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DOI : https://doi.org/10.1016/j.dsp.2018.07.011

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Abstract

This paper deals with signal processing theory related to finite impulse response (FIR) filters with linear phase. The aim is to show a new and complete proof of the equivalence between linear phase and symmetry or antisymmetry of the real coefficients of the filter. Despite numerous proofs are available in the literature, they are usually incomplete, even though the result is commonly used by the signal processing community. We hereby address a pending issue in digital signal processing: we first prove the uniqueness of the group delay for any decomposition amplitude-phase of the frequency response. Based on this first step, we then derive a complete proof of the equivalence: a FIR filter has (anti)symmetric coefficients *if and only if* the phase is linear. It must be emphasized that this brief paper deals with theoretical aspects of FIR filters.

Keywords- FIR Filter, Group Delay, Linear Phase.

I. INTRODUCTION

Analysis and design of digital finite impulse response (FIR) filters with linear phase are basics in the field of signal processing [1]–[4]. Such filters are widely used in electronic devices for their inherent stability. Furthermore, numerous techniques for FIR filter design and applications have been proposed, from early 70's [2] to the present day [5], [6]. However, unlike papers like [2], [5], [6] which deal with design and applications of FIR filters, the present paper is dedicated to theoretical aspects related to FIR filters.

It is commonly accepted that a FIR filter has a linear phase *if and only if* the taps are either symmetric or antisymmetric. However, to the best of our knowledge, although this result is commonly accepted, the proof leading to this fundamental property is not satisfactorily demonstrated in the literature [1], [7], [8], in which mathematical shortcuts are used. In particular, showing that a (anti)symmetric filter has a linear phase is straightforward (see [7] for instance), whereas the inverse implication requires more difficult developments. Otherwise, the authors of [8] made an attempt to prove the double implication, but they inherently presupposed some features regarding the group delay and the differentiability of the frequency response of FIR filters.

In this paper, we revisit the proof of the equivalence between linear phase and (anti)symmetry of FIR filters. To this end, we first prove the existence of the decomposition module/phase of the frequency response of the filter, where both the module and phase functions are at least of class $\mathcal{C}^1$. Furthermore, we show that the group delay is unique for any decomposition. To the best of our knowledge, both proofs are original contributions of this paper, and they are a mandatory prerequisite for the proof of the equivalence between linear phase and (anti)symmetry, since it justifies the existence of both the phase/module decomposition and the group delay.

Once these properties are justified, we can prove the equivalence between linear phase and (anti)symmetry of FIR filters with proper assumptions. Showing that a FIR filter has a linear phase if it is (anti)symmetric is not tackled in this paper, since it has been undertaken in [1], [7]. However, we focus on the inverse implication, *i.e.* we prove that a FIR filter is (anti)symmetric if the phase is linear. This is another original contribution of this paper. Moreover, we consider a very general approach in the sense that the filter coefficients are considered as elements of a mathematical sequence, and no assumption about FIR filters features is made. From this, the proof is based on the analysis of the linear independence of sin functions. To summarize, this paper closes the theoretical concepts related to linearity of the phase of FIR filters, (anti)symmetry of the taps, and uniqueness of the group delay.

The paper is organized as follows: in Section II, we prove the existence of a decomposition module/phase of the frequency response of any FIR filter, and the uniqueness of the group delay. In Section III, we show the following implication: if a FIR filter has a linear phase, then the coefficients are symmetric or antisymmetric, which completes the existing results concenring the

inverse implication. Finally, Section IV concludes this paper.

II. Background and Prerequisites

A. Standard Mathematical Notations

The following notations are used throughout the paper: the class $\mathcal{C}^k(\mathbb{A})$ of a function f means that the k-th derivative of f exists and is continuous on the set $\mathbb{A}$; the braces $\{.\}$ define discrete sets, and $|f|$ is the modulus of f. The mathematical equivalence between two assertions $(\mathcal{A}) \Leftrightarrow (\mathcal{B})$ means that we have the double implication $(\mathcal{A}) \Rightarrow (\mathcal{B})$ *and* $(\mathcal{A}) \Leftarrow (\mathcal{B})$.

B. Model

We consider a finite impulse response (FIR) filter composed of M real coefficients $\{h_m, 0 \leq m \leq M - 1\}$. The filter frequency response is obtained by means of the discrete-time Fourier transform (DTFT) as follows:

$$H(\nu) = \sum_{m=0}^{M-1} h_m e^{-2j\pi m\nu}, \tag{1}$$

where $\nu \in I = [-\frac{1}{2}, \frac{1}{2}]$ is the normalized frequency. In order to highlight the amplitude response of the FIR (not to be confused with the magnitude $|H|$ [2], [4]), it is more relevant to consider, if it exists, the following decomposition of $H(\nu)$:

$$H(\nu) = H_r(\nu)e^{j\phi(\nu)}, \tag{2}$$

where H_r and ϕ must be (at least) $\mathcal{C}^1(\mathbb{R})$ functions. Note that the differentiability class of H_r is rarely discussed in the literature, even it is usually stated that it is sufficient for H_r to be $\mathcal{C}^0(\mathbb{R})$. Now, it must be emphasized that the existence of the group delay, defined as $\alpha = -\frac{1}{2\pi}\frac{d\phi(\nu)}{d\nu}$, presupposes the existence of a decomposition (2) with ϕ a $\mathcal{C}^1(\mathbb{R})$ function, such that the derivative of ϕ is unique for any possible decomposition. As a consequence, we deduce that H_r is necessarily $\mathcal{C}^1(\mathbb{R})$ since $H_r(\nu) = H(\nu)e^{-j\phi(\nu)}$, where $H(\nu)$ is $\mathcal{C}^\infty$ by construction (1). Such assertions are proved hereafter.

C. Existence of the Decomposition (2)

Theorem 1. $\forall \nu \in I = [-\frac{1}{2}, \frac{1}{2}]$, $\exists H_r$ *and* ϕ *of class* $\mathcal{C}^1(\mathbb{R})$, *such that* $H(\nu)$ *defined in (1) can be decomposed as* $H(\nu) = H_r(\nu)e^{j\phi(\nu)}$.

Proof. Since any polynomial can be decomposed as a product of monomials, one can show that every monomial is $\mathcal{C}^1(\mathbb{R})$ to prove that the polynomial is $\mathcal{C}^1(\mathbb{R})$. For the sake of clarity, we rewrite (1) by using the z transform notation, where $z = e^{2j\pi\nu}$. Thus, if we suppose that $h_{M-1} \neq 0$, (1) is factorized as:

$$H(z) = \sum_{m=0}^{M-1} h_m z^{-m} = h_{M-1} \prod_{m=0}^{M-1} (z^{-1} - z_m^{-1}), \tag{3}$$

where z_m^{-1} is the root of the monomial $(z^{-1} - z_m^{-1})$. Let us denote $z_m = \rho_m e^{j\theta_m}$, where $(\rho_m, \theta_m) \in \mathbb{R}^2$. We aim to show that, for any $m \in \{0, 1, .., M-1\}$, the monomials can be expressed as

$$z^{-1} - z_m^{-1} = H_m(\nu) = H_{m,r}(\nu)e^{j\phi_m(\nu)}, \tag{4}$$

where $H_{m,r}$ and ϕ_m are $\mathcal{C}^1(\mathbb{R})$ functions, which will be proved afterward.

Two cases have to be dealt with, according to $\rho_m = 1$ or $\rho_m \neq 1$:

- if $\rho_m = 1$, some straightforward trigonometric manipulations allow us to rewrite the monomial as follows:

$$z^{-1} - z_m^{-1} = e^{-2j\pi\nu} - e^{-j\theta_m} \tag{5}$$

$$= 2\sin\left(\pi\nu - \frac{\theta_m}{2}\right)e^{-j(\pi\nu + \frac{\theta_m}{2} + \frac{\pi}{2})}, \tag{6}$$

  where we identify $H_{m,r}(\nu) = 2\sin\left(\pi\nu - \frac{\theta_m}{2}\right)$ and $\phi_m(\nu) = -(\pi\nu + \frac{\theta_m}{2} + \frac{\pi}{2})$. Note that both functions are of class $\mathcal{C}^\infty(\mathbb{R})$, hence $\mathcal{C}^1(\mathbb{R})$.

- If $\rho_m \neq 1$, H_m is of class $\mathcal{C}^\infty$ from $\mathbb{R}$ to $\mathbb{C}^*$, as $H_m(\nu) \neq 0$ for any $\nu \in I$. Then, from lifting property of the map $p : \mathbb{R} \mapsto S^1$ given by $p(\nu) = \frac{H_m(\nu)}{|H_m(\nu)|}$, where S^1 is the unit circle (see [9] for more details), we know that the polar form of $H_m(\nu) = H_{m,r}(\nu)e^{j\phi_m(\nu)}$ exists, with $H_{m,r} = |H_m|$ and ϕ_m of the same class as H_m. Hence $H_{m,r}$ and ϕ_m are of class $\mathcal{C}^1(\mathbb{R})$. This concludes the proof.

$\square$

The existence of the decomposition in (2) being proved, we can now prove that the group delay is unique for any decomposition module/phase of H. To this end, we show the uniqueness of the derivative of ϕ.

D. Uniqueness of the Derivative of ϕ

Theorem 2. *The derivative of ϕ is unique for any decomposition of H defined in (2).*

Proof. Let ν_n and ν_{n+1} be two consecutive zeros of H. We consider the interval $J_n =]\nu_n, \nu_{n+1}[$, where $H(\nu) \neq 0$. Furthermore, we denote by $(H_{r,1}, \phi_1)$ and $(H_{r,2}, \phi_2)$ two possible decompositions of H. Thus, for any $\nu \in J_n$, since $|H_{r,1}(\nu)| = |H_{r,2}(\nu)|$, we have $H_{r,1}(\nu) = \kappa(\nu) H_{r,2}(\nu)$, where $\kappa(\nu) = \{\pm 1\}$. Therefore, we deduce that:

$$e^{j(\phi_1(\nu) - \phi_2(\nu))} = \pm 1, \tag{7}$$

and then $\frac{\phi_1(\nu) - \phi_2(\nu)}{\pi} = k_n$, where $k_n \in \mathbb{Z}$. We proved that ϕ_1 and ϕ_2 are continuous on $\mathbb{R}$, hence they are on $\bar{J}_n \ni \nu_{n+1}$, thus $\frac{\phi_1(\nu_{n+1}) - \phi_2(\nu_{n+1})}{\pi} = k_n$, and on $\bar{J}_{n+1} \ni \nu_{n+1}$, thus $\frac{\phi_1(\nu_{n+1}) - \phi_2(\nu_{n+1})}{\pi} = k_{n+1}$. Then, for any n, we deduce that $k_n = k_{n+1} = k$, and $\kappa(\nu) = (-1)^k$. To summarize, the above results show that every possible decomposition of H can be deduced from a unique one (H_r, ϕ) as $((-1)^k H_r, \phi + k\pi)$. Finally, this points out that the derivative of ϕ (which is proportional to the group delay if the phase is linear) is unique and independent of the decomposition, which concludes the proof.

$\square$

Such as aforementioned, it is usual to consider $H_r(\nu) = |H(\nu)|$ in the literature, or otherwise the differentiability class of H_r and ϕ is not discussed [2], [4]. However, it can be noticed that if the $\mathcal{C}^0(\mathbb{R})$ condition is imposed to H_r instead of $\mathcal{C}^1$, as in the case of the modulus on the zeros of H, then ϕ is not $\mathcal{C}^1(\mathbb{R})$ and the group delay is not defined anymore. Furthermore, it must be noticed that all the possible decompositions $((-1)^k H_r, \phi + k\pi)$, $k \in \mathbb{Z}$, are of the same class. Since at least one is of class $\mathcal{C}^\infty$ (see proof of Theorem 1 for instance), then all decompositions are $\mathcal{C}^\infty$.

III. Symmetry of FIR Filter with Linear Phase

The aim of this section is to show that any FIR filter has a linear phase *if and only if* their real coefficients are either symmetric or antisymmetric. Mathematically, this can be expressed as

$$(\mathcal{A}) \begin{cases} H(\nu) = H_r(\nu)e^{j\phi(\nu)} \\ \\ \phi(\nu) = \beta - 2\pi\alpha\nu \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{M-1}{2} \\ \\ \beta \in \{k\pi, \frac{\pi}{2} + k\pi\} \quad (\mathcal{B}), \\ \\ h_m = \varepsilon_\beta h_{M-1-m} \end{cases} \tag{8}$$

$$\varepsilon_\beta = \begin{cases} 1, & \text{if } \beta = \frac{\pi}{2} + k\pi \\ \\ -1, & \text{if } \beta = k\pi \end{cases}, \tag{9}$$

$$k \in \mathbb{Z}, (\alpha, \beta) \in \mathbb{R}^2, \tag{10}$$

The proof of the implication $(\mathcal{A}) \Leftarrow (\mathcal{B})$ is straightforward and frequently proposed in the literature such as [1], [7], therefore it is not reported in this paper. However, we propose an original proof of the implication $(\mathcal{A}) \Rightarrow (\mathcal{B})$. This is undertaken by considering a very general context, where no *a priori* constraint on the values of the filter coefficients h_m is imposed. In particular, it is *mathematically* possible to assume $h_0 = 0$ or $h_{M-1} = 0$, even though, in practice, it is unlikely that the extremal coefficients are zero.

A. Preliminaries

Let J be an interval $J \subset I$ such as $\forall \nu \in J, \sum_{m=0}^{M-1} h_m \cos(2\pi m\nu) \neq 0$. The existence of J is guaranteed since the number of roots of a trigonometric polynomial is finite. If non-zero FIR filters with linear phase exists, for any $\nu \in J$, the decomposition of the equation (1) = (2) into real and imaginary parts yields

$$\begin{cases} H_r(\nu)\cos(\beta - 2\pi\alpha\nu) = \sum_{m=0}^{M-1} h_m \cos(2\pi m\nu) \\ \\ H_r(\nu)\sin(\beta - 2\pi\alpha\nu) = -\sum_{m=0}^{M-1} h_m \sin(2\pi m\nu) \end{cases}. \tag{11}$$

Hence, for any $\nu \in J$, we can write:

$$\tan(\beta - 2\pi\alpha\nu) = \frac{-\sum_{m=0}^{M-1} h_m \sin(2\pi m\nu)}{\sum_{m=0}^{M-1} h_m \cos(2\pi m\nu)}. \tag{12}$$

Then, after some straightforward trigonometric manipulations, we obtain:

$$\sum_{m=0}^{M-1} h_m \sin(2\pi \underbrace{(\alpha - m)}_{\alpha_m} \nu + \beta) = 0, \tag{13}$$

where α_m is defined for the sake of readability. In the following, we analyze the linear indepen-dence of the elements of set $\{\sin(2\pi\alpha_m\nu+\beta), 0 \le m \le M-1, \nu \in J\}$ in order to deduce the symmetry/antisymmetry properties of the coefficients h_m.

It is usually claimed in the literature that no additional result can be deduced from (13), except that "obvious" non-null solutions are obtained when $\beta \in \{k\pi, \frac{\pi}{2}+k\pi\}$, $k \in \mathbb{Z}$, and $\alpha = \frac{M-1}{2}$, and in that case the filter is either symmetric or antisymmetric. However, this is a shortcut that should be mathematically proved. We hereafter provide an original and complete proof of $(\mathcal{A}) \Rightarrow (\mathcal{B})$ from (13), from which we deduce the α value based on mathematical considerations.

B. Linear Independence and Notations

It must be emphasized that if the $|\alpha_m|$ elements in (13) are all distinct, then the functions $\nu \in J \mapsto \sin(2\pi\alpha_m\nu+\beta)$, $0 \le m \le M-1$, are linearly independent, except if one of the sin function is null. In any case, the only solution is either the null filter, or a trivial single-coefficient filter. As a consequence, we deduce that a non-trivial solution exists only if there are at least two different indexes m and m' such that $|\alpha_m| = |\alpha_{m'}|$, or equivalently $\alpha_m = -\alpha_{m'}$, and hence:

$$\frac{m+m'}{2} = \alpha. \tag{14}$$

This remark suggests splitting the index of the sum in (13) into two subsets: one gathering of indexes m for which a value $|\alpha_m|$ appears only once, denoted by E_1, and one for which $|\alpha_m|$ appears twice, denoted by E_2. By construction, $E_1 \cup E_2 = \{0, 1, .., M-1\}$ and $E_1 \cap E_2 = \{\emptyset\}$: it is a partition of $\{0, 1, .., M-1\}$.

Accordingly, the development of the sum in (13) leads to the following expression:

$$\sum_{m \in E_1} h_m \sin(2\pi\alpha_m\nu+\beta) \tag{15}$$

$$+ \sum_{m,m' \in E_2} \Big[(h_m+h_{m'})\cos(2\pi\alpha_m\nu)\sin(\beta) \tag{16}$$

$$+ (h_m-h_{m'})\sin(2\pi\alpha_m\nu)\cos(\beta) \Big] = 0. \tag{17}$$

C. h Symmetry and β Possible Values

E_1 *Examination:* By linear independence of the functions $\sin(2\pi\alpha_m\nu+\beta)$, we deduce that if $m \in E_1$, $h_m = 0$, except in the case of a null sin function, *i.e.* $\alpha_m = 0$ and $\beta = k\pi$, $k \in \mathbb{Z}$.
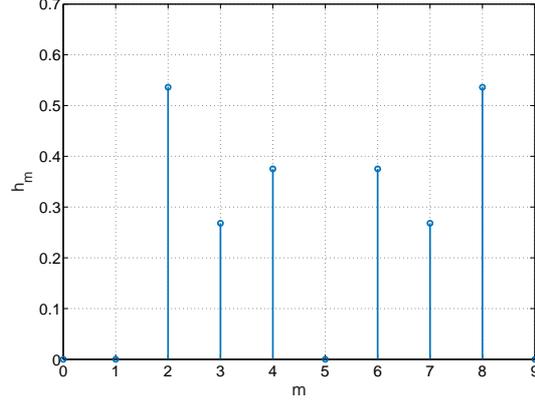
Fig. 1. Filter coefficients, $Min = 2$ and $Max = 8$.

In the latter case, $m = \alpha$ and $h_\alpha \in \mathbb{R}$. Moreover, note that, if α is an integer, from (14) $\alpha \in \{0, 1, .., M-1\}$, and $\alpha \in E_1$ because $|\alpha - m| = 0$ can appear only once.

E_2 Examination: The linear independence of the families $\sin(2\pi\alpha_m\nu)$ and $\cos(2\pi\alpha_m\nu)$, in the second sum in (17), yields: if $\cos(\beta)\sin(\beta) = \frac{1}{2}\sin(2\beta) \neq 0$, *i.e.* $\beta \neq k\frac{\pi}{2}$, $k \in \mathbb{Z}$, the expression (17) holds only if $h_m + h_{m'} = h_m - h_{m'} = 0$, *i.e.*:

$$h_m = h_{m'} = 0. \tag{18}$$

This result imposes specific values to β and coefficient pairing in order to obtain non-zero solutions, namely:

- $\beta = 0 \, [\pi]$, and in this case (17) holds if $h_m = h_{2\alpha-m}$, $m \in E_2$, which corresponds to the symmetric case;

- $\beta = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, and in this case (17) holds if $h_m = -h_{2\alpha-m}$, $m \in E_2$, which corresponds to the antisymmetric case.

D. α Value

Let Min and Max be the indexes defined such as follows:

$$Min = \arg\min_m h_m \neq 0, \tag{19}$$

$$Max = \arg\max_m h_m \neq 0. \tag{20}$$

The above definition of Min and Max is illustrated in Fig. 1, where $M = 10$. In this example, it can be observed that $Min = 2$ and $Max = 8$. We now investigate the belonging of Min and Max to E_1 and E_2 in order to derive the α value:

- *Case 1: Min and Max belong to different subsets.* For instance, suppose that $Min \in E_1$ and $Max \in E_2$. By examining E_1 in Section III-C, we know that $h_{Min} \neq 0$, then $\alpha_{Min} = 0$, which yields $\alpha = Min$ and $\beta = k\pi$, $k \in \mathbb{Z}$. Since $Max \in E_2$, $E_2 \neq \{\emptyset\}$, which contains at least two indexes m, m' necessarily larger than Min. Hence $\alpha = \frac{m+m'}{2} > Min$, which is in contradiction with $Min = \alpha$.

  The same reasoning can be carried out when $Max \in E_1$, which also leads to a contradiction. As a consequence, we deduce that both Min and Max belong to either E_1 or E_2, such as hereafter investigated.

- *Case 2: $Min, Max \in E_2$:* we deduce from (14) that $h_{Min} = h_{2\alpha-Min}$ and $h_{Max} = h_{2\alpha-Max}$, where $2\alpha - Min$ and $2\alpha - Max$ belong to E_2. Hence, we obtain $Min \leq 2\alpha - Max$ and $2\alpha - Min \leq Max$ and therefore $\frac{Min+Max}{2} \leq \alpha \leq \frac{Min+Max}{2}$, *i.e.*:

$$\alpha = \frac{Min + Max}{2}. \tag{21}$$

  Combined with those presented in Section III-C, the result in (21) highlights that the coefficients of the filter are (anti)symmetric to α.

- *Case 3: $Min, Max \in E_1$:* the unique trivial solution is $Min = Max = \alpha$, $\beta = k\pi$, $k \in \mathbb{Z}$. This result corresponds to the case where the filter is composed of a unique non-zero coefficient $h_m = h_\alpha \in \mathbb{R}$.

E. Summary

It has been shown that a constant group delay necessarily implies that:

- $\alpha = \frac{Min+Max}{2}$,

- $\beta \in \{k\pi, \frac{\pi}{2} + k\pi\}$, $k \in \mathbb{Z}$, and $\varepsilon_\beta = \begin{cases} 1, & \text{if } \beta = \frac{\pi}{2} + k\pi \\ -1, & \text{if } \beta = k\pi \end{cases}$.

- $h_m = \varepsilon_\beta h_{2\alpha-m}$, which shows that the coefficients are (anti)symmetric to α.

It must be emphasized that in practice, the extremal coefficients are non-zero values, *i.e.* $Min = 0$ and $Max = M - 1$. Eventually, even if it means discarding null coefficients at the

edges, we recognize the usual expression $\alpha = \frac{M-1}{2}$ such as defined in (10). Thus, we have shown that $(\mathcal{A}) \Rightarrow (\mathcal{B})$, which finally proves the equivalence $(\mathcal{A}) \Leftrightarrow (\mathcal{B})$.

IV. CONCLUSION

In this paper, we proposed an original proof of the equivalence between the linear phase of FIR filters and the (anti)symmetry of the coefficients. As a prerequisite, the uniqueness of the group delay for any decomposition of the filter frequency response has been proved as well. The proof of equivalence has been carried out by considering a very general model, in which the taps of the filters are elements of a mathematical sequence. From this paradigm, we analyzed the independence of sin functions in order to deduce some properties of the FIR filters, namely the (anti)symmetry property of the coefficients, as well as the values of the linear phase parameters α and β. It is worth noting that the proposed proof remains valid if FIR filters with complex coefficients are considered. In that case, however, the coefficients are conjugate (anti)symmetric.

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