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# Loop of formal diffeomorphisms and Faà di Bruno coloop bialgebra

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## Abstract

We consider a generalization of (pro)algebraic loops defined on general categories of algebras and the dual notion of a coloop bialgebra suitable to represent them as functors. Our main result is the proof that the natural loop of formal diffeomorphisms with associative coefficients is proalgebraic, and give a full description of the codivisions on its coloop bialgebra. This result provides a generalization of the Lagrange inversion formula to series with non-commutative coefficients, and a loop-theoretic explanation to the existence of the non-commutative Faà di Bruno Hopf algebra.

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## 1 Introduction

### 1.1 Presentation and overview of the results

An affine proalgebraic group  $G$  is a representable functor in groups defined on the category of commutative associative algebras over a field  $\mathbb{F}$ . The algebra representing  $G$  is the commutative Hopf algebra  $\mathbb{F}[G]$  of regular functions. In this paper we consider two generalizations of proalgebraic groups, on one side to representable functors *on categories of non-commutative algebras*, on the other side to functors taking values *in non-associative groups with divisions*, that is, *loops*.

Our main motivation comes from two proalgebraic groups of formal series appearing in renormalization in quantum field theory: the group of invertible series with constant term equal to 1, represented by the Hopf algebra of symmetric functions, and that of formal diffeomorphisms tangent to the identity, represented by the Faà di Bruno Hopf algebra. Details on the role played by these series in quantum field theory are given in a separate section below.

Both types of series make sens with non-commutative coefficients, and both representative Hopf algebras admit a non-commutative version [7]. We are interested in the relationship

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between the non-commutative algebras and the sets of series. For this, we first consider generalizations of proalgebraic groups to categories of non-commutative algebras.

Functors in groups on general categories have been studied by algebraic topologists in the late 50's. D. Kan considered them on the category of groups [21], and B. Eckmann and P. Hilton [14] introduced them on general categories. Their representative Hopf-type object is called a *cogroup*. In a category with coproduct  $\amalg$  and initial object, a cogroup is an object  $H$  endowed with a comultiplication, a counit and an antipode satisfying the usual properties of Hopf algebras, where the comultiplication takes values in  $H \amalg H$  instead of  $H \otimes H$  (which is not necessarily defined). Cogroups are then generalizations of commutative Hopf algebras which, contrarily to quantum groups in the case of associative algebras, preserve the functorial properties and the adjoint constructions. They have proved to be very fruitful in homotopy theory, where they appear as special  $H$ -spaces [22], as shown by I. Berstein [3]. A comprehensive study of cogroups in many varieties of algebras can be found in G. Bergman and A. Hausknecht's book [2].

However, not all proalgebraic groups admit an extension to non-commutative algebras. For instance, while the group of invertible formal series naturally extends as a proalgebraic group to the category of associative algebras, the group of formal diffeomorphism does not. We show, on this example, that the extension of the functor is sometimes possible if we regard the original group as a *loop*.

Loops are multiplicative sets with unit and with a left and a right division instead of two-sided inverses. They first appeared, with some extra properties, in the work of R. Moufang [32] on alternative rings. Associative loops are groups. Similarly to Lie groups, the tangent space of a smooth loop carries a particular algebraic structure called a Sabinin algebra [37, 29], which reduces to a Mal'cev algebra [28] for smooth Moufang loops. The notion of universal enveloping algebra has been extended to Sabinin algebras by I. Shestakov, U. U. Umirbaev [39] and J. Mostovoy, J. M. Pérez-Izquierdo [30].

In this paper we consider functors in loops on a general category  $\mathbf{C}$  with coproduct and initial object and call their representative objects *coloops in  $\mathbf{C}$* . We specialise  $\mathbf{C}$  to be a variety of algebras over a field  $\mathbb{F}$  to have a reasonable notion of generalized (pro)algebraic loop. The first simple example is the extension of the functors of invertible elements in a unital algebra and that of unitary elements in a unital involutive algebra. As expected, the largest category on which these functors are representable as loops turn out to be respectively that of alternative and of alternative involutive algebras (Prop. 3.4 and Prop. 3.9). We also show that the loop of unitary elements in the Cayley-Dickson extension of an involutive algebra is not representable on non-commutative algebras (Prop. 3.11), even if examples of such loops exist. Then we turn to loops of formal series with coefficients in a non-commutative algebra. First we consider the set of invertible series (with constant term equal to 1). The algebra of series with coefficients in an alternative algebra is alternative. Surprisingly, in contrast to the previous results, we find that the set of invertible series is a proalgebraic loop on all algebras, not necessarily alternative (Thm. 4.4). Finally, our main result concerns the natural loop of formal diffeomorphisms (tangent to the identity) with associative coefficients. We show that it is proalgebraic, and give the closed formulas of the codivisions on its representative Faà di Bruno coloop bialgebra (Def. 5.5 and Thm. 5.25). For this, we express the co-operations in terms of some recursive operators defined on any positively graded algebra (Thm. 5.17), which extend the natural pre-Lie product of the Witt Lie algebra (cf. [8, 15]) but not as a multibrace product (cf. [24]), and which turn out to be very rich in combinatorial properties. The coefficients appearing in the divisions show up sequences of integer numbers typical of the Lagrange inversion formula (as Catalan numbers) and some new ones, that we call (*labeled*) *Lagrange coefficients* (Def. 5.3 and 5.4). This result is a generalization of the Lagrange inversion formula to series with non-commutative coefficients, and gives a loop-theoretic explanation to the existence of the non-commutative Faà di Bruno Hopf algebra [7].

## 1.2 Motivation: formal series in quantum field theory

In perturbative quantum field theory, the correlation functions, which give the probability amplitude of an event, are asymptotic series in the powers of a measurable parameter  $\lambda$ , such as the electric charge, called the coupling constant. For instance, for a self-interacting field  $\phi$  with coupling  $\lambda$  and mass  $m$ , the  $k$ -point correlation function is a series

$$G^{(k)}(x_1, \dots, x_k) = \langle \phi(x_1) \cdots \phi(x_k) \rangle = \sum_{n=0}^{\infty} G_n^{(k)}(x_1, \dots, x_k; m, \hbar) \lambda^n$$

where the  $n$ th coefficient is a finite sum of amplitudes of suitable Feynman graphs with  $k$  fixed external legs, which depend on the mass  $m$  and on the Planck constant  $\hbar$ , and  $n$  is related to the number of internal vertices of the graph.

The computation of the correlation functions gives rise to some divergent integrals, or ill-defined product of singular distributions. Giving a meaning to such terms requires a renormalization procedure, which globally amounts to suitably multiply and compose the correlation functions with some others series, called renormalization factors, obtained by assembling the counterterms needed to cure each divergency [13], [19]. Given an ambient algebra  $A$ , typically  $\mathbb{C}$  or the algebra  $\mathbb{C}((\varepsilon))$  of Laurent series in a regularization parameter  $\varepsilon$ , in renormalization theory there appear two groups of formal series in the variable  $\lambda$  and coefficients in  $A$ :

- the set  $\text{Inv}(A) = \left\{ a(\lambda) = \sum_{n \geq 0} a_n \lambda^n \mid a_0 = 1, a_n \in A \right\}$  of invertible series, endowed with the pointwise multiplication  $(ab)(\lambda) = a(\lambda) b(\lambda)$  and the unit  $1(\lambda) = 1$ , which represent the Green's functions (up to an invertible factor) and the renormalization factors;
- the set  $\text{Diff}(A) = \left\{ a(\lambda) = \sum_{n \geq 0} a_n \lambda^{n+1} \mid a_0 = 1, a_n \in A \right\}$  of formal diffeomorphisms, endowed with the composition law  $(a \circ b)(\lambda) = a(b(\lambda))$  and the unit  $e(\lambda) = \lambda$ , which represent the bare coupling constants.

Dyson's renormalization formulas [13] are modeled by the semi-direct product  $\text{Diff}(A) \ltimes \text{Inv}(A)$ , endowed with the law

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \circ a_2, (b_1 \circ a_2) b_2),$$

where  $a_1, a_2 \in \text{Diff}(A)$  and  $b_1, b_2 \in \text{Inv}(A)$ , which is well defined because formal diffeomorphisms act on invertible series from the right, by composition.

These groups are proalgebraic on commutative algebras. Physically, this means that the overall renormalization procedure (except the scheme which says how to compute the counterterms) is independent of the chosen field theory, whenever the latter leads to commutative amplitudes. The recent results on the Renormalization Hopf algebras, initiated by A. Connes and D. Kreimer [10, 11], show even a stronger result: co-operations dual to the multiplication and the composition of series exist even on Hopf algebras generated by Feynman graphs. In other words, there exist proalgebraic groups of series expanded over Feynman graphs or various types of trees, which turns out to be extremely efficient in handling the combinatorial content of renormalization procedures [5, 6, 7, 40, 36].

The toy model  $\phi^3$  theory used by Connes-Kreimer is a scalar field theory and leads to the commutative algebra  $A = \mathbb{C}$  of amplitudes. However, interesting situations involve non-commutative algebras. In fact, Feynman amplitudes are complex numbers for single scalar fields, the coupling constants and the renormalization factors, but they are  $4 \times 4$  complex matrices for the fermionic or bosonic fields, and may be represented by higher order matrices for theories involving several interacting fields. In this case, forcing the final counterterms to be scalar, as imposed by the fact that the renormalization factors act on the Lagrangian, prevents to describe

the renormalization in a functorial way, as shown by the results in [41], where the Hopf algebra does not represent a functorial group on  $A = M_4(\mathbb{C})$ . In order to preserve this functoriality, there is a need to understand Dyson's formulas for sets of series  $\text{Inv}(A)$  and  $\text{Diff}(A)$  also when  $A$  is not a commutative algebra. This is the motivation for the present work.

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## 2 Loops and coloops

### 2.1 Loops and functors in loops

A **loop** is a non-empty set  $Q$  endowed with a **multiplication**  $Q \times Q \longrightarrow Q$ ,  $(a, b) \longmapsto a \cdot b$ , a (two-sided) **unit**  $1 \in Q$ , a **left division**  $\backslash : Q \times Q \longrightarrow Q$  and a **right division**  $/ : Q \times Q \longrightarrow Q$  satisfying the **cancellation properties**

$$a \cdot (a \backslash b) = b, \quad a \backslash (a \cdot b) = b, \quad (2.1)$$

$$(a/b) \cdot b = a, \quad (a \cdot b)/b = a. \quad (2.2)$$

Given two loops  $Q$  and  $Q'$ , a **homomorphism of loops**  $f : Q \longrightarrow Q'$  is of course a map which preserves the multiplication, and therefore the unit and the divisions.

The multiplication in a loop  $Q$  is not necessarily associative, if it is associative then the loop is a group. Any element  $a$  in a loop  $Q$  has a **right inverse**  $1/a$  and a **left inverse**  $a \setminus 1$ , which do not necessarily coincide and do not necessarily determine the divisions, in the sense that they do not satisfy the identities

$$(a \setminus 1) \cdot b = a \setminus b \quad \text{and} \quad b \cdot (1/a) = b/a \quad (2.3)$$

for any  $a, b \in Q$ , which hold in any group. Examples of loops which are not groups are known since a long time, see for instance [9] or [35].

Denote by **Loop** the category of loops and let  $F : \mathbf{Loop} \longrightarrow \mathbf{Set}_*$  be the forgetful functor to the category of pointed sets. Given a category  $\mathbf{C}$ , a functor  $Q : \mathbf{C} \longrightarrow \mathbf{Loop}$  is said to be **representable** if the composite functor  $FQ$  is representable. This means that  $Q$  is naturally isomorphic to a hom-set functor  $A \mapsto \text{Hom}_{\mathbf{C}}(H, A)$  for a given object  $H$  in  $\mathbf{C}$ . Following an established terminology on cogroups, the representative object  $H$  can then be called a **coloop in  $\mathbf{C}$** . A reasonable notion of **(pro)algebraic loop** is obtained for  $\mathbf{C}$  being a variety of algebras over a field  $\mathbb{F}$ , its representative coloop then being a sort of bialgebra.

In this section we describe coloops in an axiomatic way. In the next sections we give some easy examples of algebraic and non-algebraic loops on associative and non-associative algebras, and then study extensively the loop of invertible series and that of formal diffeomorphisms.

## 2.2 Coloops in general categories

Given a category  $\mathbf{C}$ , Yoneda Lemma says that the category of representable functors from  $\mathbf{C}$  to  $\mathbf{Set}$ , with natural transformations, is equivalent to  $\mathbf{C}$ . The equivalence is realized by the contravariant Yoneda functor  $Y$  from  $\mathbf{C}$  to the functor category  $\mathbf{Set}^{\mathbf{C}}$ , defined on any object  $H$  in  $\mathbf{C}$  by the functor  $Y(H) = \text{Hom}_{\mathbf{C}}(H, \_)$ , and on any map  $\phi : H' \rightarrow H$  by the natural transformation  $Y(\phi) : Y(H) \rightarrow Y(H') : \alpha \mapsto \alpha \phi$  (cf. [27] for details). In this section we characterise the subcategory of  $\mathbf{C}$  equivalent to representable functors from  $\mathbf{C}$  to **Loop**.

The cartesian product of two functors  $Y(H_1)$  and  $Y(H_2)$  is known to be represented by the categorical coproduct  $H_1 \amalg H_2$ , i.e.  $Y(H_1) \times Y(H_2) = Y(H_1 \amalg H_2)$ , and the constant functor to the base point is known to be represented by an initial object  $I$ , i.e. it is of the form  $Y(I)$ . We recall the categorical notations about the coproduct, the initial object and some related categorical maps we need to define coloops.

The coproduct in a category  $\mathbf{C}$  is a bifunctor  $\amalg$  defined on two objects  $A$  and  $B$  as the unique object  $A \amalg B$  together with two maps  $i_1 : A \rightarrow A \amalg B$  and  $i_2 : B \rightarrow A \amalg B$  satisfying the following universal property: for any maps  $f : A \rightarrow C$  and  $g : B \rightarrow C$ , there exists a unique map  $\langle f, g \rangle : A \amalg B \rightarrow C$  such that  $\langle f, g \rangle i_1 = f$  and  $\langle f, g \rangle i_2 = g$ . On two maps  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$ , the bifunctor is defined as the map  $f \amalg g = \langle i'_1 f, i'_2 g \rangle : A \amalg B \rightarrow A' \amalg B'$ . The coproduct can be extended to several objects and maps with similar universal constructions, and turns out to be an associative bifunctor, in the sense that  $(A \amalg B) \amalg C = A \amalg (B \amalg C) = A \amalg B \amalg C$  for any three objects and  $(f \amalg g) \amalg h = f \amalg (g \amalg h) = f \amalg g \amalg h$  for any three maps in  $\mathbf{C}$ .

An initial object in  $\mathbf{C}$  is an object  $I$  together with a unique map  $u_A : I \rightarrow A$  on any object, which commutes with any map  $f : A \rightarrow B$ , that is,  $f u_A = u_B$ . Then, there are canonical isomorphisms  $A \amalg I \cong A \cong I \amalg A$  given by the maps

$$\begin{aligned} \varphi_1 : A \longrightarrow A \amalg I & \quad \text{with inverse} & \quad \psi_1 = \langle \text{id}_A, u_A \rangle : A \amalg I \longrightarrow A, \\ \varphi_2 : A \longrightarrow I \amalg A & \quad \text{with inverse} & \quad \psi_2 = \langle u_A, \text{id}_A \rangle : I \amalg A \longrightarrow A. \end{aligned}$$

In particular, we have  $I \amalg I \cong I$ ,  $\langle u_A, u_A \rangle = u_A$  and therefore also  $u_{A \amalg B} = u_A \amalg u_B$ .

For any objects  $A$  and  $B$ , there is a canonical symmetry operator  $\tau_{A,B} = \langle i_2, i_1 \rangle : A \amalg B \rightarrow B \amalg A$  such that  $\tau_{A,B}^{-1} = \tau_{B,A}$ . Note that  $B \amalg A = A \amalg B$  as objects in  $\mathbf{C}$ , but the maps  $i_1$  and  $i_2$  are inverted. The twist  $\tau$  is precisely the map which identifies  $A \amalg B$  and  $B \amalg A$  as universal objects. To sum up,  $(\mathbf{C}, \amalg, I, \tau)$  is a *strict symmetric monoidal category*.

Furthermore, for any  $A$ , there exists a canonical folding map  $\mu_A = \langle \text{id}_A, \text{id}_A \rangle : A \amalg A \rightarrow A$  such that, for any maps  $f, g : A \rightarrow B$ , we have

$$\langle f, g \rangle = \mu_B (f \amalg g).$$

It follows that  $\mu$  preserves the unit, i.e.  $\mu_A(u_A \amalg u_A) = \langle u_A, u_A \rangle = u_A$ , that it is associative, i.e.  $\mu_A(\mu_A \amalg \text{id}_A) = \mu_A(\text{id}_A \amalg \mu_A)$ , and that it is commutative, i.e.  $\mu_A \tau_{A,A} = \mu_A$ . It also follows that  $\mu$  commutes with any map  $f : A \rightarrow B$  in  $\mathbf{C}$ , i.e.  $\mu_B (f \amalg f) = f \mu_A$ . To sum up, we can say that *any object  $(A, \mu_A, u_A)$  is a commutative monoid in  $\mathbf{C}$* , with respect to the monoidal product  $\amalg$ , and that *any map  $f : A \rightarrow B$  in  $\mathbf{C}$  is a morphism of monoids*. Finally, one can prove that the folding map on  $A \amalg B$  is given by  $\mu_{A \amalg B} = (\mu_A \amalg \mu_B) (\text{id}_A \amalg \tau_{B,A} \amalg \text{id}_B)$ .

**Definition 2.1** Let us call **coloop in  $\mathbf{C}$**  an object  $H$  endowed with the following maps in  $\mathbf{C}$ :

i) a **comultiplication**  $\Delta : H \rightarrow H \amalg H$ ;

ii) a **count**  $\varepsilon : H \rightarrow I$  satisfying the **countary** property

$$(\varepsilon \amalg \text{id}) \Delta = \varphi_2 \quad \text{and} \quad (\text{id} \amalg \varepsilon) \Delta = \varphi_1, \quad (2.4)$$

where  $\varphi_1 : H \rightarrow H \amalg I$  and  $\varphi_2 : H \rightarrow I \amalg H$  are the canonical isomorphisms;

iii) a **right codivision**  $\delta_r : H \rightarrow H \amalg H$  satisfying the **right cocancellation** properties

$$(\text{id} \amalg \mu) (\delta_r \amalg \text{id}) \Delta = i_1 \quad \text{and} \quad (\text{id} \amalg \mu) (\Delta \amalg \text{id}) \delta_r = i_1, \quad (2.5)$$

where  $i_1 : H \rightarrow H \amalg H$  can be factorized as  $i_1 = (\text{id} \amalg u) \varphi_1$ ;

iv) a **left codivision**  $\delta_l : H \rightarrow H \amalg H$  satisfying the **left cocancellation** properties

$$(\mu \amalg \text{id}) (\text{id} \amalg \delta_l) \Delta = i_2 \quad \text{and} \quad (\mu \amalg \text{id}) (\text{id} \amalg \Delta) \delta_l = i_2, \quad (2.6)$$

where  $i_2 : H \rightarrow H \amalg H$  can be factorized as  $i_2 = (u \amalg \text{id}) \varphi_2$ .

If  $H$  and  $H'$  are two coloops in  $\mathbf{C}$ , we say that a map  $f : H \rightarrow H'$  is a **homomorphism of coloops** if it commutes with the coproducts, the counits and the codivisions.

**Proposition 2.2** *Let  $H$  be a coloop in  $\mathbf{C}$ .*

1. *The codivisions verify the identities*

$$\mu \delta_r = u \varepsilon \quad \text{and} \quad \mu \delta_l = u \varepsilon, \quad (2.7)$$

*and the following partial counitality properties*

$$(\text{id} \amalg \varepsilon) \delta_r = \varphi_1 \quad \text{and} \quad (\varepsilon \amalg \text{id}) \delta_l = \varphi_2. \quad (2.8)$$

2. We can define a **right antipode**  $S_r : H \longrightarrow H$  and a **left antipode**  $S_l : H \longrightarrow H$  by setting

$$S_r := \psi_2(\varepsilon \amalg \text{id}) \delta_r \quad \text{and} \quad S_l := \psi_1(\text{id} \amalg \varepsilon) \delta_l, \quad (2.9)$$

where  $\psi_1 = \langle \text{id}, u_H \rangle : H \amalg I \rightarrow H$  and  $\psi_2 = \langle u_H, \text{id} \rangle : I \amalg H \rightarrow H$  are isomorphisms. The antipodes satisfy the following **left and right five-terms identities**

$$\mu(S_r \amalg \text{id}) \Delta = u \varepsilon \quad \text{and} \quad \mu(\text{id} \amalg S_l) \Delta = u \varepsilon. \quad (2.10)$$

These properties are easily verified. A proof using tangle diagrams is given in the Appendix.

**Theorem 2.3** *Let  $\mathbf{C}$  be a category with coproduct and initial object. Then the Yoneda functor is a contravariant equivalence of categories from the category of coloops in  $\mathbf{C}$  to that of covariant representable functors  $Q : \mathbf{C} \longrightarrow \mathbf{Loop}$ .*

**Proof.** We follow the ideas of Eckmann-Hilton [14], who characterized the subcategories of  $\mathbf{C}$  equivalent to the category of representable functors from  $\mathbf{C}$  respectively to the category  $\mathbf{Mag}$  of unital multiplicative sets, called unital magmas in [38, 4], and to the category  $\mathbf{Grp}$  of groups<sup>1</sup>.

Let us first prove that the Yoneda functor, applied to coloops in  $\mathbf{C}$ , gives rise to a functor in loops. On a given coloop  $H$ , let us call  $Q = Y(H)$ . We define the multiplication and the divisions on each set  $Q(A) = \text{Hom}_{\mathbf{C}}(H, A)$  as usual convolution with the coproduct and the codivisions in  $H$ , namely

$$\begin{aligned} \alpha \cdot \beta &= \langle \alpha, \beta \rangle \Delta = \mu_A(\alpha \amalg \beta) \Delta \\ \alpha / \beta &= \langle \alpha, \beta \rangle \delta_r = \mu_A(\alpha \amalg \beta) \delta_r \\ \alpha \setminus \beta &= \langle \alpha, \beta \rangle \delta_l = \mu_A(\alpha \amalg \beta) \delta_l, \end{aligned} \quad (2.11)$$

for any  $\alpha, \beta \in Q(A)$ . The unit in  $Q(A)$  is given, as usual, by the map  $1_A = u_A \varepsilon$ , and the left and right inverses of  $\alpha$  are then easily described as  $\alpha \setminus 1 = \alpha S_l$  and  $1 / \alpha = \alpha S_r$ . Then, using the cocancellation identities (2.5) and (2.6), and because  $\mu_A$  is associative and commutes with  $\mathbf{C}$ -maps, it is easy to verify that the divisions given by (2.11) satisfy the cancellation properties (2.2) and (2.1).

Now fix a homomorphism of coloops  $\phi : H' \longrightarrow H$ , and call  $Q = Y(H)$ ,  $Q' = Y(H')$  and  $\Phi = Y(\phi)$ . Yoneda Lemma tells us already that  $\Phi$  is given on an object  $A$  by the map

$$\begin{aligned} \Phi_A : Q(A) &\longrightarrow Q'(A) \\ \alpha &\longmapsto \Phi_A(\alpha) = \alpha \phi, \end{aligned}$$

and that, for any  $f : A \rightarrow B$ ,  $\Phi$  acts on the map  $Q(f) : Q(A) \rightarrow Q(B)$  given by  $Q(f)(\alpha) = f \alpha$  as a natural transformation, i.e.

$$\Phi_B(Q(f)(\alpha)) = f \alpha \phi = Q'(f)(\Phi_A(\alpha)).$$

It is then easy to verify that  $\Phi_A$  is a homomorphism of loops, that is, for any  $\alpha, \beta \in Q(A)$ , we have

$$\Phi_A(\alpha \cdot \beta) = \Phi_A(\alpha) \cdot \Phi_A(\beta),$$

and similarly for the other co-operations.

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<sup>1</sup>Eckmann-Hilton require  $\mathbf{C}$  to have zero-maps, we replace them with an initial object.



Viceversa, let us describe how a functor in loops  $Q$  gives rise to a coloop structure on its representative object  $H$ . Suppose that the covariant functor  $Q$  is represented by an object  $H$ , i.e.  $Q = Y(H)$ , that the set  $Q(A)$  is a loop for any  $A$  in  $\mathbf{C}$ , and that for any map  $f : A \rightarrow B$  the induced map  $Q(f) : Q(A) \rightarrow Q(B)$  given by  $\alpha \mapsto Q(f)(\alpha) = f\alpha$  is a loop homomorphism. We use repeatedly the fact that, given  $\alpha, \beta \in Q(A)$ , for the composite maps  $f(\alpha \cdot \beta), f\alpha, f\beta \in Q(B)$  we have

$$f(\alpha \cdot \beta) = Q(f)(\alpha \cdot \beta) = Q(f)(\alpha) \cdot Q(f)(\beta) = (f\alpha) \cdot (f\beta) \quad (2.12)$$

and similarly for the operations  $/$  and  $\backslash$ . Seeing  $i_1, i_2 : H \rightarrow H \amalg H$  as elements of  $Q(H \amalg H)$ , we define the comultiplication and the codivisions on  $H$  by

$$\Delta = i_1 \cdot i_2, \quad \delta_r = i_1/i_2, \quad \delta_l = i_1 \backslash i_2$$

and the counit  $\varepsilon$  as the unit  $1_I$  in  $Q(I)$ . It follows that the antipodes are the inverses of the identity map,  $S_r = 1_H/\text{id}_H$  and  $S_l = \text{id}_H \backslash 1_H$ .

Let us show that these maps give a coloop structure to  $H$ , and that the functor  $Q \mapsto H$  is inverse to the Yoneda one,  $H \mapsto Q = Y(H)$ . For any  $\alpha, \beta \in Q(A)$ , we apply (2.12) to  $A = H \amalg H$ ,  $B = A$ ,  $f = \langle \alpha, \beta \rangle : H \amalg H \rightarrow A$  and to the elements  $\alpha = i_1, \beta = i_2$  of  $Q(H \amalg H)$ , and get

$$\begin{aligned} \langle \alpha, \beta \rangle \Delta &= \langle \alpha, \beta \rangle (i_1 \cdot i_2) \\ &= (\langle \alpha, \beta \rangle i_1) \cdot (\langle \alpha, \beta \rangle i_2) = \alpha \cdot \beta \end{aligned} \quad (2.13)$$

and similarly for the operations  $/$  and  $\backslash$ . Now apply  $Q$  to a unit map  $u_A : I \rightarrow A$ . Since  $Q(u_A) : Q(I) \rightarrow Q(A)$  is a homomorphism of loops, it preserves the units, and therefore, for  $\varepsilon = 1_I \in Q(I)$ , we have

$$Q(u_A)(\varepsilon) = u_A \varepsilon = 1_A.$$

In particular we have  $u_H \varepsilon = 1_H$ , and therefore, using (2.13), we have

$$\langle u_H \varepsilon, \text{id}_H \rangle \Delta = 1_H \cdot \text{id}_H = \text{id}_H.$$

On the other side, we have

$$\begin{aligned} \langle u_H \varepsilon, \text{id}_H \rangle \Delta &= \langle u_H, \text{id}_H \rangle (\varepsilon \amalg \text{id}) \Delta \\ &= \psi_2 (\varepsilon \amalg \text{id}) \Delta, \end{aligned}$$

and we obtain the equality  $\psi_2 (\varepsilon \amalg \text{id}) \Delta = \text{id}$ . Since  $\psi_2$  is the inverse map to  $\varphi_2$ , we obtain  $(\varepsilon \amalg \text{id}) \Delta = \varphi_2$ , which proves (2.4). Let us show equalities (2.5). Firstly, we have trivially that

$$(\text{id} \amalg u_H) i_1 = \langle i_1, i_2 u_H \rangle i_1 = i_1.$$

Secondly, note that  $\delta_r \cdot i_2 = (i_1/i_2) \cdot i_2 = i_1$ , therefore

$$\begin{aligned} (\text{id} \amalg \mu) (\delta_r \amalg \text{id}) \Delta &= \langle i_1, i_2 \mu \rangle \langle i_1 \delta_r, i_2 \rangle (i_1 \cdot i_2) \\ &= \langle i_1, i_2 \mu \rangle (\delta_r \cdot i_2) \\ &= \langle i_1, i_2 \mu \rangle i_1 = i_1. \end{aligned}$$

Finally, since  $\langle \alpha, \beta \rangle (i_1/i_2) = \alpha/\beta$  and  $(i_1 \cdot i_2)/i_2 = i_1$ , we also have

$$\begin{aligned} (\text{id} \amalg \mu) (\Delta \amalg \text{id}) \delta_r &= \langle i_1, i_2 \mu \rangle \langle i_1 \Delta, i_2 \rangle (i_1/i_2) \\ &= \langle i_1, i_2 \mu \rangle ((i_1 \cdot i_2)/i_2) \\ &= \langle i_1, i_2 \mu \rangle i_1 = i_1. \end{aligned}$$

The same arguments apply to the left codivision.  $\square$

The relationship between coloops and cogroups is straightforward. As usual, a coloop  $H$  is **coassociative** if

$$(\Delta \amalg \text{id}) \Delta = (\text{id} \amalg \Delta) \Delta, \quad (2.14)$$

and  $H$  is **cocommutative** if  $\tau \Delta = \Delta$ .

We say that  $H$  has the **left** and **right coinverse property** if the codivisions are determined by the antipodes, that is,

$$\delta_l = (S_l \amalg \text{id}) \Delta. \quad \text{and} \quad \delta_r = (\text{id} \amalg S_r) \Delta \quad (2.15)$$

These identities correspond to the analogues (2.3) in the loop  $Q = Y(H)$ .

Furthermore, an **antipode** on  $H$  is a map  $S : H \rightarrow H$  satisfying the **five-terms identity**

$$\mu(S \amalg \text{id}) \Delta = \mu(\text{id} \amalg S) \Delta = u \varepsilon. \quad (2.16)$$

This happens if and only if  $S_r = S_l$ . Note that the unicity of the antipode does not imply that the coinverse properties are verified. A counterexample is given by the coloop of formal diffeomorphisms, cf. Section 5.

A cogroup in a category  $\mathbf{C}$  is an object  $H$  endowed with a coassociative comultiplication  $\Delta$ , a counit  $\varepsilon$  satisfying the counitary property (2.4) and an antipode  $S$  satisfying the five-terms identity (2.16), cf. [3].

**Proposition 2.4** *If  $H$  is a coassociative coloop, then it is a cogroup.*

In fact, in the Appendix we prove with tangles that coassociativity implies that the left and the right antipodes coincide, and therefore  $H$  has an antipode satisfying the five-terms identity (2.16), and moreover that it satisfies the coinverse properties (2.15).

### 2.3 (Pro)algebraic loops

Let  $\mathbf{A}$  be a variety of unital algebras over a field  $\mathbb{F}$ , that is, the subcategory of vector spaces over  $\mathbb{F}$  which collects all algebras of a certain *type*, given by a set of operations of various arities, included the unit of arity 0, defined by a set of identities (cf. [27] ch. V). For instance,  $\mathbf{A}$  can be the category of  $\mathcal{P}$ -algebras, where  $\mathcal{P}$  is an algebraic operad with  $\mathcal{P}(0) = \{\text{unit map}\}$  (cf. [25]).

Then,  $\mathbf{A}$  has a coproduct and an initial object (cf. [27] ch. IX), therefore we can apply to  $\mathbf{A}$  the results of the previous section. More precisely, the initial object is given by the trivial unital algebra  $\mathbb{F}$ . Suppose that in  $\mathbf{A}$  there are operations  $p$  of arity  $n > 0$ , and let  $\bar{A}$  denote the subalgebra of such operations, then  $A \amalg B$  is the quotient of the free algebra  $\mathbf{A}(\bar{A} \oplus \bar{B})$  (which always exists, cf. [27] ch. V) by the ideal generated by the identities

$$\begin{aligned} p_{\mathbf{A}(A \oplus B)}(a_1, \dots, a_n) &= p_A(a_1, \dots, a_n) \in \bar{A}, & \text{for any } a_1, \dots, a_n \in \bar{A} \\ p_{\mathbf{A}(A \oplus B)}(b_1, \dots, b_n) &= p_B(b_1, \dots, b_n) \in \bar{B}, & \text{for any } b_1, \dots, b_n \in \bar{B} \\ 1_A &= 1_B = 1_{\mathbf{A}(A \oplus B)}, \end{aligned} \quad (2.17)$$

for all the operations  $p$  admitted in  $\mathbf{A}$ . The universal properties of  $\amalg$  follow from the universal properties of the free algebra  $\mathbf{A}(\bar{A} \oplus \bar{B})$ .

**Examples 2.5** 1. In the category  $\mathbf{Com}_{\mathbb{F}}$  of unital commutative and associative algebras over  $\mathbb{F}$ , the free algebra  $\mathbf{Com}_{\mathbb{F}}(V)$  on a vector space  $V$  is the symmetric algebra  $S(V)$ , and the coproduct of two algebras  $A$  and  $B$  is the tensor product  $A \otimes B$ .

2. In the category  $\mathbf{As}_{\mathbb{F}}$  of unital associative algebras over  $\mathbb{F}$ , the free algebra  $\mathbf{As}_{\mathbb{F}}(V)$  is the tensor algebra  $T(V)$ , and the coproduct<sup>2</sup>  $A \amalg B$  of two algebras  $A$  and  $B$  is the tensor algebra  $T(\bar{A} \oplus \bar{B})$  modulo relations (2.17), which mean that  $a \otimes a' = a a'$  whenever  $a$  and  $a'$  are both in  $\bar{A}$  or both in  $\bar{B}$ . As a vector space, we then have

$$A \amalg B = \mathbb{F} \oplus \bigoplus_{n \geq 1} \left[ \underbrace{\bar{A} \otimes \bar{B} \otimes \bar{A} \otimes \cdots}_n \oplus \underbrace{\bar{B} \otimes \bar{A} \otimes \bar{B} \otimes \cdots}_n \right],$$

and the multiplication in  $A \amalg B$  is given by the concatenation modulo the above relations. For instance, if we denote the multiplication in  $A \amalg B$  by  $\bullet$ , we have

$$\begin{aligned} (b \otimes a) \bullet (b' \otimes a' \otimes b'') &= b \otimes a \otimes b' \otimes a' \otimes b'' \\ (b \otimes a) \bullet (a' \otimes b' \otimes a'') &= b \otimes (a a') \otimes b' \otimes a''. \end{aligned}$$

3. Let  $\mathbf{Alg}_{\mathbb{F}}$  be the category of unital algebras (not necessarily associative, also called magmatic) over  $\mathbb{F}$ . The free unital algebra on a vector space  $V$  is the tensor algebra with parenthesizing  $T\{V\}$ , and the coproduct  $A \amalg B$  of two algebras  $A$  and  $B$  is the quotient of  $T\{\bar{A} \oplus \bar{B}\}$  modulo relations (2.17), which again mean that  $a \otimes a' = a a'$  whenever  $a$  and  $a'$  are both in  $\bar{A}$  or both in  $\bar{B}$ . The multiplication in  $A \amalg B$  is the concatenation with parenthesis.
4. An **alternative algebra** is an algebra  $A$  such that the associator  $(a, b, c) = (ab)c - a(bc)$  is skew-symmetric, that is

$$(b, a, c) = -(a, b, c) \quad \text{and} \quad (a, c, b) = -(a, b, c) \quad (2.18)$$

for any  $a, b, c \in A$ . This is equivalent to require that

$$(ab)b = a(bb) \quad \text{and} \quad (aa)b = a(ab)$$

for any  $a, b \in A$ . Unital alternative algebras over a field  $\mathbb{F}$  form a subcategory of  $\mathbf{Alg}_{\mathbb{F}}$ , denoted by  $\mathbf{Alt}_{\mathbb{F}}$ , which is a variety with initial object  $\mathbb{F}$ . The coproduct  $A \amalg B$  of two unital alternative algebras is the quotient of the coproduct in  $\mathbf{Alg}_{\mathbb{F}}$  by the relations (2.18). For details see [43].

5. In a category  $\mathbf{A}$ , an (anti) **involution** is a unary linear operation  $*$  :  $A \rightarrow A$  such that

$$(a^*)^* = a \quad \text{and} \quad (a_1 a_2)^* = a_2^* a_1^*, \quad (2.19)$$

for any  $a, a_1, a_2 \in A$ . Each of the four previous categories of algebras can be considered with involution, and denote by  $\mathbf{A}^*$ . For such algebras, the initial object and the coproduct are the same as in  $\mathbf{A}$ , the involution on  $A \amalg B$  is automatically defined from the involutions on  $A$  and  $B$  by properties (2.19). Note that, in  $\mathbf{Alg}_{\mathbb{F}}$  and in  $\mathbf{Alt}_{\mathbb{F}}$ , the parenthesizing of a word  $a_1 \otimes \cdots \otimes a_n$  is inverted from left to right by the involution, together with the single letters of the word.

**Definition 2.6** A coloop  $H$  in a variety of unital algebras  $\mathbf{A}$  is called a **coloop  $\mathbf{A}$ -bialgebra**. Its associated functor in loops  $Q = Y(H)$  is then called an **algebraic loop on  $\mathbf{A}$**  if  $H$  is a finitely generated algebra, and a **pralgebraic loop on  $\mathbf{A}$**  if  $H$  is not finitely generated. In this case, it is an inductive limit of finitely generated coloop  $\mathbf{A}$ -bialgebras.

<sup>2</sup>In associative algebras, the coproduct is usually called *free product* and denoted by  $*$ .

There are not many examples of algebraic loops. In section 3, we give an example of an algebraic group on commutative algebras which can be extended to associative algebras (the group of unitary elements), and another one which can not be extended to a functor on associative algebras even as a loop (the Cayley-Dickson loop). Viceversa, in section 5 we give the example of a proalgebraic group which can be extended to associative algebras as a proalgebraic loop (the loop of formal diffeomorphisms). Finally, in section 4, we also give an example of an algebraic group which can be extended to associative algebras as a group, and to non-associative algebras as a loop (the loop of invertible series).

**Remark 2.7** All these examples of cogroups and coloops have free underlying algebra structure. The fact that this should hold in any category (under certain completeness hypothesis) has not been proved, but it was proved for cogroups in several categories: by D. Kan [21] in the category of groups, by I. Berstein [3] (and later reproved by J. Zhang in [44]) in the category of graded connected associative algebras, and by B. Fresse [16] in the category of complete algebra over any operad. For coloops, this result is proved by G. Bergman and A.O. Hausknecht [2] in the category of graded connected associative rings.

Before giving the examples, we mention two maps which allow to compare coloop and cogroup bialgebras to usual Hopf algebras. A coloop  $\mathbf{A}$ -bialgebra has the operations  $p : H^{\otimes n} \rightarrow H$  from  $\mathbf{A}$ , and the categorical folding map  $\mu : H \amalg H \rightarrow H$  needed to describe the coloop axioms, which can be iterated on  $n$  copies of  $H$ . In general, there is no relationship between these two types of operations, since  $H^{\otimes n}$  need not being an algebra in  $\mathbf{A}$ .

Assume that  $\mathbf{A}$  is a category of algebras such that, for any  $\mathbf{A}$ -algebras  $A$  and  $B$ , the tensor product  $A \otimes B$  is again an  $\mathbf{A}$ -algebra with componentwise operations

$$p_{A \otimes B}^{(n)}(a_1 \otimes b_1, \dots, a_n \otimes b_n) = p_A^{(n)}(a_1, \dots, a_n) \otimes p_B^{(n)}(b_1, \dots, b_n)$$

and unit  $1_{A \otimes B} = 1_A \otimes 1_B$ .

**Definition 2.8** For any  $n \geq 2$  and for any  $n$  algebras  $A_k$ , with  $k = 1, \dots, n$ , we call **canonical projection** of  $A_1 \amalg \dots \amalg A_n$  onto  $A_1 \otimes \dots \otimes A_n$  the algebra homomorphism

$$\pi := \langle j_1, \dots, j_n \rangle : A_1 \amalg \dots \amalg A_n \rightarrow A_1 \otimes \dots \otimes A_n$$

induced by the injective algebra maps  $j_k : A_k \rightarrow A_1 \otimes \dots \otimes A_n$  given by

$$j_k(a_k) = 1_{A_1} \otimes \dots \otimes a_k \otimes \dots \otimes 1_{A_n}.$$

The map  $\pi$  reorders the elements of  $A_1 \amalg \dots \amalg A_n$  and then multiplies them within each  $A_k$  to get elements in  $A_1 \otimes \dots \otimes A_n$ . For instance, if we denote by  $a^{(k)}$  an element  $a \in A_k$  seen in the coproduct  $A_1 \amalg \dots \amalg A_n$ , we have

$$\pi(a^{(1)}b^{(2)}c^{(1)}d^{(2)}) = (ac) \otimes (bd).$$

This map is surjective, because a preimage of any  $a_1 \otimes a_2 \otimes \dots \otimes a_n \in A_1 \otimes \dots \otimes A_n$  by  $\pi$  is given by  $a_1^{(1)}a_2^{(2)} \dots a_n^{(n)} \in A_1 \amalg \dots \amalg A_n$ .

Note that, when all  $A_k$  coincide and we are given an operation  $p$  of arity  $n$ , the map  $p \pi : A^{\amalg n} \rightarrow A$  is not, in general, an algebra homomorphism (because  $p$  is not), and therefore it surely differs from the folding map  $\mu = \langle \text{id}_A, \dots, \text{id}_A \rangle$ . In fact,  $\mu$  multiplies the elements of  $A$  in the order they appear in  $A^{\amalg n}$  (it is a *concatenation*), while  $p \pi$  first reorders the factors in  $A^{\amalg n}$  with  $\pi$ , as explained above, then multiplies them (it is a *componentwise operation*).

**Definition 2.9** On the other side, for any  $n \geq 2$  and for any  $n$  algebras  $A_k$ , with  $k = 1, \dots, n$ , there are categorical maps  $i_k : A_k \rightarrow A_1 \amalg \dots \amalg A_n$ . For any operation  $p$  of arity  $n$  in  $\mathbf{A}$ , we call **canonical inclusion** of  $A_1 \otimes \dots \otimes A_n$  in  $A_1 \amalg \dots \amalg A_n$  the linear map  $\iota_p : A_1 \otimes \dots \otimes A_n \rightarrow A_1 \amalg \dots \amalg A_n$  defined by

$$\iota_p(a_1 \otimes \dots \otimes a_n) := p_{\amalg}(i_1(a_1), \dots, i_n(a_n)),$$

where  $p_{\amalg} : (A_1 \amalg \dots \amalg A_n)^{\otimes n} \rightarrow A_1 \amalg \dots \amalg A_n$  denotes the operation  $p$  on the coproduct algebra  $A_1 \amalg \dots \amalg A_n$ . It follows from the definition of  $\amalg$  that this map is injective.

Note that  $\iota_p$  is not, in general, an algebra homomorphism, because the operation  $p$  in  $\mathbf{A}$  is not. However, when all  $A_k$  coincide (say, with  $A$ ), the map  $\iota_p$  allows to recover the operation  $p_A : A^{\otimes n} \rightarrow A$  from the folding map  $\mu$ , in the sense that  $\mu \iota_p = p_A$ , because

$$\mu p_{\amalg}(i_1(a_1), \dots, i_n(a_n)) = p_A(a_1, \dots, a_n)$$

for any  $a_1, \dots, a_n \in A$ .

**Proposition 2.10** *When the map  $\iota_p$  is well defined, we have  $\pi \iota_p = \text{id}_{A_1 \otimes \dots \otimes A_n}$ .*

**Proof.** Denote by  $p_{\otimes}$  the operation  $p$  on the tensor algebra  $A_1 \otimes \dots \otimes A_n$ . Since  $\pi$  is an algebra homomorphism, for any  $a_k \in A_k$ , with  $k = 1, \dots, n$ , we have

$$\begin{aligned} \pi \iota_p(a_1 \otimes \dots \otimes a_n) &= \pi p_{\amalg}(i_1(a_1), \dots, i_n(a_n)) = p_{\otimes}(\pi(i_1(a_1)), \dots, \pi(i_n(a_n))) \\ &= p_{\otimes}(\langle j_1, \dots, j_n \rangle(i_1(a_1)), \dots, \langle j_1, \dots, j_n \rangle(i_n(a_n))) \\ &= p_{\otimes}(j_1(a_1), \dots, j_n(a_n)) \\ &= a_1 \otimes \dots \otimes a_n. \end{aligned}$$

□

**Remark 2.11** These maps allow in particular to compare the coloop bialgebra representing some loop to other types of bialgebras related to it which appear in the literature. In particular, the universal enveloping algebra of the Sabinin algebra associated to the loop has been studied in [33, 30, 31]. Because of the axioms, it is clear that the graded dual of this universal enveloping algebra does not coincide with the bialgebra  $H^{\otimes}$  induced by a coloop bialgebra  $H$ , nor in  $\mathbf{Alg}_{\mathbb{F}}$  nor in  $\mathbf{As}_{\mathbb{F}}$ .

Finally, let us use these maps to compare associative coloop bialgebras and Hopf algebras. Let  $H$  be a coloop bialgebra in  $\mathbf{As}_{\mathbb{F}}$ . Denote by  $H^{\otimes}$  the algebra  $H$  endowed with the usual co-operations

$$\Delta^{\otimes} = \pi \Delta, \quad \delta_r^{\otimes} = \pi \delta_r, \quad \delta_l^{\otimes} = \pi \delta_r : H^{\otimes} \rightarrow H^{\otimes} \otimes H^{\otimes},$$

the counit  $\varepsilon$  and the antipodes  $S_r, S_l$ , which are all still algebra homomorphisms on  $H^{\otimes}$ .

**Proposition 2.12** *If  $\Delta$  is coassociative, then  $\Delta^{\otimes}$  is coassociative. Moreover, we have*

$$S_r = (\varepsilon \otimes \text{id}) \delta_r^{\otimes} \quad \text{and} \quad S_l = (\text{id} \otimes \varepsilon) \delta_l^{\otimes}.$$

**Proof.** If  $\Delta$  is coassociative, the two terms

$$(\Delta^\otimes \otimes \text{id}) \Delta^\otimes = (\pi \otimes \text{id}) \pi_{(H \amalg H) \amalg H} (\Delta \amalg \text{id}) \Delta$$

and

$$(\text{id} \otimes \Delta^\otimes) \Delta^\otimes = (\text{id} \otimes \pi) \pi_{H \amalg (H \amalg H)} (\text{id} \amalg \Delta) \Delta$$

coincide, because  $\Delta$  is coassociative and because the two maps  $(\pi \otimes \text{id}) \otimes \pi_{(H \amalg H) \amalg H}$  and  $(\text{id} \otimes \pi) \otimes \pi_{H \amalg (H \amalg H)}$  coincide with the standard projection  $\pi : H \amalg H \amalg H \longrightarrow H \otimes H \otimes H$ . Therefore  $\Delta^\otimes$  is coassociative.

For any  $a \in H$ , the term  $\delta_r(a) \in H^{(1)} \amalg H^{(2)}$  is a finite sum of products of elements of  $H^{(1)}$  and of  $H^{(2)}$  in alternative order. The right antipode  $S(a)_r = (\varepsilon \amalg \text{id}) \delta_r(a)$  turns all the factors belonging to  $H^{(1)}$  into scalars, which can then be positioned on the lefthand side of all the remaining elements belonging to  $H^{(2)}$ . Therefore the result is the same that we obtain if we first reorder the factors in  $H^{(1)}$  all at the leftmost position by applying  $\delta_r(a)^\otimes$ . Same with  $S_l$  by putting all the scalars on the rightmost position.  $\square$

Note however that  $S_r$  and  $S_l$  do not necessarily satisfy the left and right five-terms identities for  $\Delta^\otimes$  on  $H^\otimes$ , because

$$m(S_r \otimes \text{id}) \Delta^\otimes = m \pi (S_r \amalg \text{id}) i \Delta^\otimes = \mu \iota \pi (S_r \amalg \text{id}) \iota \pi \Delta$$

and  $\iota \pi$  is not the identity map on  $H \amalg H$ . Therefore, even if  $H$  is a cogroup bialgebra,  $H^\otimes$  is not necessarily a Hopf algebra.

### 3 Coloops of invertible and unitary elements

#### 3.1 Loop of invertible elements

In this section we give an example of an abelian algebraic group which can be extended to associative algebras as a group, to alternative algebras as a loop, but not to non-associative algebras, even as a loop.

Let  $\mathbb{F}$  be a field. For any unital commutative algebra  $A$  over  $\mathbb{F}$ , the set

$$I(A) = \{a \in A \mid a \text{ admits an inverse } a^{-1}\}$$

is the abelian group of invertible elements in  $A$ . The functor  $I$  is represented on  $\mathbf{Com}_{\mathbb{F}}$  by the commutative (and cocommutative) Hopf algebra of Laurent polynomials  $H_{\mathbb{I}} = \mathbb{F}[x, x^{-1}]$ , with evident co-operations

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1, \quad S(x) = x^{-1}.$$

We show that the functor  $I$  admits an extension to associative algebras as a group, and that it admits an extension to non-associative algebras, as a loop, only on alternative algebras.

**Definition 3.1** Let us call **invertible coloop bialgebra** on  $\mathbb{F}$  the associative algebra  $H_{\mathbb{I}}^{\amalg} = \mathbb{F}[x, x^{-1}]$  endowed with the following co-operations with values in the coproduct  $H_{\mathbb{I}}^{\amalg} \amalg H_{\mathbb{I}}^{\amalg}$  of the category  $\mathbf{As}_{\mathbb{F}}$ :

$$\begin{aligned} \Delta(x) &= x^{(1)} x^{(2)} & \Delta(x^{-1}) &= (x^{-1})^{(2)} (x^{-1})^{(1)}, \\ \varepsilon(x) &= 1 & \varepsilon(x^{-1}) &= 1, \\ \delta_r(x) &= x^{(1)} (x^{-1})^{(2)} & \delta_r(x^{-1}) &= x^{(2)} (x^{-1})^{(1)}, \\ \delta_l(x) &= (x^{-1})^{(1)} x^{(2)} & \delta_l(x^{-1}) &= (x^{-1})^{(2)} x^{(1)}. \end{aligned}$$

It follows that there is a two-sided antipode given by  $S(x) = x^{-1}$  and  $S(x^{-1}) = x$ .

**Proposition 3.2** *The algebra  $H_1^{\mathbb{H}}$  is a cogroup bialgebra in  $\mathbf{As}_{\mathbb{F}}$  and represents, for any associative algebra  $A$ , the group*

$$I(A) = \text{Hom}_{\mathbf{As}_{\mathbb{F}}}(H_1^{\mathbb{H}}, A)$$

*of invertible elements of  $A$ . Moreover, the group  $I(A)$  is abelian if and only if  $A$  is commutative.*

**Proof.** The axioms of a coloop bialgebra for the codivisions are easily verified. The first claim is then ensured by the fact that  $\Delta$  is clearly coassociative. Thus, the second statement is evident.

The fact that the group  $I(A)$  is abelian if and only if  $A$  is commutative is less evident because  $\Delta$  is not cocommutative. In fact, we have

$$\tau \Delta(x) = \tau(x^{(1)} x^{(2)}) = x^{(2)} x^{(1)} \neq x^{(1)} x^{(2)} = \Delta(x).$$

It is however true because the generators  $x$  and  $x^{-1}$  are group-like, and therefore the commutativity of the convolution product only depends on that of the multiplication in  $A$ .  $\square$

**Example 3.3** An evident example is  $I(M_n(\mathbb{F})) = GL_n(\mathbb{F})$ .

**Proposition 3.4** *The algebraic group  $I$  can be extended as a loop to a variety of algebras  $\mathbf{A} \subset \mathbf{Alg}_{\mathbb{F}}$  if and only if  $\mathbf{A}$  is a subcategory of alternative algebras  $\mathbf{Alt}_{\mathbb{F}}$  admitting coproduct and initial object. In particular, it is an algebraic loop on  $\mathbf{Alt}_{\mathbb{F}}$ .*

**Proof.** If  $I$  could be extended as an algebraic loop to  $\mathbf{Alg}$ , its representative coloop bialgebra should be the algebra  $H_1^{\mathbb{H}} = \mathbb{F}[x, x^{-1}]$  with co-operations defined on generators as in Def. 3.1 but taking values in the coproduct  $H_1^{\mathbb{H}} \amalg H_1^{\mathbb{H}}$  of the category  $\mathbf{Alg}$ . This algebra is not a coloop bialgebra in  $\mathbf{Alg}$ , because the codivisions do not satisfy the cocancellation properties (2.6) and (2.5). In fact, the element

$$(\text{id} \amalg \mu) (\delta_r \amalg \text{id}) \Delta(x) = (x^{(1)} (x^{-1})^{(2)}) x^{(2)}$$

can not coincide with  $i_1(x) = x^{(1)}$  in  $H_1^{\mathbb{H}} \amalg H_1^{\mathbb{H}}$ .

However, the conditions which make the cocancellation being satisfied, all similar to the one above, are precisely equivalent to the axioms of alternative algebras, when all non-zero elements are invertible (cf. [43]).  $\square$

**Example 3.5** The octonions  $\mathbb{O}$  form an alternative algebra, where the set  $I(\mathbb{O})$  of invertible elements is well known to be a Moufang loop (cf. [9]), that is, it is a loop satisfying the Moufang identities

$$\begin{aligned} a(b(ca)) &= ((ab)a)c & (ab)(ca) &= (a(bc))a \\ a(b(cb)) &= ((ab)c)b & (ab)(ca) &= a((bc)a) \end{aligned}$$

for any elements  $a, b, c$ .

### 3.2 Loop of unitary elements

Consider now involutive algebras  $A$ , and the subgroup of  $I(A)$  made of unitary elements in  $A$ , namely

$$U(A) = \{a \in A \mid a a^* = 1\},$$

when  $A$  is commutative. Exactly as for  $I$ , the functor  $U$  is represented on  $\mathbf{Com}_{\mathbb{F}}^*$  by the commutative Hopf algebra  $H_U = \mathbb{F}[x, x^* \mid x x^* = 1]$ , with co-operations

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1, \quad S(x) = x^*.$$

**Definition 3.6** Let us call **unitary coloop bialgebra** on  $\mathbb{F}$  the associative algebra  $H_U^{\mathbb{H}} = \mathbb{F}[x, x^* \mid x x^* = 1]$  endowed with the co-operations defined on generators exactly as those in Def. 3.1, where the generator  $x^{-1}$  is replaced by  $x^*$ .

As for the invertible coloop bialgebra, one can prove that

**Proposition 3.7** *The algebra  $H_U^{\mathbb{H}}$  is a cogroup bialgebra in  $\mathbf{As}_{\mathbb{F}}^*$  and represents, for any involutive associative algebra  $A$ , the group*

$$U(A) = \mathrm{Hom}_{\mathbf{As}_{\mathbb{F}}^*}(H_U^{\mathbb{H}}, A)$$

*of unitary elements of  $A$ . Moreover, the group  $U(A)$  is abelian if and only if  $A$  is commutative.*

**Examples 3.8** For  $\mathbb{F} = \mathbb{R}$ , this functor allows to describe several groups of unitary matrices.

1. Applied to the algebra  $M_n(\mathbb{R})$ , if we take the transposition of matrices as involution, it gives  $U(\mathbb{R}) = \{1, -1\}$ , and the orthogonal group  $U(M_n(\mathbb{R})) = O(n)$  for  $n > 1$ .
2. On  $M_n(\mathbb{C})$ , we take as involution the complex conjugate of the transposition. Then  $U(\mathbb{C}) = U(1) \cong \mathbb{S}^1$  and  $U(M_n(\mathbb{C})) = U(n)$  is the unitary group.
3. Let  $\mathbb{H}$  be the algebra of quaternions. On  $M_n(\mathbb{H})$ , we take as involution the quaternionic conjugate of the transposition. Then  $U(\mathbb{H}) \cong Sp(1) \cong SU(2) \cong \mathbb{S}^3$  and  $U(M_n(\mathbb{H})) \cong U(n, \mathbb{H}) \cong Sp(n)$  is the compact symplectic group, also called the hyperunitary group.

Again exactly as for the invertible coloop bialgebra, one can prove the next result.

**Proposition 3.9** *The algebraic group  $U$  can be extended as a loop to a variety of algebras  $\mathbf{A} \subset \mathbf{Alg}_{\mathbb{F}}^*$  if and only if  $\mathbf{A}$  is a subcategory of involutive alternative algebras  $\mathbf{Alt}_{\mathbb{F}}^*$  admitting coproduct and initial object. In particular,  $U$  is an algebraic loop on  $\mathbf{Alt}_{\mathbb{F}}^*$ .*

**Example 3.10** On the alternative algebra of octonions  $\mathbb{O}$ , the octonian conjugaison is an involution, and the set  $U(\mathbb{O})$  of unitary elements is again a Moufang loop, homomorphic to  $\mathbb{S}^7$ .

### 3.3 Unitary Cayley-Dickson loops

In this section we give an example of a loop which is not algebraic on associative algebras.

Let  $\mathbb{F}$  be a field and  $j$  denote an imaginary unit. For any involutive commutative algebra  $A$  over  $\mathbb{F}$ , the set

$$U_{CD}(A) = \{a + b j \in A + A j \mid a a^* + b b^* = 1\}$$



gives the group of unitary elements in the Cayley-Dickson algebra  $A + Aj$  with multiplication

$$(a + bj)(c + dj) = (ac - d^*b) + (da + bc^*)j,$$

unit 1, and involution  $(a + bj)^* = a^* - bj$ .

The functor  $A \longmapsto U_{CD}(A)$  is representable on  $\mathbf{Com}_{\mathbb{F}}^*$ , by the commutative Hopf algebra

$$H_{UCD} = \mathbb{F}[x, x^*, y, y^* \mid xx^* + yy^* = 1]$$

with co-operations

$$\begin{aligned} \Delta(x) &= x \otimes x - y \otimes y^* & \Delta(y) &= x \otimes y + y \otimes x^*, \\ \varepsilon(x) &= 1 & \varepsilon(y) &= 0, \\ S(x) &= x^* & S(y) &= -y. \end{aligned}$$

**Proposition 3.11** *The algebraic group  $U_{CD}$  can not be extended as a loop to the category of involutive associative algebras.*

**Proof.** If  $U_{CD}$  could be extended to an algebraic loop to  $\mathbf{As}_{\mathbb{F}}^*$ , its representative coloop bialgebra  $H_{UCD}^{\Pi}$  should be an associative algebra generated by  $x, x^*, y$  and  $y^*$  submitted to conditions which give  $xx^* + yy^* = 1$  if the variables commute. The co-operations should then be defined on generators exactly as in the commutative case, but taking values in the coproduct  $H_{UCD}^{\Pi} \amalg H_{UCD}^{\Pi}$  of the category  $\mathbf{As}_{\mathbb{F}}^*$ .

The conditions  $xx^* = x^*x$  and  $xx^* + yy^* = 1$  are enough to guarantee that the algebra  $H_{UCD}^{\Pi}$  has a well defined comultiplication, a counit and an antipode satisfying the five-terms identities. However, the codivisions, defined according to the coinverse properties (2.15) as

$$\begin{aligned} \delta_r(x) &= x^{(1)}(x^*)^{(2)} + (y^*)^{(2)}y^{(1)} & \delta_r(y) &= -y^{(2)}x^{(1)} + y^{(1)}x^{(2)} \\ \delta_l(x) &= (x^*)^{(1)}x^{(2)} + (y^*)^{(2)}y^{(1)} & \delta_l(y) &= y^{(2)}(x^*)^{(1)} - y^{(1)}(x^*)^{(2)}, \end{aligned}$$

satisfy the cocancellation identities (2.5) and (2.6) if and only if

$$x^{(1)}(x^*x)^{(2)} = (x^*x)^{(2)}x^{(1)} \quad \text{and} \quad y^{(1)}(y^*y)^{(2)} = (y^*y)^{(2)}y^{(1)}$$

in  $H_{UCD}^{\Pi} \amalg H_{UCD}^{\Pi}$ . This could happen for two reasons. The first is that the map  $N : H_{UCD}^{\Pi} \longrightarrow H_{UCD}^{\Pi}$  given by  $N(a) = aa^* = a^*a$  has scalar values, i.e. its image is in  $u(\mathbb{F}) \subset H_{UCD}^{\Pi}$ . This is the case if  $H_{UCD}^{\Pi}$  is a *composition algebra*, cf. [1]. But composition algebras do not have a coproduct. The second possibility to verify these conditions is that the identity  $a^{(1)}b^{(2)} = b^{(2)}a^{(1)}$  holds in  $H_{UCD}^{\Pi} \amalg H_{UCD}^{\Pi}$  for any elements  $a, b \in H_{UCD}^{\Pi}$ . This means that  $\amalg = \otimes$  and therefore it is only possible in the category  $\mathbf{Com}_{\mathbb{F}}^*$ .  $\square$

**Examples 3.12** In agreement with this result, we find few examples of loops arising as sets of unitary elements in the Cayley-Dickson algebra constructed on an associative algebra:

1.  $U_{CD}(\mathbb{R}) = U(\mathbb{C}) \cong \mathbb{S}^1$  is an abelian group.  $U_{CD}(M_n(\mathbb{R}))$  is a loop only for  $n = 2$ .
2.  $U_{CD}(\mathbb{C}) = U(\mathbb{H}) \cong \mathbb{S}^3$  is a group.  $U_{CD}(M_n(\mathbb{C}))$  is a loop only for  $n = 2$ .
3.  $U_{CD}(\mathbb{H}) \cong U(\mathbb{O}) \cong \mathbb{S}^7$  is a Moufang loop.  $U_{CD}(M_n(\mathbb{H}))$  is not a loop for  $n > 1$ .

## 4 Coloop of invertible series

The group of invertible series (with constant term equal to 1), is the set of formal series

$$\text{Inv}(A) = \left\{ a(\lambda) = \sum_{n \geq 0} a_n \lambda^n \mid a_0 = 1, a_n \in A \right\}$$

with coefficients  $a_n$  taken in a commutative algebra  $A$ , endowed with the pointwise multiplication  $(ab)(\lambda) = a(\lambda) b(\lambda)$ , unit  $1(\lambda) = 1$ , and where the inverse of a series  $a(\lambda)$  is found by recursion. It is an abelian proalgebraic group on **Com**, represented by the cocommutative Hopf algebra

$$\begin{aligned} H_{\text{inv}} &= \mathbb{F}[x_n, n \geq 1] \quad (x_0 = 1) \\ \Delta_{\text{inv}}(x_n) &= \sum_{m=0}^n x_m \otimes x_{n-m} \end{aligned}$$

known as Hopf algebra of symmetric functions [17].

The functor  $\text{Inv}$  admits an evident extension to associative algebras as a functor in groups (but not abelian), represented by the cogroup bialgebra [7]

$$\begin{aligned} H_{\text{inv}} &= \mathbb{F}\langle x_n, n \geq 1 \rangle \quad (x_0 = 1) \\ \Delta_{\text{inv}}(x_n) &= \sum_{m=0}^n x_m^{(1)} x_{n-m}^{(2)} \end{aligned}$$

with antipode defined recursively. The projection of this bialgebra by the canonical map  $\pi$  given in Def. 2.8 coincides with the Hopf algebra of non-commutative symmetric functions (cf. [18]).

In this section we show that the functor  $\text{Inv}$  can be extended to non-associative algebras, as a proalgebraic loop.

### 4.1 Loop of invertible series

**Definition 4.1** Let  $A$  be a unital algebra and let  $\lambda$  be a formal variable. We call invertible series in  $\lambda$  with coefficients in  $A$  the formal series in the set

$$\text{Inv}(A) = \left\{ a = \sum_{n \geq 0} a_n \lambda^n \mid a_0 = 1, a_n \in A \right\},$$

endowed with the multiplication

$$a \cdot b = \sum_{n \geq 0} \sum_{m=0}^n a_m b_{n-m} \lambda^n$$

and the unit  $e$  given by  $e_0 = 1$  and  $e_n = 0$  for all  $n > 1$ . For instance,

$$\begin{aligned} (a \cdot b)_1 &= a_1 + b_1, \\ (a \cdot b)_2 &= a_2 + a_1 b_1 + b_2, \\ (a \cdot b)_3 &= a_3 + a_2 b_1 + a_1 b_2 + b_3. \end{aligned}$$

**Proposition 4.2** *For any unital algebra  $A$ , the set of invertible series  $\text{Inv}(A)$  is a loop.*

**Proof.** It is clear that the series  $e$  is a unit for the given multiplication, so we only have to show that there exist a left and a right divisions satisfying the cancellation properties (2.2) and (2.1). Since the multiplication is completely symmetric in the the two variables, the proof for the two divisions is exactly the same. We do it for the right division.

Given two series  $a = \sum a_n \lambda^n$  and  $b = \sum b_n \lambda^n$ , we define the right division  $a/b = \sum (a/b)_n \lambda^n$  so that  $(a/b)\dot{b} = a$ , that is

$$\sum_{m=0}^n (a/b)_m b_{n-m} = a_n \quad \text{for any } n \geq 0.$$

These equations are solved recursively from  $(a/b)_0 = 1$ , and give the  $n$ th term

$$(a/b)_n = a_n - \sum_{m=0}^{n-1} (a/b)_m b_{n-m}.$$

Let us then prove by induction that  $(a \cdot b)/b = a$ , that is,  $((a \cdot b)/b)_n = a_n$  for any  $n \geq 0$ . We have  $((a \cdot b)/b)_0 = a_0 = 1$  and, for any  $n \geq 1$ ,

$$\begin{aligned} ((a \cdot b)/b)_n &= (a \cdot b)_n - \sum_{m=0}^{n-1} (a \cdot b)_m b_{n-m} \\ &= a_n + \sum_{m=0}^{n-1} (a_m - (a \cdot b)_m) b_{n-m}, \end{aligned}$$

so if we suppose that  $((a \cdot b)/b)_m = a_m$  for any  $m \geq n - 1$ , we have  $((a \cdot b)/b)_n = a_n$ . □

For instance, for the right division we find

$$\begin{aligned} (a/b)_1 &= a_1 - b_1, \\ (a/b)_2 &= a_2 - a_1 b_1 - b_2 + b_1 b_1, \\ (a/b)_3 &= a_3 - (a_1 b_2 + a_2 b_1) + (a_1 b_1) b_1 - b_3 + (b_1 b_2 + b_2 b_1) - (b_1 b_1) b_1, \end{aligned}$$

and for the left division we find

$$\begin{aligned} (a \setminus b)_1 &= b_1 - a_1, \\ (a \setminus b)_2 &= b_2 - a_1 b_1 - a_2 + a_1 a_1, \\ (a \setminus b)_3 &= b_3 - (a_1 b_2 + a_2 b_1) + a_1 (a_1 b_1) - b_3 + (a_1 a_2 + a_2 a_1) - a_1 (a_1 a_1). \end{aligned}$$

## 4.2 Coloop bialgebra of invertible series

For any integer  $n \geq 1$  and any  $1 \leq \ell \leq n$ , let  $\mathcal{C}_n^\ell$  denote the set of compositions of  $n$  of length  $\ell$ , that is, the set of ordered sequences  $\mathbf{n} = (n_1, \dots, n_\ell)$  such that

$$n_1 + \dots + n_\ell = n, \quad \text{and} \quad n_1, \dots, n_\ell \geq 1. \quad (4.1)$$

For instance, for  $\ell = 1, 2, 3$ , we have

$$\begin{aligned} \mathcal{C}_1^1 &= \{(1)\}, & \mathcal{C}_2^1 &= \{(2)\}, & \mathcal{C}_2^2 &= \{(1, 1)\}, \\ \mathcal{C}_3^1 &= \{(3)\}, & \mathcal{C}_3^2 &= \{(2, 1), (1, 2)\}, & \mathcal{C}_3^3 &= \{(1, 1, 1)\}. \end{aligned}$$

**Definition 4.3** Let us call **coloop bialgebra of invertible series** the free unital algebra

$$H_{\text{inv}}^{\text{H}} = T\{x_n \mid n \geq 1\}$$

with the following graded co-operations:

- comultiplication  $\Delta_{\text{inv}}^{\text{H}} : H_{\text{inv}}^{\text{H}} \longrightarrow H_{\text{inv}}^{\text{H}} \amalg H_{\text{inv}}^{\text{H}}$  given by

$$\Delta_{\text{inv}}^{\text{H}}(x_n) = \sum_{m=0}^n x_m y_{n-m};$$

- counit  $\varepsilon : H_{\text{inv}}^{\text{H}} \longrightarrow \mathbb{F}$  given by  $\varepsilon(x_n) = \delta_{n,0}$ ;
- right codivision  $\delta_r : H_{\text{inv}}^{\text{H}} \longrightarrow H_{\text{inv}}^{\text{H}} \amalg H_{\text{inv}}^{\text{H}}$  given by

$$\delta_r(x_n) = x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( (((x_{n_1} - y_{n_1}) y_{n_2}) y_{n_3}) \cdots \right) y_{n_{\ell+1}},$$

where  $\mathcal{C}_n^{\ell+1}$  is the set of compositions of  $n$  of length  $\ell + 1$ , cf. (4.1);

- left codivision  $\delta_l : H_{\text{inv}}^{\text{H}} \longrightarrow H_{\text{inv}}^{\text{H}} \amalg H_{\text{inv}}^{\text{H}}$  given by

$$\delta_l(x_n) = y_n - x_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} x_{n_1} \left( \cdots (x_{n_2} (x_{n_\ell} (y_{n_{\ell+1}} - x_{n_{\ell+1}}))) \right).$$

**Theorem 4.4** *The algebra  $H_{\text{inv}}^{\text{H}}$  is a coloop bialgebra and represents the loop of invertible series as a functor  $\text{Inv} : \mathbf{Alg} \longrightarrow \mathbf{Loop}$ .*

*As a consequence, given an algebra  $A$ , a series  $a = \sum_{n \geq 0} a_n \lambda^n \in \text{Inv}(A)$  can be seen as an algebra homomorphism  $a : H_{\text{inv}}^{\text{H}} \longrightarrow A$  defined on the generators of  $H_{\text{inv}}^{\text{H}}$  by  $a(x_n) = a_n$ , and the right and left division  $a/b$  and  $a \setminus b$  are given at any order  $n$  by the following closed formulas:*

$$\begin{aligned} (a/b)_n &= \mu_A(a \amalg b) \delta_r(x_n) \\ &= a_n - b_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( (((a_{n_1} - b_{n_1}) b_{n_2}) b_{n_3}) \cdots \right) b_{n_{\ell+1}}, \\ (a \setminus b)_n &= \mu_A(a \amalg b) \delta_l(x_n) \\ &= b_n - a_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} a_{n_1} \left( \cdots (a_{n_2} (a_{n_\ell} (b_{n_{\ell+1}} - a_{n_{\ell+1}}))) \right). \end{aligned}$$

**Proof.** The algebra  $H_{\text{inv}}^{\text{H}}$  clearly represents the functor  $\text{Inv}$  with values in sets, and the comultiplication  $\Delta_{\text{inv}}^{\text{H}}$  represents the pointwise multiplication of series. The only thing which should be proved is that  $H_{\text{inv}}^{\text{H}}$  is a coloop bialgebra with the given codivisions. The formulas for the left and for the right codivisions are perfectly symmetric, in the sense that  $\delta_l = \tau \delta_r$ , so it suffices to give the details for one codivision. Let us then show that the right codivision satisfies the two equations (2.5).

Concerning the first one, we have

$$(\delta_r \amalg \text{id}) \Delta_{\text{inv}}^{\text{H}}(x_n) = \delta_r(x_n) + z_n + \sum_{m=1}^{n-1} \delta_r(x_m) z_{n-m}$$

and therefore

$$\begin{aligned}
(\text{id} \amalg \mu)(\delta_r \amalg \text{id})\Delta_{\text{inv}}^{\text{H}}(x_n) &= \delta_r(x_n) + y_n + \sum_{m=1}^{n-1} \delta_r(x_m) y_{n-m} \\
&= x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( ((u_{n_1} y_{n_2}) y_{n_3}) \cdots \right) y_{n_{\ell+1}} + y_n \\
&\quad + \sum_{m=1}^{n-1} u_m y_{n-m} + \sum_{m=1}^{n-1} \sum_{\lambda=1}^{m-1} (-1)^\lambda \sum_{\mathbf{m} \in \mathcal{C}_m^{\lambda+1}} \left( ((u_{m_1} y_{m_2}) \cdots) y_{m_{\lambda+1}} \right) y_{n-m}
\end{aligned}$$

where we set  $u_n := x_n - y_n$  and therefore we have

$$\sum_{m=1}^{n-1} u_m y_{n-m} = \sum_{\mathbf{n} \in \mathcal{C}_n^2} u_{n_1} y_{n_2}.$$

Setting  $\ell = \lambda + 1$  in the last sum, we have  $2 \leq \ell \leq n - 1$  and  $\ell \leq m \leq n - 1$  with

$$\bigcup_{m=\lambda}^{n-1} \mathcal{C}_m^\ell \times \mathcal{C}_{n-m}^1 = \mathcal{C}_n^{\ell+1},$$

therefore

$$\sum_{m=1}^{n-1} \sum_{\lambda=1}^{m-1} (-1)^\lambda \sum_{\mathbf{m} \in \mathcal{C}_m^{\lambda+1}} \left( ((u_{m_1} y_{m_2}) \cdots) y_{m_{\lambda+1}} \right) y_{n-m} = \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( ((u_{n_1} y_{n_2}) y_{n_3}) \cdots \right) y_{n_{\ell+1}}.$$

Thus, we finally obtain

$$(\delta_r \amalg \text{id})\Delta_{\text{inv}}^{\text{H}}(x_n) = x_n.$$

For the second identity, we rewrite the comultiplication as

$$\Delta_{\text{inv}}^{\text{H}}(x_n) = x_n + y_n + \sum_{\mathbf{n} \in \mathcal{C}_n^2} x_{n_1} y_{n_2}$$

and using the fact that

$$\mathcal{C}_n^{\ell+1} = \bigcup_{m=1}^{n-1} \mathcal{C}_m^1 \times \mathcal{C}_{n-m}^\ell,$$

and setting  $\mu = \ell$ , we rewrite the right codivision as

$$\delta_r(x_n) = u_n + \sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} \left( ((u_m y_{k_1}) y_{k_2}) \cdots \right) y_{k_\mu}.$$

We then have

$$\begin{aligned}
(\Delta_{\text{inv}}^{\text{II}} \amalg \text{id})\delta_r(x_n) &= \Delta_{\text{inv}}^{\text{II}}(x_n) - z_n + \sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} \left( ((\Delta_{\text{inv}}^{\text{II}}(x_m) - z_m) z_{k_1}) \cdots \right) z_{k_\mu} \\
&= x_n + y_n + \sum_{\mathbf{n} \in \mathcal{C}_n^2} x_{n_1} y_{n_2} + \sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} ((x_m z_{k_1}) \cdots) z_{k_\mu} - z_n \\
&\quad + \sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} ((y_m z_{k_1}) \cdots) z_{k_\mu} \\
&\quad + \sum_{m=1}^{n-1} \sum_{\mathbf{m} \in \mathcal{C}_m^2} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} \left( ((x_{m_1} y_{m_2}) z_{k_1}) \cdots \right) z_{k_\mu} \\
&\quad - \sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} ((z_m z_{k_1}) \cdots) z_{k_\mu}.
\end{aligned}$$

When we then apply  $\text{id} \amalg \mu$ , we identify  $z_m = y_m$  and  $z_{k_i} = y_{k_i}$  for  $i = 1, \dots, \mu$ , and therefore we have

$$\begin{aligned}
(\text{id} \amalg \mu)(\Delta_{\text{inv}}^{\text{II}} \amalg \text{id})\delta_r(x_n) &= x_n + \sum_{\mathbf{n} \in \mathcal{C}_n^2} x_{n_1} y_{n_2} + \sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} ((x_m y_{k_1}) \cdots) y_{k_\mu} \\
&\quad + \sum_{m=1}^{n-1} \sum_{\mathbf{m} \in \mathcal{C}_m^2} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} \left( ((x_{m_1} y_{m_2}) y_{k_1}) \cdots \right) y_{k_\mu}
\end{aligned}$$

where

$$\sum_{m=1}^{n-1} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} ((x_m y_{k_1}) \cdots) y_{k_\mu} = \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} ((x_{n_1} y_{n_2}) \cdots) y_{n_{\ell+1}},$$

and

$$\begin{aligned}
\sum_{\mathbf{n} \in \mathcal{C}_n^2} x_{n_1} y_{n_2} + \sum_{m=1}^{n-1} \sum_{\mathbf{m} \in \mathcal{C}_m^2} \sum_{\mu=1}^{n-m} (-1)^\mu \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\mu} \left( ((x_{m_1} y_{m_2}) y_{k_1}) \cdots \right) y_{k_\mu} \\
= - \sum_{\mathbf{n} \in \mathcal{C}_n^{1+1}} (-1)^1 x_{n_1} y_{n_{1+1}} - \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} ((x_{n_1} y_{n_2}) \cdots) y_{n_{\ell+1}}.
\end{aligned}$$

Then we finally have

$$(\text{id} \amalg \mu)(\Delta_{\text{inv}}^{\text{II}} \amalg \text{id})\delta_r(x_n) = x_n.$$

□

### 4.3 Properties of the loop of invertible series

**Corollary 4.5** 1. *The proalgebraic loop  $\text{Inv}$  on the category  $\mathbf{Alg}$  is not right alternative nor power associative.*

2. The left and the right inverses of any  $a \in \text{Inv}(A)$  do not coincide, that is

$$a \setminus e \neq e/a.$$

3. The left and right inversions do not allow to construct the divisions, that is,

$$a/b \neq a(e/b) \quad \text{and} \quad a \setminus b \neq (a \setminus e)b$$

for any  $a, b \in \text{Inv}(A)$ .

**Proof.** 1. It suffices to show that the coloop bialgebra  $H_{\text{inv}}^{\text{II}}$  is not right coalternative, that is  $(\text{id} \amalg \mu)K \neq 0$ , where  $K = (\Delta_{\text{inv}}^{\text{II}} \amalg \text{id})\Delta_{\text{inv}}^{\text{II}} - (\text{id} \amalg \Delta_{\text{inv}}^{\text{II}})\Delta_{\text{inv}}^{\text{II}}$  is the coassociator.

The first default from right alternativity appears on the generator  $x_3$ , since we have

$$\begin{aligned} K(x_3) &= (x_1 y_1) z_1 - x_1 (y_1 z_1) \\ (\text{id} \amalg \mu)K(x_3) &= (x_1 y_1) y_1 - x_1 (y_1 y_1) \neq 0. \end{aligned}$$

For instance, if  $A$  is the algebra of sedenions, spanned by 1 and the imaginary units  $e_i$  for  $i = 1, \dots, 15$ , the default from right alternativity can be seen comparing  $(ab)b$  and  $a(bb)$  for the two series

$$a = 1 + (e_1 + e_{10})\lambda \quad \text{and} \quad b = 1 + (e_5 + e_{14})\lambda$$

because  $(e_1 + e_{10})(e_5 + e_{14}) = 0$  and therefore

$$(ab)b - a(bb) = -(e_1 + e_{10})(e_5 + e_{14})^2 \lambda^3 = 2(e_1 + e_{10}) \lambda^3.$$

Similarly,  $\text{Inv}(A)$  is not power associative because

$$\mu(\text{id} \amalg \mu)K(x_3) = (x_1 x_1) x_1 - x_1 (x_1 x_1) \neq 0.$$

For instance, if we take  $A$  to be the algebra of  $2 \times 2$  matrices with coefficients in the sedenion algebra, for the series  $c = 1 + c_1 \lambda$  with

$$c_1 = \begin{pmatrix} e_1 + e_{10} & e_5 + e_{14} \\ 0 & 1 \end{pmatrix}$$

we have

$$c_1^2 c_1 = \begin{pmatrix} -2(e_1 + e_{10}) & -(e_5 + e_{14}) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad c_1 c_1^2 = \begin{pmatrix} -2(e_1 + e_{10}) & e_5 + e_{14} \\ 0 & 1 \end{pmatrix}$$

and therefore

$$(cc)c - c(cc) = \begin{pmatrix} 0 & -2(e_5 + e_{14}) \\ 0 & 0 \end{pmatrix} \lambda^3.$$

2. The left and right inverses in  $\text{Inv}(A)$  coincide if and only if the left antipode  $S_l$  and the right antipode  $S_r$  of  $H_{\text{inv}}^{\text{II}}$  coincide. Applying equations (2.9), we find

$$\begin{aligned} S_r(x_n) &= (\varepsilon \amalg \text{id}) \delta_r(x_n) \\ &= -x_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( ((x_{n_1} x_{n_2}) x_{n_3}) \cdots \right) x_{n_{\ell+1}} \end{aligned}$$

and

$$\begin{aligned} S_l(x_n) &= (\text{id} \amalg \varepsilon) \delta_l(x_n) \\ &= -x_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} x_{n_1} \left( \cdots (x_{n_2} (x_{n_\ell} x_{n_{\ell+1}})) \right), \end{aligned}$$

therefore the two antipodes do not coincide.

For instance, for a series  $c = 1 + c_1 \lambda$ , we have

$$\begin{aligned} e/c &= c S_r = 1 - c_1 \lambda + c_1 c_1 \lambda^2 - (c_1 c_1) c_1 \lambda^3 + ((c_1 c_1) c_1) c_1 \lambda^4 + \cdots \\ c \setminus e &= c S_l = 1 - c_1 \lambda + c_1 c_1 \lambda^2 - c_1 (c_1 c_1) \lambda^3 + c_1 (c_1 (c_1 c_1)) \lambda^4 + \cdots \end{aligned}$$

If  $c$  is the series considered above, with coefficients in the algebra of  $2 \times 2$  matrices over sedenions and

$$c_1 = \begin{pmatrix} e_1 + e_{10} & e_5 + e_{14} \\ 0 & 1 \end{pmatrix},$$

we have

$$e/c - c \setminus e = \begin{pmatrix} 0 & -2(e_5 + e_{14}) \\ 0 & 0 \end{pmatrix} \lambda^3 + O(\lambda^4).$$

3. To show that  $a/b \neq a(e/b)$  and  $a \setminus b \neq (a \setminus e)b$  in the loop  $\text{Inv}(A)$  is equivalent to show that

$$\delta_r \neq (\text{id} \amalg S_r) \Delta_{\text{inv}}^{\text{H}} \quad \text{and} \quad \delta_l \neq (S_l \amalg \text{id}) \Delta_{\text{inv}}^{\text{H}}$$

in the coloop bialgebra  $H_{\text{inv}}^{\text{H}}$ . Let us show it for the right codivision. For any generator  $x_n$ , we have

$$\begin{aligned} (\text{id} \amalg S_r) \Delta_{\text{inv}}^{\text{H}}(x_n) &= x_n + S_r(y_n) + \sum_{m=1}^{n-1} x_m S_r(y_{n-m}) \\ &= x_n - y_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} ((y_{n_1} y_{n_2}) \cdots) y_{n_{\ell+1}} \\ &\quad - \sum_{m=1}^{n-1} x_m y_{n-m} - \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m-1} (-1)^\lambda \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^{\lambda+1}} x_m \left( ((y_{k_1} y_{k_2}) \cdots) y_{k_{\lambda+1}} \right). \end{aligned}$$

Writing the last two sums in terms of compositions of  $n$  yields

$$(\text{id} \amalg S_r) \Delta_{\text{inv}}^{\text{H}}(x_n) = x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( x_{n_1} \left( ((y_{n_2} y_{n_3}) \cdots) y_{n_{\ell+1}} \right) - ((y_{n_1} y_{n_2}) \cdots) y_{n_{\ell+1}} \right),$$

which is clearly different from the expression of  $\delta_r(x_n)$ .  $\square$

**Proposition 4.6** *Given an algebra  $A$ , the loop  $\text{Inv}(A)$  satisfies a polynomial identity*

$$(*) \quad F(a, b, \dots, c) = 0$$

for any series  $a, b, \dots, c \in \text{Inv}(A)$  if and only if the identity  $(*)$  is satisfied in  $A$ , that is, for any elements  $a, b, \dots, c \in A$ .



This implies in particular that  $\text{Inv}(A)$  is a Moufang loop if and only if  $A$  is left and right alternative, and that  $\text{Inv}(A)$  is a group if and only if  $A$  is associative.

**Proof.** Roughly speaking, this result follows from the fact that the comultiplication  $\Delta_{\text{inv}}^{\text{H}}$  is linear on both sides on generators. More precisely, if  $(*)$  holds in  $A$ , then it holds in the algebra of formal series  $A[[\lambda]]$  and therefore in its multiplicative subset  $\text{Inv}(A)$ . Viceversa, if  $(*)$  holds in  $\text{Inv}(A)$ , it suffices to consider series of the form  $a = 1 + (a_1 - 1)\lambda$ ,  $b = 1 + (b_1 - 1)\lambda, \dots$ ,  $c = 1 + (c_1 - 1)\lambda$ , apply  $(*)$  and evaluate at  $\lambda = 1$ , the result is  $(*)$  on the elements  $a_1, b_1, \dots, c_1 \in A$ .  $\square$

## 5 Coloop of formal diffeomorphisms

The group of formal diffeomorphisms (tangent to the identity) is the set of series

$$\text{Diff}(A) = \left\{ a = \sum_{n \geq 0} a_n \lambda^{n+1} \mid a_0 = 1, a_n \in A \right\}$$

with coefficients  $a_n$  taken in a commutative algebra  $A$ , endowed with the composition law  $(a \circ b)(\lambda) = a(b(\lambda))$ , unit  $e(\lambda) = \lambda$ , and where the inverse of a series  $a(\lambda)$  is given by the Lagrange inversion formula [23]. It is a proalgebraic group on **Com**, represented by the Faà di Bruno Hopf algebra [12], [20]

$$H_{\text{FdB}} = \mathbb{F}[x_n, n \geq 1] \quad (x_0 = 1)$$

$$\Delta_{\text{FdB}}(x_n) = \sum_{m=0}^n x_m \otimes \sum_{(p)} \frac{(m+1)!}{p_0! p_1! \cdots p_n!} x_1^{p_1} \cdots x_n^{p_n}$$

where the sum is done over the set of tuples  $(p_0, p_1, p_2, \dots, p_n)$  of non-negative integers such that  $p_0 + p_1 + p_2 + \cdots + p_n = m + 1$  and  $p_1 + 2p_2 + \cdots + np_n = n - m$ . In this section we show that this group can be extended as a proalgebraic loop to the category **As**.

### 5.1 Loop of formal diffeomorphisms

**Definition 5.1** Let  $A$  be a unital associative algebra, non necessarily commutative, and let  $\lambda$  be a formal variable. We call formal diffeomorphisms in  $\lambda$  with coefficients in  $A$  the formal series in the set

$$\text{Diff}(A) = \left\{ a = \sum_{n \geq 0} a_n \lambda^{n+1} \mid a_0 = 1, a_n \in A \right\},$$

endowed with the composition law

$$a \circ b = \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k_0 + \cdots + k_m = n - m \\ k_0, \dots, k_m \geq 0}} a_m b_{k_0} \cdots b_{k_m} \lambda^{n+1}$$

$$= \sum_{n \geq 0} \left( a_n + b_n + \sum_{m=1}^{n-1} a_m \sum_{l=1}^n \binom{m+1}{l} \sum_{\substack{k_1 + \cdots + k_m = n - m \\ k_1, \dots, k_m \geq 1}} b_{k_1} \cdots b_{k_m} \right) \lambda^{n+1}$$

and the unit  $e$  given by  $e_0 = 1$  and  $e_n = 0$  for all  $n > 1$ . For instance,

$$(a \circ b)_1 = a_1 + b_1,$$

$$(a \circ b)_2 = a_2 + 2a_1 b_1 + b_2,$$

$$(a \circ b)_3 = a_3 + 3a_2 b_1 + a_1(2b_2 + b_1^2) + b_3.$$

The indeterminate  $\lambda$  is not necessary to define the loop law, but helps to keep track of the degree of the terms in the sum.

**Proposition 5.2** *For any unital associative algebra  $A$ , the set  $\text{Diff}(A)$  is a loop.*

**Proof.** It is clear that the composition is a well-defined operation, and that  $e$  is a unit. Let us show that the left and right divisions exist.

i) Let us prove that there exists a right division  $/$  satisfying the two equations (2.2). Given two series  $a = \sum a_n \lambda^{n+1}$  and  $b = \sum b_n \lambda^{n+1}$ , let us define the series  $a/b = \sum (a/b)_n \lambda^{n+1}$  so that  $(a/b) \circ b = a$ , that is

$$\sum_{m=0}^n \sum_{\substack{k_0+\dots+k_m=n-m \\ k_0, \dots, k_m \geq 0}} (a/b)_m b_{k_0} \cdots b_{k_m} = a_n \quad \text{for any } n \geq 0.$$

From now on, in the sum over the integers  $k_0, \dots, k_p$  we omit to write that all integers can be zero. These equations are solved recursively, starting from  $(a/b)_0 = a_0 = 1$ . The  $n$ th term is given by

$$(a/b)_n = a_n - \sum_{m=0}^{n-1} \sum_{k_0+\dots+k_m=n-m} (a/b)_m b_{k_0} \cdots b_{k_m}.$$

To prove that  $(a \circ b)/b = a$ , i.e. that  $\left((a \circ b)/b\right)_n = a_n$  for any  $n \geq 0$ , we proceed by induction. We have  $\left((a \circ b)/b\right)_0 = (a \circ b)_0 = 1$ , therefore

$$\left((a \circ b)/b\right)_1 = (a \circ b)_1 - \left((a \circ b)/b\right)_0 b_1 = a_1 + b_1 - b_1 = a_1$$

and

$$\begin{aligned} \left((a \circ b)/b\right)_n &= (a \circ b)_n - \sum_{m=0}^{n-1} \sum_{k_0+\dots+k_m=n-m} \left((a \circ b)/b\right)_m b_{k_0} \cdots b_{k_m} \\ &= \sum_{m=0}^n \sum_{k_0+\dots+k_m=n-m} a_p b_{k_0} \cdots b_{k_m} - \sum_{m=0}^{n-1} \sum_{k_0+\dots+k_m=n-m} \left((a \circ b)/b\right)_m b_{k_0} \cdots b_{k_m} \\ &= a_n + \sum_{m=0}^{n-1} \sum_{k_0+\dots+k_m=n-m} \left(a_m - \left((a \circ b)/b\right)_m\right) b_{k_0} \cdots b_{k_m}, \end{aligned}$$

so, if we suppose that  $\left((a \circ b)/b\right)_m = a_m$  for any  $m \leq n-1$ , we have  $\left((a \circ b)/b\right)_n = a_n$ .

ii) To prove the existence of the left division we proceed in the same way: the series  $a \setminus b$  that satisfies the identity  $a \circ (a \setminus b) = b$  of equations (2.1), that is,

$$\sum_{m=0}^n \sum_{k_0+\dots+k_m=n-m} a_m (a \setminus b)_{k_0} \cdots (a \setminus b)_{k_m} = b_n \quad \text{for any } n \geq 0,$$

is given recursively by  $(a \setminus b)_0 = 1$  and

$$(a \setminus b)_n = b_n - \sum_{m=1}^n \sum_{k_0 + \dots + k_m = n-m} a_m (a \setminus b)_{k_0} \cdots (a \setminus b)_{k_m}.$$

The identity  $a \setminus (a \circ b) = b$  means that, for any  $n \geq 0$ , we have  $(a \setminus (a \circ b))_n = b_n$ . This is proved by induction. We have  $(a \setminus (a \circ b))_0 = 1$ , therefore

$$(a \setminus (a \circ b))_1 = (a \circ b)_1 - a_1 (a \setminus (a \circ b))_0 (a \setminus (a \circ b))_0 = a_1 + b_1 - a_1 = b_1$$

and

$$\begin{aligned} (a \setminus (a \circ b))_n &= (a \circ b)_n - \sum_{m=1}^n \sum_{k_0 + \dots + k_m = n-m} a_m (a \setminus (a \circ b))_{k_0} \cdots (a \setminus (a \circ b))_{k_m} \\ &= \sum_{m=0}^n \sum_{k_0 + \dots + k_m = n-m} a_m b_{k_0} \cdots b_{k_m} - \sum_{m=1}^n \sum_{k_0 + \dots + k_m = n-m} a_m (a \setminus (a \circ b))_{k_0} \cdots (a \setminus (a \circ b))_{k_m} \\ &= a_0 b_n + \sum_{m=1}^n \sum_{k_0 + \dots + k_m = n-m} a_m \left( b_{k_0} \cdots b_{k_m} - (a \setminus (a \circ b))_{k_0} \cdots (a \setminus (a \circ b))_{k_m} \right) \\ &= b_n + \sum_{m=1}^{n-1} \sum_{k_0 + \dots + k_m = n-m} a_m \left( b_{k_0} \cdots b_{k_m} - (a \setminus (a \circ b))_{k_0} \cdots (a \setminus (a \circ b))_{k_m} \right) \\ &\quad + a_n \left( b_0 \cdots b_0 - (a \setminus (a \circ b))_0 \cdots (a \setminus (a \circ b))_0 \right) \\ &= b_n + \sum_{m=1}^{n-1} \sum_{k_0 + \dots + k_m = n-m} a_m \left( b_{k_0} \cdots b_{k_m} - (a \setminus (a \circ b))_{k_0} \cdots (a \setminus (a \circ b))_{k_m} \right), \end{aligned}$$

so, if we suppose that  $(a \setminus (a \circ b))_m = b_m$  for any  $m \leq n-1$ , we have  $(a \setminus (a \circ b))_n = b_n$ .  $\square$

For instance, the first terms of the right division are

$$\begin{aligned} (a/b)_1 &= a_1 - b_1, \\ (a/b)_2 &= a_2 - [b_2 + 2(a/b)_1 b_1] = a_2 - 2a_1 b_1 - (b_2 - 2b_1^2), \\ (a/b)_3 &= a_3 - [b_3 + (a/b)_1(2b_2 + b_1^2) + 3(a/b)_2 b_1] \\ &= a_3 - (2a_1 b_2 + 3a_2 b_1) + 5a_1 b_1^2 - [b_3 - (2b_1 b_2 + 3b_2 b_1) + 5b_1^3], \end{aligned}$$

and the first terms of the left division are

$$\begin{aligned} (a \setminus b)_1 &= b_1 - a_1, \\ (a \setminus b)_2 &= b_2 - [2a_1(a \setminus b)_1 + a_2] = b_2 - 2a_1 b_1 - (a_2 - 2a_1^2), \\ (a \setminus b)_3 &= b_3 - [a_1(2(a \setminus b)_2 + (a \setminus b)_1^2) + a_2(3(a \setminus b)_1) + a_3] \\ &= b_3 - (2a_1 b_2 + 3a_2 b_1) + (5a_1^2 b_1 + a_1 b_1 a_1 - a_1 b_1^2) - [a_3 - (2a_1 a_2 + 3a_2 a_1) + 5a_1^3]. \end{aligned}$$

We now prove that the loop of formal diffeomorphisms is proalgebraic over associative algebras, and give its representative coloop bialgebra.

## 5.2 Faà di Bruno coloop bialgebra

To describe the codivisions we need to introduce some sets of sequences and two types of related integer coefficients.

**Definition 5.3** For any  $\ell \geq 1$ , let  $\mathcal{M}_\ell$  denote the set of sequences  $\mathbf{m} = (m_1, \dots, m_\ell)$  such that

$$m_1 + \dots + m_\ell = \ell, \quad \text{and} \quad m_1 + \dots + m_j \geq j \quad \text{for} \quad j = 1, \dots, \ell - 1. \quad (5.1)$$

For instance, for  $\ell = 1, 2, 3$ , we have

$$\mathcal{M}_1 = \{(1)\}, \quad \mathcal{M}_2 = \{(2, 0), (1, 1)\}, \quad \mathcal{M}_3 = \{(3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1)\}.$$

For any  $\ell \geq 1$  and any sequence  $(n_1, \dots, n_{\ell+1})$  of positive integers, we call **Lagrange coefficient**<sup>3</sup> the number

$$d_{\ell+1}(n_1, \dots, n_{\ell+1}) = \sum_{\mathbf{m} \in \mathcal{M}_\ell} \binom{n_1 + 1}{m_1} \cdots \binom{n_\ell + 1}{m_\ell} \binom{n_{\ell+1} + 1}{0}.$$

For  $\ell = 0$ ,  $\mathcal{M}_0$  is empty and we set

$$d_1(n_1) = \binom{n_1 + 1}{0} = 1.$$

For instance, for  $\ell = 1, 2, 3$ , we have

$$\begin{aligned} d_2(n_1, n_2) &= \binom{n_1 + 1}{1}, \\ d_3(n_1, n_2, n_3) &= \binom{n_1 + 1}{2} + \binom{n_1 + 1}{1} \binom{n_2 + 1}{1}, \\ d_4(n_1, n_2, n_3, n_4) &= \binom{n_1 + 1}{3} + \binom{n_1 + 1}{2} \binom{n_2 + 1}{1} + \binom{n_1 + 1}{2} \binom{n_3 + 1}{1} \\ &\quad + \binom{n_1 + 1}{1} \binom{n_2 + 1}{2} + \binom{n_1 + 1}{1} \binom{n_2 + 1}{1} \binom{n_3 + 1}{1}. \end{aligned}$$

**Definition 5.4** For any  $\ell \geq 2$ , let  $\mathcal{E}_{\ell-1} = \{1, 2\}^{\ell-1}$  be the set of sequences  $\mathbf{e} = (e_1, \dots, e_{\ell-1})$  such that

$$e_i \in \{1, 2\} \quad \text{for} \quad i = 1, \dots, \ell - 1. \quad (5.2)$$

For any  $\mathbf{e} \in \mathcal{E}_{\ell-1}$ , let  $\mathcal{M}_\ell^{\mathbf{e}}$  be the set of sequences  $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathcal{M}_\ell$  such that

$$m_j = 0 \quad \text{if} \quad e_{j-1} = 2, \quad \text{for} \quad j = 2, \dots, \ell.$$

In particular, if  $\mathbf{e} = (1, 1, \dots, 1)$  then  $\mathcal{M}_\ell^{\mathbf{e}} = \mathcal{M}_\ell$ , while if  $\mathbf{e}$  contains at least one value 2, the set  $\mathcal{M}_\ell^{\mathbf{e}}$  is a proper subset of  $\mathcal{M}_\ell$ .

For instance, for  $\ell = 2$ , we have

$$\begin{aligned} \mathcal{M}_2^{(1)} &= \mathcal{M}_2 = \{(2, 0), (1, 1)\}, \\ \mathcal{M}_2^{(2)} &= \{(2, 0)\}. \end{aligned}$$

---

<sup>3</sup>These coefficients appear in the Lagrange inversion formula [23], cf. [7].

For  $\ell = 3$ , we have

$$\begin{aligned}\mathcal{M}_3^{(1,1)} &= \mathcal{M}_3 = \{(3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1)\} \\ \mathcal{M}_3^{(1,2)} &= \{(3, 0, 0), (2, 1, 0), (1, 2, 0)\} \\ \mathcal{M}_3^{(2,1)} &= \{(3, 0, 0), (2, 0, 1)\} \\ \mathcal{M}_3^{(2,2)} &= \{(3, 0, 0)\}.\end{aligned}$$

For any  $\ell \geq 1$ , any sequence  $\mathbf{e} \in \mathcal{E}_{\ell-1}$  and any sequence  $(n_1, \dots, n_{\ell+1})$  of positive integers, we call **labeled Lagrange coefficient** the number

$$d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) = \sum_{\mathbf{m} \in \mathcal{M}_{\ell}^{\mathbf{e}}} \binom{n_1+1}{m_1} \cdots \binom{n_{\ell}+1}{m_{\ell}} \binom{n_{\ell+1}+1}{0}.$$

Of course, if  $\mathbf{e} = (1, 1, \dots, 1)$  then  $d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) = d_{\ell+1}(n_1, \dots, n_{\ell+1})$ , and if  $\mathbf{e}$  contains at least one value 2 then  $d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) < d_{\ell+1}(n_1, \dots, n_{\ell+1})$ .

Here are the values of  $d_{\ell+1}^{\mathbf{e}}$  for  $\ell = 1, 2, 3$ :

$$\begin{aligned}d_2^{(1)}(n_1, n_2) &= d_2(n_1, n_2) = \binom{n_1+1}{1}; \\ d_3^{(1,1)}(n_1, n_2, n_3) &= d_3(n_1, n_2, n_3) = \binom{n_1+1}{2} + \binom{n_1+1}{1} \binom{n_2+1}{1}, \\ d_3^{(1,2)}(n_1, n_2, n_3) &= \binom{n_1+1}{2}; \\ d_4^{(1,1,1)}(n_1, n_2, n_3, n_4) &= d_4(n_1, n_2, n_3, n_4) \\ &= \binom{n_1+1}{3} + \binom{n_1+1}{2} \binom{n_2+1}{1} + \binom{n_1+1}{2} \binom{n_3+1}{1} \\ &\quad + \binom{n_1+1}{1} \binom{n_2+1}{2} + \binom{n_1+1}{1} \binom{n_2+1}{1} \binom{n_3+1}{1}, \\ d_4^{(1,1,2)}(n_1, n_2, n_3, n_4) &= \binom{n_1+1}{3} + \binom{n_1+1}{2} \binom{n_2+1}{1} + \binom{n_1+1}{1} \binom{n_2+1}{2}, \\ d_4^{(1,2,1)}(n_1, n_2, n_3, n_4) &= \binom{n_1+1}{3} + \binom{n_1+1}{2} \binom{n_3+1}{1}, \\ d_4^{(1,2,2)}(n_1, n_2, n_3, n_4) &= \binom{n_1+1}{3}.\end{aligned}$$

For any  $n \geq 1$ , let  $x_n$  be a graded variable, with degree  $n$ . For  $X = \text{Span}_{\mathbb{F}}\{x_n, n \geq 1\}$ , the tensor algebra  $H = T(X)$  can be seen as the set of non-commutative polynomials in the variables  $x_1, x_2, \dots$ , that we denote by  $\mathbb{F}\langle x_n, n \geq 1 \rangle$ . It is then useful to denote the unit 1 of  $H$  by  $x_0$ .

The unital associative coproduct algebra  $H \amalg H$  is then the tensor algebra  $T(X^{(1)} \oplus X^{(2)})$  on two identical sets of variables. To simplify the notations, in this section we denote by  $x_n = x_n^{(1)}$  and  $y_n = x_n^{(2)}$  the generators taken in the two sets.

**Definition 5.5** We call **Faà di Bruno coloop bialgebra** the free unital associative algebra

$$H_{\text{FdB}}^{\amalg} = \mathbb{F}\langle x_n, n \geq 1 \rangle, \quad x_0 = 1$$

of non-commutative polynomials in the graded variables  $x_n$ , with the following graded co-operations:

- comultiplication  $\Delta_{\text{FdB}}^{\text{II}} : H_{\text{FdB}}^{\text{II}} \longrightarrow H_{\text{FdB}}^{\text{II}} \amalg H_{\text{FdB}}^{\text{II}}$  given by

$$\Delta_{\text{FdB}}^{\text{II}}(x_n) = x_n + y_n + \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \binom{n_1+1}{\ell} x_{n_1} y_{n_2} \cdots y_{n_{\ell+1}},$$

where  $\mathcal{C}_n^{\ell+1}$  is the set of compositions of  $n$  of length  $\ell+1$ , cf. (4.1);

- counit  $\varepsilon : H_{\text{FdB}}^{\text{II}} \longrightarrow \mathbb{F}$  given by  $\varepsilon(x_n) = \delta_{n,0}$ ;
- right codivision  $\delta_r : H_{\text{FdB}}^{\text{II}} \longrightarrow H_{\text{FdB}}^{\text{II}} \amalg H_{\text{FdB}}^{\text{II}}$  given by

$$\delta_r(x_n) = \sum_{\ell=0}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) (x_{n_1} - y_{n_1}) y_{n_2} \cdots y_{n_{\ell+1}},$$

where the Lagrange coefficients  $d_{\ell+1}$  are given in Def. 5.3;

- left codivision  $\delta_l : H_{\text{FdB}}^{\text{II}} \longrightarrow H_{\text{FdB}}^{\text{II}} \amalg H_{\text{FdB}}^{\text{II}}$  given by

$$\begin{aligned} \delta_l(x_n) &= (y_n - x_n) - \sum_{\mathbf{n} \in \mathcal{C}_n^2} d_2(n_1, n_2) x_{n_1} (y_{n_2} - x_{n_2}) \\ &\quad + \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) x_{n_1} x_{n_2}^{(e_1)} \cdots x_{n_\ell}^{(e_{\ell-1})} (y_{n_{\ell+1}} - x_{n_{\ell+1}}) \end{aligned}$$

where the set of sequences  $\mathcal{E}_{\ell-1}$  and the labeled Lagrange coefficients  $d_{\ell+1}^{\mathbf{e}}$  are given in Def. 5.4, and where we set

$$(-1)^{\mathbf{e}} = (-1)^{e_1 + \cdots + e_{\ell-1} - (\ell-1)}$$

and, according to the previous convention, we set

$$x_n^{(e_i)} = \begin{cases} x_n & \text{if } e_i = 1 \\ y_n & \text{if } e_i = 2 \end{cases}.$$

For instance, on the first five generators, the comultiplication is

$$\begin{aligned} \Delta_{\text{FdB}}^{\text{II}}(x_1) &= x_1 + y_1 \\ \Delta_{\text{FdB}}^{\text{II}}(x_2) &= x_2 + y_2 + 2x_1y_1 \\ \Delta_{\text{FdB}}^{\text{II}}(x_3) &= x_3 + y_3 + (2x_1y_2 + 3x_2y_1) + x_1y_1^2 \\ \Delta_{\text{FdB}}^{\text{II}}(x_4) &= x_4 + y_4 + (2x_1y_3 + 3x_2y_2 + 4x_3y_1) + (x_1(y_1y_2 + y_2y_1) + 3x_2y_1^2) \\ \Delta_{\text{FdB}}^{\text{II}}(x_5) &= x_5 + y_5 + (2x_1y_4 + 3x_2y_3 + 4x_3y_2 + 5x_4y_1) \\ &\quad + (x_1(y_1y_3 + y_2^2 + y_3y_1) + 3x_2(y_1y_2 + y_2y_1) + 6x_3y_1^2) + x_2y_1^3 \end{aligned}$$

the right codivision, with  $u_n = x_n - y_n$ , is

$$\begin{aligned} \delta_r(x_1) &= u_1 \\ \delta_r(x_2) &= u_2 - 2u_1y_1 \\ \delta_r(x_3) &= u_3 - (2u_1y_2 + 3u_2y_1) + 5u_1y_1^2 \\ \delta_r(x_4) &= u_4 - (2u_1y_3 + 3u_2y_2 + 4u_3y_1) + (5u_1y_1y_2 + 7u_1y_2y_1 + 9u_2y_1^2) - 14u_1y_1^3 \\ \delta_r(x_5) &= u_5 - (2u_1y_4 + 3u_2y_3 + 4u_3y_2 + 5u_4y_1) \\ &\quad + (5u_1y_1y_3 + 7u_1y_2^2 + 9u_1y_3y_1 + 9u_2y_1y_2 + 12u_2y_2y_1 + 14u_3y_1^2) \\ &\quad - (14u_1y_1^2y_2 + 19u_1y_1y_2y_1 + 23u_1y_2y_1^2 + 28u_2y_1^3) + 42u_1y_1^4 \end{aligned}$$

and the left codivision has additional terms which contain both variables  $x$  and  $y$  in alternative order beside the first position which is always  $x$ , and last position which is always  $v_n = y_n - x_n$ :

$$\begin{aligned}
\delta_l(x_1) &= v_1 \\
\delta_l(x_2) &= v_2 - 2x_1v_1 \\
\delta_l(x_3) &= v_3 - (2x_1v_2 + 3x_2v_1) + 5x_1^2v_1 - x_1y_1v_1 \\
\delta_l(x_4) &= v_4 - (2x_1v_3 + 3x_2v_2 + 4x_3v_1) + (5x_1^2v_2 + 7x_1x_2v_1 + 9x_2x_1v_1) - 14x_1^3v_1 \\
&\quad - (x_1y_1v_2 + x_1y_2v_1 + 3x_2y_1v_1) + (4x_1^2y_1v_1 + 2x_1y_1x_1v_1) \\
\delta_l(x_5) &= v_5 - (2x_1v_4 + 3x_2v_3 + 4x_3v_2 + 5x_4v_1) \\
&\quad + (5x_1^2v_3 + 7x_1x_2v_2 + 9x_1x_3v_1 + 9x_2x_1v_2 + 12x_2^2v_1 + 14x_3x_1v_1) \\
&\quad - (14x_1^3v_2 + 19x_1^2x_2v_1 + 23x_1x_2x_1v_1 + 28x_2x_1^2v_1) + 42x_1^4v_1 \\
&\quad - (x_1y_1v_3 + x_1y_2v_2 + x_1y_3v_1 + 3x_2y_1v_2 + 3x_2y_2v_1 + 6x_3y_1v_1) \\
&\quad + (4x_1^2y_1v_2 + 4x_1^2y_2v_1 + 9x_1x_2y_1v_1 + 10x_2x_1y_1v_1 \\
&\quad\quad + 2x_1y_1x_1v_2 + 3x_1y_1x_2v_1 + 2x_1y_2x_1v_1 + 7x_2y_1x_1v_1 - x_2y_1^2v_1) \\
&\quad - (14x_1^3y_1v_1 + 9x_1^2y_1x_1v_1 + 5x_1y_1x_1^2v_1 - x_1^2y_1^2v_1 - x_1y_1x_1y_1v_1).
\end{aligned}$$

We now want to prove that the algebra given above is indeed a coloop bialgebra. The only difficulty is to prove that the codivisions satisfy the cocancellation properties (2.6) and (2.5), which are equivalent to some recurrence relations on the Lagrange coefficients  $d_{\ell+1}$  and  $d_{\ell+1}^e$ .

We prove in fact a stronger result, namely, that there exist some operators  $R_{\ell+1}$  and  $R_{\ell+1}^e$  defined on the tensor space  $T(A)$  over any positively graded algebra  $A$ , which produce the Lagrange coefficients and which satisfy the wished recurrence relations. These operators provide an alternative definition of the Faà di Bruno codivisions when applied to the non-unital associative coproduct algebra  $A = \overline{H_{\text{fdB}}^{\text{II}}} \amalg \overline{H_{\text{fdB}}^{\text{I}}} = \overline{T}(X^{(1)} \oplus X^{(2)})$ .

### 5.3 Faà di Bruno co-operations in terms of recursive operators

Let  $A = \bigoplus_{n \geq 1} A_n$  be a positively graded associative algebra over a field  $\mathbb{F}$ , and let us denote by  $|a|$  the degree of an element  $a \in A$ , that is, the integer  $n$  such that  $a \in A_n$ . The tensor algebra  $T(A) = \bigoplus_{\ell \geq 0} A^{\otimes \ell}$  is then bigraded, on one side by the tensor power  $\ell$ , that we call **length**, and on the other side by the grading induced by that of  $A$ , that we call **degree**,

$$|a_1 \otimes a_2 \otimes \cdots \otimes a_\ell| = \sum_{i=1}^{\ell} |a_i|.$$

Then  $T(A)$  can be decomposed into the following direct sum with respect to the degree<sup>4</sup>:

$$T(A) = \mathbb{F} \oplus \bigoplus_{n \geq 1} \left( \bigoplus_{\ell=1}^n \bigoplus_{\mathbf{n} \in \mathcal{C}_n^\ell} A_{n_1} \otimes \cdots \otimes A_{n_\ell} \right),$$

where the compositions  $\mathbf{n} \in \mathcal{C}_n^\ell$  are defined by eq. (4.1).

---

<sup>4</sup>Note that if  $A$  had a null degree component  $A_0$ , then  $T(A)$  would contain an infinite sum of terms in each degree, namely  $T(A)_0 = \bigoplus_{p \geq 0} A_0^{\otimes p}$  and  $T(A)_n = \bigoplus_{\ell=1}^n \bigoplus_{\mathbf{n} \in \mathcal{C}_n^\ell} \bigoplus_{p \geq 0} \binom{\ell+p}{\ell} A_0^{\otimes p} \otimes A_{n_1} \otimes \cdots \otimes A_{n_\ell}$  for  $n \geq 1$ .

**Definition 5.6** Let us define a graded linear operation

$$\triangleright : T(A) \otimes T(A) \longrightarrow \mathbb{F} \oplus A$$

by setting

$$\begin{aligned} a \triangleright (b_1 \otimes \cdots \otimes b_\ell) &= \binom{|a|+1}{\ell} a b_1 \cdots b_\ell \\ (a_1 \otimes \cdots \otimes a_{\ell'}) \triangleright (b_1 \otimes \cdots \otimes b_{\ell''}) &= a_1 \triangleright (a_2 \otimes \cdots \otimes a_{\ell'} \otimes b_1 \otimes \cdots \otimes b_{\ell''}) \\ &= \binom{|a_1|+1}{\ell'+\ell''-1} a_1 \cdots a_{\ell'} b_1 \cdots b_{\ell''} \end{aligned}$$

where the expressions on the right-hand side mean the product in the algebra  $A$ .

In particular, if we apply these rules to  $1 \in \mathbb{F} = A^{\otimes 0}$ , we have

$$\begin{aligned} 1 \triangleright 1 &= 1 \\ 1 \triangleright b &= b \\ 1 \triangleright (b_1 \otimes \cdots \otimes b_\ell) &= 0 \quad \text{if } \ell > 1 \\ a \triangleright 1 &= a \\ (a_1 \otimes \cdots \otimes a_\ell) \triangleright 1 &= \binom{|a_1|+1}{\ell-1} a_1 \cdots a_\ell. \end{aligned}$$

**Remark 5.7** The restriction  $\triangleright : A \otimes T(A) \longrightarrow A$  is a brace product on  $A$  which is symmetric if  $A$  is commutative and generalises the natural pre-Lie product of the Lie subalgebra of strictly positive generators in the Witt algebra (cf. [8, 15]). Note however that  $\triangleright$  on  $T(A) \otimes T(A)$  is not a multibrace product (cf. [24]), even excluding the scalar component, because the first non-trivial multibrace identity  $M_{21}(a \otimes b + b \otimes a; c) + M_{11}(M_{11}(a; b); c) = M_{11}(a; M_{11}(b; c)) + M_{12}(a; b \otimes c + c \otimes b)$  is not satisfied. Moreover, a unit for  $\triangleright$  can not exist, because of length's arguments, and  $\triangleright$  is not associative, since for any  $a, b, c \in A$  we have

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (|a|+1)|a| abc \neq 0.$$

The algebraic structure described by the operator  $\triangleright$  in terms of generators and relations is an open question.

**Definition 5.8** We call **left recursive operator**  $L : T(A) \longrightarrow T(A)$  the collection  $L = \{L_\ell, \ell \geq 0\}$  of (non homogeneous) operations

$$\begin{aligned} L_0 : \mathbb{F} &\longrightarrow \mathbb{F} \\ L_\ell : A^{\otimes \ell} &\longrightarrow \bigoplus_{\lambda=1}^{\ell} A^{\otimes \lambda}, \quad \ell \geq 1 \end{aligned}$$

defined recursively by  $L_0(1) = 1$  and, for any  $\ell \geq 1$ , by

$$L_\ell(a_1, \dots, a_\ell) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \left( L_i(a_1, \dots, a_i) \triangleright a_{i+1} \right) \otimes a_{i+2} \otimes \cdots \otimes a_\ell,$$

where we denote  $L_\ell(a_1, \dots, a_\ell) := L_\ell(a_1 \otimes \cdots \otimes a_\ell)$ .



The first left operators give

$$\begin{aligned}
L_1(a) &= a, \\
L_2(a, b) &= a \triangleright b - a \otimes b \\
&= \binom{|a|+1}{1} ab - a \otimes b, \\
L_3(a, b, c) &= (a \triangleright b) \triangleright c - (a \otimes b) \triangleright c - (a \triangleright b) \otimes c + a \otimes b \otimes c \\
&= \left( \binom{|a|+1}{1} \binom{|a|+|b|+1}{1} - \binom{|a|+1}{2} \right) abc - \binom{|a|+1}{1} ab \otimes c + a \otimes b \otimes c.
\end{aligned}$$

The left operators  $L_\ell$  can be easily described in a closed way.

**Lemma 5.9** *For any  $\ell \geq 2$  and any  $a_1, \dots, a_\ell \in A$  we have*

$$L_\ell(a_1, \dots, a_\ell) = L_{\ell-1}(a_1, \dots, a_{\ell-1}) \triangleright a_\ell - L_{\ell-1}(a_1, \dots, a_{\ell-1}) \otimes a_\ell.$$

*As a consequence,  $L_\ell(a_1, \dots, a_\ell)$  is the sum of the  $2^{\ell-1}$  possible multi-monomials obtained by combining the operations  $\triangleright$  and  $\otimes$  with fixed parenthesizing on the left, namely*

$$L_\ell(a_1, \dots, a_\ell) = \sum_{\mu_1, \dots, \mu_{\ell-1} \in \{0,1\}} (-1)^{\mu_1 + \dots + \mu_{\ell-1}} \left( \dots \left( (a_1 *_{\mu_1} a_2) *_{\mu_2} a_3 \right) *_{\mu_3} \dots *_{\mu_{\ell-2}} a_{\ell-1} \right) *_{\mu_{\ell-1}} a_\ell$$

where we set

$$*_{\mu} = \begin{cases} \triangleright & \text{if } \mu = 0 \\ \otimes & \text{if } \mu = 1 \end{cases}.$$

**Proof.** By induction on  $\ell$ . For  $\ell = 2$ , we have

$$L_1(a_1) \triangleright a_2 - L_1(a_1) \otimes a_2 = a_1 \triangleright a_2 - a_1 \otimes a_2 = L_2(a_1, a_2).$$

Now suppose that for any  $i = 2, \dots, \ell - 1$  we have

$$L_i(a_1, \dots, a_i) = L_{i-1}(a_1, \dots, a_{i-1}) \triangleright a_i - L_{i-1}(a_1, \dots, a_{i-1}) \otimes a_i.$$

Let us develop the sum defining  $L_\ell(a_1, \dots, a_\ell)$ . At each step, we separate the first two terms of

the sum over  $i = 0, \dots, \ell - 1$ :

$$\begin{aligned}
L_\ell(a_1, \dots, a_\ell) &= \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \left( L_i(a_1, \dots, a_i) \triangleright a_{i+1} \right) \otimes a_{i+2} \otimes \dots \otimes a_\ell \\
&= (-1)^{\ell-1} a_1 \otimes a_2 \otimes \dots \otimes a_\ell + (-1)^{\ell-2} (a_1 \triangleright a_2) \otimes \dots \otimes a_\ell \\
&\quad + \sum_{i=2}^{\ell-1} (-1)^{\ell-1-i} \left( L_i(a_1, \dots, a_i) \triangleright a_{i+1} \right) \otimes a_{i+2} \otimes \dots \otimes a_\ell \\
&= (-1)^{\ell-2} L_2(a_1, a_2) \otimes a_3 \otimes \dots \otimes a_\ell + (-1)^{\ell-3} \left( L_2(a_1, a_2) \triangleright a_3 \right) \otimes \dots \otimes a_\ell \\
&\quad + \sum_{i=3}^{\ell-1} (-1)^{\ell-1-i} \left( L_i(a_1, \dots, a_i) \triangleright a_{i+1} \right) \otimes a_{i+2} \otimes \dots \otimes a_\ell \\
&= (-1)^{\ell-3} L_3(a_1, a_2, a_3) \otimes a_4 \otimes \dots \otimes a_\ell + (-1)^{\ell-4} \left( L_3(a_1, a_2, a_3) \triangleright a_4 \right) \otimes \dots \otimes a_\ell \\
&\quad + \sum_{i=4}^{\ell-1} (-1)^{\ell-1-i} \left( L_i(a_1, \dots, a_i) \triangleright a_{i+1} \right) \otimes a_{i+2} \otimes \dots \otimes a_\ell \\
&= \dots \\
&= (-1)^{\ell-(\ell-1)} L_{\ell-1}(a_1, \dots, a_{\ell-1}) \otimes a_\ell + (-1)^{\ell-\ell} L_{\ell-1}(a_1, \dots, a_{\ell-1}) \triangleright a_\ell \\
&= L_{\ell-1}(a_1, \dots, a_{\ell-1}) \triangleright a_\ell - L_{\ell-1}(a_1, \dots, a_{\ell-1}) \otimes a_\ell.
\end{aligned}$$

□

**Definition 5.10** We call **right recursive operator**  $R : T(A) \longrightarrow T(A)$  the collection  $R = \{R_\ell, \ell \geq 0\}$  of (non homogeneous) operations

$$\begin{aligned}
R_0 : \mathbb{F} &\longrightarrow \mathbb{F} \\
R_\ell : A^{\otimes \ell} &\longrightarrow \bigoplus_{\lambda=1}^{\ell} A^{\otimes \lambda}, \quad \ell \geq 1
\end{aligned}$$

defined recursively by  $R_0(1) = 1$  and, for any  $\ell \geq 1$ , by

$$\begin{aligned}
R_\ell(a_1, \dots, a_\ell) &= \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_\ell^j} (a_1 \triangleright R_{p_1-1}(a_2, \dots, a_{p_1})) \\
&\quad \otimes (a_{p_1+1} \triangleright R_{p_2-1}(a_{p_1+2}, \dots, a_{p_1+p_2})) \otimes \dots \\
&\quad \otimes (a_{p_1+\dots+p_{j-1}+1} \triangleright R_{p_j-1}(a_{p_1+\dots+p_{j-1}+2}, \dots, a_{p_1+\dots+p_j})),
\end{aligned} \tag{5.3}$$

where we denote  $R_\ell(a_1, \dots, a_\ell) := R_\ell(a_1 \otimes \dots \otimes a_\ell)$ .

For instance, the first right operators are

$$\begin{aligned}
R_1(a) &= a, \\
R_2(a, b) &= a \triangleright b + a \otimes b \\
&= \binom{|a|+1}{1} ab + a \otimes b, \\
R_3(a, b, c) &= a \triangleright (b \triangleright c) + a \triangleright (b \otimes c) + a \otimes (b \triangleright c) + (a \triangleright b) \otimes c + a \otimes b \otimes c \\
&= \left( \binom{|a|+1}{1} \binom{|b|+1}{1} + \binom{|a|+1}{2} \right) abc + \binom{|b|+1}{1} a \otimes bc \\
&\quad + \binom{|a|+1}{1} ab \otimes c + a \otimes b \otimes c.
\end{aligned}$$

The right operators  $R_\ell$  can also be described in a closed way.

**Definition 5.11** Let  $\mathcal{M}_\ell$  be the set of sequences satisfying (5.1). For any  $\mathbf{m} \in \mathcal{M}_\ell$ , let  $R_{\mathbf{m}} = R_{\mathbf{m}}^\ell : A^{\otimes \ell} \longrightarrow T(A)$  be the operator which nests the operation  $\triangleright$  in an element  $a_1 \otimes \cdots \otimes a_\ell$  according to the sequence  $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathcal{M}_\ell$  with the following algorithm:

- The length of  $R_{\mathbf{m}}^\ell(a_1, \dots, a_\ell)$  is given by  $m_1$ , that is,  $R_{\mathbf{m}}^\ell(a_1, \dots, a_\ell) \in H^{\otimes m_1}$ .
- Start with  $a_1$  on the leftmost position and proceed for increasing  $i = 1, \dots, \ell - 1$  to nest a factor  $a_i \triangleright Q_i(a_{i+1}, a_{i+2}, \dots)$  where  $Q_i$  is a multi-monomial of length  $m_{i+1}$ .
- For  $i = 2, \dots, \ell - 1$ , the role of the variable  $a_i$  in the monomial is either determined by the action of the previous variables  $a_1, \dots, a_{i-1}$ , or it is the first variable of a new tensor factor.
- The last variable  $a_\ell$  is ruled by a value  $m_{n_\ell+1} = 0$ , omitted in the sequence  $\mathbf{m}$ , which says that  $a_\ell$  acts on 0 variables by  $\triangleright$  and allows it to close the monomial.

**Example 5.12** Let us give some examples of this algorithm, for  $\ell = 5$ . Fix  $a_1, \dots, a_5 \in A$  and set  $n_i = |a_i|$  for  $i = 1, \dots, 5$ . For  $\mathbf{m} = (2, 1, 0, 2, 0)$ , the multi-monomial  $R_{(2,1,0,2,0)}^5(a_1, \dots, a_5)$  is composed of two tensor factors (because  $m_1 = 2$ ). The variable  $a_1$  acts by  $\triangleright$  on a multi-monomial of length 1 (because  $m_2 = 1$ ) which starts necessarily by  $a_2$ , and since  $a_2$  does not act by  $\triangleright$  (because  $m_3 = 0$ ), the first tensor factor is necessarily of the form  $a_1 \triangleright a_2$ . Then the second tensor factor starts with  $a_3$  acting by  $\triangleright$  on a multi-monomial of length 2 (because  $m_4 = 2$ ), which starts necessarily by  $a_4$ . Since  $a_4$  does not act by  $\triangleright$  (because  $m_5 = 0$ ), the second tensor factor is necessarily of the form  $a_3 \triangleright (a_4 \otimes a_5)$ . Therefore we finally have

$$\begin{aligned} R_{(2,1,0,2,0)}^5(a_1, \dots, a_5) &= (a_1 \triangleright a_2) \otimes (a_3 \triangleright (a_4 \otimes a_5)) \\ &= \binom{n_1 + 1}{1} \binom{n_3 + 1}{2} a_1 a_2 \otimes a_3 a_4 a_5. \end{aligned}$$

For  $\mathbf{m} = (2, 1, 2, 0, 0)$ , the variable  $a_1$  still acts by  $\triangleright$  on a multi-monomial of length 1 which starts necessarily by  $a_2$ , but this time  $a_2$  itself acts by  $\triangleright$  on a multi-monomial of length 2, and this exhausts the possible  $\triangleright$  operations. Finally, this time we have

$$\begin{aligned} R_{(2,1,2,0,0)}^5(a_1, \dots, a_5) &= (a_1 \triangleright (a_2 \triangleright (a_3 \otimes a_4))) \otimes a_5 \\ &= \binom{n_1 + 1}{1} \binom{n_2 + 1}{2} a_1 a_2 a_3 a_4 \otimes a_5. \end{aligned}$$

Note that the binomial coefficients given by a sequence  $\mathbf{m} \in \mathcal{M}_\ell$  can be determined directly from the last  $\ell - 1$  digits, plus an extra null value. For  $(2, 1, 0, 2, 0)$  we have exactly

$$\binom{n_1 + 1}{1} \binom{n_2 + 1}{0} \binom{n_3 + 1}{2} \binom{n_4 + 1}{0} \binom{n_5 + 1}{0}.$$

and for  $(2, 1, 2, 0, 0)$  we have

$$\binom{n_1 + 1}{1} \binom{n_2 + 1}{2} \binom{n_3 + 1}{0} \binom{n_4 + 1}{0} \binom{n_5 + 1}{0}.$$

Two more examples of the algorithm: for  $\mathbf{m} = (3, 0, 2, 0, 0)$ ,

$$\begin{aligned} R_{(3,0,2,0,0)}^5(a_1, \dots, a_5) &= a_1 \otimes (a_2 \triangleright (a_3 \otimes a_4)) \otimes a_5 \\ &= \binom{n_2 + 1}{2} a_1 \otimes a_2 a_3 a_4 \otimes a_5, \end{aligned}$$

and for  $\mathbf{m} = (4, 0, 1, 0, 0)$ ,

$$\begin{aligned} R_{(4,0,1,0,0)}^5(a_1, \dots, a_5) &= a_1 \otimes (a_2 \triangleright a_3) \otimes a_4 \otimes a_5 \\ &= \binom{n_2 + 1}{1} a_1 \otimes a_2 a_3 \otimes a_4 \otimes a_5. \end{aligned}$$

**Lemma 5.13** *For any  $\ell \geq 1$  and any  $(a_1, \dots, a_\ell)$ , we have*

$$R_\ell(a_1, \dots, a_\ell) = \sum_{\mathbf{m} \in \mathcal{M}_\ell} R_{\mathbf{m}}^\ell(a_1, \dots, a_\ell).$$

For instance, for  $\ell = 1, 2, 3$  we have

$$\begin{aligned} R_{(1)}^1(a) &= a, \\ R_{(2,0)}^2(a, b) &= a \otimes b \\ R_{(1,1)}^2(a, b) &= a \triangleright b = \binom{n_1 + 1}{1} a b \\ R_{(3,0,0)}^3(a, b, c) &= a \otimes b \otimes c \\ R_{(2,1,0)}^3(a, b, c) &= (a \triangleright b) \otimes c = \binom{n_1 + 1}{1} a b \otimes c \\ R_{(2,0,1)}^3(a, b, c) &= a \otimes (b \triangleright c) = \binom{n_2 + 1}{1} a \otimes b c \\ R_{(1,2,0)}^3(a, b, c) &= a \triangleright (b \otimes c) = \binom{n_1 + 1}{2} a b c \\ R_{(1,1,1)}^3(a, b, c) &= a \triangleright (b \triangleright c) = \binom{n_1 + 1}{1} \binom{n_2 + 1}{1} a b c. \end{aligned}$$

Comparing with the value of  $R_1$ ,  $R_2$  and  $R_3$  given above, the assertion is easily verified.

**Proof.** Let us call  $\tilde{R}_\ell(a_1, \dots, a_\ell)$  the sum over  $\mathbf{m} \in \mathcal{M}_\ell$  of Lemma 5.13, and prove that it solves equation (5.3) by induction on  $\ell$ .

For  $\ell = 1, 2, 3$  the assertion was proved in the examples. For any  $\ell \geq 1$ , we then suppose that on the right-hand side of eq. (5.3) we have  $R_{p_{i-1}} = \tilde{R}_{p_{i-1}}$  for any  $1 \leq i \leq j$ , and we set

$$P_i = p_1 + p_2 + \dots + p_i,$$

so that

$$\begin{aligned} R_\ell(a_1, \dots, a_\ell) &= \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_\ell^j} \sum_{i=1}^j \sum_{\mathbf{q}^{(i)} \in \mathcal{M}_{p_{i-1}}} (a_1 \triangleright \tilde{R}_{\mathbf{q}^{(1)}}^{p_1-1}(a_2, \dots, a_{p_1})) \otimes \\ &\quad \otimes (a_{p_1+1} \triangleright \tilde{R}_{\mathbf{q}^{(2)}}^{p_2-1}(a_{p_1+2}, \dots, a_{p_2})) \otimes \dots \otimes (a_{P_{j-1}+1} \triangleright \tilde{R}_{\mathbf{q}^{(j)}}^{p_j-1}(a_{P_{j-1}+2}, \dots, a_{P_j})). \end{aligned}$$

In this sum, we can note the following things:

- The running value  $j$  gives the length of the corresponding multi-monomial.
- In the first tensor factor, the value  $q_1^{(1)}$  represents the length of  $\tilde{R}_{\mathbf{q}^{(1)}}^{p_1-1}(a_2, \dots, a_{p_1})$ , that is, a sequence number associated to  $a_1$ , and more generally  $\mathbf{q}^{(1)}$  rules the nested operations up to the variable  $a_{p_1}$ . The last variable  $a_{p_1}$  does not act on further variables and so it should be associated to a missing value 0. Therefore, the nested operations in the whole first tensor factor are ruled by the sequence  $(\mathbf{q}^{(1)}, 0)$ .

- Similarly, for any  $i \leq j$ , the nested operations in the  $i$ th tensor factor are ruled by the sequence  $(\mathbf{q}^{(i)}, 0)$ .

Let us then associate to this expression the sequence

$$\mathbf{m} = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, \dots, \mathbf{q}^{(j)}),$$

that is,

$$\begin{aligned} m_1 &= j \\ m_2 &= q_1^{(1)}, \dots, m_{p_1} = q_{p_1-1}^{(1)}, m_{p_1+1} = 0, \\ m_{P_{i-1}+2} &= q_1^{(i)}, \dots, m_{P_i} = q_{p_i-1}^{(i)}, m_{P_i+1} = 0, \quad \text{for } 1 \leq i \leq j-1 \\ m_{P_{j-1}+2} &= q_1^{(j)}, \dots, m_{P_j} = q_{p_j-1}^{(j)}, \end{aligned}$$

which has precisely length

$$1 + (p_1 - 1) + 1 + (p_2 - 1) + \dots + 1 + (p_j - 1) = p_1 + \dots + p_j = P_j = \ell.$$

Note that in the sum over the sequences  $\mathbf{p} = (p_1, \dots, p_j) \in \mathcal{C}_\ell^j$ , where  $p_i \geq 1$  for  $i = 1, \dots, j$ , there occur the terms with  $p_i - 1 = 0$ . In this case the multipolynomial  $\tilde{R}_{p_i-1} = R_0 = 1$  has no variables, and the set  $\mathcal{M}_{p_i-1} = \mathcal{M}_0$  is empty. The corresponding sequence  $\mathbf{q}^{(i)}$  is then absent in  $\mathbf{m}$ , but its associated null value must be present, for any  $i = 1, \dots, j-1$ , to preserve the total length  $\ell$ . Following the rules of the algorithm given in Def. 5.11, we can therefore write

$$\begin{aligned} (a_1 \triangleright \tilde{R}_{\mathbf{q}^{(1)}}^{p_1-1}(a_2, \dots, a_{p_1})) \otimes (a_{p_1+1} \triangleright \tilde{R}_{\mathbf{q}^{(2)}}^{p_2-1}(a_{p_1+2}, \dots, a_{P_2})) \otimes \dots \\ \dots \otimes (a_{P_{j-1}+1} \triangleright \tilde{R}_{\mathbf{q}^{(j)}}^{p_j-1}(a_{P_{j-1}+2}, \dots, a_{P_j})) = R_{\mathbf{m}}(a_1, \dots, a_\ell). \end{aligned}$$

Let us call

$$\mathcal{N}_\ell = \{\mathbf{m} = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, \dots, \mathbf{q}^{(j)}) \mid 1 \leq j \leq \ell, \mathbf{p} \in \mathcal{C}_\ell^j, \mathbf{q}^{(i)} \in \mathcal{M}_{p_i-1} \text{ for } 1 \leq i \leq j\}$$

the set of sequences obtained in this way. Then the equality

$$R_\ell(a_1, a_2, \dots, a_\ell) = \sum_{\mathbf{m} \in \mathcal{N}_\ell} R_{\mathbf{m}}^\ell(a_1, \dots, a_\ell) = \tilde{R}_\ell(a_1, a_2, \dots, a_\ell)$$

holds if we show that  $\mathcal{N}_\ell = \mathcal{M}_\ell$ .

Let us first show that  $\mathcal{N}_\ell \subset \mathcal{M}_\ell$ . For fixed  $j, \mathbf{p}$  and  $\mathbf{q}^{(1)}, \dots, \mathbf{q}^{(j)}$ , we have

$$\begin{aligned} m_1 + \dots + m_\ell &= j + \sum_{i=1}^j (q_1^{(i)} + \dots + q_{p_i-1}^{(i)}) \\ &= j + (p_1 - 1) + (p_2 - 1) + \dots + (p_j - 1) \\ &= j + p_1 + \dots + p_j - j = \ell. \end{aligned}$$

For any  $h = 1, \dots, \ell$ , suppose that  $h$  belongs to the  $r$ th block, for some  $r \leq j$ , that is,

$$h = P_{r-1} + 1 + k = p_1 + \dots + p_{r-1} + 1 + k,$$

with  $1 \leq k \leq p_r - 1$ . Then we have

$$\begin{aligned} m_1 + \dots + m_h &= j + \sum_{i=1}^{r-1} (q_1^{(i)} + \dots + q_{p_i-1}^{(i)}) + (q_1^{(r)} + \dots + q_k^{(r)}) \\ &= j + (p_1 - 1) + (p_2 - 1) + \dots + (p_{r-1} - 1) + (q_1^{(r)} + \dots + q_k^{(r)}) \\ &= (p_1 + \dots + p_{r-1}) + (q_1^{(r)} + \dots + q_k^{(r)}) + (j - r) + 1 \\ &\geq P_{r-1} + 1 + k = h \end{aligned}$$

because  $q_1^{(r)} + \dots + q_k^{(r)} \geq k$  and  $j - r \geq 0$ .

Finally, let us show that there is a bijection between  $\mathcal{N}_\ell$  and  $\mathcal{M}_\ell$ . The set  $\mathcal{M}_\ell$  is well known to be in bijection with the set  $PBT_{\ell+1}$  of planar binary trees with  $\ell + 1$  leaves (and a root). An explicit bijection  $\Phi : \mathcal{M}_\ell \rightarrow PBT_{\ell+1}$  is described in [7], Definition 2.16, using the *over* and *under* grafting operations on trees, namely

$$t/s = \begin{array}{c} t \\ \diagdown \\ s \end{array} \quad \text{and} \quad t \setminus s = \begin{array}{c} t \\ \diagup \\ s \end{array}.$$

The first values of  $\Phi$ , for the empty sequence in  $\mathcal{M}_0$  and for  $(1) \in \mathcal{M}_1$  and  $(2, 0), (1, 1) \in \mathcal{M}_2$ , are

$$\Phi(\ ) = |, \quad \Phi(1) = \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad \Phi(2, 0) = \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad \Phi(1, 1) = \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array}.$$

So, for our purpose, it is enough to show that  $\mathcal{N}_\ell$  is in bijection with  $PBT_{\ell+1}$ . For this, since  $\mathcal{N}_\ell \subset \mathcal{M}_\ell$ , consider the map  $\Phi$  restricted to  $\mathcal{N}_\ell$  and let us show that the image  $\Phi(\mathcal{N}_\ell)$  coincides with  $PBT_{\ell+1}$ .

For a given sequence  $\mathbf{m} = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, \dots, \mathbf{q}^{(j)}) \in \mathcal{N}_\ell$ , we have:

- The sequence  $\mathbf{m}$  is decomposable as  $\mathbf{m} = (\mathbf{m}', \mathbf{q}^{(j)})$  into the two well-defined sequences  $\mathbf{m}' = (j, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, \dots, \mathbf{q}^{(j-1)}, 0) \in \mathcal{M}_{P_{j-1}+1}$  and  $\mathbf{q}^{(j)} \in \mathcal{M}_{p_{j-1}}$ . In fact, if we set  $\ell' = P_{j-1} + 1$ , we have

$$m'_1 + \dots + m'_{\ell'} = j + (p_1 - 1) + \dots + (p_{j-1} - 1) = j + P_{j-1} - (j - 1) = \ell',$$

and for any  $h = 1, \dots, \ell'$  one can see that  $m'_1 + \dots + m'_h \geq h$  with a computation similar to that used to show that  $\mathbf{m} \in \mathcal{M}_\ell$ .

According to the definition of  $\Phi$ , we then have  $\Phi(\mathbf{m}) = \Phi(\mathbf{m}') \setminus \Phi(\mathbf{q}^{(j)})$ . Graphically, if we denote the trees by  $t = \Phi(\mathbf{m})$ ,  $t' = \Phi(\mathbf{m}')$  and  $t_j = \Phi(\mathbf{q}^{(j)})$ , this means that

$$t = \begin{array}{c} t_j \\ \diagup \\ t' \end{array}.$$

- The sequence  $\mathbf{m}'$  is surely not decomposable because it is of the form

$$\mathbf{m}' = (m''_1 + 1, m''_2, \dots, m''_{\ell''}, 0)$$

with

$$\mathbf{m}'' = (j-1, \mathbf{q}^{(1)}, 0, \mathbf{q}^{(2)}, 0, \dots, \mathbf{q}^{(j-1)}) \in \mathcal{M}_{\ell''}, \quad \ell'' = \ell' - 1.$$

The sequence  $\mathbf{m}''$  indeed belongs to  $\mathcal{M}_{\ell''}$  for the same reason used to show that  $\mathbf{m} \in \mathcal{M}_\ell$ . Then, the sequence  $\mathbf{m}'$  is not decomposable in position 1 because  $m'_1 = m''_1 + 1 \geq 2$ , and it is not decomposable in any position  $h = 2, \dots, \ell''$  because  $\mathbf{m}'' \in \mathcal{M}_{\ell''}$  implies that  $m''_1 + \dots + m''_h = m''_1 + 1 + m''_2 + \dots + m''_h \geq h + 1$ , and therefore surely  $m'_1 + \dots + m'_h \neq h$ .

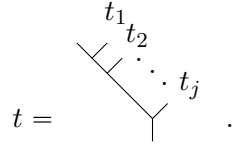
Finally, according to the definition of  $\Phi$ , we then have  $\Phi(\mathbf{m}') = \Phi(\mathbf{m}'') / \begin{array}{c} \diagup \\ \diagdown \end{array}$ . If we set  $t'' = \Phi(\mathbf{m}'')$ , this means that

$$t' = \begin{array}{c} t'' \\ \diagup \\ \diagdown \end{array}$$

and therefore

$$t = \begin{array}{c} t'' \\ \diagup \\ \diagdown \\ t_j \end{array}.$$

- The same arguments can be applied to the sequence  $\mathbf{m}''$  and its new components, until we reach a full description of the tree  $t = \Phi(\mathbf{m})$  in terms of the trees  $t_i = \Phi(\mathbf{q}^{(i)})$ , for  $i = 1, \dots, j$ , namely



Let us denote this tree by  $G^j(t_1, \dots, t_j)$ .

In conclusion, if we let  $j$  run from 1 to  $\ell$ , we consider all possible sequences  $\mathbf{p} \in \mathcal{C}_\ell^j$  and for any  $i = 1, \dots, j$  all trees  $t_i \in \Phi(\mathcal{M}_{p_i-1}) = PBT_{p_i}$ , the result  $G^j(t_1, \dots, t_j)$  is any possible tree with number of leaves given by

$$|G^j(t_1, \dots, t_j)| = 1 + |t_1| + \dots + |t_j| = 1 + p_1 + \dots + p_j = \ell + 1.$$

In other words, we have

$$\Phi(\mathcal{N}_\ell) = \{t = G^j(t_1, \dots, t_j) \mid j = 1, \dots, \ell, \mathbf{p} \in \mathcal{C}_\ell^j, t_i \in PBT_{p_i}, 1 \leq i \leq j\} = PBT_{\ell+1}.$$

□

**Corollary 5.14** *For any  $\ell \geq 0$  and any  $a_1, \dots, a_{\ell+1} \in A$ , set  $n_i = |a_i| \geq 1$  for  $i = 1, \dots, \ell + 1$ . Then, for any sequence  $\mathbf{m} \in \mathcal{M}_\ell$ , we have*

$$a_1 \triangleright R_{\mathbf{m}}(a_2, \dots, a_{\ell+1}) = \binom{n_1 + 1}{m_1} \dots \binom{n_\ell + 1}{m_\ell} \binom{n_{\ell+1} + 1}{0} a_1 a_2 \dots a_{\ell+1}.$$

Therefore

$$a_1 \triangleright R_\ell(a_2, \dots, a_{\ell+1}) = d(n_1, \dots, n_{\ell+1}) a_1 a_2 \dots a_{\ell+1},$$

where the Lagrange coefficients  $d(n_1, \dots, n_{\ell+1})$  are given in Def. 5.3.

To describe the left codivision we introduce a last set of operators corresponding to the labeled Lagrange coefficients.

**Definition 5.15** For any  $\ell \geq 2$ , let  $\mathcal{E}_{\ell-1}$  be the set of sequences given by (5.2). We call **labeled right recursive operator**  $R^{\mathbf{e}} : T(A) \rightarrow T(A)$  the collection of operations  $R^{\mathbf{e}} = \{R_0, R_1, R_\ell^{\mathbf{e}}, \ell \geq 2, \mathbf{e} \in \mathcal{E}_{\ell-1}\}$ , where  $R_0$  and  $R_1$  are the unlabeled right recursive operations and, for  $\ell \geq 2$  and  $\mathbf{e} \in \mathcal{E}_{\ell-1}$ , the operation

$$R_\ell^{\mathbf{e}} : A^{\otimes \ell} \longrightarrow \bigoplus_{\lambda=1}^{\ell} A^{\otimes \lambda}$$

is defined recursively by

$$\begin{aligned} R_\ell^{\mathbf{e}}(a_1, \dots, a_\ell) &= \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_\ell^j} \delta_{e_1, 1} (a_1 \triangleright R_{p_1-1}^{(e_2, \dots, e_{p_1-1})}(a_2, \dots, a_{p_1})) \\ &\quad \otimes \delta_{e_{p_1+1}, 1} (a_{p_1+1} \triangleright R_{p_2-1}^{(e_{p_1+2}, \dots, e_{p_1+p_2-1})}(a_{p_1+2}, \dots, a_{p_1+p_2})) \otimes \dots \\ &\quad \dots \otimes \delta_{e_{P_{j-1}+1}, 1} (a_{P_{j-1}+1} \triangleright R_{p_j-1}^{(e_{P_{j-1}+2}, \dots, e_{P_j-1})}(a_{P_{j-1}+2}, \dots, a_{P_j})), \end{aligned} \tag{5.4}$$

where we set  $P_i = p_1 + \dots + p_i$  for  $i = 1, \dots, j$ , and where  $\delta_{e_i, 1}$  is the Kronecker delta, equal to 1 if  $e_i = 1$  and equal to 0 if  $e_i = 2$ .

If  $\mathbf{e} = (1, 1, \dots, 1)$ , then  $R_\ell^{\mathbf{e}} = R_\ell$ . If  $\mathbf{e}$  contains at least one value  $e_i = 2$ , then  $R_\ell^{\mathbf{e}}$  is obtained from  $R_\ell$  by removing the terms which contain any factor of the form  $a_i \triangleright Q_i$ .

For instance, starting with  $R_1(a) = a$ , the first terms of the recursion are as follows. For  $\ell = 2$ , we have  $\mathcal{E}_1 = \{(1), (2)\}$  and

$$\begin{aligned} R_2^{(1)}(a, b) &= a \triangleright R_1(b) + a \otimes b \\ &= a \triangleright b + a \otimes b \end{aligned}$$

$$R_2^{(2)}(a, b) = a \otimes b.$$

For  $\ell = 3$ , we have  $\mathcal{E}_2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$  and therefore

$$\begin{aligned} R_3^{(1,1)}(a, b, c) &= a \triangleright R_2^{(1)}(b, c) + (a \triangleright R_1(b)) \otimes c + a \otimes (b \triangleright R_1(c)) + a \otimes b \otimes c \\ &= a \triangleright (b \triangleright c) + a \triangleright (b \otimes c) + (a \triangleright b) \otimes c + a \otimes (b \triangleright c) + a \otimes b \otimes c \end{aligned}$$

$$\begin{aligned} R_3^{(1,2)}(a, b, c) &= a \triangleright R_2^{(2)}(b, c) + (a \triangleright R_1(b)) \otimes c + a \otimes b \otimes c \\ &= a \triangleright (b \otimes c) + (a \triangleright b) \otimes c + a \otimes b \otimes c \end{aligned}$$

$$\begin{aligned} R_3^{(2,1)}(a, b, c) &= a \otimes (b \triangleright R_1(c)) + a \otimes b \otimes c \\ &= a \otimes (b \triangleright c) + a \otimes b \otimes c \end{aligned}$$

$$R_3^{(2,2)}(a, b, c) = a \otimes b \otimes c.$$

For  $\ell = 4$ , the set  $\mathcal{E}_3$  contains 8 sequences, and we have

$$\begin{aligned} R_4^{(1,1,1)}(a, b, c, d) &= a \triangleright R_3^{(1,1)}(b, c, d) \\ &\quad + (a \triangleright R_2^{(1)}(b, c)) \otimes d + (a \triangleright R_1(b)) \otimes (c \triangleright R_1(d)) + a \otimes (b \triangleright R_2^{(1)}(c, d)) \\ &\quad + (a \triangleright R_1(b)) \otimes c \otimes d + a \otimes (b \triangleright R_1(c)) \otimes d + a \otimes b \otimes (c \triangleright R_1(d)) \\ &\quad + a \otimes b \otimes c \otimes d \end{aligned}$$

$$\begin{aligned} R_4^{(1,1,2)}(a, b, c, d) &= a \triangleright R_3^{(1,2)}(b, c, d) + (a \triangleright R_2^{(1)}(b, c)) \otimes d + a \otimes (b \triangleright R_2^{(2)}(c, d)) \\ &\quad + (a \triangleright R_1(b)) \otimes c \otimes d + a \otimes (b \triangleright R_1(c)) \otimes d + a \otimes b \otimes c \otimes d \end{aligned}$$

$$\begin{aligned} R_4^{(1,2,1)}(a, b, c, d) &= a \triangleright R_3^{(2,1)}(b, c, d) + (a \triangleright R_2^{(2)}(b, c)) \otimes d + (a \triangleright R_1(b)) \otimes (c \triangleright R_1(d)) \\ &\quad + (a \triangleright R_1(b)) \otimes c \otimes d + a \otimes b \otimes (c \triangleright R_1(d)) + a \otimes b \otimes c \otimes d \end{aligned}$$

$$\begin{aligned} R_4^{(2,1,1)}(a, b, c, d) &= a \otimes (b \triangleright R_2^{(1)}(c, d)) + a \otimes (b \triangleright R_1(c)) \otimes d + a \otimes b \otimes (c \triangleright R_1(d)) \\ &\quad + a \otimes b \otimes c \otimes d \end{aligned}$$

$$R_4^{(1,2,2)}(a, b, c, d) = a \triangleright R_3^{(2,2)}(b, c, d) + (a \triangleright R_2^{(2)}(b, c)) \otimes d + (a \triangleright R_1(b)) \otimes c \otimes d + a \otimes b \otimes c \otimes d$$

$$R_4^{(2,1,2)}(a, b, c, d) = a \otimes (b \triangleright R_2^{(2)}(c, d)) + a \otimes (b \triangleright R_1(c)) \otimes d + a \otimes b \otimes c \otimes d$$

$$R_4^{(2,2,1)}(a, b, c, d) = a \otimes b \otimes (c \triangleright R_1(d)) + a \otimes b \otimes c \otimes d$$

$$R_4^{(2,2,2)}(a, b, c, d) = a \otimes b \otimes c \otimes d$$



The labeled right operations can also be given by a closed formula.

**Lemma 5.16** *For any  $\ell \geq 2$ , for any sequence  $\mathbf{e} \in \mathcal{E}_{\ell-1}$  and for any  $a_1, \dots, a_\ell \in A$ , we have*

$$R_\ell^{\mathbf{e}}(a_1, \dots, a_\ell) = \sum_{\mathbf{m} \in \mathcal{M}_\ell^{\mathbf{e}}} R_{\mathbf{m}}^\ell(a_1, \dots, a_\ell).$$

*As a consequence, if for  $a_1, \dots, a_{\ell+1} \in A$  we denote  $n_i = |a_i|$  for  $i = 1, \dots, \ell + 1$ , we then have*

$$a_1 \triangleright R_\ell^{\mathbf{e}}(a_2, \dots, a_{\ell+1}) = d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) a_1 \cdots a_{\ell+1} \in A,$$

*where the labeled Lagrange coefficients  $d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1})$  are given in Def. 5.4.*

**Proof.** If  $\mathbf{e} = (1, 1, \dots, 1)$  there is nothing to prove. Otherwise, for any value  $e_i = 2$  in  $\mathbf{e}$ , we obtain  $R_\ell^{\mathbf{e}}(a_1, \dots, a_\ell)$  from  $R_\ell(a_1, \dots, a_\ell)$  by removing the terms containing a factor  $a_i \triangleright Q_i$ . By Lemma 5.13, such a term is associated to a sequence  $\mathbf{m} \in \mathcal{M}_\ell$ , and by Def. 5.11 the factor  $a_i \triangleright Q_i$  corresponds to a non-zero value  $m_{i+1}$ . Therefore, in order to remove such terms, it suffices to consider sequences  $\mathbf{m}$  with  $m_{i+1} = 0$  whenever  $e_i = 2$ .  $\square$

**Theorem 5.17** *The co-operations of the Faà di Bruno coloop bialgebra can be equivalently defined in terms of the recursive operators as follows:*

$$\begin{aligned} \Delta_{\text{FdB}}^{\text{H}}(x_n) &= x_n + y_n + \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} x_{n_1} \triangleright (y_{n_2} \otimes \cdots \otimes y_{n_{\ell+1}}), \\ \delta_r(x_n) &= (x_n - y_n) + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} L_\ell(x_{n_1} - y_{n_1}, y_{n_2}, \dots, y_{n_\ell}) \triangleright y_{n_{\ell+1}} \\ &= (x_n - y_n) + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (x_{n_1} - y_{n_1}) \triangleright R_\ell(y_{n_2}, \dots, y_{n_{\ell+1}}), \\ \delta_l(x_n) &= (y_n - x_n) - \sum_{\mathbf{n} \in \mathcal{C}_n^2} x_{n_1} \triangleright (y_{n_2} - x_{n_2}) \\ &\quad + \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} x_{n_1} \triangleright R_\ell^{\mathbf{e}}(x_{n_2}^{(e_1)}, \dots, x_{n_\ell}^{(e_{\ell-1})}, y_{n_{\ell+1}} - x_{n_{\ell+1}}) \end{aligned}$$

*where the set of sequences  $\mathcal{E}_{\ell-1}$  is given in Def. 5.15 and the conventions on the sign  $(-1)^{\mathbf{e}}$  and the value of  $x_n^{(e_i)}$  is given in Def. 5.5.*

**Proof.** It follows from the definition of  $\triangleright$  given in Def. 5.6, the expression of  $R_\ell$  given in Cor. 5.14, and that of  $R_\ell^{\mathbf{e}}$  given in Lemma 5.16. The equivalence of the presentations of the right codivision in terms of  $R_\ell$  and  $L_\ell$  is proved in Cor. 5.20 in next section.  $\square$

## 5.4 Functoriality of the diffeomorphisms loop

To prove the main theorem of this section we need some preliminary recurrence relations for the recursive operators, and consequently for the Lagrange coefficients.

**Corollary 5.18** For any  $\ell \geq 1$  and any sequence  $(n_1, \dots, n_{\ell+1})$  of positive integers, the coefficients  $d_{\ell+1}(a_1, \dots, a_\ell)$  satisfy the following recursive equation:

$$d_{\ell+1}(n_1, \dots, n_\ell) = \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_j^\ell} \binom{n_1+1}{j} d_{p_1}(n_2, \dots, n_{p_1+1}) d_{p_2}(n_{p_1+2}, \dots, n_{p_1+p_2+1}) \cdots \\ \cdots d_{p_j}(n_{p_1+\dots+p_{j-1}+2}, \dots, n_{\ell+1}).$$

**Proof.** Applying  $a_1 \triangleright R_\ell(a_2, a_3, \dots, a_{\ell+1})$  to the recursive equation (5.3) immediately gives the result.  $\square$

**Lemma 5.19** For any  $\ell \geq 0$  and any  $a_1, \dots, a_{\ell+1} \in A$ , the following recursive equation holds:

$$a_1 \triangleright R_\ell(a_2, \dots, a_{\ell+1}) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} (a_1 \triangleright R_i(a_2, \dots, a_{i+1})) \triangleright (a_{i+2} \otimes \cdots \otimes a_{\ell+1}).$$

Modulo the factor  $a_1 a_2 \cdots a_{\ell+1}$ , this means that

$$d_{\ell+1}(n_1, \dots, n_{\ell+1}) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \binom{n_1 + \cdots + n_{i+1} + 1}{\ell - i} d_{i+1}(n_1, \dots, n_{i+1}).$$

**Proof.** The two assertions are equivalent, and the second one appears as recursion for the coefficients in the non-commutative Lagrange inversion formula. It is essentially based on the Chu-Vandermonde identity and can be proved<sup>5</sup> using the hypergeometric function  ${}_2F_1$  or using some trick as in [7], Lemma 2.15.  $\square$

**Corollary 5.20** For any  $\ell \geq 0$  and any  $a_1, \dots, a_{\ell+1} \in A$  we have

$$a_1 \triangleright R_\ell(a_2, \dots, a_{\ell+1}) = L_\ell(a_1, \dots, a_\ell) \triangleright a_{\ell+1}$$

**Proof.** By induction on  $\ell$ . For  $\ell = 0, 1$  the identity is easily verified, because

$$a_1 \triangleright 1 = a_1 = 1 \triangleright a_1 \\ a_1 \triangleright R_1(a_2) = a_1 \triangleright a_2 = L_1(a_1) \triangleright a_2.$$

Now suppose that for  $i = 1, \dots, \ell - 1$  we have  $a_1 \triangleright R_i(a_2, \dots, a_{i+1}) = L_i(a_1, \dots, a_i) \triangleright a_{i+1}$ . Then by Lemma 5.19 and Def. 5.8 we have

$$a_1 \triangleright R_\ell(a_2, \dots, a_{\ell+1}) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} (a_1 \triangleright R_i(a_2, \dots, a_{i+1})) \triangleright (a_{i+2} \otimes \cdots \otimes a_{\ell+1}) \\ = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} (L_i(a_1, \dots, a_i) \triangleright a_{i+1}) \triangleright (a_{i+2} \otimes \cdots \otimes a_{\ell+1}) \\ = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \left( (L_i(a_1, \dots, a_i) \triangleright a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_{\ell+1} \right) \triangleright a_{\ell+1} \\ = L_\ell(a_1, \dots, a_\ell) \triangleright a_{\ell+1}.$$

$\square$

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<sup>5</sup>We warmly thank Jiang Zeng for pointing out this method to us.

**Remark 5.21** In the case  $\mathbf{e} \neq (1, 1, \dots, 1)$ , whether there exists an operator  $L_\ell^{\mathbf{e}} : A^{\otimes \ell} \longrightarrow T(A)$  such that

$$a_1 \triangleright R_\ell^{\mathbf{e}}(a_2, \dots, a_{\ell+1}) = L_\ell^{\mathbf{e}}(a_1, \dots, a_\ell) \triangleright a_{\ell+1}$$

is an open question.

**Lemma 5.22** For any  $\ell \geq 1$  and any  $a_1, \dots, a_{\ell+1} \in A$ , the following recursive equation holds:

$$a_1 \triangleright R_\ell(a_2, \dots, a_{\ell+1}) = \sum_{i=1}^{\ell} (-1)^{i-1} (a_1 \triangleright (a_2 \otimes \cdots \otimes a_{i+1})) \triangleright R_{\ell-i}(a_{i+2} \otimes \cdots \otimes a_{\ell+1}).$$

Modulo the factor  $a_1 a_2 \cdots a_{\ell+1}$ , and if we call  $n_i = |a_i|$  for  $i = 1, \dots, \ell + 1$ , this means that

$$d_{\ell+1}(n_1, \dots, n_{\ell+1}) = \sum_{i=1}^{\ell} (-1)^{i-1} \binom{n_1+1}{i} d_{\ell+1-i}(n_1+\cdots+n_{i+1}, n_{i+2}, \dots, n_{\ell+1}). \quad (5.5)$$

**Proof.** The two assertions are equivalent. Let us prove the second one by induction on  $\ell$ . Let us call  $\tilde{d}_{\ell+1}(n_1, \dots, n_{\ell+1})$  the right-hand side of equation (5.5).

For  $\ell = 1$ , since  $\mathcal{M}_1 = \{(1)\}$  by definition we have

$$d_2(n_1, n_2) = \binom{n_1+1}{1} \binom{n_2+1}{0} = \binom{n_1+1}{1},$$

and the sum in  $\tilde{d}_2(n_1, n_2)$  has only one term for  $i = 1$ , which gives

$$\begin{aligned} \tilde{d}_2(n_1, n_2) &= \binom{n_1+1}{1} d_1(n_1+n_2) \\ &= \binom{n_1+1}{1} \binom{n_1+n_2+1}{0} = \binom{n_1+1}{1}, \end{aligned}$$

therefore  $d_2(n_1, n_2) = \tilde{d}_2(n_1, n_2)$ .

Now suppose that eq. (5.5) holds for any  $1 \leq k \leq \ell - 1$ , that is, we have

$$d_{k+1}(n_1, \dots, n_{k+1}) = \sum_{i=1}^k (-1)^{i-1} \binom{n_1+1}{i} d_{k+1-i}(n_1+\cdots+n_{i+1}, n_{i+2}, \dots, n_{k+1}),$$

and prove it for  $\ell$ . For this, we write  $d_{\ell+1}(n_1, \dots, n_{\ell+1})$  using the recursion given in Lemma 5.19 as a sum over  $0 \leq k \leq \ell - 1$ , and separate the term  $k = 0$  to which we can not apply the inductive hypothesis. Then we expand the factor  $d_{k+1}(n_1, \dots, n_{k+1})$  using the inductive hypothesis and exchange the sums over  $k$  and  $i$ . We finally obtain

$$\begin{aligned} d_{\ell+1}(n_1, \dots, n_{\ell+1}) &= \sum_{k=1}^{\ell-1} (-1)^{\ell-1-k} \binom{n_1+\cdots+n_{k+1}+1}{\ell-k} d_{k+1}(n_1, \dots, n_{k+1}) + (-1)^{\ell-1} \binom{n_1+1}{\ell} \\ &= \sum_{i=1}^{\ell-1} (-1)^{i-1} \binom{n_1+1}{i} \sum_{k=i}^{\ell-1} (-1)^{\ell-1-k} \binom{n_1+\cdots+n_{k+1}+1}{\ell-k} d_{k+1-i}(n_1+\cdots+n_{i+1}, n_{i+2}, \dots, n_{k+1}) \\ &\quad + (-1)^{\ell-1} \binom{n_1+1}{\ell}. \end{aligned}$$

Then,  $d_{\ell+1}(n_1, \dots, n_{\ell+1})$  is equal to

$$\begin{aligned} \tilde{d}_{\ell+1}(n_1, \dots, n_{\ell+1}) &= \sum_{i=1}^{\ell-1} (-1)^{i-1} \binom{n_1+1}{i} d_{\ell+1-i}(n_1+\dots+n_{i+1}, n_{i+2}, \dots, n_{\ell+1}) \\ &\quad + (-1)^{\ell-1} \binom{n_1+1}{\ell}, \end{aligned}$$

if and only if, for any  $1 \leq i \leq \ell - 1$ , we have

$$\begin{aligned} \sum_{k=i}^{\ell-1} (-1)^{\ell-1-k} \binom{n_1+\dots+n_{k+1}+1}{\ell-k} d_{k+1-i}(n_1+\dots+n_{i+1}, n_{i+2}, \dots, n_{k+1}) \\ = d_{\ell+1-i}(n_1+\dots+n_{i+1}, n_{i+2}, \dots, n_{\ell+1}). \end{aligned}$$

This identity is easily verified by setting  $j = k - i$ ,  $p_1 = n_1 + \dots + n_{i+1}$  and  $p_j = n_{i+j}$  for  $2 \leq j \leq \ell - 1 - i$ , since it gives

$$\sum_{j=0}^{\ell-i-1} (-1)^{\ell-i-1-j} \binom{p_1+\dots+p_{j+1}+1}{\ell-i-j} d_{j+1}(p_1, p_2, \dots, p_{j+1}) = d_{\ell-i+1}(p_1, p_2, \dots, p_{\ell-i+1})$$

which holds again by Lemma 5.19.  $\square$

**Corollary 5.23** *For any  $\ell \geq 1$  and any  $a_1, \dots, a_\ell \in A$ , the following recursive equation holds:*

$$L_\ell(a_1, \dots, a_\ell) = \sum_{i=1}^{\ell-1} (-1)^{i-1} L_{\ell-i}(a_1 \triangleright (a_2 \otimes \dots \otimes a_{i+1}), a_{i+2}, \dots, a_\ell) + (-1)^{\ell-1} a_1 \otimes \dots \otimes a_\ell.$$

**Proof.** It suffices to write

$$\begin{aligned} a_1 \triangleright R_\ell(a_2, \dots, a_{\ell+1}) &= \sum_{i=1}^{\ell-1} (-1)^{i-1} (a_1 \triangleright (a_2 \otimes \dots \otimes a_{i+1})) \triangleright R_{\ell-i}(a_{i+2} \otimes \dots \otimes a_{\ell+1}) \\ &\quad + (-1)^\ell a_1 \triangleright (a_2 \otimes \dots \otimes a_{\ell+1}) \end{aligned}$$

after Lemma 5.22, and to apply the equality  $b_1 \triangleright R_j(b_2, \dots, b_{j+1}) = L_j(b_1, \dots, b_j) \triangleright b_{j+1}$  everywhere.  $\square$

**Lemma 5.24** *For any  $\ell \geq 3$ , any  $\mathbf{e} \in \mathcal{E}_{\ell-1}$  and any  $a_1, \dots, a_\ell \in A$ , we have*

$$\begin{aligned} R_\ell^{\mathbf{e}}(a_1, \dots, a_\ell) &= \delta_{e_1,1} \left( a_1 \triangleright R_{\ell-1}^{(e_2, \dots, e_{\ell-1})}(a_2, \dots, a_\ell) \right) \\ &\quad + \sum_{i=1}^{\ell-1} R_i^{(e_1, \dots, e_{i-1})}(a_1, \dots, a_i) \otimes \delta_{e_{i+1},1} \left( a_{i+1} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})}(a_{i+2}, \dots, a_\ell) \right). \end{aligned}$$

**Proof.** The term  $j = 1$  in the defining recursion (5.4) gives

$$\delta_{e_1,1} \left( a_1 \triangleright R_{\ell-1}^{(e_2, \dots, e_{\ell-1})}(a_2, \dots, a_\ell) \right),$$

se it remains to prove that

$$\begin{aligned}
& \sum_{j=2}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_{\ell}^j} \delta_{e_1,1} \left( a_1 \triangleright R_{p_1-1}^{(e_2, \dots, e_{p_1-1})} (a_2, \dots, a_{p_1}) \right) \\
& \quad \otimes \delta_{e_{p_1+1},1} \left( a_{p_1+1} \triangleright R_{p_2-1}^{(e_{p_1+2}, \dots, e_{p_1+p_2-1})} (a_{p_1+2}, \dots, a_{p_1+p_2}) \right) \otimes \cdots \\
& \quad \cdots \otimes \delta_{e_{P_{j-1}+1},1} \left( a_{P_{j-1}+1} \triangleright R_{p_j-1}^{(e_{P_{j-1}+2}, \dots, e_{P_j-1})} (a_{P_{j-1}+2}, \dots, a_{P_j}) \right) = \\
& \quad = \sum_{i=1}^{\ell-1} R_i^{(e_1, \dots, e_{i-1})} (a_1, \dots, a_i) \otimes \delta_{e_{i+1},1} \left( a_{i+1} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})} (a_{i+2}, \dots, a_{\ell}) \right).
\end{aligned} \tag{5.6}$$

Let us prove this identity by induction. For  $\ell = 3$  and  $\ell = 4$ , it is easy to verify on the above examples that

$$\begin{aligned}
R_3^e(a, b, c) &= \delta_{e_1,1} a \triangleright R_2^{(e_2)}(b, c) + R_1(a) \otimes \delta_{e_2,1} (b \triangleright R_1(c)) + R_2^{(e_1)}(a, b) \otimes c, \\
R_4^e(a, b, c, d) &= \delta_{e_1,1} a \triangleright R_3^{(e_2, e_3)}(b, c, d) + a \otimes \delta_{e_2,1} (b \triangleright R_2^{(e_3)}(c, d)) \\
& \quad + R_2^{(e_1)}(a, b) \otimes \delta_{e_3,1} (c \triangleright R_1(d)) + R_3^{(e_1, e_2)}(a, b, c) \otimes d.
\end{aligned}$$

Now suppose it holds up to order  $\ell - 1$ , and let us prove at order  $\ell$ .

Consider the left-hand side of eq. (5.6). Since  $j \geq 2$ , we can write

$$\mathcal{C}_{\ell}^j = \bigcup_{i=j-1}^{\ell-1} \mathcal{C}_i^{j-1} \times \mathcal{C}_{\ell-i}^1$$

and decompose  $\mathbf{p} \in \mathcal{C}_{\ell}^j$  into  $(p_1, \dots, p_{j-1}) \in \mathcal{C}_i^{j-1}$  and  $(p_j) \in \mathcal{C}_{\ell-i}^1$  for any value  $i = j-1, \dots, \ell-1$ . We then have  $p_1 + \dots + p_{j-1} = i$  and  $p_j = \ell - (p_1 + \dots + p_{j-1}) = \ell - i$ . Therefore the left-hand side can be written as

$$\begin{aligned}
& \sum_{j=2}^{\ell} \sum_{i=j-1}^{\ell-1} \sum_{(p_1, \dots, p_{j-1}) \in \mathcal{C}_i^{j-1}} \delta_{e_1,1} \left( a_1 \triangleright R_{p_1-1}^{(e_2, \dots, e_{p_1-1})} (a_2, \dots, a_{p_1}) \right) \otimes \cdots \\
& \quad \cdots \otimes \delta_{e_{P_{j-2}+1},1} \left( a_{P_{j-2}+1} \triangleright R_{p_{j-1}-1}^{(e_{P_{j-2}+2}, \dots, e_{P_{j-1}-1})} (a_{P_{j-2}+2}, \dots, a_i) \right) \otimes \\
& \quad \otimes \delta_{e_{i+1},1} \left( a_{i+1} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})} (a_{i+2}, \dots, a_{\ell}) \right) = \\
& = \sum_{i=1}^{\ell-1} \left( \sum_{k=1}^i \sum_{(p_1, \dots, p_k) \in \mathcal{C}_i^k} \delta_{e_1,1} \left( a_1 \triangleright R_{p_1-1}^{(e_2, \dots, e_{p_1-1})} (a_2, \dots, a_{p_1}) \right) \otimes \cdots \right. \\
& \quad \left. \cdots \otimes \delta_{e_{P_{j-2}+1},1} \left( a_{P_{j-2}+1} \triangleright R_{p_k-1}^{(e_{P_{j-2}+2}, \dots, e_{P_{k-1}})} (a_{P_{j-2}+2}, \dots, a_i) \right) \right) \otimes \\
& \quad \otimes \delta_{e_{i+1},1} \left( a_{i+1} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})} (a_{i+2}, \dots, a_{\ell}) \right).
\end{aligned}$$

Applying the inductive hypothesis to the sum over  $k$  leads to the result.  $\square$

**Theorem 5.25** *The associative algebra  $H_{\text{FD B}}^{\text{H}}$  is indeed a coloop bialgebra and represents the loop of formal diffeomorphisms  $\text{Diff}$  as a functor  $\text{Diff} : \mathbf{As} \rightarrow \mathbf{Loop}$ .*

*As a consequence, given an associative algebra  $A$ , a series  $a = \sum_{n \geq 0} a_n \lambda^{n+1} \in \text{Diff}(A)$  can be seen as an algebra homomorphism  $a : H_{\text{FD B}}^{\text{H}} \rightarrow A$  defined on the generators of  $H_{\text{FD B}}^{\text{H}}$  by*

$a(x_n) = a_n$ , and the right and left division  $a/b$  and  $a \setminus b$  are given at any order  $n$  by the following closed formulas:

$$\begin{aligned} (a/b)_n &= \mu_A (a \amalg b) \delta_r(x_n) \\ &= a_n - b_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) (a_{n_1} - b_{n_1}) b_{n_2} \cdots b_{n_{\ell+1}}, \\ (a \setminus b)_n &= \mu_A (a \amalg b) \delta_l(x_n) \\ &= b_n - a_n + \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) a_{n_1} c_{n_2}^{(e_1)} \cdots c_{n_\ell}^{(e_{\ell-1})} (b_{n_{\ell+1}} - a_{n_{\ell+1}}), \end{aligned}$$

where  $c_{n_i}^{(e_i)} = a_{n_i}$  if  $e_i = 1$  and  $c_{n_i}^{(e_i)} = b_{n_i}$  if  $e_i = 2$ .

**Proof.** The free associative algebra  $H_{\text{FdB}}^{\text{H}}$  clearly represents the sets  $\text{Diff}(A)$  over associative algebras  $A$ , and the comultiplication  $\Delta_{\text{FdB}}^{\text{H}}$  is just the Faà di Bruno comultiplication  $\Delta_{\text{FdB}}$  seen with values in  $H_{\text{FdB}} \amalg H_{\text{FdB}}$  instead of  $H_{\text{FdB}} \otimes H_{\text{FdB}}$ , therefore it clearly represents the loop law given in Definition 5.1. Thus, the theorem is proved if we show that  $H_{\text{FdB}}^{\text{H}}$  is indeed a coloop bialgebra.

The comultiplication  $\Delta_{\text{FdB}}^{\text{H}}$  satisfies the compatibility relation with the standard counit, because  $\Delta_{\text{FdB}}$  does, and coassociativity is not required. So it remains to check that the codivisions  $\delta_r$  and  $\delta_l$  given in Def. 5.1 satisfy the identities (2.5) and (2.6). Since these maps are algebra morphisms, it suffices to verify these identities on the generators  $x_n$ , for any  $n \geq 1$ .

i) Let us start with the right codivision and show that it satisfies the first identity (2.5), namely

$$(\text{id} \amalg \mu) (\delta_r \amalg \text{id}) \Delta_{\text{FdB}}^{\text{H}}(x_n) = x_n,$$

which explicitly gives the recurrence

$$\delta_r(x_n) = u_n - \sum_{m=1}^{n-1} \sum_{\ell=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\ell} \delta_r(x_m) \triangleright (y_{k_1} \otimes \cdots \otimes y_{k_\ell}). \quad (5.7)$$

Expanding  $\delta_r(x_n)$  in terms of the left recursive operators, this equation becomes

$$\begin{aligned} u_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} L_\ell(u_{n_1}, y_{n_2}, \dots, y_{n_\ell}) \triangleright y_{n_{\ell+1}} \\ = u_n - \sum_{m=1}^{n-1} \sum_{j=1}^{n-m} \sum_{\mathbf{q} \in \mathcal{C}_{n-m}^j} \sum_{i=0}^{m-1} (-1)^i \sum_{\mathbf{p} \in \mathcal{C}_m^{i+1}} \left( L_i(u_{p_1}, y_{p_2}, \dots, y_{p_i}) \triangleright y_{p_{i+1}} \right) \triangleright (y_{q_1} \otimes \cdots \otimes y_{q_j}) \\ = u_n + \sum_{\ell=i+j=1}^{n-1} \sum_{i=0}^{\ell-1} (-1)^{i+1} \sum_{m=i+1}^{n-1} \sum_{\mathbf{p} \in \mathcal{C}_m^{i+1}} \sum_{\mathbf{q} \in \mathcal{C}_{n-m}^{\ell-i}} \left( L_i(u_{p_1}, y_{p_2}, \dots, y_{p_i}) \triangleright y_{p_{i+1}} \right) \triangleright (y_{q_1} \otimes \cdots \otimes y_{q_{\ell-i}}). \end{aligned}$$

Now, since

$$\bigcup_{m=i+1}^{n-1} \mathcal{C}_m^{i+1} \times \mathcal{C}_{n-m}^{\ell-i} \cong \mathcal{C}_n^{\ell+1},$$

let us call  $\mathbf{n} = (\mathbf{p}, \mathbf{q})$ , that is,

$$(n_1, n_2, \dots, n_{\ell+1}) = (p_1, \dots, p_{i+1}, q_1, \dots, q_j).$$

Then, the recursion (5.7) is equivalent, for any  $n \geq 1$ , any  $1 \leq \ell \leq n - 1$  and any  $\mathbf{n} \in \mathcal{C}_n^{\ell+1}$ , to the equation

$$\begin{aligned} L_\ell(u_{n_1}, y_{n_2}, \dots, y_{n_\ell}) \triangleright y_{n_{\ell+1}} &= \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} (L_i(u_{n_1}, y_{n_2}, \dots, y_{n_i}) \triangleright y_{n_{i+1}}) \triangleright (y_{n_{i+2}} \otimes \dots \otimes y_{n_{\ell+1}}) \\ &= \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i} \left( (L_i(u_{n_1}, y_{n_2}, \dots, y_{n_i}) \triangleright y_{n_{i+1}}) \otimes y_{n_{i+2}} \otimes \dots \otimes y_{n_\ell} \right) \triangleright y_{n_{\ell+1}}, \end{aligned}$$

which holds by definition of  $L_\ell$ .

The second identity (2.5), namely

$$(\text{id} \amalg \mu) (\Delta_{\text{FdB}}^{\amalg} \amalg \text{id}) \delta_r(x_n) = x_n,$$

is better developed using the expansion over the right recursive operators, and explicitly gives the recurrence

$$\begin{aligned} \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (-1)^\ell x_{n_1} \triangleright R_\ell(y_{n_2}, \dots, y_{n_{\ell+1}}) & \quad (5.8) \\ &= \sum_{m=1}^{n-1} \sum_{p=1}^{n-m} \sum_{i=1}^p \sum_{\mathbf{p} \in \mathcal{C}_p^i} \sum_{j=0}^{n-m-p} \sum_{\mathbf{q} \in \mathcal{C}_{n-m-p}^j} (-1)^{j+1} (x_m \triangleright (y_{p_1} \otimes \dots \otimes y_{p_i})) \triangleright R_j(y_{q_1}, \dots, y_{q_j}). \end{aligned}$$

Rewriting the sums in terms of  $m = 1, \dots, n-1$ ,  $\ell = i+j = 1, \dots, n-m$ ,  $i = 1, \dots, \ell$  and  $j = \ell - i$ , this gives a sum over  $\mathbf{p} \in \mathcal{C}_p^i$  and  $\mathbf{q} \in \mathcal{C}_{n-m-p}^{\ell-i}$  for  $p = i, \dots, n-m$ . That is, we get a sum over  $\mathbf{k} = (\mathbf{p}, \mathbf{q}) \in \mathcal{C}_{n-m}^\ell$  and consequently a sum over  $\mathbf{n} = (m, \mathbf{k}) \in \mathcal{C}_n^{\ell+1}$ :

$$\begin{aligned} \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (-1)^\ell x_{n_1} \triangleright R_\ell(y_{n_2}, \dots, y_{n_{\ell+1}}) & \\ &= \sum_{m=1}^{n-1} \sum_{\ell=1}^{n-m} \sum_{i=1}^{\ell} \sum_{p=i}^{n-m} \sum_{\mathbf{p} \in \mathcal{C}_p^i} \sum_{\mathbf{q} \in \mathcal{C}_{n-m-p}^{\ell-i}} (-1)^{\ell-i+1} (x_m \triangleright (y_{p_1} \otimes \dots \otimes y_{p_i})) \triangleright R_j(y_{q_1}, \dots, y_{q_{\ell-i}}) \\ &= \sum_{m=1}^{n-1} \sum_{\ell=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\ell} \sum_{i=1}^{\ell} (-1)^{\ell-i+1} (x_m \triangleright (y_{k_1} \otimes \dots \otimes y_{k_i})) \triangleright R_{\ell-i}(y_{k_{i+1}}, \dots, y_{k_\ell}) \\ &= \sum_{\ell=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} (-1)^\ell \sum_{i=1}^{\ell} (-1)^{i-1} (x_{n_1} \triangleright (y_{n_2} \otimes \dots \otimes y_{n_{i+1}})) \triangleright R_{\ell-i}(y_{n_{i+2}}, \dots, y_{n_{\ell+1}}). \end{aligned}$$

Therefore, for any  $n \geq 2$ , any  $\ell = 1, \dots, n-1$  and any sequence  $\mathbf{n} \in \mathcal{C}_n^{\ell+1}$ , eq. (5.8) is equivalent to the recurrence equation

$$x_{n_1} \triangleright R_\ell(y_{n_2}, \dots, y_{n_{\ell+1}}) = \sum_{i=1}^{\ell} (-1)^{i-1} (x_{n_1} \triangleright (y_{n_2} \otimes \dots \otimes y_{n_{i+1}})) \triangleright R_{\ell-i}(y_{n_{i+2}}, \dots, y_{n_{\ell+1}}),$$

which is proved in Lemma 5.22.

ii) Let us show now that the left codivision given in Def. (5.5) satisfies the identities (2.6). The first identity (2.6), namely

$$(\mu \amalg \text{id}) (\text{id} \amalg \delta_l) \Delta_{\text{FdB}}^{\amalg}(x_n) = y_n,$$

explicitely gives the recurrence

$$\delta_l(x_n) = v_n - \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\lambda} x_m \triangleright (\delta_l(y_{k_1}) \otimes \cdots \otimes \delta_l(y_{k_\lambda})), \quad (5.9)$$

where  $y_k$  is just the  $k$ th generator  $x_k$  in the second copy of the free product algebra  $H_{\text{FdB}}^{\text{II}} \amalg H_{\text{FdB}}^{\text{II}}$ , therefore the formula for  $\delta_l(y_k)$  is just the same as for  $\delta_l(x_k)$ .

Let us rewrite the expansion of  $\delta_l(x_n)$  over the labeled right recursive operators as

$$\delta_l(x_n) = v_n + \sum_{m=1}^{n-1} x_m \triangleright \left( -v_{n-m} + \sum_{\ell=2}^{n-m} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_{n-m}^\ell} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} R_\ell^{\mathbf{e}}(x_{n_1}^{(e_1)}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_\ell}) \right),$$

then eq. (5.9) is clearly verified for  $n = 1$ , because  $\delta_l(x_1) = v_1$ , and for any  $n \geq 2$  and any  $m = 1, \dots, n-1$ , it is equivalent to the equation

$$\begin{aligned} \sum_{\lambda=1}^{\mu} \sum_{\mathbf{k} \in \mathcal{C}_\mu^\lambda} \delta_l(x_{k_1}) \otimes \cdots \otimes \delta_l(x_{k_\lambda}) &= \\ &= v_\mu - \sum_{\ell=2}^{\mu} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_\mu^\ell} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} R_\ell^{\mathbf{e}}(x_{n_1}^{(e_1)}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_\ell}), \end{aligned} \quad (5.10)$$

where we set  $\mu = n - m = 1, \dots, n - 1$ .

Let us prove this equation by induction on  $\mu$ . For  $\mu = 1$  we again have  $\delta_l(x_1) = v_1$ . So, suppose that eq. (5.10) holds up to order  $\mu - 1$  and prove it at order  $\mu$ .

On the left-hand side of eq. (5.10), we separate the term  $\lambda = 1$  and observe that, for  $\lambda \geq 2$ , we can decompose  $\mathbf{k} = (k_1, \dots, k_{\lambda-1}, k_\lambda) \in \mathcal{C}_\mu^\lambda$  into  $(\mathbf{q}, \nu) \in \mathcal{C}_{\mu-\nu}^{\lambda-1} \times \mathcal{C}_\nu^1$  with

$$\begin{aligned} q_i &= k_i \quad \text{for } i = 1, \dots, \lambda - 1 \\ \nu &= k_\lambda. \end{aligned}$$

Since

$$\mathcal{C}_\mu^\lambda = \bigcup_{\nu=1}^{\mu-\lambda+1} \mathcal{C}_{\mu-\nu}^{\lambda-1} \times \mathcal{C}_\nu^1,$$

the left-hand side of eq. (5.10) can then be written as

$$\delta_l(x_\mu) + \sum_{\nu=1}^{\mu-1} \left( \sum_{i=1}^{\mu-\nu} \sum_{\mathbf{q} \in \mathcal{C}_{\mu-\nu}^i} \delta_l(x_{q_1}) \otimes \cdots \otimes \delta_l(x_{q_i}) \right) \otimes \delta_l(x_\nu).$$

We can then apply the inductive hypothesis (5.10) on the sum over  $i = 1, \dots, \mu - \nu$ , and develop the single factors  $\delta_l(x_\mu)$  and  $\delta_l(x_\nu)$  in terms of the labeled right recursive operations, thus



obtaining

$$\begin{aligned}
& \sum_{\lambda=1}^{\mu} \sum_{\mathbf{k} \in \mathcal{C}_{\mu}^{\lambda}} \delta_l(x_{k_1}) \otimes \cdots \otimes \delta_l(x_{k_{\lambda}}) = \\
& = v_{\mu} - \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^2} x_{n_1} \triangleright v_{n_2} + \sum_{\lambda=2}^{\mu-1} (-1)^{\lambda} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\lambda+1}} \sum_{\mathbf{e} \in \mathcal{E}_{\lambda-1}} (-1)^{\mathbf{e}} x_{n_1} \triangleright R_{\lambda}^{\mathbf{e}}(x_{n_2}^{(e_1)}, \dots, x_{n_{\lambda}}^{(e_{\lambda-1})}, v_{n_{\lambda+1}}) \\
& + \sum_{\nu=1}^{\mu-1} \left( v_{\mu-\nu} - \sum_{i=2}^{\mu-\nu} (-1)^i \sum_{\mathbf{p} \in \mathcal{C}_{\mu-\nu}^i} \sum_{\mathbf{e}' \in \mathcal{E}_{i-1}} (-1)^{\mathbf{e}'} R_i^{\mathbf{e}'}(x_{p_1}^{(e'_1)}, \dots, x_{p_{i-1}}^{(e'_{i-1})}, v_{p_i}) \right) \otimes \\
& \otimes \left( v_{\nu} - \sum_{\mathbf{q} \in \mathcal{C}_{\nu}^2} x_{q_1} \triangleright v_{q_2} + \sum_{j=2}^{\nu-1} (-1)^j \sum_{\mathbf{q} \in \mathcal{C}_{\nu}^{j+1}} \sum_{\mathbf{e}'' \in \mathcal{E}_{j-1}} (-1)^{\mathbf{e}''} x_{q_1} \triangleright R_j^{\mathbf{e}''}(x_{q_2}^{(e''_1)}, \dots, x_{q_j}^{(e''_{j-1})}, v_{q_{j+1}}) \right).
\end{aligned}$$

Now we develop the tensor product, and expand its left-hand side factors with respect to the variables  $v_r = y_r - x_r$ . Beside  $v_{\mu}$ , which is the first term of eq. (5.10), the other terms are:

$$\begin{aligned}
A & = - \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^2} x_{n_1} \triangleright v_{n_2} - \sum_{\nu=1}^{\mu-1} x_{\mu-\nu} \otimes v_{\nu} + \sum_{\nu=1}^{\mu-1} y_{\mu-\nu} \otimes v_{\nu} \\
& = -(-1)^2 \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^2} \sum_{\mathbf{e}=(e_1) \in \mathcal{E}_1} (-1)^{\mathbf{e}} R_2^{\mathbf{e}}(x_{n_1}^{(e_1)}, v_{n_2}),
\end{aligned}$$

which corresponds to the term  $\ell = 2$  of eq. (5.10); then

$$\begin{aligned}
B & = \sum_{\lambda=2}^{\mu-1} (-1)^{\lambda} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\lambda+1}} \sum_{\mathbf{e} \in \mathcal{E}_{\lambda-1}} (-1)^{\mathbf{e}} x_{n_1} \triangleright R_{\lambda}^{\mathbf{e}}(x_{n_2}^{(e_1)}, \dots, x_{n_{\lambda}}^{(e_{\lambda-1})}, v_{n_{\lambda+1}}) \\
& = - \sum_{\ell=3}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\ell}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} \delta_{e_1,1} \left( x_{n_1}^{(e_1)} \triangleright R_{\ell-1}^{(e_2, \dots, e_{\ell-1})}(x_{n_2}^{(e_2)}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}}) \right).
\end{aligned}$$

Furthermore, setting  $(\mathbf{p}, \nu) = \mathbf{n}$  and  $\lambda = i$ , we have

$$\begin{aligned}
C & = - \sum_{\nu=1}^{\mu-1} \sum_{i=2}^{\mu-\nu} (-1)^i \sum_{\mathbf{p} \in \mathcal{C}_{\mu-\nu}^i} \sum_{\mathbf{e}' \in \mathcal{E}_{i-1}} (-1)^{\mathbf{e}'} R_i^{\mathbf{e}'}(x_{p_1}^{(e'_1)}, \dots, x_{p_{i-1}}^{(e'_{i-1})}, v_{p_i}) \otimes v_{\nu} \\
& = - \sum_{\ell=3}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\ell}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} R_{\ell-1}^{(e_1, \dots, e_{\ell-2})}(x_{n_1}^{(e_1)}, \dots, x_{n_{\ell-2}}^{(e_{\ell-2})}, x_{n_{\ell-1}}^{(e_{\ell-1})}) \otimes v_{\ell}
\end{aligned}$$

because  $-v_{p_i} = x_{p_i}^{(1)} - x_{p_i}^{(2)}$  and  $\mathcal{E}_{\lambda} = \mathcal{E}_{\lambda-1} \times \mathcal{E}_1$ . With similar manipulations, we then have the terms

$$\begin{aligned}
D & = - \sum_{\nu=1}^{\mu-1} \sum_{\mathbf{q} \in \mathcal{C}_{\nu}^2} v_{\mu-\nu} \otimes (x_{q_1} \triangleright v_{q_2}) \\
& = -(-1)^3 \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^3} \sum_{\mathbf{e} \in \mathcal{E}_2} (-1)^{\mathbf{e}} x_{n_1}^{(e_1)} \otimes \delta_{e_2,1} (x_{n_2}^{(e_2)} \triangleright v_{n_3}),
\end{aligned}$$

and

$$\begin{aligned}
E &= \sum_{\nu=1}^{\mu-1} \sum_{i=2}^{\mu-\nu} (-1)^i \sum_{\mathbf{p} \in \mathcal{C}_{\mu-\nu}^i} \sum_{\mathbf{e}' \in \mathcal{E}_{i-1}} (-1)^{\mathbf{e}'} \sum_{\mathbf{q} \in \mathcal{C}_2^{\nu}} R_i^{\mathbf{e}'}(x_{p_1}^{(e'_1)}, \dots, x_{p_{i-1}}^{(e'_{i-1})}, v_{p_i}) \otimes (x_{q_1} \triangleright v_{q_2}) \\
&= - \sum_{\ell=4}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\ell}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} R_{\ell-2}^{(e_1, \dots, e_{\ell-3})}(x_{n_1}^{(e_1)}, \dots, x_{n_{\ell-2}}^{(e_{\ell-2})}) \otimes \delta_{e_{\ell-1}, 1}(x_{n_{\ell-1}}^{(e_{\ell-1})} \triangleright v_{n_{\ell}}),
\end{aligned}$$

then also

$$\begin{aligned}
F &= \sum_{\nu=1}^{\mu-1} \sum_{j=2}^{\nu-1} (-1)^j \sum_{\mathbf{e}'' \in \mathcal{E}_{j-1}} (-1)^{\mathbf{e}''} \sum_{\mathbf{q} \in \mathcal{C}_{\nu}^{j+1}} v_{\mu-\nu} \otimes (x_{q_1} \triangleright R_j^{\mathbf{e}''}(x_{q_2}^{(e''_1)}, \dots, x_{q_j}^{(e''_{j-1})}, v_{q_{j+1}})) \\
&= - \sum_{\ell=4}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\ell}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} x_{n_1}^{(e_1)} \otimes \delta_{e_2, 1}(x_{n_2}^{(e_2)} \triangleright R_{\ell-2}^{(e_3, \dots, e_{\ell-1})}(x_{n_3}^{(e_3)}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}}))
\end{aligned}$$

and finally

$$\begin{aligned}
G &= - \sum_{i=2}^{\mu-\nu} (-1)^i \sum_{\mathbf{p} \in \mathcal{C}_{\mu-\nu}^i} \sum_{\mathbf{e}' \in \mathcal{E}_{i-1}} (-1)^{\mathbf{e}'} \sum_{j=2}^{\nu-1} (-1)^j \sum_{\mathbf{q} \in \mathcal{C}_{\nu}^{j+1}} \sum_{\mathbf{e}'' \in \mathcal{E}_{j-1}} (-1)^{\mathbf{e}''} \\
&\quad R_i^{\mathbf{e}'}(x_{p_1}^{(e'_1)}, \dots, x_{p_{i-1}}^{(e'_{i-1})}, v_{p_i}) \otimes (x_{q_1} \triangleright R_j^{\mathbf{e}''}(x_{q_2}^{(e''_1)}, \dots, x_{q_j}^{(e''_{j-1})}, v_{q_{j+1}})) \\
&= - \sum_{\ell=5}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\ell}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} \sum_{i=2}^{\ell-3} R_i^{(e_1, \dots, e_{i-1})}(x_{n_1}^{(e_1)}, \dots, x_{n_i}^{(e_i)}) \otimes \\
&\quad \otimes \delta_{e_{i+1}, 1}(x_{n_{i+1}}^{(e_{i+1})} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})}(x_{n_{i+2}}^{(e_{i+2})}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}})).
\end{aligned}$$

We now observe that the sum  $C$  extends  $E$  to the value  $\ell = 3$ , and the sum  $C + E$  extends  $G$  to the value  $\ell \geq 3$ ,  $i = \ell - 2$  and  $i = \ell - 1$ . Similarly,  $D$  extends  $F$  to the value  $\ell = 3$ , and the sum  $D + F$  extends  $G$  to the value  $\ell \geq 3$  and  $i = 1$ . Therefore, we have

$$\begin{aligned}
C + E + D + F + G &= - \sum_{\ell=3}^{\mu} (-1)^{\ell} \sum_{\mathbf{n} \in \mathcal{C}_{\mu}^{\ell}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} \sum_{i=1}^{\ell-1} R_i^{(e_1, \dots, e_{i-1})}(x_{n_1}^{(e_1)}, \dots, x_{n_i}^{(e_i)}) \otimes \\
&\quad \otimes \delta_{e_{i+1}, 1}(x_{n_{i+1}}^{(e_{i+1})} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})}(x_{n_{i+2}}^{(e_{i+2})}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}})).
\end{aligned}$$

Finally, eq. (5.10) is then equivalent, for any  $\ell \geq 3$ , any  $\mathbf{n} \in \mathcal{C}_{\nu}^{\ell}$  and any  $\mathbf{e} \in \mathcal{E}_{\ell-1}$ , to the following recursion

$$\begin{aligned}
R_{\ell}^{\mathbf{e}}(x_{n_1}^{(e_1)}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}}) &= \delta_{e_1, 1}(x_{n_1}^{(e_1)} \triangleright R_{\ell-1}^{(e_2, \dots, e_{\ell-1})}(x_{n_2}^{(e_2)}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}})) \\
&\quad + \sum_{i=1}^{\ell-1} R_i^{(e_1, \dots, e_{i-1})}(x_{n_1}^{(e_1)}, \dots, x_{n_i}^{(e_i)}) \otimes \\
&\quad \otimes \delta_{e_{i+1}, 1}(x_{n_{i+1}}^{(e_{i+1})} \triangleright R_{\ell-i-1}^{(e_{i+2}, \dots, e_{\ell-1})}(x_{n_{i+2}}^{(e_{i+2})}, \dots, x_{n_{\ell-1}}^{(e_{\ell-1})}, v_{n_{\ell}})),
\end{aligned}$$

which is proved in Lemma 5.24.

The second identity (2.6), namely

$$(\mu \amalg \text{id})(\text{id} \amalg \Delta_{\text{FdB}}^{\text{H}}) \delta_l(x_n) = y_n,$$

can not be expressed as a recurrence on  $R_\ell^e$ , because these operators do not show up explicitly to which factor of  $H_{\text{FdB}}^{\text{II}} \amalg H_{\text{FdB}}^{\text{II}}$  the variables belong. Then, let us use the recursion (5.9) to describe  $\delta_l$  and prove the second identity by induction on  $n$ .

The identity is verified for  $n = 1$  because we have

$$(\mu \amalg \text{id}) (\text{id} \amalg \Delta_{\text{FdB}}^{\text{II}}) \delta_l(x_1) = x_1 + y_1 - x_1 = y_1.$$

Then, suppose it holds up to the degree  $n - 1$ . Since  $\Delta_{\text{FdB}}^{\text{II}}$  and  $\mu$  are algebra homomorphisms, if we apply the operator  $D = (\mu \amalg \text{id}) (\text{id} \amalg \Delta_{\text{FdB}}^{\text{II}})$  to the expression

$$\delta_l(x_n) = v_n - \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\lambda} x_m \triangleright (\delta_l(y_{k_1}) \otimes \cdots \otimes \delta_l(y_{k_\lambda})),$$

we obtain, for  $D(\delta_l(x_n))$ , the sum of

$$D(v_n) = y_n + \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\lambda} x_m \triangleright (y_{k_1} \otimes \cdots \otimes y_{k_\lambda})$$

and of

$$- \sum_{m=1}^{n-1} \sum_{\lambda=1}^{n-m} \sum_{\mathbf{k} \in \mathcal{C}_{n-m}^\lambda} x_m \triangleright (D(\delta_l(y_{k_1})) \otimes \cdots \otimes D(\delta_l(y_{k_\lambda}))).$$

Therefore the second identity is satisfied if, for any  $m = 1, \dots, n - 1$ , any  $\lambda = 1, \dots, n - m$  and any  $\mathbf{k} \in \mathcal{C}_{n-m}^\lambda$ , we have

$$D(\delta_l(y_{k_1})) \otimes \cdots \otimes D(\delta_l(y_{k_\lambda})) = y_{k_1} \otimes \cdots \otimes y_{k_\lambda},$$

which is true by inductive hypothesis. □

## 5.5 Properties of the diffeomorphisms loop

**Proposition 5.26** *The coloop bialgebra  $H_{\text{FdB}}^{\text{II}}$  has a two-sided antipode  $S$  such that*

$$\delta_r = (\text{id} \amalg S) \Delta_{\text{FdB}}^{\text{II}},$$

while the identity  $\delta_l = (S \amalg \text{id}) \Delta_{\text{FdB}}^{\text{II}}$  does not hold. Moreover, the antipode in the Faà di Bruno coloop bialgebra coincide with that in the non-commutative Faà di Bruno Hopf algebra given in [7], that is,

$$S(x_n) = - \sum_{\ell=0}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) x_{n_1} x_{n_2} \cdots x_{n_{\ell+1}}.$$

**Proof.** i) In a coloop bialgebra, the left and right antipodes are given respectively by

$$S_l = (\text{id} \amalg \varepsilon) \delta_l \quad \text{and} \quad S_r = (\varepsilon \amalg \text{id}) \delta_r,$$

cf. (2.9). Let us show that for  $H_{\text{FdB}}^{\text{II}}$  these two operators coincide, and therefore the two-sided antipode is well defined by  $S := S_l = S_r$ .

Indeed, let us fix  $n \geq 1$ . For the right antipode we have

$$\begin{aligned} S_r(x_n) &= \sum_{\ell=0}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) (\varepsilon(x_{n_1}) - y_{n_1}) y_{n_2} \cdots y_{n_{\ell+1}} \\ &= - \sum_{\ell=0}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) x_{n_1} x_{n_2} \cdots x_{n_{\ell+1}}, \end{aligned}$$

where  $n_1 > 0$  implies  $\varepsilon(x_{n_1}) = 0$ , and where we renamed the variables  $y$  as  $x$  because  $S_r$  takes values in  $H_{\text{FdB}}^{\text{H}}$ . For the left antipode we have

$$\begin{aligned} S_l(x_n) &= \varepsilon(y_n) - x_n - \sum_{\mathbf{n} \in \mathcal{C}_n^2} d_1(n_1) x_{n_1} (\varepsilon(y_{n_2}) - x_{n_2}) \\ &\quad + \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \sum_{\mathbf{e} \in \mathcal{E}_{\ell-1}} (-1)^{\mathbf{e}} d_{\ell+1}^{\mathbf{e}}(n_1, \dots, n_{\ell+1}) (\text{id} \amalg \varepsilon)(x_{n_1} x_{n_2}^{(e_1)} \cdots x_{n_\ell}^{(e_{\ell-1})} v_{n_{\ell+1}}). \end{aligned}$$

Since  $\varepsilon$  kills the terms where some  $y$  appears, in the sum over the sequences  $\mathbf{e} \in \mathcal{E}_{\ell-1}$  there only remains the sequence  $\mathbf{e} = (1, 1, \dots, 1)$ , for which  $d_{\ell+1}^{\mathbf{e}} = d_{\ell+1}$  and  $(-1)^{\mathbf{e}} = +$ , and therefore we have

$$\begin{aligned} S_l(x_n) &= -x_n + \sum_{\mathbf{n} \in \mathcal{C}_n^2} d_1(n_1) x_{n_1} x_{n_2} \\ &\quad - \sum_{\ell=2}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) x_{n_1} x_{n_2} \cdots x_{n_{\ell+1}} \\ &= S_r(x_n). \end{aligned}$$

ii) Let us now prove the identity  $\delta_r = (\text{id} \amalg S) \Delta_{\text{FdB}}^{\text{H}}$ . For any generator  $x_n$  of  $H_{\text{FdB}}^{\text{H}}$ , we have

$$\begin{aligned} (\text{id} \amalg S) \Delta_{\text{FdB}}^{\text{H}}(x_n) &= x_n + S(y_n) + \sum_{j=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{j+1}} \binom{n_1+1}{j} x_{n_1} S(y_{n_2}) \cdots S(y_{n_{j+1}}) \\ &= x_n - y_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) y_{n_1} y_{n_2} \cdots y_{n_{\ell+1}} \\ &\quad + \sum_{j=1}^{n-1} \sum_{\mathbf{n} \in \mathcal{C}_n^{j+1}} \sum_{p_1=1}^{n_2} \cdots \sum_{p_j=1}^{n_{j+1}} (-1)^{p_1+\cdots+p_j} \sum_{\mathbf{q}^1 \in \mathcal{C}_{n_2}^{p_1}} \cdots \sum_{\mathbf{q}^j \in \mathcal{C}_{n_{j+1}}^{p_j}} \binom{n_1+1}{j} d_{p_1}(q_1^1, \dots, q_{p_1}^1) \cdots \\ &\quad \cdots d_{p_j}(q_1^j, \dots, q_{p_j}^j) x_{n_1} y_{q_1^1} \cdots y_{q_{p_1}^1} \cdots y_{q_1^j} \cdots y_{q_{p_j}^j}. \end{aligned}$$

Set  $\ell = p_1 + \cdots + p_j$ , then  $1 \leq j \leq \ell \leq n-1$ . Since

$$\bigcup_{n_1=1}^{n-\ell} \bigcup_{p_1=1}^{n_2} \cdots \bigcup_{p_j=1}^{n_{j+1}} \mathcal{C}_{n_1}^1 \times \mathcal{C}_{n_2}^{p_1} \times \cdots \times \mathcal{C}_{n_{j+1}}^{p_j} = \mathcal{C}_n^{\ell+1},$$

because  $n_1 + n_2 + \cdots + n_{j+1} = n$ , if we rename the sequence  $(n_1, q_1^1, \dots, q_{p_1}^1, \dots, q_1^j, \dots, q_{p_j}^j)$  as  $\mathbf{n} \in \mathcal{C}_n^{\ell+1}$ , the sum over  $j$  becomes

$$\sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} \left( \sum_{j=1}^{\ell} \sum_{\mathbf{p} \in \mathcal{C}_j^j} \binom{n_1+1}{j} d_{p_1}(n_2, \dots, n_{p_1+1}) \cdots d_{p_j}(n_{p_{j-1}+2}, \dots, n_{\ell+1}) \right) x_{n_1} y_{n_2} \cdots y_{n_{\ell+1}},$$

where  $Pi = p_1 + \dots + p_i$  for  $i = 1, \dots, j$ . Using the recurrence proved in Corollary 5.18, we finally obtain

$$\begin{aligned} (\text{id} \amalg S) \Delta_{\text{FdB}}^{\text{H}}(x_n) &= x_n - y_n - \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) y_{n_1} y_{n_2} \cdots y_{n_{\ell+1}} \\ &\quad + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) x_{n_1} y_{n_2} \cdots y_{n_{\ell+1}} \\ &= \delta_r(x_n). \end{aligned}$$

iii) The first counterexample to the analogue identity  $\delta_l = (S \amalg \text{id}) \Delta_{\text{FdB}}^{\text{H}}$  is on the generator  $x_3$ , for which we have

$$(S \amalg \text{id}) \Delta_{\text{FdB}}^{\text{H}}(x_3) = v_3 - (2x_1v_2 + 3x_2v_1) + 5x_1^2v_1 - x_1v_1y_1,$$

where  $v_n = y_n - x_n$ , while

$$\delta_l(x_3) = v_3 - (2x_1v_2 + 3x_2v_1) + 5x_1^2v_1 - x_1y_1v_1.$$

□

This result allows on one side to deduce some properties of the loop of formal diffeomorphisms, and on the other side to compare the Faà di Bruno coloop bialgebra with the non-commutative Faà di Bruno Hopf algebra.

**Corollary 5.27** 1. *The proalgebraic loop Diff is not right alternative, nor power associative.*

2. *Nevertheless, Diff has two-sided inverses and, for a given an associative algebra A and an element  $a \in \text{Diff}(A)$ , the inverse  $a^{-1} = a \setminus e = e / a$  is given by the usual Lagrange coefficients, namely*

$$(a^{-1})_n = - \sum_{\ell=0}^{n-1} (-1)^\ell \sum_{\mathbf{n} \in \mathcal{C}_n^{\ell+1}} d_{\ell+1}(n_1, \dots, n_{\ell+1}) a_{n_1} a_{n_2} \cdots a_{n_{\ell+1}}.$$

3. *The inversion allows to construct the right division, that is,  $a/b = a \circ (e/b)$  for any  $a, b \in \text{Diff}(A)$ , but it does not allow to construct the left division, because  $b \setminus a \neq (b \setminus e) \circ a$  if  $a_n \neq 0$  and  $b_m \neq 0$  for some  $n, m \geq 1$ .*

**Proof.** 1. The loop  $\text{Diff}(A)$  is right alternative if and only if the coloop bialgebra  $H_{\text{FdB}}^{\text{H}}$  is right coalternative, that is  $(\text{id} \amalg \mu) K = 0$ , where  $K = (\Delta_{\text{FdB}}^{\text{H}} \amalg \text{id}) \Delta_{\text{FdB}}^{\text{H}} - (\text{id} \amalg \Delta_{\text{FdB}}^{\text{H}}) \Delta_{\text{FdB}}^{\text{H}}$  is the coassociator.

The first default from right alternativity appears on the generator  $x_5$ . If we temporarily denote by  $x_n = x_n^{(1)}$ ,  $y_n = x_n^{(2)}$  and  $z_n = x_n^{(3)}$  the three copies of the generators in  $H \amalg H \amalg H$ , we get

$$\begin{aligned} K(x_5) &= 6x_3(y_1z_1 - z_1y_1) \\ &\quad + x_2[3(y_2z_1 - z_2y_1) + 3(y_1z_2 - z_1y_2) + (8y_1^2z_1 - 7y_1z_1y_1 - z_1y_1^2) \\ &\quad \quad + (8y_1z_1^2 - 7z_1y_1z_1 - z_1^2y_1)] \\ &\quad + x_1[(y_3z_1 - z_3y_1) + (y_2z_2 - z_2y_2) + (y_1z_3 - z_1y_3) + 3(y_2y_1z_1 - y_2z_1y_1) \\ &\quad \quad + 2(y_1y_2z_1 - y_1z_2y_1) + 2(y_1^2z_2 - y_1z_1y_2) + 2(y_1z_1z_2 - z_1y_1z_2) \\ &\quad \quad + (3y_2z_1^2 - 2z_2y_1z_1) + (2y_1z_2z_1 - 3z_1y_2z_1) \\ &\quad \quad + (5y_1^2z_1^2 + y_1z_1^3 - 4y_1z_1y_1z_1 - y_1z_1^2y_1 - z_1y_1z_1^2)] \end{aligned}$$

and therefore

$$(\text{id} \amalg \mu) K(x_5) = x_1(y_2y_1^2 - y_1y_2y_1) \neq 0.$$

The default from right alternativity on the generator  $x_5$  corresponds to the power  $\lambda^6$  of usual series with substitution law, and can be detected comparing  $(a \circ b) \circ b$  and  $a \circ (b \circ b)$  for the two series

$$a(\lambda) = \lambda + a_1\lambda^2 \quad \text{and} \quad b(\lambda) = \lambda + b_1\lambda^2 + b_2\lambda^3$$

with  $a_1 = 1$  and  $b_2b_1^2 \neq b_1b_2b_1$ . For instance<sup>6</sup>, by taking the  $2 \times 2$  elementary matrices  $b_1 = E_{11}$  and  $b_2 = E_{21}$ , for which  $b_2b_1^2 = b_2$  and  $b_1b_2b_1 = 0$ .

The same computation shows that  $\text{Diff}(A)$  is not power associative, because

$$\mu(\text{id} \amalg \mu) K(x_5) = x_1x_2x_1^2 - x_1^2x_2x_1 \neq 0.$$

For a series  $c(\lambda) = \lambda + c_1\lambda^2 + c_2\lambda^3$ , we then have  $(c \circ c) \circ c \neq c \circ (c \circ c)$  if  $c_1c_2c_1^2 \neq c_1^2c_2c_1$ . For instance, this is verified for the two  $2 \times 2$  matrices

$$c_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad c_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

for which we have

$$c_1c_2c_1^2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad c_1^2c_2c_1 = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}.$$

2. The left and right inverses of  $a \in \text{Diff}(A)$  can be found using respectively the left antipode  $S_l$  and right antipode  $S_r$  of  $H_{\text{FdB}}^{\text{II}}$ , according to the standard rule

$$(e/a)_n = a(S_l(x_n)) \quad \text{and} \quad (a \setminus e)_n = a(S_r(x_n)).$$

By Proposition 5.26 we have  $S_r = S_l$ , therefore  $e/a = a \setminus e$ .

3. The identity  $a/b = a \circ (e/b)$  in the loop  $\text{Diff}(A)$  is equivalent to the identity  $\delta_r = (\text{id} \amalg S_r) \Delta_{\text{FdB}}^{\text{II}}$  in  $H_{\text{FdB}}^{\text{II}}$ , proved in Proposition 5.26. The analogue identity for the left division does not hold.  $\square$

The commutative Faà di Bruno Hopf algebra which represents the classical proalgebraic group  $\text{Diff}$ , mentioned at the beginning of section 5, admits a non-commutative lift [7]

$$H_{\text{FdB}}^{\text{nc}} = \mathbb{F}\langle x_n, n \geq 1 \rangle, \quad (x_0 = 1)$$

$$\Delta_{\text{FdB}}^{\text{nc}}(x_n) = \sum_{m=0}^n x_m \otimes \sum_{(k)} x_{k_0} \cdots x_{k_m},$$

where the sum is over the set of tuples  $(k_0, k_1, k_2, \dots, k_m)$  of non-negative integers such that  $k_0 + k_1 + k_2 + \cdots + k_m = n - m$ . Since  $\text{Diff}$  is not a group over associative algebras, the existence of this Hopf algebra is not *a priori* ensured by the extension of the functor  $\text{Diff}$  from  $\mathbf{Com}_{\mathbb{F}}$  to  $\mathbf{As}_{\mathbb{F}}$ .

**Corollary 5.28** *The image of the coloop bialgebra  $H_{\text{FdB}}^{\text{II}}$  under the canonical projection  $\pi$  given in Def. 2.8 is the non-commutative Faà di Bruno Hopf algebra  $H_{\text{FdB}}^{\text{nc}}$ , that is,*

$$(H_{\text{FdB}}^{\text{II}})^{\otimes} = H_{\text{FdB}}^{\text{nc}}.$$

---

<sup>6</sup>The authors warmly thank J. M. Pérez-Izquierdo for communicating this example.

**Proof.** Indeed, we have  $(H_{\text{FdB}}^{\text{H}})^{\otimes} = H_{\text{FdB}}^{\text{nc}}$  as an algebra, and eventhough  $\Delta_{\text{FdB}}^{\text{H}}$  is not coassociative, the comultiplication  $(\Delta_{\text{FdB}}^{\text{H}})^{\otimes}$  coincides with  $\Delta_{\text{FdB}}^{\text{nc}}$  and therefore it is coassociative with respect to the component-wise multiplication in  $H_{\text{FdB}}^{\text{nc}} \otimes H_{\text{FdB}}^{\text{nc}}$ .


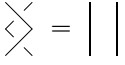

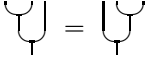
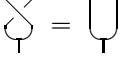
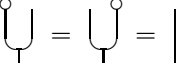


The assertion is then proved because, by Prop. 5.26, the antipode in  $H_{\text{FdB}}^{\text{H}}$  is unique and coincides with that in  $H_{\text{FdB}}^{\text{nc}}$  on generators.  $\square$

## 6 Appendix: Categorical proofs with tangles


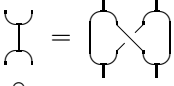
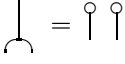
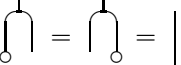

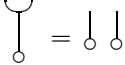

Tangle diagrams are an efficient tool to prove formal (categorical) properties. Tangles are drawings suitable to represent operations and co-operations in a monoidal category, cf. [26] [42], and therefore can be used to encode the structure of coloops in a category  $(\mathbf{C}, \text{H}, I)$ . In the context of non-associative algebras they have been used in [34] to code deformations of the enveloping algebra of a Malcev algebra, seen as the infinitesimal structure of a Moufang loop.

Tangles are drawings to be read from the top to the bottom as concatenation of operations acting on objects related by the monoidal product, and not by a cartesian (or tensor) product. Here is the list of the tangles needed to represent all the operations and the co-operations in coloops, with their defining identities.

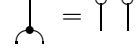
### Categorical maps

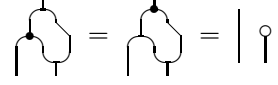
$\tau$	twist		invertible		Sect. 2.2
$\mu$	folding map		associative		Sect. 2.2
			commutative		Sect. 2.2
			unital		Sect. 2.2
$u$	unit		folding morphism		Sect. 2.2

### Coloop structure maps

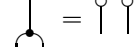
$\Delta$	comultiplication		folding morphism		Sect. 2.2
			unital		Sect. 2.2
			counital		Eq. (2.4)
$\varepsilon$	counit		folding morphism		Sect. 2.2
			unital		Sect. 2.2

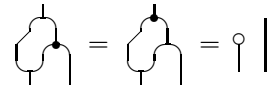
$\delta_r$  right codivision  folding morphism  Sect. 2.2

unital  Sect. 2.2

right cocancellation  Eq. (2.5)

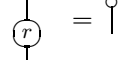
$\delta_l$  left codivision  folding morphism  Sect. 2.2

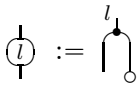
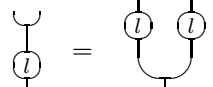
unital  Sect. 2.2

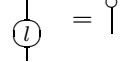
left cocancellation  Eq. (2.6)

### Further coloop maps

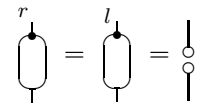
$S_r$  right antipode  folding morphism  Sect. 2.2

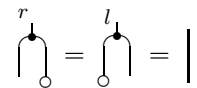
unital  Sect. 2.2

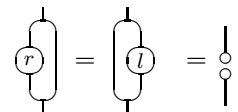
$S_l$  left antipode  folding morphism  Sect. 2.2

unital  Sect. 2.2

### Properties of coloops

$\mu \delta_r = \mu \delta_l = u \varepsilon$   Eq. (2.7)

partial counitality  $(\text{id} \amalg \varepsilon) \delta_r = (\varepsilon \amalg \text{id}) \delta_l = \text{id}$   Eq. (2.8)

left and right five-terms identities  $\mu (S_r \amalg \text{id}) \Delta = \mu (\text{id} \amalg S_l) \Delta = u \varepsilon$   Eq. (2.10)



**Proof of Eq. (2.7):** For  $\delta_r$

$$\begin{array}{c} r \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \downarrow \end{array}$$

and similarly for  $\delta_l$ .

**Proof of Eq. (2.8):** For  $\delta_r$

$$\begin{array}{c} r \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \downarrow \end{array}$$

and similarly for  $\delta_l$ .

**Proof of Eq. (2.10):** For  $S_r$

$$\begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \downarrow \end{array}$$

and similarly for  $S_l$ .

### Properties of cogroups

coassociative	$(\Delta \amalg \text{id}) \Delta = (\text{id} \amalg \Delta) \Delta$	$\begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array}$	Eq. (2.14)
---------------	---	---	------------

unique antipode	$S_r = S_l =: S$	$\begin{array}{c} r \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} l \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} =: \begin{array}{c} \text{---} \\   \\ \text{---} \\   \\ \text{---} \\ \downarrow \end{array}$	Prop. 2.4
-----------------	------------------	--	-----------

five-terms identity	$\mu(S \amalg \text{id}) \Delta = \mu(\text{id} \amalg S) \Delta = u \varepsilon$	$\begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\   \\ \text{---} \\   \\ \text{---} \\ \downarrow \end{array}$	Eq. (2.16)
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right coinverse property	$\delta_r = (\text{id} \amalg S) \Delta$	$\begin{array}{c} r \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array}$	Eq. (2.15)
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left coinverse property	$\delta_l = (S \amalg \text{id}) \Delta$	$\begin{array}{c} l \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array}$	Eq. (2.15)
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**Lemma 6.1 (cf. Prop. 2.4)** *If a coloop  $H$  is coassociative, then it has a two-sided antipode satisfying the five-terms identity (2.16) and the coinverse properties (2.15). Therefore it is a cogroup.*

**Proof.** Assume that  $\Delta$  is coassociative. Then we have  $S_l = S_r$  because

$$\begin{array}{c} l \\ \downarrow \\ \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \downarrow \end{array}$$

and therefore  $S := S_l = S_r$  satisfies the five-terms identity because of (2.10).

For the the coinverse properties, let us show that the operator  $R := (\text{id} \amalg S)\Delta$  satisfies the right cocancellations (2.6), and therefore it coincides with  $\delta_r$ . In fact, we have

and

Similarly, the operator  $(S \amalg \text{id})\Delta$  satisfies the left-cocancellations (2.6), and therefore it coincides with  $\delta_l$ .  $\square$

## References

- [1] A. A. Albert, *Quadratic forms permitting composition*, Ann. of Math. Second Series, **43** (1) (1942), 161–177.
- [2] G. M. Bergman, A.-O. Hausknecht, *Cogroups and co-rings in categories of associative algebras*, Math Surveys and Monographs **45**, AMS Providence, RI, 1996.
- [3] I. Berstein, *On co-groups in the category of associative algebras*, Trans. Amer. Math. Soc. **115** (1965), 257–269.
- [4] N. Bourbaki, *Algebra I: Chapters 13*, Springer-Verlag, 1989 [1970].
- [5] Ch. Brouder, A. Frabetti, *Renormalization of QED with planar binary trees*, Eur. Phys. J. C **19** (2001), 715–741.
- [6] Ch. Brouder, A. Frabetti, *QED Hopf algebras on planar binary trees*, J. Alg. **267** (2003), 298–322.
- [7] Ch. Brouder, A. Frabetti, Ch. Krattenthaler, *Non-commutative Hopf algebra of formal diffeomorphisms*, Adv. Math. **200** (2006), 479–524.
- [8] Ch. Brouder, A. Frabetti, F. Menous, *Combinatorial Hopf algebras from renormalization*, Journal of Algebraic Combinatorics **32** (2010), 557–578.
- [9] R. H. Bruck, *A survey of binary systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1958.
- [10] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*. Comm. Math. Phys. **210** (2000), 249–273.
- [11] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem. II. The  $\beta$ -function, diffeomorphisms and the renormalization group*. Comm. Math. Phys. **216** (2001), 215–241.
- [12] P. Doubilet, *A Hopf algebra arising from the lattice of partitions of a set*, J. Algebra **28** (1986), 127–132.

- [13] F. J. Dyson, *The S matrix in quantum electrodynamics*, Phys. Rev. **76** (1949), 1736–1755.
- [14] B. Eckmann, P. J. Hilton, *Group-like structures in general categories I, II, III*, Math. Ann. **143** (1962), 227–255; **151** (1963), 150–186; **150** (1963), 165–187.
- [15] A. Frabetti, D. Manchon, *Five interpretations of Faà di Brunos formula*, in IRMA Lect. in Math. and Theor. Physics **21**, 91–148, K. Ebrahimi-Fard and F. Fauvet ed., European Math. Soc., 2015.
- [16] B. Fresse, *Cogroups in algebras over an operad are free algebras*, Comment. Math. Helv. **73** (1998), 637–676.
- [17] L. Geissinger, *Hopf algebras of symmetric functions and class functions*, in *Combinatoire et représentations du groupe symétrique*, D. Foata ed., Lect. Notes in Math., **579**, Springer, (1977), 168–181.
- [18] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, J.-Y. Thibon, *Non-commutative symmetric functions*, Adv. in Math. **112** (1995), 218–348.
- [19] C. Itzykson, J.-P. Zuber, *Quantum Field Theory*, McGraw Hill 1980.
- [20] S. Joni, G.-C. Rota, *Coalgebras and bialgebras in combinatorics*, Stud. Appl. Math. **61** (1979), 93–139.
- [21] D. M. Kan, *On monoids and their dual*, Bol. Soc. Mat. Mexicana **3** (1958), 52–61.
- [22] D. M. Kan, *A combinatorial definition of homotopy groups*, Ann. of Math **67** (1958), 282–312.
- [23] J.-L. Lagrange, *Nouvelle méthode pour résoudre les équations littérales par le moyen des séries*, Mémoires de l’Académie Royale des Sciences et Belles-Lettres de Berlin **24** (1770), 251–326.
- [24] J.-L. Loday, M. *Combinatorial Hopf algebras*, Clay Mathematics Proceedings **11** (2010), 347–383.
- [25] J.-L. Loday, B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften, Vol. **346**, Springer-Verlag, 2012.
- [26] V. Lyubashenko, *Tangles and Hopf algebras in braided categories*, Journal of Pure and Applied Algebra **98** (1995), 245–278.
- [27] S. Mac lane, *Categories for the working mathematician*, Graduate Text in Mathematics Vol.5, Springer, 1978 [1971].
- [28] A. I. Mal’cev, (1955), *Analytic loops*, Mat. Sb. N.S. (in Russian) **36** (1955), 569–576.
- [29] P. O. Mikheev, L. V. Sabinin, *Quasigroups and differential geometry*, in *Quasigroups and Loops: Theory and Applications*, Sigma Series in Pure Mathematics **8**, Heldermann, Berlin, 1990, 357–430.
- [30] J. Mostovoy, J. M. Pérez-Izquierdo, *Formal multiplications, bialgebras of distributions and non-associative Lie theory*, Transformation Groups **15** Issue 3 (2010), 625–653.
- [31] J. Mostovoy, J. M. Perez-Izquierdo, I. P. Shestakov, *Hopf algebras in non-associative Lie theory*, Bull. Math. Sci. **4** no. 1 (2014), 129–173.

- [32] R. Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. **110** (1935), 416–430.
- [33] J. M. Pérez-Izquierdo, *Algebras, hyperalgebras, nonassociative bialgebras and loops*, Adv. Math. **208** (2007), 834–876.
- [34] J. M. Pérez-Izquierdo, I. P. Shestakov, *Co-Moufang deformations of the universal enveloping algebra of the algebra of traceless octonions*, Alg. and Rep. Th. **18** issue 5 (2015), 1247-1265.
- [35] H. O. Pflugfelder, *Quasigroups and Loops: Introduction*, Heldermann, Berlin 1990.
- [36] G. Pinter, *The Hopf Algebra Structure of Connes and Kreimer in Epstein-Glaser Renormalization*, Lett. Math. Phys. **54** (2000), 227–233.
- [37] L. V. Sabinin, P. O. Mikheev, *On the infinitesimal theory of local analytic loops*, Sov. Math. Dokl. **36** (1988), 545–548.
- [38] J.-P. Serre, *Lie Algebras and Lie Groups*, Benjamin, 1965.
- [39] I. P. Shestakov, U. U. Umirbaev, *Free Akinis algebras, primitive elements, and hyperalgebras*, J. Algebra **250** (2002), 533–548.
- [40] W. van Suijlekom, *Renormalization of gauge fields using Hopf algebras*. In Quantum Field Theory, Eds. B. Fauser, J. Tolksdorf and E. Zeidler, Birkhauser Verlag, Basel, 2008.
- [41] W. van Suijlekom, *Multiplicative renormalization and Hopf algebras*. In Arithmetic and Geometry Around Quantization, Eds. O. Ceyhan et al., Progress in Mathematics 279, Springer Science + Business Media LLC, 2010.
- [42] D. N. Yetter, *Quantum groups and representations of monoidal categories*. Math. Proc. of the Cambridge Phil. Soc. **108** (1990), 261–290.
- [43] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, A. I. Shirshov, *Rings that are nearly associative*. Academic Press, 1982 [1978].
- [44] J. J. Zhang, *H-algebras*, Adv. Math. **89** (1991), 144–191.