\end{aligned}$$

Now, since

$$\bigcup_{m=i+1}^{n-1} \mathcal{C}_m^{i+1} \times \mathcal{C}_{n-m}^{\ell-i} \cong \mathcal{C}_n^{\ell+1},$$

let us call $\mathbf{n} = (\mathbf{p}, \mathbf{q})$, that is,

$$(n_1, n_2, ..., n_{\ell+1}) = (p_1, ..., p_{i+1}, q_1, ..., q_j).$$

Then, the recursion (5.4.11) is equivalent, for any $n \geqslant 1$, any $1 \leqslant \ell \leqslant n-1$ and any $\mathbf{n} \in \mathcal{C}_n^{\ell+1}$, to the equation

$$\begin{aligned}
&L_\ell(u_{n_1}, y_{n_2}, ..., y_{n_\ell}) \rhd y_{n_{\ell+1}} \\
&\qquad = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \big(L_i(u_{n_1}, y_{n_2}, ..., y_{n_i}) \rhd y_{n_{i+1}} \big) \rhd (y_{n_{i+2}} \otimes \cdots \otimes y_{n_{\ell+1}}) \\
&\qquad = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \Big(\big(L_i(u_{n_1}, y_{n_2}, ..., y_{n_i}) \rhd y_{n_{i+1}} \big) \otimes y_{n_{i+2}} \otimes \cdots \otimes y_{n_\ell} \Big) \rhd y_{n_{\ell+1}},
\end{aligned}$$

which holds by definition of L_ℓ.

The second identity (2.2.3), namely

$$(\mathrm{id} \amalg \mu)\, (\Delta_{\mathrm{FdB}}^{\amalg} \amalg \mathrm{id})\, \delta_r(x_n) = x_n,$$

is better developed using the expansion over the right recursive operators, and explicitely gives the recurrence

$$\begin{aligned}
&\sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (-1)^{\ell}\ x_{n_1} \rhd R_\ell(y_{n_2}, ..., y_{n_{\ell+1}}) \qquad (5.4.12) \\
&\qquad = \sum_{m=1}^{n-1} \sum_{p=1}^{n-m} \sum_{i=1}^{p} \sum_{\mathbf{p} \in \mathcal{C}_p^i} \sum_{j=0}^{n-m-p} \sum_{\mathbf{q} \in \mathcal{C}_{n-m-p}^{j}} (-1)^{j+1} \big(x_m \rhd (y_{p_1} \otimes \cdots \otimes y_{p_i}) \big) \rhd R_j(y_{q_1}, ..., y_{q_j}).
\end{aligned}$$

Rewriting the sums in terms of $m = 1, ..., n-1$, $\ell = i + j = 1, ..., n-m$, $i = 1, ..., \ell$ and $j = \ell - i$, this gives a sum over $\mathbf{p} \in \mathcal{C}_p^i$ and $\mathbf{q} \in \mathcal{C}_{n-m-p}^{\ell-i}$ for $p = i, ..., n-m$. That is, we get a sum over $\mathbf{k} = (\mathbf{p}, \mathbf{q}) \in \mathcal{C}_{n-m}^{\ell}$ and consequently a sum over $\mathbf{n} = (m, \mathbf{k}) \in \mathcal{C}_n^{\ell+1}$:

$$\begin{aligned}
&\sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (-1)^{\ell}\, x_{n_1} \triangleright R_\ell(y_{n_2}, ..., y_{n_{\ell+1}}) \\
&\quad = \sum_{m=1}^{n-1} \sum_{\ell=1}^{n-m} \sum_{i=1}^{\ell} \sum_{p=i}^{n-m} \sum_{\mathbf{p} \in \mathcal{C}_p^i} \sum_{\mathbf{q} \in \mathcal{C}_{n-m-p}^{\ell-i}} (-1)^{\ell-i+1} \big(x_m \triangleright (y_{p_1} \otimes \cdots \otimes y_{p_i})\big) \triangleright R_{\ell-i}(y_{q_1}, ..., y_{q_{\ell-i}}) \\
&\quad = \sum_{m=1}^{n-1} \sum_{\ell=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^{\ell}} \sum_{i=1}^{\ell} (-1)^{\ell-i+1} \big(x_m \triangleright (y_{k_1} \otimes \cdots \otimes y_{k_i})\big) \triangleright R_{\ell-i}(y_{k_{i+1}}, ..., y_{k_\ell}) \\
&\quad = \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (-1)^{\ell} \sum_{i=1}^{\ell} (-1)^{i-1} \big(x_{n_1} \triangleright (y_{n_2} \otimes \cdots \otimes y_{n_{i+1}})\big) \triangleright R_{\ell-i}(y_{n_{i+2}}, ..., y_{n_{\ell+1}}).
\end{aligned}$$

Therefore, for any $n \geqslant 2$, any $\ell = 1, ..., n-1$ and any sequence $\mathbf{n} \in \mathcal{C}_n^{\ell+1}$, eq. (5.4.12) is equivalent to the recurrence equation

$$x_{n_1} \triangleright R_\ell(y_{n_2}, ..., y_{n_{\ell+1}}) = \sum_{i=1}^{\ell} (-1)^{i-1} \big(x_{n_1} \triangleright (y_{n_2} \otimes \cdots \otimes y_{n_{i+1}})\big) \triangleright R_{\ell-i}(y_{n_{i+2}}, ..., y_{n_{\ell+1}}),$$

which is proved in Lemma 5.4.5.

ii) Let us show now that the left codivision given in Def. (5.2.6) satisfies the identities (2.2.4). The first identity (2.2.4), namely

$$(\mu \amalg \mathrm{id})\, (\mathrm{id} \amalg \delta_l)\, \Delta_{\mathrm{FdB}}^{\amalg}(x_n) = y_n,$$

explicitely gives the recurrence (with $v_n = y_n - x_n$)

$$\delta_l(x_n) = v_n - \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^{\lambda}} x_m \triangleright \big(\delta_l(y_{k_1}) \otimes \cdots \otimes \delta_l(y_{k_\lambda})\big), \tag{5.4.13}$$

where y_k is just the kth generator x_k in the second copy of the free product algebra $H_{\mathrm{FdB}}^{\amalg} \amalg H_{\mathrm{FdB}}^{\amalg}$, therefore the formula for $\delta_l(y_k)$ is just the same as for $\delta_l(x_k)$.

To show this, we consider the expansion (5.3.15) of $\delta_l(x_n)$ given in Thm. 5.3.14. Since $R_\ell^{\mathbf{e}} = 0$ when $e_1 = 2$, we have $x_{n_1}^{(e_1)} = x_{n_1}$ and we can rewrite (5.3.15) as

$$\delta_l(x_n) = v_n + \sum_{m=1}^{n-1} x_m \triangleright \sum_{\ell=1}^{n-m} (-1)^{\ell} \sum_{\substack{\mathbf{n} \in \mathcal{C}_{n-m}^{\ell} \\ \mathbf{e} \in \mathcal{E}_\ell}} (-1)^{\mathbf{e}} R_\ell^{\mathbf{e}}(x_{n_1}^{(e_2)}, ..., x_{n_{\ell-1}}^{(e_\ell)}, v_{n_\ell}).$$

Then eq. (5.4.13) is clearly verified for $n = 1$, because $\delta_l(x_1) = v_1$, and for any $n \geqslant 2$ and any $m = 1, ..., n-1$, it is equivalent to the equation

$$\begin{aligned}
&\sum_{\lambda=1}^{\mu} \sum_{\mathbf{k} \in \mathcal{C}_\mu^\lambda} \delta_l(x_{k_1}) \otimes \cdots \otimes \delta_l(x_{k_\lambda}) = \qquad (5.4.14) \\
&\qquad = v_\mu - \sum_{\ell=2}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_\mu^\ell} \sum_{\mathbf{e} \in \mathcal{E}_\ell} (-1)^{\mathbf{e}}\; R_\ell^{\mathbf{e}}(x_{n_1}^{(e_2)}, ..., x_{n_{\ell-1}}^{(e_\ell)}, v_{n_\ell}),
\end{aligned}$$

for any $\mu = n - m = 1, ..., n-1$.

Let us prove this equation by induction on μ. For $\mu = 1$ we again have $\delta_l(x_1) = v_1$. So, suppose that eq. (5.4.14) holds up to order $\mu - 1$ and prove it at order μ.

On the left-hand side of eq. (5.4.14), we separate the term $\lambda = 1$ and observe that, for $\lambda \geqslant 2$, we can decompose $\mathbf{k} = (k_1, ..., k_{\lambda-1}, k_\lambda) \in \mathcal{C}^\lambda_\mu$ into $(\mathbf{q}, \nu) \in \mathcal{C}^{\lambda-1}_{\mu-\nu} \times \mathcal{C}^1_\nu$ with

$$\begin{aligned} q_i &= k_i \qquad \text{for } i = 1, ..., \lambda - 1 \\ \nu &= k_\lambda. \end{aligned}$$

Since

$$\mathcal{C}^\lambda_\mu = \bigcup_{\nu=1}^{\mu-\lambda+1} \mathcal{C}^{\lambda-1}_{\mu-\nu} \times \mathcal{C}^1_\nu,$$

the left-hand side of eq. (5.4.14) can then be written as

$$\delta_l(x_\mu) + \sum_{\nu=1}^{\mu-1} \left(\sum_{i=1}^{\mu-\nu} \sum_{\mathbf{q} \in \mathcal{C}^i_{\mu-\nu}} \delta_l(x_{q_1}) \otimes \cdots \otimes \delta_l(x_{q_i}) \right) \otimes \delta_l(x_\nu).$$

We then apply the inductive hypothesis (5.4.14) to the sum over $i = 1, ..., \mu - \nu$, and expand the single factors $\delta_l(x_\mu)$ and $\delta_l(x_\nu)$ as in (5.3.15), thus obtaining

$$\begin{aligned} &\sum_{\lambda=1}^{\mu} \sum_{\mathbf{k} \in \mathcal{C}^\lambda_\mu} \delta_l(x_{k_1}) \otimes \cdots \otimes \delta_l(x_{k_\lambda}) = \\ &\quad = v_\mu + \sum_{\lambda=1}^{\mu-1} (-1)^\lambda \sum_{\mathbf{n} \in \mathcal{C}^{\lambda+1}_\mu} \sum_{\mathbf{e} \in \mathcal{E}_\lambda} (-1)^{\mathbf{e}} \; x^{(e_1)}_{n_1} \rhd R^{\mathbf{e}}_\lambda(x^{(e_2)}_{n_2}, ..., x^{(e_\lambda)}_{n_\lambda}, v_{n_{\lambda+1}}) \qquad (5.4.15) \\ &\qquad + \sum_{\nu=1}^{\mu-1} \left(v_{\mu-\nu} - \sum_{i=2}^{\mu-\nu} (-1)^i \sum_{\mathbf{p} \in \mathcal{C}^i_{\mu-\nu}} \sum_{\mathbf{e}' \in \mathcal{E}_i} (-1)^{\mathbf{e}'} \; R^{\mathbf{e}'}_i(x^{(e'_2)}_{p_1}, ..., x^{(e'_i)}_{p_{i-1}}, v_{p_i}) \right) \otimes \\ &\qquad\qquad \otimes \left(v_\nu + \sum_{j=1}^{\nu-1} (-1)^j \sum_{\mathbf{q} \in \mathcal{C}^{j+1}_\nu} \sum_{\mathbf{e}'' \in \mathcal{E}_j} (-1)^{\mathbf{e}''} \; x^{(e''_1)}_{q_1} \rhd R^{\mathbf{e}''}_j(x^{(e''_2)}_{q_2}, ..., x^{(e''_j)}_{q_j}, v_{q_{j+1}}) \right). \end{aligned}$$

Finally, it remains to prove that the right-hand side of eq. (5.4.14) coincides with the right-hand side of eq. (5.4.15). The first term v_μ appears in both formulas, let us compare the other terms.

The first term in eq. (5.4.15) is

$$A = \sum_{\lambda=1}^{\mu-1} (-1)^\lambda \sum_{\mathbf{n} \in \mathcal{C}^{\lambda+1}_\mu} \sum_{\mathbf{e}'' \in \mathcal{E}_\lambda} (-1)^{\mathbf{e}''} \; x^{(e''_1)}_{n_1} \rhd R^{\mathbf{e}''}_\lambda(x^{(e''_2)}_{n_2}, ..., x^{(e''_\lambda)}_{n_\lambda}, v_{n_{\lambda+1}}).$$

We apply the trick

$$a = R^{(1)}_1(a) = \sum_{(e') \in \mathcal{E}_1} R^{(e')}_1(a) \qquad (5.4.16)$$

to the element $x^{(e''_1)}_{n_1}$, then set $\ell = \lambda + 1$ and $\mathbf{e} = (e', \mathbf{e}'') = (e', e''_1, ..., e''_\lambda)$, and get

$$A = -\sum_{\ell=2}^{\mu} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}^\ell_\mu} \sum_{\mathbf{e} \in \mathcal{E}_\ell} (-1)^{\mathbf{e}} \; \left(R^{(e_1)}_1(x^{(e_2)}_{n_1}) \rhd R^{(e_2, ..., e_\ell)}_{\ell-1}(x^{(e_3)}_{n_2}, ..., x^{(e_\ell)}_{n_{\ell-1}}, v_{n_\ell}) \right).$$

The second term in eq. (5.4.15) is

$$B = \sum_{\nu=1}^{\mu-1} v_{\mu-\nu} \otimes v_\nu .$$

We apply the second trick

$$\begin{aligned} v_n &= -(x_n^{(1)} - x_n^{(2)}) = - \sum_{(e')\in \mathcal{E}_1} (-1)^{e'-1} x_n^{(e')} \\ &= - \sum_{\mathbf{e}=(e_1,e_2)\in \mathcal{E}_2} (-1)^{\mathbf{e}} R_1^{(e_1)}(x_n^{(e_2)}) \end{aligned} \tag{5.4.17}$$

to the element $v_{\mu-\nu}$, and get

$$\begin{aligned} B &= - \sum_{\nu=1}^{\mu-1} \sum_{\mathbf{e}\in \mathcal{E}_2} (-1)^{\mathbf{e}} \; R_1^{(e_1)}(x_n^{(e_2)}) \otimes v_\nu \\ &= - \sum_{\mathbf{n}\in \mathcal{C}_\mu^2} \sum_{\mathbf{e}\in \mathcal{E}_2} (-1)^{\mathbf{e}} \; R_1^{(e_1)}(x_{n_1}^{(e_1)}) \otimes v_{n_2}. \end{aligned}$$

Using again (5.4.17), the third term in eq. (5.4.15) becomes

$$\begin{aligned} C &= \sum_{\nu=1}^{\mu-1} \sum_{j=2}^{\nu-1} (-1)^j \sum_{\mathbf{q}\in \mathcal{C}_\nu^{j+1}} \sum_{\mathbf{e}'''\in \mathcal{E}_j} (-1)^{\mathbf{e}'''} v_{\mu-\nu} \otimes \Big(x_{q_1}^{(e_1''')} \rhd R_j^{\mathbf{e}'''}(x_{q_2}^{(e_2''')}, ..., x_{q_j}^{(e_j''')}, v_{q_{j+1}}) \Big) \\ &= - \sum_{\nu=1}^{\mu-1} \sum_{j=2}^{\nu-1} (-1)^j \sum_{\mathbf{q}\in \mathcal{C}_\nu^{j+1}} \sum_{\substack{\mathbf{e}'\in \mathcal{E}_1 \\ \mathbf{e}''\in \mathcal{E}_1 \\ \mathbf{e}'''\in \mathcal{E}_j}} (-1)^{\mathbf{e}'} (-1)^{\mathbf{e}''} (-1)^{\mathbf{e}'''} \; R_1^{(e')}(x_{\mu-\nu}^{(e'')}) \otimes \Big(x_{q_1}^{(e_1''')} \rhd R_j^{\mathbf{e}'''}(x_{q_2}^{(e_2''')}, ..., x_{q_j}^{(e_j''')}, v_{q_{j+1}}) \Big). \end{aligned}$$

We set $\ell = 1+1+j$, $\mathbf{n} = (\mu-\nu, \mathbf{q}) \in \mathcal{C}_\mu^\ell$ and $\mathbf{e} = (e', e'', \mathbf{e}''') \in \mathcal{E}_\ell$, and obtain

$$C = - \sum_{\ell=3}^{\mu-1} (-1)^\ell \sum_{\mathbf{n}\in \mathcal{C}_\mu^\ell} \sum_{\mathbf{e}\in \mathcal{E}_\ell} (-1)^{\mathbf{e}} \; R_1^{(e_1)}(x_{n_1}^{(e_2)}) \otimes \Big(x_{n_2}^{(e_3)} \rhd R_{\ell-2}^{(e_3,...,e_\ell)}(x_{n_3}^{(e_4)}, ..., x_{n_{\ell-1}}^{(e_\ell)}, v_{n_\ell}) \Big).$$

The fourth term in eq. (5.4.15) is

$$D = - \sum_{\nu=1}^{\mu-1} \sum_{i=2}^{\mu-\nu} (-1)^i \sum_{\mathbf{p}\in \mathcal{C}_{\mu-\nu}^i} \sum_{\mathbf{e}'\in \mathcal{E}_i} (-1)^{\mathbf{e}'} \; R_i^{\mathbf{e}'}(x_{p_1}^{(e_2')}, ..., x_{p_{i-1}}^{(e_i')}, v_{p_i}) \otimes v_\nu .$$

We write $v_{p_i} = -\sum_{\mathbf{e}''\in \mathcal{E}_1} (-1)^{\mathbf{e}''} x_n^{(e'')}$ using (5.4.16), and set $\ell = i+1$, $\mathbf{n} = (\mathbf{p}, \nu) \in \mathcal{C}_\mu^\ell$ and $\mathbf{e} = (\mathbf{e}', \mathbf{e}'') \in \mathcal{E}_\ell$. Then we have

$$D = - \sum_{\ell=3}^{\mu} (-1)^\ell \sum_{\mathbf{n}\in \mathcal{C}_\mu^\ell} \sum_{\mathbf{e}\in \mathcal{E}_\ell} (-1)^{\mathbf{e}} \; \sum_{i=2}^{\ell-1} R_i^{(e_1,...,e_i)}(x_{n_1}^{(e_2)}, ..., x_{n_i}^{(e_{\ell-1})}) \otimes v_\ell .$$

With similar manipulations, setting $\ell = i+j+1$ and $\mathbf{n} = (\mathbf{p}, \mathbf{q}) \in C_\mu^\ell$, the last term in eq. (5.4.15)

is

$$E = -\sum_{\nu=1}^{\mu-1}\sum_{i=2}^{\mu-\nu}\sum_{j=2}^{\nu-1}(-1)^{i+j}\sum_{\mathbf{p}\in\mathcal{C}^i_{\mu-\nu}}\sum_{\mathbf{q}\in\mathcal{C}^{j+1}_{\nu}}\sum_{\mathbf{e}'\in\mathcal{E}_i}\sum_{\mathbf{e}'''\in\mathcal{E}_j}(-1)^{\mathbf{e}'}(-1)^{\mathbf{e}'''}$$
$$R_i^{\mathbf{e}'}(x_{p_1}^{(e'_2)},...,x_{p_{i-1}}^{(e'_i)},v_{p_i})\otimes\Big(x_{q_1}\rhd R_j^{\mathbf{e}'''}(x_{q_2}^{(e'''_2)},...,x_{q_j}^{(e'''_j)},v_{q_{j+1}})\Big)$$
$$= -\sum_{\ell=4}^{\mu}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}^{\ell}_{\mu}}\sum_{\mathbf{e}\in\mathcal{E}_\ell}(-1)^{\mathbf{e}}\sum_{i=2}^{\ell-2}R_i^{(e_1,...,e_i)}(x_{n_1}^{(e_2)},...,x_{n_i}^{(e_{i+1})})\otimes$$
$$\otimes\Big(x_{n_{i+1}}^{(e_{i+2})}\rhd R_{\ell-i-1}^{(e_{i+2},...,e_\ell)}(x_{n_{i+2}}^{(e_{i+3})},...,x_{n_{\ell-1}}^{(e_\ell)},v_{n_\ell})\Big).$$

We now observe that the sum B extends C to the value $\ell = 2$, and that D extends E to the value $\ell = 3$. Alltogether, we have

$$B+C+D+E = -\sum_{\ell=3}^{\mu}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}^{\ell}_{\mu}}\sum_{\mathbf{e}\in\mathcal{E}_\ell}(-1)^{\mathbf{e}}\sum_{i=1}^{\ell-1}R_i^{(e_1,...,e_i)}(x_{n_1}^{(e_2)},...,x_{n_i}^{(e_{i+1})})\otimes$$
$$\otimes\Big(x_{n_{i+1}}^{(e_{i+2})}\rhd R_{\ell-i-1}^{(e_{i+2},...,e_\ell)}(x_{n_{i+2}}^{(e_{i+3})},...,x_{n_{\ell-1}}^{(e_\ell)},v_{n_\ell})\Big).$$

Therefore, eq. (5.4.14) is then equivalent, for any $\ell \geqslant 3$, any $\mathbf{n} \in \mathcal{C}^{\ell}_{\nu}$ and any $\mathbf{e} \in \mathcal{E}_\ell$, to the following recursion

$$R_\ell^{\mathbf{e}}(x_{n_1}^{(e_2)},...,x_{n_{\ell-1}}^{(e_\ell)},v_{n_\ell}) = R_1^{(e_1)}(x_{n_1}^{(e_2)})\rhd R_{\ell-1}^{(e_2,...,e_\ell)}(x_{n_2}^{(e_3)},...,x_{n_{\ell-1}}^{(e_\ell)},v_{n_\ell})$$
$$+\sum_{i=1}^{\ell-1}R_i^{(e_1,...,e_i)}(x_{n_1}^{(e_2)},...,x_{n_i}^{(e_{i+1})})\otimes\Big(x_{n_{i+1}}^{(e_{i+2})}\rhd R_{\ell-i-1}^{(e_{i+2},...,e_\ell)}(x_{n_{i+2}}^{(e_{i+3})},...,x_{n_{\ell-1}}^{(e_\ell)},v_{n_\ell})\Big),$$

which is proved in Lemma 5.4.8.

The second identity (2.2.4), namely

$$(\mu\amalg\mathrm{id})\,(\mathrm{id}\amalg\Delta^{\amalg}_{\mathrm{FdB}})\,\delta_l(x_n) = y_n,$$

can not be expressed as a recurrence on $R^{\mathbf{e}}_\ell$, because these operators do not show up explicitely to which factor of $H^{\amalg}_{\mathrm{FdB}}\amalg H^{\amalg}_{\mathrm{FdB}}$ the variables belong. Then, let us use the recursion (5.4.13) to describe δ_l and prove the second identity by induction on n.

The identity is verified for $n = 1$ because we have

$$(\mu\amalg\mathrm{id})\,(\mathrm{id}\amalg\Delta^{\amalg}_{\mathrm{FdB}})\,\delta_l(x_1) = x_1 + y_1 - x_1 = y_1.$$

Then, suppose it holds up to the degree $n-1$ Since $\Delta^{\amalg}_{\mathrm{FdB}}$ and μ are algebra homomorphisms, if we apply the operator $D = (\mu\amalg\mathrm{id})\,(\mathrm{id}\amalg\Delta^{\amalg}_{\mathrm{FdB}})$ to the expression

$$\delta_l(x_n) = v_n - \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}x_m\rhd\big(\delta_l(y_{k_1})\otimes\cdots\otimes\delta_l(y_{k_\lambda})\big),$$

we obtain, for $D\big(\delta_l(x_n)\big)$, the sum of

$$D(v_n) = y_n + \sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}\sum_{\mathbf{k}\in\mathcal{C}^{\lambda}_{n-m}}x_m\rhd(y_{k_1}\otimes\cdots\otimes y_{k_\ell})$$

and of

$$-\sum_{m=1}^{n-1}\sum_{\lambda=1}^{n-m}\sum_{\mathbf{k}\in\mathcal{C}_{n-m}^{\lambda}} x_m \rhd \Big(D\big(\delta_l(y_{k_1})\big)\otimes\cdots\otimes D\big(\delta_l(y_{k_\lambda})\big)\Big).$$

Therefore the second identity is satisfied if, for any $m=1,...,n-1$, any $\lambda=1,...,n-m$ and any $\mathbf{k}\in\mathcal{C}_{n-m}^{\lambda}$, we have

$$D\big(\delta_l(y_{k_1})\big)\otimes\cdots\otimes D\big(\delta_l(y_{k_\lambda}) = y_{k_1}\otimes\cdots\otimes y_{k_\ell},$$

which is true by inductive hypothesis. □

5.5 Properties of the diffeomorphisms loop

Proposition 5.5.1 *The coloop bialgebra $H_{\mathrm{FdB}}^{\amalg}$ has a two-sided antipode S such that*

$$\delta_r = (\mathrm{id}\amalg S)\,\Delta_{\mathrm{FdB}}^{\amalg},$$

while the identity $\delta_l = (S\amalg \mathrm{id})\,\Delta_{\mathrm{FdB}}^{\amalg}$ does not hold. Moreover, the antipode in the Faà di Bruno coloop bialgebra coincides with that in the non-commutative Faà di Bruno Hopf algebra given in [8], that is,

$$S(x_n) = -\sum_{\ell=0}^{n-1}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1,...,n_{\ell+1})\; x_{n_1}x_{n_2}\cdots x_{n_{\ell+1}}.$$

Proof. i) In a coloop bialgebra, the left and right antipodes are given respectively by

$$S_l = (\mathrm{id}\amalg\varepsilon)\,\delta_l \qquad \text{and} \qquad S_r = (\varepsilon\amalg\mathrm{id})\,\delta_r,$$

cf. (2.2.8). Let us show that for $H_{\mathrm{FdB}}^{\amalg}$ these two operators coincide, and therefore the two-sided antipode is well defined by $S := S_l = S_r$.

Indeed, let us fix $n\geqslant 1$. For the right antipode we have

$$\begin{aligned} S_r(x_n) &= \sum_{\ell=0}^{n-1}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1,...,n_{\ell+1})\;\big(\varepsilon(x_{n_1})-y_{n_1}\big)\,y_{n_2}\cdots y_{n_{\ell+1}} \\ &= -\sum_{\ell=0}^{n-1}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1,...,n_{\ell+1})\;x_{n_1}x_{n_2}\cdots x_{n_{\ell+1}}, \end{aligned}$$

where $n_1>0$ implies $\varepsilon(x_{n_1})=0$, and where we renamed the variables y as x because S_r takes values in $H_{\mathrm{FdB}}^{\amalg}$. For the left antipode we have

$$\begin{aligned} S_l(x_n) = \varepsilon(y_n) - x_n &- \sum_{\mathbf{n}\in\mathcal{C}_n^2} d_1(n_1)\;x_{n_1}\big(\varepsilon(y_{n_2})-x_{n_2}\big) \\ &+\sum_{\ell=2}^{n-1}(-1)^{\ell}\sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}}\sum_{\mathbf{e}\in\mathcal{E}_{\ell-1}}(-1)^{\mathbf{e}}\;d_{\ell+1}^{\mathbf{e}}(n_1,...,n_{\ell+1})\;(\mathrm{id}\amalg\varepsilon)\big(x_{n_1}x_{n_2}^{(e_1)}\cdots x_{n_\ell}^{(e_{\ell-1})}\,v_{n_{\ell+1}}\big). \end{aligned}$$

Since ε kills the terms where some y appears, in the sum over the sequences $\mathbf{e} \in \mathcal{E}_{\ell-1}$ there only remains the sequence $\mathbf{e} = (1, 1, ..., 1)$, for which $d^{\mathbf{e}}_{\ell+1} = d_{\ell+1}$ and $(-1)^{\mathbf{e}} = +$, and therefore we have

$$\begin{aligned} S_l(x_n) &= -x_n + \sum_{\mathbf{n} \in \mathcal{C}^2_n} d_1(n_1)\ x_{n_1}\, x_{n_2} \\ &\qquad - \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}^{\ell+1}_n} d_{\ell+1}(n_1, ..., n_{\ell+1})\ x_{n_1} x_{n_2} \cdots x_{n_{\ell+1}} \\ &= S_r(x_n). \end{aligned}$$

ii) Let us now prove the identity $\delta_r = (\mathrm{id} \amalg S)\, \Delta^{\amalg}_{\mathrm{FdB}}$. For any generator x_n of $H^{\amalg}_{\mathrm{FdB}}$, we have

$$(\mathrm{id} \amalg S)\, \Delta^{\amalg}_{\mathrm{FdB}}(x_n) = x_n + S(y_n) + \sum_{j=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}^{j+1}_n} \binom{n_1 + 1}{j}\ x_{n_1}\ S(y_{n_2}) \cdots S(y_{n_{j+1}})$$

$$\begin{aligned} &= x_n - y_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}^{\ell+1}_n} d_{\ell+1}(n_1, ..., n_{\ell+1})\ y_{n_1}\, y_{n_2} \cdots y_{n_{\ell+1}} \\ &\quad + \sum_{j=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}^{j+1}_n} \sum_{p_1=1}^{n_2} \cdots \sum_{p_j=1}^{n_{j+1}} (-1)^{p_1 + \cdots + p_j} \sum_{\mathbf{q}^1 \in \mathcal{C}^{p_1}_{n_2}} \cdots \sum_{\mathbf{q}^j \in \mathcal{C}^{p_j}_{n_{j+1}}} \binom{n_1 + 1}{j}\ d_{p_1}(q^1_1, ..., q^1_{p_1}) \cdots \\ &\qquad \cdots d_{p_j}(q^j_1, ..., q^j_{p_j})\ x_{n_1}\ y_{q^1_1} \cdots y_{q^1_{p_1}} \cdots y_{q^j_1} \cdots y_{q^j_{p_j}}. \end{aligned}$$

Set $\ell = p_1 + \cdots + p_j$, then $1 \leqslant j \leqslant \ell \leqslant n - 1$. Since

$$\bigcup_{n_1=1}^{n-\ell} \bigcup_{p_1=1}^{n_2} \cdots \bigcup_{p_j=1}^{n_{j+1}} C^1_{n_1} \times C^{p_1}_{n_2} \times \cdots \times C^{p_j}_{n_{j+1}} = C^{\ell+1}_n,$$

because $n_1 + n_2 + \cdots + n_{j+1} = n$, if we rename the sequence $(n_1, q^1_1, ..., q^1_{p_1}, ..., q^j_1, ..., q^j_{p_j})$ as $\mathbf{n} \in C^{\ell+1}_n$, the sum over j becomes

$$\sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}^{\ell+1}_n} \left(\sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}^j_\ell} \binom{n_1 + 1}{j}\ d_{p_1}(n_2, ..., n_{p_1+1}) \cdots d_{p_j}(n_{P_{j-1}+2}, ..., n_{\ell+1}) \right) x_{n_1}\ y_{n_2} \cdots y_{n_{\ell+1}},$$

where $Pi = p_1 + \cdots + p_i$ for $i = 1, ..., j$. Using the recurrence proved in Corollary 5.4.1, we finally obtain

$$\begin{aligned} (\mathrm{id} \amalg S)\, \Delta^{\amalg}_{\mathrm{FdB}}(x_n) &= x_n - y_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}^{\ell+1}_n} d_{\ell+1}(n_1, ..., n_{\ell+1})\ y_{n_1}\, y_{n_2} \cdots y_{n_{\ell+1}} \\ &\qquad + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}^{\ell+1}_n} d_{\ell+1}(n_1, ..., n_{\ell+1})\ x_{n_1}\, y_{n_2} \cdots y_{n_{\ell+1}} \\ &= \delta_r(x_n). \end{aligned}$$

iii) The first counterexample to the analogue identity $\delta_l = (S \amalg \mathrm{id})\, \Delta^{\amalg}_{\mathrm{FdB}}$ is on the generator x_3, for which we have

$$(S \amalg \mathrm{id})\, \Delta^{\amalg}_{\mathrm{FdB}}(x_3) = v_3 - (2x_1 v_2 + 3x_2 v_1) + 5x_1^2 v_1 - x_1 v_1 y_1,$$

where $v_n = y_n - x_n$, while

$$\delta_l(x_3) = v_3 - (2x_1v_2 + 3x_2v_1) + 5x_1^2v_1 - x_1y_1v_1.$$

□

This result allows us on one side to deduce some properties of the loop of formal diffeomorphisms, and on the other side to compare the Faà di Bruno coloop bialgebra with the non-commutative Faà di Bruno Hopf algebra.

Corollary 5.5.2 *1. The proalgebraic loop* Diff *is not right alternative, nor power associative.*

2. Nevertheless, Diff *has two-sided inverses and, for a given an associative algebra A and an element $a \in \mathrm{Diff}(A)$, the inverse $a^{-1} = a\backslash e = e/a$ is given by the usual Lagrange coefficients, namely*

$$\big(a^{-1}\big)_n = -\sum_{\ell=0}^{n-1}(-1)^\ell \sum_{\mathbf{n}\in\mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1,...,n_{\ell+1})\; a_{n_1}\, a_{n_2}\cdots a_{n_{\ell+1}}.$$

3. The inversion allows us to construct the right division, that is, $a/b = a \circ (e/b)$ for any $a, b \in \mathrm{Diff}(A)$, but it does not allow us to construct the left division, because $b\backslash a \neq (b\backslash e)\circ a$ if $a_n \neq 0$ and $b_m \neq 0$ for some $n, m \geqslant 1$.

Proof. 1. The loop $\mathrm{Diff}(A)$ is right alternative if and only if the coloop bialgebra $H_{\mathrm{FdB}}^{\amalg}$ is right coalternative, that is $(\mathrm{id}\amalg\mu)\,K = 0$, where $K = (\Delta_{\mathrm{FdB}}^{\amalg}\amalg\mathrm{id})\,\Delta_{\mathrm{FdB}}^{\amalg} - (\mathrm{id}\amalg\Delta_{\mathrm{FdB}}^{\amalg})\,\Delta_{\mathrm{FdB}}^{\amalg}$ is the coassociator.

The first deviation from right alternativity appears on the generator x_5. If we temporarily denote by $x_n = x_n^{(1)}$, $y_n = x_n^{(2)}$ and $z_n = x_n^{(3)}$ the three copies of the generators in $H\amalg H\amalg H$, we get

$$\begin{aligned}
K(x_5) = {} & 6x_3(y_1z_1 - z_1y_1) \\
& + x_2\big[3(y_2z_1 - z_2y_1) + 3(y_1z_2 - z_1y_2) + (8y_1^2z_1 - 7y_1z_1y_1 - z_1y_1^2) \\
& \qquad + (8y_1z_1^2 - 7z_1y_1z_1 - z_1^2y_1)\big] \\
& + x_1\big[(y_3z_1 - z_3y_1) + (y_2z_2 - z_2y_2) + (y_1z_3 - z_1y_3) + 3(y_2y_1z_1 - y_2z_1y_1) \\
& \quad + 2(y_1y_2z_1 - y_1z_2y_1) + 2(y_1^2z_2 - y_1z_1y_2) + 2(y_1z_1z_2 - z_1y_1z_2) \\
& \quad + (3y_2z_1^2 - 2z_2y_1z_1) + (2y_1z_2z_1 - 3z_1y_2z_1) \\
& \qquad + (5y_1^2z_1^2 + y_1z_1^3 - 4y_1z_1y_1z_1 - y_1z_1^2y_1 - z_1y_1z_1^2)\big]
\end{aligned}$$

and therefore

$$(\mathrm{id}\amalg\mu)\,K(x_5) = x_1(y_2y_1^2 - y_1y_2y_1) \neq 0.$$

The generator x_5 corresponds to the power λ^6 of usual series with substitution law, therefore the deviation from right alternativity computed on the generator x_5 can be detected by comparing the values $(a\circ b)\circ b$ and $a\circ(b\circ b)$ for the two series

$$a(\lambda) = \lambda + a_1\lambda^2 \qquad \text{and} \qquad b(\lambda) = \lambda + b_1\lambda^2 + b_2\lambda^3$$

with $a_1 = 1$ and $b_2b_1^2 \neq b_1b_2b_1$. For instance[6], by taking the 2×2 elementary matrices $b_1 = E_{11}$ and $b_2 = E_{21}$, for which $b_2b_1^2 = b_2$ and $b_1b_2b_1 = 0$.

[6]The authors warmly thank J. M. Pérez-Izquierdo for comunicating this example.

The same computation shows that $\mathrm{Diff}(A)$ is not power associative, because

$$\mu\,(\mathrm{id}\amalg\mu)\,K(x_5) = x_1x_2x_1^2 - x_1^2x_2x_1 \neq 0.$$

For a series $c(\lambda) = \lambda + c_1\lambda^2 + c_2\lambda^3$, we then have $(c\circ c)\circ c \neq c\circ(c\circ c)$ if $c_1c_2c_1^2 \neq c_1^2c_2c_1$. For instance, this is verified for the two 2×2 matrices

$$c_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad c_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

for which we have

$$c_1c_2c_1^2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad c_1^2c_2c_1 = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}.$$

2. The left and right inverses of $a \in \mathrm{Diff}(A)$ can be found using respectively the left antipode S_l and right antipode S_r of $H^{\amalg}_{\mathrm{FdB}}$, according to the standard rule

$$(e/a)_n = a\big(S_l(x_n)\big) \qquad \text{and} \qquad (a\backslash e)_n = a\big(S_r(x_n)\big).$$

By Proposition 5.5.1 we have $S_r = S_l$, therefore $e/a = a\backslash e$.

3. The identity $a/b = a\circ(e/b)$ in the loop $\mathrm{Diff}(A)$ is equivalent to the identity $\delta_r = (\mathrm{id}\amalg S_r)\,\Delta^{\amalg}_{\mathrm{FdB}}$ in $H^{\amalg}_{\mathrm{FdB}}$, proved in Proposition 5.5.1. The analogue identity for the left division does not hold. □

The commutative Faà di Bruno Hopf algebra which represents the classical proalgebraic group Diff, mentioned at the beginning of section 5, admits a non-commutative lift [8]

$$H^{\mathrm{nc}}_{\mathrm{FdB}} = \mathbb{F}\langle x_n,\ n\geqslant 1\rangle, \qquad (x_0 = 1)$$
$$\Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{(k)} x_{k_0}\cdots x_{k_m},$$

where the sum is over the set of tuples $(k_0, k_1, k_2, ..., k_m)$ of non-negative integers such that $k_0 + k_1 + k_2 + \cdots + k_m = n - m$. Since Diff is not a group over associative algebras, the existence of this Hopf algebra is not *a priori* ensured by the extention of the functor Diff from $\mathbf{Com}_{\mathbb{F}}$ to $\mathbf{As}_{\mathbb{F}}$.

Corollary 5.5.3 *The image of the coloop bialgebra $H^{\amalg}_{\mathrm{FdB}}$ under the canonical projection π given in Def. 2.3.7 is the non-commutative Faà di Bruno Hopf algebra $H^{\mathrm{nc}}_{\mathrm{FdB}}$, that is,*

$$(H^{\amalg}_{\mathrm{FdB}})^{\otimes} = H^{\mathrm{nc}}_{\mathrm{FdB}}.$$

Proof. Indeed, we have $(H^{\amalg}_{\mathrm{FdB}})^{\otimes} = H^{\mathrm{nc}}_{\mathrm{FdB}}$ as an algebra, and eventhough $\Delta^{\amalg}_{\mathrm{FdB}}$ is not coassociative, the comultiplication $(\Delta^{\amalg}_{\mathrm{FdB}})^{\otimes}$ coincides with $\Delta^{\mathrm{nc}}_{\mathrm{FdB}}$ and therefore it is coassociative with respect to the component-wise multiplication in $H^{\mathrm{nc}}_{\mathrm{FdB}}\otimes H^{\mathrm{nc}}_{\mathrm{FdB}}$.

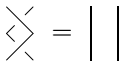
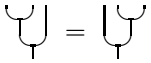
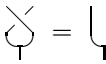
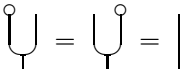
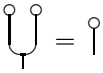
The assertion is then proved because, by Prop. 5.5.1, the antipode in $H^{\amalg}_{\mathrm{FdB}}$ is unique and coincides with that in $H^{\mathrm{nc}}_{\mathrm{FdB}}$ on generators. □

6 Appendix: Categorical proofs with tangles

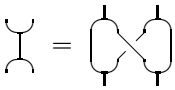
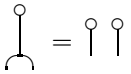
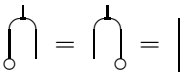
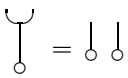
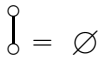
Tangle diagrams are an efficient tool to prove formal (categorical) properties. Tangles are drawings suitable to represent operations and co-operations in a monoidal category, cf. [29] [49], and therefore can be used to encode the structure of coloops in a category $(\mathbf{C}, \amalg, I)$. In the context of non-associative algebras they have been used in [40] to code deformations of the enveloping algebra of a Malcev algebra, seen as the infinitesimal structure of a Moufang loop.

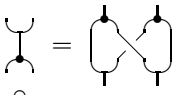
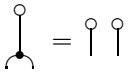
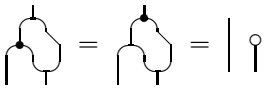
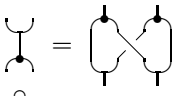
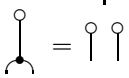
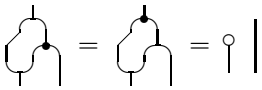
Tangles are drawings to be read from the top to the bottom as concatenation of operations acting on objects related by the monoidal product, and not by a cartesian (or tensor) product. Here is the list of the tangles needed to represent all the operations and the co-operations in coloops, with their defining identities.

Categorical maps

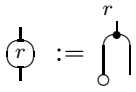
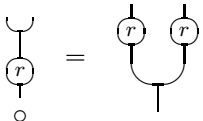
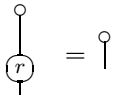
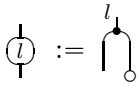
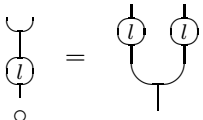
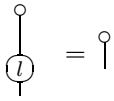
τ	twist		invertible		Sect. 2.2
μ	folding map		associative		Sect. 2.2
			commutative		Sect. 2.2
			unital		Sect. 2.2
u	unit		folding morphism		Sect. 2.2

Coloop structure maps

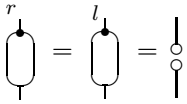
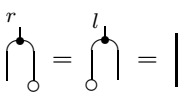
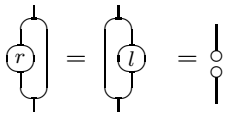
Δ	comultiplication		folding morphism		Sect. 2.2
			unital		Sect. 2.2
			counital		Eq. (2.2.2)
ε	counit		folding morphism		Sect. 2.2
			unital		Sect. 2.2

δ_r	right codivision		folding morphism		Sect. 2.2
			unital		Sect. 2.2
			right cocancellation		Eq. (2.2.3)
δ_l	left codivision		folding morphism		Sect. 2.2
			unital		Sect. 2.2
			left cocancellation		Eq. (2.2.4)

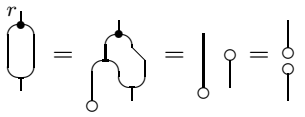
Further coloop maps

S_r	right antipode		folding morphism		Sect. 2.2
			unital		Sect. 2.2
S_l	left antipode		folding morphism		Sect. 2.2
			unital		Sect. 2.2

Properties of coloops

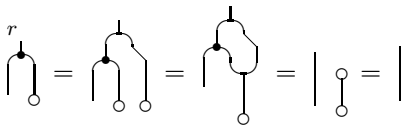
	$\mu\,\delta_r = \mu\,\delta_l = u\,\varepsilon$		Eq. (2.2.6)
partial counitality	$(\mathrm{id} \amalg \varepsilon)\,\delta_r = (\varepsilon \amalg \mathrm{id})\,\delta_l = \mathrm{id}$		Eq. (2.2.7)
left and right 5-terms identities	$\mu\,(S_r \amalg \mathrm{id})\,\Delta = \mu\,(\mathrm{id} \amalg S_l)\,\Delta = u\,\varepsilon$		Eq. (2.2.9)

Proof of Eq. (2.2.6): For δ_r



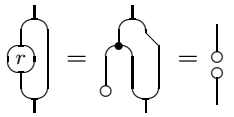
and similarly for δ_l.

Proof of Eq. (2.2.7): For δ_r



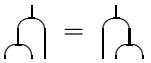
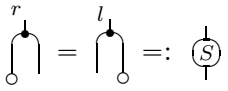
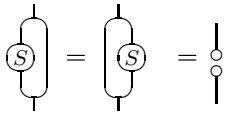
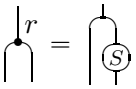
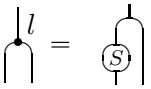
and similarly for δ_l.

Proof of Eq. (2.2.9): For S_r



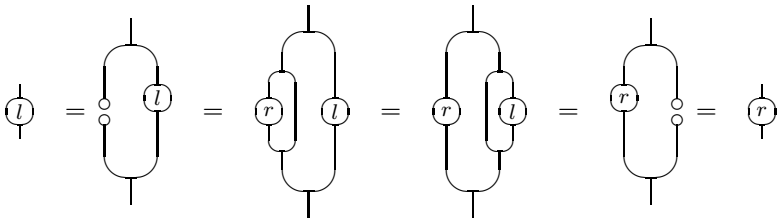
and similarly for S_l.

Properties of cogroups

coassociative	$(\Delta \amalg \mathrm{id})\,\Delta = (\mathrm{id} \amalg \Delta)\,\Delta$		Eq. (2.2.14)
unique antipode	$S_r = S_l =: S$		Prop. 2.2.17
5-terms identity	$\mu\,(S \amalg \mathrm{id})\,\Delta = \mu\,(\mathrm{id} \amalg S)\,\Delta = u\,\varepsilon$		Eq. (2.2.16)
right coinverse property	$\delta_r = (\mathrm{id} \amalg S)\,\Delta$		Eq. (2.2.15)
left coinverse property	$\delta_l = (S \amalg \mathrm{id})\,\Delta$		Eq. (2.2.15)

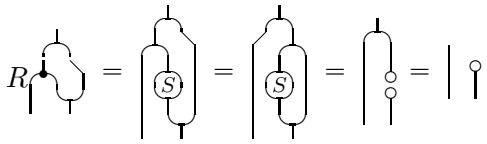
Lemma 6.0.1 (cf. Prop. 2.2.17) *If a coloop H is coassociative, then it has a two-sided antipode satisfying the 5-terms identity (2.2.16) and the coinverse properties (2.2.15). Therefore it is a cogroup.*

Proof. Assume that Δ is coassociative. then we have $S_l = S_r$ because

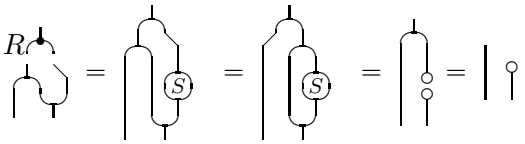


and therefore $S := S_l = S_r$ satisfies the 5-terms identity becuase of (2.2.9).

For the the coinverse properties, let us show that the operator $R := (\mathrm{id} \amalg S)\Delta$ satisfies the right cocancellations (2.2.4), and therefore it coincides with δ_r. In fact, we have



and



Similarly, the operator $(S \amalg \mathrm{id})\Delta$ satisfies the left-cocancellations (2.2.4), and therefore it coincides with δ_l. □

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