The heavy range of randomly biased walks on trees

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Abstract: We focus on recurrent random walks in random environment (RWRE) on Galton-Watson trees. The range of these walks, that is the number of sites visited at some fixed time, has been studied in three different papers [AC18], [AdR17] and [dR16]. Here we study the heavy range: the number of edges visited at least α times for some real α. The asymptotic behavior of this process when α is a power of the number of steps of the walk is given for all the recurrent cases. It turns out that this heavy range plays a crucial role in the rate of convergence of an estimator of the environment from a single trajectory of the RWRE.

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1. Introduction

Here we are interested in random walks in random environment (RWRE) on a supercritical Galton-Watson tree, the random environment is then a branching random walk.The aim of this paper is first to study the number of edges visited by the walk more than α times for some $\alpha > 0$; we call this random variable *heavy range* in the paper. This can be seen in some sense as an extension of the works of [AC18], generalized to the whole class of recurrent walks on trees. A second aim is to apply our control on the heavy range to a problem of non-parametric estimation for the distribution of the environment extending this time the work of [DL17] in the one-dimensional case.

Let us first give a precise definition of the process we are interested in. Consider a supercritical Galton-Watson tree $\mathbb{T}$ with offspring distributed as a random variable ν, such that $\mathbb{E}[\nu] > 1$ as $\mathbb{T}$ is supercritical. In the paper, we adopt the following usual notations for tree-related quantities: the root of $\mathbb{T}$ is denoted by e, for any $x \in \mathbb{T}$, ν_x denotes the number of descendants of x, the parent of a vertex x is denoted by x^* and its children by $\{x_i,\ 1 \leq i \leq \nu_x\}$. For technical reasons, we add to the root e, a parent e^* which is not considered as a vertex of the tree. We also denote by $[\![x, y]\!]$ the sequence of vertices realizing the unique shortest path between x and y, by $|x|$ the generation of x, that is the length of the path $[\![e, x]\!]$ and we write $x < y$ when y is a descendant of x that is when x is an element of $[\![e, y]\!]$. Finally, we write $\mathbb{T}_n$ for the tree truncated at generation n. We then introduce a real-valued branching random walk indexed by $\mathbb{T}$: $(V(x), x \in \mathbb{T})$. We suppose that $V(e) = 0$ and we denote the increments of $V(x)$ by $\omega_x := V(x) - V(x_*)$. For any generation n and conditionally to $\mathcal{E}_n = \{\mathbb{T}_n, (V(x), x \in \mathbb{T}_n)\}$, the variables $(\omega_{x_i},\ i \leq \nu_x)$ where x is a vertex of $\mathbb{T}$ such that $|x| = n$ are assumed to be i.i.d. distributed as a random variable $(\omega_i, i \leq \nu)$. We denote by $\mathbf{P}$ the distribution of $\mathcal{E} = \{\mathbb{T}, (V(x), x \in \mathbb{T})\}$.

For a given realization of the environment $\mathcal{E}$, we consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{T} \cup \{e^*\}$

which transition probabilities are defined by the following relations:

$$\begin{aligned}
&\mathbb{P}^{\mathcal{E}}(X_{n+1}=e|X_n=e^*)=1\ ,\\
\forall x\in\mathbb{T}\setminus\{e^*\},\quad &\mathbb{P}^{\mathcal{E}}(X_{n+1}=x^*|X_n=x)=\frac{e^{-V(x)}}{e^{-V(x)}+\sum_{i=1}^{\nu_x}e^{-V(x_i)}}=\frac{1}{1+\sum_{i=1}^{\nu_x}e^{-\omega_{x_i}}}\ ,\\
\forall j\le\nu_x,\quad &\mathbb{P}^{\mathcal{E}}(X_{n+1}=x_j|X_n=x)=\frac{e^{-V(x_j)}}{e^{-V(x)}+\sum_{i=1}^{\nu_x}e^{-V(x_i)}}=\frac{e^{-\omega_{x_j}}}{1+\sum_{i=1}^{\nu_x}e^{-\omega_{x_i}}}\ .
\end{aligned}$$

The measure $\mathbb{P}^{\mathcal{E}}$ is usually referred to as the quenched distribution of the walk $(X_n)_{n\in\mathbb{N}}$ in contrast to the annealed distribution $\mathbb{P}$, the measure $\mathbb{P}^{\mathcal{E}}$ integrated with respect to the law of $\mathcal{E}$:

$$\mathbb{P}(\,\cdot\,)=\int\mathbb{P}^{\mathcal{E}}(\,\cdot\,)\,\mathbf{P}(\,\mathrm{d}\mathcal{E}\,)\ .$$

For $x\in\mathbb{T}\cup\{e^*\}$, we use the notation $\mathbb{P}_x^{\mathcal{E}}$ for the conditional probability $\mathbb{P}_x^{\mathcal{E}}(\cdot|X_0=x)$; when there is no subscript, the walk is supposed to start at the root e. We finally introduce $\mathbb{P}^*$, the annealed probability conditioned on the survival set of the tree $\mathbb{T}$.

The walk $(X_n)_{n\in\mathbb{N}}$, called biased random walk on a tree, was first introduced by R. Lyons (see [Lyo90] and [Lyo92]). In our case where the bias is random, the first references go back to R. Lyons and R. Pemantle [LP92] and M.V. Menshikov and D. Petritis [MP02]. Random walks in random environment on trees form a subclass of canonical models in the more general framework of random motions in random media that are widely used in physics. They are a natural extension of the one dimensional model, originally introduced in the works of [Che62]. These models have been intensively studied in the last four decades, mostly in the physics and probability theory literature.

The behaviors of randomly biased walks on trees differ deeply from the behaviors of the RWRE in the one-dimensional case. In particular there are several regimes for both recurrent and transient cases. A complete classification for the recurrent cases is given by G. Faraud [Far11] (for the transient cases, see E. Aidekon [Aïd08]). It can be determined from the fluctuations of log-Laplace transform $\psi(s):=\log\mathbb{E}\left[\sum_{|z|=1}e^{-sV(z)}\right]$ as resumed in Figure 1.

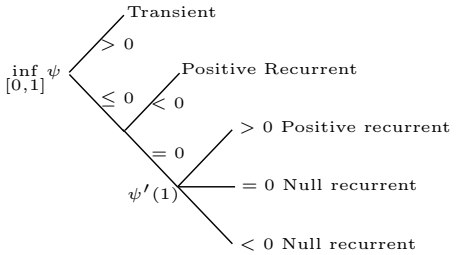

Figure 1: Recurrence criteria for $(X_n)_{n\in\mathbb{N}}$ on Galton-Watson trees

Here we focus on the recurrent cases that is to say when $\inf_{s\in[0,1]}\psi(s)\le 0$. They present essentially three different asymptotic regimes which depend on the fluctuation of ψ. In the paper, we need a bunch of classical assumptions which are summarized here:

Assumption A1.

- *the Galton-Watson tree is supercritical:* $\mathbb{E}\Big[\sum_{|x|=1} 1\Big] = \mathbb{E}[\nu] \in (1,\infty)$.
- *the log-Laplace transform* ψ *is well defined on a neighborhood of* $[0,1]$*:*

$$\exists r_1, r_2 > 0,\ \forall s \in [-r_1, 1+r_2], \quad \psi(s) < \infty\ . \tag{1}$$

- $\inf_{s\in[0,1]} \psi(s) \leq 0$ *(so* $(X_n)_{n\in\mathbb{N}}$ *is recurrent).*
- *let* $t_0 := \inf\{\, s \geq 0,\ \psi(s) = 0\,\}$ *then:*

$$\mathbb{E}\Big[\Big(\sum_{|z|=1} e^{-t_0 V(z)}\Big)^2\Big] < +\infty. \tag{2}$$

and if $t_0 = 1$*:*

- *if* $\psi'(1) \geq 0$*, there exists* $\delta > 0$ *such that* $\mathbb{E}[\nu^{1+\delta}] < +\infty$,
- *if* $\psi'(1) < 0$*, let*

$$\kappa := \inf\{s > 1, \psi(s) = 0\} \in (1, +\infty], \tag{3}$$

when $\kappa < \infty$*,*

$$\mathbb{E}\Big[\sum_{|z|=1} e^{-\kappa V(z)} \cdot \max(-V(z), 0)\Big] < +\infty\ . \tag{4}$$

Let us briefly describe the different recurrent cases (assuming the above conditions).

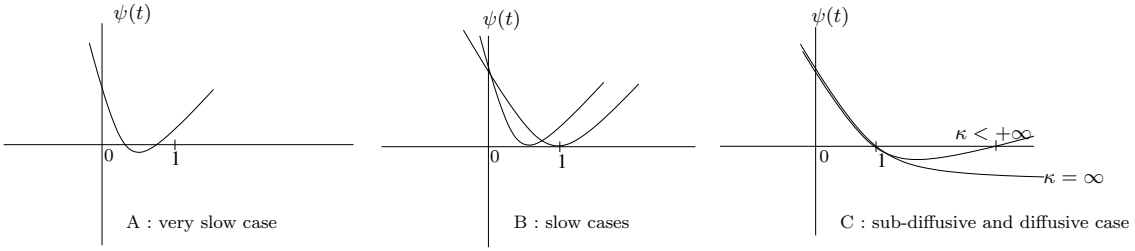

Figure 2: Log-Laplace ψ and corresponding behaviors of $(X_n)_{n\in\mathbb{N}}$.

First, when $\inf_{s\in[0,1]} \psi(s) < 0$ (Figure 2: A), the walk is positive recurrent and the largest generation visited up to an instant n, $X_n^* := \max_{k\leq n} |X_k|$ is of the order $\log n$ ([HS07b]), this is the slowest case, even slower than one dimensional Sinai's random walk ([Sin82]).

If $\inf_{s\in[0,1]} \psi(s) = 0$ (Figure 2: B and C), then different behaviors appear depending on the sign of $\psi'(1)$. When $\psi'(1) \geq 0$ (Figure 2: B), there are again two possible cases: if $\psi'(1) > 0$ then the walk is positive recurrent whereas when $\psi'(1) = 0$ the walk is null recurrent. However these two cases lead to the same asymptotic behavior for X_n^* (up to some multiplicative constant) which is of the order $(\log n)^3$ (see [HS07a] and [FHS11]).

Finally, when $\psi'(1) < 0$ (Figure 2: C), the walk is null recurrent and X_n^* is of the order $n^{1-1/\min(\kappa,2)}$ where κ is defined in (3) (see [HS07b], [Far11], [AdR17], [dR16]).

In this paper we study the heavy range: the number of edges of the tree which are visited more than α times in n steps. Precisely, for $x \in \mathbb{T}$, let N_x^n be the number of times, the walk $(X_n)_{n\in\mathbb{N}}$ visits the edge (x^*, x) before time n:

$$N_x^n := \sum_{k=1}^{n} \mathbf{1}_{X_{k-1}=x^*, X_k=x} \ .$$

Then, for any $\alpha > 0$ the *heavy range* is the random variable

$$R_\alpha^n := \sum_{x\in\mathbb{T}} \mathbf{1}_{N_x^n \geq \alpha} \ . \tag{5}$$

Note that for $\alpha = 1$, the above random variable is the usual range. In this case, the asymptotic behavior is known for all null recurrent cases, see [AC18], [AdR17] and [dR16]. The heavy range is then a natural generalization of the usual range which helps to understand how the random walk spreads on the tree. In the following subsection we describe first the asymptotic behavior of the heavy range at random time and then the behavior of R_α^n properly. Then in subsection 1.2, we see how the heavy range controls the error rate of a natural estimator of the distribution of the environment via a single trajectory of the RWRE.

1.1. The behavior of heavy range $R_{n^\theta}^n$

Let $T^{(n)}$ denote the n^{th} return to e^*:

$$T^{(0)} := 0 \quad \text{and} \quad \forall n \geq 1,\ T^{(n)} := \inf\{k > T^{(n-1)},\ X_k = e^*\} \ .$$

The following theorem gives the behavior of the heavy range taken at time $T^{(n)}$ in all recurrent cases. Note that in these cases, $T^{(n)}$ is almost surely finite for every integer n. For the sake of simplicity, in all the paper we write $R_\alpha^{(n)}$ for $R_\alpha^{T^{(n)}}$ and $N_x^{(n)}$ for $N_x^{T^{(n)}}$, we also use the notation $\log^+ x$ for $\log(\max(1, x))$.

Theorem 1.1. *Assume A1. For any $\theta \geq 0$,*

$$\frac{\log^+ R_{n^\theta}^{(n)}}{\log n} \text{ converges in } \mathbb{P}^*\text{-probability to a constant } \xi_\theta.$$

Moreover, when $\theta \geq 1$, $\xi_\theta = 0$ and, when $\theta < 1$,

- *if $\inf_{s\in[0,1]} \psi(s) < 0$ or $\inf_{s\in[0,1]} \psi(s) = 0$ with $\psi'(1) \geq 0$, then*

$$\xi_\theta = t_0(1-\theta), \text{ with recall } t_0 = \inf\{s > 0,\ \psi(s) = 0\} \leq 1, \tag{6}$$

- *if $\inf_{s\in[0,1]} \psi(s) = 0$ with $\psi'(1) < 0$, then*

$$\xi_\theta = \kappa(1-\theta), \text{ if } 1 < \kappa \leq 2, \tag{7}$$

$$\xi_\theta = \max\Big(2 - \kappa\theta, 1-\theta\Big), \text{ if } \kappa \in (2, \infty]. \tag{8}$$

The proof of Theorem 1.1 for $\theta < 1$ is given in Section 3. The cases $\theta \geq 1$ are then easily obtained using the results in the cases $\theta < 1$ and the fact that, for any $n \geq 1$, $R_{n^\theta}^{(n)}$ is a non-increasing function of θ.

Note that (6) corresponds to the slower cases, that is when X_n behaves like a power of $\log n$. Also $t_0 < 1$ if and only if the random walk is positive recurrent whereas $t_0 = 1$ corresponds to the so called boundary case for the branching potential ($\psi(1) = \psi'(1) = 0$).

For the diffusive cases, that is if $\kappa > 2$ for small $\theta \leq 1/(\kappa-1)$, the heavy range is of the order $n^{2-\kappa\theta+o(1)}$, larger than $n^{1-\theta}$. Conversely, for larger values of θ, that is to say if we are interested on sites often visited by the walk then the range is of the order $n^{1-\theta}$. We will see in the proof that this fact depends deeply where the edges sufficiently visited are localized on the tree.

The heavy range in deterministic times will be easier to interpret in terms of the behavior of the walk. For this purpose it is important to control $T^{(n)}$, the time of the n^{th} return to e^*. It has been studied in [AD14b] and more precisely in [HS16] and [Hu17], for completeness we recall these results in the following remark; note that in these papers the results are given for the n^{th}-return time to the root e and not to e^* but this does not change the rates of convergence.

Remark 1.2.

1. *If* $\inf_{s\in[0,1]}\psi(s) = 0$, *with* $\psi'(1) = 0$ *then in probability* $T^{(n)}/(n\log n)$ *converges in probability to a positive limit (see [HS16], Proposition 2.5).*
2. *If* $\inf_{s\in[0,1]}\psi(s) = 0$, $\psi'(1) < 0$ *and* $\kappa \neq 2$, $T^{(n)}/n^{\min(\kappa,2)}$ *converges in law to a positive limit and if* $\kappa = 2$, $(T^{(n)}\log n)/n^2$ *converges in law to a positive limit (see [Hu17], Corollary 1.2).*
3. *Finally if* $\psi'(1) > 0$ *with* $\inf_{s\in[0,1]}\psi(s) = 0$ *or* $\inf_{s\in[0,1]}\psi(s) < 0$ *then* $T^{(n)}/n$ *converges in probability to a positive limit (both are positive recurrent random walks).*

We can now obtain the behavior of the heavy range in deterministic time:

Corollary 1.3. *Assume A1. For any* $\theta \in [0,1]$,

$$\frac{\log^+ R^n_{n^\theta}}{\log n} \text{ converges in } \mathbb{P}^*\text{-probability to a constant } \tilde{\xi}_\theta.$$

Moreover,

- *if* $\inf_{s\in[0,1]}\psi(s) \leq 0$ *with* $\psi'(1) \geq 0$, *then* $\tilde{\xi}_\theta = \xi_\theta = t_0(1-\theta)$,
- *if* $\inf_{s\in[0,1]}\psi(s) = 0$ *with* $\psi'(1) < 0$ *and*
  - *if* $1 < \kappa \leq 2$, *then* $\tilde{\xi}_\theta = 1-\kappa\theta$ *for* $\theta \leq 1/\kappa$ *and 0 otherwise,*
  - *if* $\kappa \in (2,\infty]$, *then* $\tilde{\xi}_\theta = \max(1-\kappa\theta\ ,\ 1/2-\theta)$ *for* $\theta \leq 1/2$ *and 0 otherwise.*

Let us compare, for null recurrent cases, the heavy range $R^n_{n^\theta}$ for $\theta > 0$ and regular range, that is R^n_1. It is proved in [AC18] that for the slow case, when $\psi(1) = \psi'(1) = 0$, $\frac{\log n}{n} R_1^{(n)}$ converges in probability to an explicit positive constant, whereas for the sub-diffusive and diffusive case, when $\psi(1) = 0$, $\psi'(1) < 0$, it is proved in [AdR17] and [dR16] that the correct normalization for the convergence in probability of $R_1^{(n)}$ is simply $1/n$. So for the regular range we observe that sub-diffusive and diffusive cases spread more than in the slow case. For the heavy range the opposite appears, indeed when $\psi(1) = \psi'(1) = 0$, $t_0 = 1$ and $\tilde{\xi}_\theta = 1-\theta$ which is larger than $1-\kappa\theta$ for $\kappa > 1$. This tells us that the environment of the slow null recurrent case creates much more vertices where the walk spends larger amount of time than in the other cases.

As this corollary is a direct consequence of the above theorem and remark, we only give a short proof in one case.

Proof. We only give a proof for the case $\inf_{s\in[0,1]}\psi(s) = 0$ with $\psi'(1) < 0$ and $\kappa \in (2,\infty]$ for $\theta > 0$ such that $1-\kappa\theta > 1/2-\theta$; the others ones can be treated with the same arguments. According to Remark 1.2, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}^*\Big(T^{(\lfloor n^{(1+\epsilon)/2}\rfloor)} \geq n\Big) = 1. \tag{9}$$

Then fix some $\epsilon > 0$ and denote $\theta_\epsilon = 2\theta/(1+\epsilon)$. Suppose that ϵ is small enough to have $2 - \kappa\theta_\epsilon \geq 1 - \theta_\epsilon$. For $n \geq 2$,

$$\mathbb{P}^*\left(\frac{\log^+ R^n_{n^\theta}}{\log n} \geq 1 - \kappa\theta + 2\epsilon\right) \leq \mathbb{P}^*\left(\frac{\log^+ R^{(\lfloor n^{(1+\epsilon)/2}\rfloor)}_{n^\theta}}{\log n} \geq 1 - \kappa\theta + 2\epsilon\right) + \mathbb{P}^*\left(T^{(\lfloor n^{(1+\epsilon)/2}\rfloor)} < n\right).$$

And, as $\theta = \frac{1+\epsilon}{2}\theta_\epsilon$,

$$\begin{aligned}\mathbb{P}^*\left(\frac{\log^+ R^{(\lfloor n^{1/2+\epsilon}\rfloor)}_{n^\theta}}{\log n} \geq 1 - \kappa\theta + 2\epsilon\right) &= \mathbb{P}^*\left(\frac{\log^+ R^{(\lfloor n^{(1+\epsilon)/2}\rfloor)}_{\left(n^{(1+\epsilon)/2}\right)^{\theta_\epsilon}}}{\log n^{(1+\epsilon)/2}} \geq \frac{2(1 - \kappa((1+\epsilon)/2)\theta_\epsilon + 2\epsilon)}{1+\epsilon}\right) \\ &\leq \mathbb{P}^*\left(\frac{\log^+ R^{(\lfloor n^{(1+\epsilon)/2}\rfloor)}_{\lfloor n^{(1+\epsilon)/2}\rfloor^{\theta_\epsilon}}}{\log n^{(1+\epsilon)/2}} \geq 2 - \kappa\theta_\epsilon + \frac{2\epsilon}{1+\epsilon}\right).\end{aligned}$$

Then (9) and Theorem 1.1 show that for any ϵ small enough,

$$\lim_{n\to\infty} \mathbb{P}^*\left(\frac{\log^+ R^n_{n^\theta}}{\log n} \geq (1 - \kappa\theta + 2\epsilon)\right) = 0\ .$$

The lower bound and the other cases can be obtained with similar arguments. □

Here we obtain the first order for the asymptotic expansion of the heavy range for any recurrent cases and of course a natural question is now to obtain the correct normalization for this range. Even if we succeed to treat all the cases at once here, we do not expect that this can be reproduced for more precise normalization due the deep difference that appears between cases.

1.2. Non-parametric estimation of the law of the environment

The study of the heavy range has been partly motivated by the statistical problem we present here. The statistical study of random processes in random environment has been overlooked in the literature until recently, when new biophysics experiments produced data that can be modeled (at least in an ideal-case setup) by RWRE. For example in [VBM06], RWRE are used as a mathematical model of a mechanical denaturation of DNA. Consequently, a new series of works appeared on statistical procedures aiming at estimating the distribution of the environment from a RWRE trajectory.

We start with a brief resume of what has been done recently in the one dimensional discrete case. Recall that for RWRE on $\mathbb{Z}$, the environment is a sequence of random variables $(\rho_x)_{x\in\mathbb{Z}}$. When the environment is i.i.d., the estimation of the distribution of ρ_0 observing only the RWRE $(X_n)_{n\in\mathbb{N}}$ is a natural problem. The first theoretical result appears in [AE04] which treats the problem in a general settings with no quantitative purpose, then in [And11] and [AD12] particular cases are detailed. Recently more attention has been paid to the one dimensional case: the aim was to provide a parametric estimation of the distribution of the environment with the help of a single trajectory of random walk $(X_n)_{n\in\mathbb{N}}$ ([CFL$^+$14], [FLM14], [FGL14], [CFLL16]). When the environment is a Markov chain a parametric approach can also be found in [ALM15]. The problem of non-parametric estimation has been studied in [DL17], the aim of the result we present below is to extend their studies to the more delicate case of randomly biased random walks on supercritical Galton-Watson trees.

Indeed, the next step after the model of RWRE for mechanical denaturation of DNA in [VBM06], is to construct a similar model for mechanical denaturation of RNA. But while DNA molecules are quite linear, RNA molecules present a more complicated geometry with secondary structure (see [GBH01]). It seems then natural to describe RNA molecules with trees and to mimic its mechanical denaturation by a random walk in random environment on a tree.

However, the estimation of the distribution of the environment for a random walk on a tree is much more complicated as we have to study the law of the transitions of a branching random walk and not only a real random variable. Therefore, we only consider here a very particular and simple case; this part of the paper has to be seen as a first step to understand how the methods used for the one-dimensional case can be generalized. So, for all the statistical results in this part and in Section 4, we add to A1 the following assumptions:

Assumption A2.

- *the reproduction law of the Galton-Watson is bounded:* $\exists K > 0, \quad \mathbb{P}(\nu \leq K) = 1$.
- *given the tree up to generation* n *and the number of children* ν_x *of some* $x \in \mathbb{T}$ *such that* $|x| = n$, *the variables* $(\omega_{x_i})_{1\leq i\leq \nu_x}$ *are i.i.d. with the same distribution as some variable* ω.

Our aim is then to estimate the distribution of ω given the observation of a single trajectory of the walk $(X_n)_{n\in\mathbb{N}}$ up to time $T^{(n)}$. In particular we need $T^{(n)}$ to be finite which is the case by recurrence of $\mathbf{X}$. Precisely, we give an estimation of the c.d.f F of

$$\rho := (1 + e^{-\omega})^{-1} \tag{10}$$

instead of the one of ω, but this is of course equivalent as conversely $\omega = \log[\rho/(1-\rho)]$.

Remark 1.4. *It is possible to relax the condition on* ν*: if we suppose that the distribution is not bounded but only subgaussian, we still have the same rates of convergence in the following theorems. However the proof is more technical and for the sake of clarity we have chosen to present only the bounded case.*

Our first theorem gives the existence of the estimator and the rates of convergences, in a second theorem we give more details on how this estimator is constructed.

Theorem 1.5. *Assume A1, A2 and that the c.d.f.* F *of* ρ *is* γ*-Hölder for some* $\gamma \in (0, 2]$. *There exists an estimator* $\widehat{F}_n$ *of* F, *which is a function of the trajectory* $(X_k)_{0\leq k\leq T^{(n)}}$, *such that for any* $\epsilon > 0$,

$$n^{r-\epsilon}\|\widehat{F}_n - F\|_\infty \text{ tends to 0 in } \mathbb{P}^*\text{-probability where}$$

i) $r = \frac{\gamma t_0}{\gamma + t_0}$ *if* $\inf_{s\in[0,1]} \psi(s) < 0$, $\inf_{s\in[0,1]} \psi(s) = 0$, *with* $\psi'(1) \geq 0$,
ii) $r = \frac{\gamma\kappa}{\gamma+\kappa}$ *if* $\inf_{s\in[0,1]} \psi(s) = 0$, $\psi'(1) < 0$ *and* $\kappa \leq 2$,
iii) $r = \frac{2\gamma}{\gamma+\kappa}$ *if* $\inf_{s\in[0,1]} \psi(s) = 0$, $\psi'(1) < 0$ *and* $2 < \kappa \leq 2 + \gamma$,
iv) $r = \frac{\gamma}{\gamma+1}$ *if* $\inf_{s\in[0,1]} \psi(s) = 0$, $\psi'(1) < 0$ *and* $\kappa > 2 + \gamma$.

In the previous theorem, the rate of convergence is given as a function of the parameter n, it is more natural to give a rate depending on $T^{(n)}$. This is done in the following corollary which is a direct consequence of Theorem 1.5 and Remark 1.2.

Corollary 1.6. *Assume conditions of Theorem 1.5. The estimator* $\widehat{F}_n$ *is such that for any* $\epsilon > 0$,

$$\left(T^{(n)}\right)^{r-\epsilon} \|\widehat{F}_n - F\|_\infty \text{ tends to 0 in } \mathbb{P}^*\text{-probability where}$$

i) $r = \frac{\gamma t_0}{\gamma + t_0}$ *if* $\inf_{s\in[0,1]} \psi(s) < 0$, $\inf_{s\in[0,1]} \psi(s) = 0$, *with* $\psi'(1) \geq 0$,

ii) $r = \frac{\gamma}{\gamma+\kappa}$ *if* $\inf_{s\in[0,1]} \psi(s) = 0$, $\psi'(1) < 0$ *and* $\kappa \leq 2+\gamma$,
iii) $r = \frac{\gamma}{2(\gamma+1)}$ *if* $\inf_{s\in[0,1]} \psi(s) = 0$, $\psi'(1) < 0$ *and* $\kappa > 2+\gamma$.

The corollary shows that for our estimator the best rate is obtained in the limit case $t_0 = 1$ which corresponds to the best compromise between the number of visited sites and the number of times most of the sites are visited. A remarkable difference if we compare our result on a tree to the recurrent one dimensional random walk of [DL17] is the error rate. Indeed, in this paper, the time of observation used instead of $T^{(n)}$ is τ_n, the first time the coordinate n is reached by the walk. They obtain (see Corollary 2 in [DL17]) for the recurrent case an error rate of the order $\log\log\tau_n/\log\tau_n$. This is very large compared to what is obtained here in Corollary 1.6, so (recurrent) walk on the tree naturally yields a better rate for the error. The reason for that comes from the range of the different walks: for the one dimensional recurrent case very few coordinates are visited before a given instant (around $(\log n)^2$, for an instant n), whereas (see Theorem 1) for the walk on the tree the range is much larger (of the order of a power of n, for the same instant n).

We now give a more explicit description of $\widehat{F}_n$. For this purpose, let us introduce the following family of estimators

$$\widehat{F}_n^{\alpha}(u) := \frac{1}{R_\alpha^{(n)}\mathbb{E}[\nu]} \sum_{x\in\mathbb{T}} \psi_\alpha^{\lfloor \alpha u\rfloor}\left(N_{x^*}^{(n)}, N_x^{(n)}\right) \tag{11}$$

where

$$\psi_\alpha^l(i,j) := \frac{\mathbf{1}_{i\geq\alpha}}{\binom{i-1+j}{\alpha-1}} \sum_{k=0}^{l-1} \binom{i}{k}\binom{j}{\alpha-1-k} \ ,$$

using the conventions $0/0 = 0$ and $\binom{n}{k} = 0$ if $0 \leq n < k$. The logic behind the definition of the $\widehat{F}_n^{\alpha}$ will become clear in Section 4. The following theorem shows that $(\widehat{F}_n^{\alpha})_n$ are estimators of the c.d.f. F with an error rate depending on the variables $R_\alpha^{(n)}$.

Theorem 1.7. *Assume A1, A2 and that the c.d.f. F of ρ is in $\mathcal{C}^\gamma$ for some $\gamma \in (0,2]$. Then for any integers $\alpha, n \geq 1$, and any real $z > 0$, we have*

$$\mathbb{P}\left(\|\widehat{F}_n^{\alpha} - F\|_\infty \geq \frac{K}{\mathbb{E}[\nu]} \sqrt{\frac{z + \log\alpha + 2\log R_\alpha^{(n)}}{2R_\alpha^{(n)}}} + \frac{2\|F\|_\gamma}{\alpha^{\gamma/2}} \right) \leq Ce^{-z} \ .$$

Theorem 1.7 shows that the random part of the error rate our estimator is a function of the heavy range introduced in the previous subsection. To obtain the optimal $\widehat{F}_n^{\alpha}$ for the estimation of F, a compromise must be done between considering sites which have been sufficiently visited, that is choosing a large α, and considering a sufficiently large number of sites, that is choosing α small enough.

Remark that, in this theorem, we work with probability $\mathbb{P}$ and not $\mathbb{P}^*$. Theorem 1.7 will be proved in Section 4. But we can directly explain how it implies 1.5.

Proof of Theorem 1.5. The estimator $\widehat{F}_n$ given in Theorem 1.5 is obtained from the collection $\left(\widehat{F}_n^{\alpha}\right)_{\alpha\geq 1}$ via Goldenshluger-Lepski's method (see [GL08]). Using arguments similar to the ones presented in the proof of Lemma 3 in [DL17], we can show that for any integer $n \geq 1$, and any real $z > 0$, there exists a r.v. $\widehat{\alpha}_{z,n}$ depending only on $(X_k)_{0\leq k\leq T^{(n)}}$ such that

$$\mathbb{P}\left(\|\widehat{F}_n^{\widehat{\alpha}_{z,n}} - F\|_\infty \geq \inf_{\alpha\geq 1} \frac{4K}{\mathbb{E}[\nu]} \sqrt{\frac{z + 3\log\alpha + 2\log R_\alpha^{(n)}}{2R_\alpha^{(n)}}} + \frac{6\|F\|_\gamma}{\alpha^{\gamma/2}} \right) \leq Ce^{-z} \ .$$

Theorem 1.5 is now easily obtained using the estimations of $R_\alpha^{(n)}$ given in Theorem 1.1. □

The rest of the paper is organized as follows. In Section 2 we present tools related to the environment together with estimates for potential V that will be used in Section 3. More specifically we study the number of vertices with low potential which have a great importance for the slow cases. Section 3 is the heart of the paper, we study the heavy range and prove Theorem 1.1. This section is decomposed into two subsections the first one deals with an upper bound for the heavy range and the second one with a lower bound. Note that the proof of the lower bound is more technical and need in particular to estimate the fluctuations of the environment, these estimates are more complex for the sub-diffusive and diffusive cases than for the slow cases. Then, in Section 4, we prove Theorem 1.5 on the non-parametric estimation problem of the environment.

Finally, throughout the paper, the letter C stands for a universal constant which value can change from line to line.

2. Preliminaries on the environment

In this section we present results related to the environment which are used in Section 3 to prove Theorem 1.1. Like in the whole paper, we assume in this section that Assumptions A1 are valid. In a first subsection we give useful technical lemmata and in a second one, we state and prove a result concerning the number of vertices with low potential V. We also introduce some notation specific to the environment. For $u \in \mathbb{T}$, we denote the maximum of V between e and u by $\overline{V}$ and the minimum of V between e and u by $\underline{V}$:

$$\overline{V}(u) := \max_{x \leq u} V(x) \quad \text{and} \quad \underline{V}(u) := \min_{x \leq u} V(x).$$

We also denote by V_u the environment centered at u: for any $x \geq u$, $V_u(x) := V(x) - V(u)$. Remark that if u and v are different vertices of the same generation ℓ, then, given $\mathbb{T}_\ell$, $(V_u(x))_{x \geq u}$ and $(V_v(x))_{x \geq v}$ are i.i.d. and distributed as V under $\mathbb{P}$.

2.1. Technical estimates

First recall the many-to-one Formula (see [Shi15] Chapter 1, and [FHS11] equation 2.1). It is used several times in the paper to compute different expectations related to the environment.

Lemma 2.1 (Many-to-one Formula)**.** *For any $t > 0$,*

$$\mathbb{E}\Big[\sum_{|x|=m} f(V(x_i), 1 \leq i \leq m) \Big] = \mathbb{E}\Big[e^{tS_m + \psi(t)m} f(S_i, 1 \leq i \leq m) \Big].$$

where $(S_n)_{n \in \mathbb{N}}$ is the random walk starting at 0 and such that the increments $(S_{n+1} - S_n)_{n \in \mathbb{N}}$ are i.i.d. and for any measurable function $h : \mathbb{R}^m \to [0, \infty)$,

$$\mathbb{E}[h(S_1)] = e^{-\psi(t)} \mathbb{E}\left[\sum_{|x|=1} e^{-tV(x)} h(V(x)) \right].$$

The next lemma deals with a key random variable which appears in the study of the heavy range (via the edge local time of random walk $(X_n)_{n \in \mathbb{N}}$, see below (27)) after that Many-to-one Formula is applied.

Lemma 2.2. *Assume $\psi(1) = 0$ and $\psi'(1) < 0$ and that the parameter κ defined in (3) is finite. Let us consider the random walk $(S_n)_{n\in\mathbb{N}}$ of the Many-to-one Lemma with $t = 1$ and define for any $\ell \geq 0$, the random variable:*

$$H_\ell^S := \sum_{k=0}^{\ell} e^{S_k - S_\ell} \ . \tag{12}$$

Then, we can find three constants $C \geq c$ and $A > 0$ such that

$$\forall m \geq 1,\ \forall \ell \in \mathbb{N}, \quad \mathbb{P}\big(H_\ell^S \geq m\big) \leq \frac{C}{m^{\kappa-1}},$$

and

$$\forall m \geq 1,\ \forall \ell \geq A \log m, \quad \mathbb{P}\big(H_\ell^S \geq m\big) \geq \frac{c}{m^{\kappa-1}} \ .$$

Proof. By definition of S_1, $\mathbb{E}[-S_1] = \psi'(1) < 0$ and $\mathbb{E}\big[e^{-(\kappa-1)S_1}\big] = e^{\psi(\kappa)} = 1$. According to Lemma 1 in [KKS75] for the lattice case and to Theorem 2 in [Gri75] for the non-lattice case, $\sum_{j=0}^{+\infty} e^{-S_j}$ is $\mathbb{P}$-a.s. finite and there are some constants c, C, such that for $m \geq 1$,

$$\frac{c}{m^{\kappa-1}} \leq \mathbb{P}\Big(\sum_{j=0}^{+\infty} e^{-S_j} \geq m\Big) \leq \frac{C}{m^{\kappa-1}} \ .$$

As for any $\ell \in \mathbb{N}$, $H_\ell^S \overset{\mathcal{L}}{=} \sum_{k=0}^{\ell} e^{-S_k}$ where $\overset{\mathcal{L}}{=}$ stands for the equality in law, this leads directly to the upper bound of the lemma.

For the lower bound, we remark that

$$\mathbb{P}\big(H_\ell^S \geq m\big) \geq \mathbb{P}\Big(\sum_{k=0}^{\infty} e^{-S_k} \geq \frac{3}{2}m\Big) - \mathbb{P}\Big(\sum_{k=\ell+1}^{\infty} e^{-S_k} > \frac{m}{2}\Big).$$

And we only have to prove that for ℓ large enough, $\mathbb{P}\big(\sum_{k=\ell+1}^{\infty} e^{-S_k} > m/2\big)$ is negligible compared to $1/m^{\kappa-1}$. Indeed, consider the event $A_\ell := \{\forall k \geq \ell+1, e^{-S_k} \leq \frac{m}{2k^2}\}$. On A_ℓ,

$$\sum_{k=\ell+1}^{\infty} e^{-S_k} \leq \frac{m}{2} \sum_{k=\ell+1}^{\infty} \frac{1}{k^2} \leq \frac{m}{2\ell}.$$

Therefore, Markov inequality yields

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{k=\ell+1}^{\infty} e^{-S_k} > m/2\Big) \leq \mathbb{P}\big(\overline{A}_\ell\big) = \mathbb{P}\Big(\bigcup_{k\geq\ell+1} \Big\{e^{-S_k} > \frac{m}{2k^2}\Big\}\Big) \\
&\leq \sum_{k\geq\ell+1} \mathbb{P}\Big(e^{-(\kappa-1)S_k/2} > \Big(\frac{m}{2k^2}\Big)^{(\kappa-1)/2}\Big) \\
&\leq \Big(\frac{2}{m}\Big)^{(\kappa-1)/2} \sum_{k\geq\ell+1} k^{\kappa-1} e^{k\psi(\frac{\kappa+1}{2})} \leq C m^{-(\kappa-1)/2} \ell^{\kappa-1} e^{\ell\psi(\frac{\kappa+1}{2})},
\end{aligned}$$

As $\psi(\frac{\kappa+1}{2}) < 0$, we can find a constant A such that for $\ell \geq A \log m$, the above expression is $o(m^{-(\kappa-1)})$. This concludes the proof of the lemma. □

We give now a control of the maximum of V at the very first generations of the tree.

Lemma 2.3. *For any $\delta > 0$, we can find two constants ϵ_1, ϵ_2 such that for n large enough,*

$$\mathbb{P}\Big(\max_{|u| \le \epsilon_1 \log n} |V(u)| \ge \delta \log n \Big) \le n^{-\epsilon_2} \ .$$

Note that the optimal bound, that is to say the one that leads to the almost-sure behavior of $\max_{|u|\le \epsilon \log n} V(u)$, is well known (see for example [AD14a]) but its exact value has no importance here.

Proof. Recall the definition of r_1 in (1), for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\Big(\max_{|u| \le \epsilon \log n} V(u) \ge a\epsilon \log n \Big) &\le \sum_{j \le \epsilon \log n} \mathbb{E}\Big[\sum_{|v|=j} \mathbf{1}_{V(v) \ge a\epsilon \log n} \Big] \le \sum_{j \le \epsilon \log n} \mathbb{E}\Big[\sum_{|v|=j} e^{r_1 V(v) - r_1 a\epsilon \log n} \Big] \\ &\le \sum_{j \le \epsilon \log n} e^{-r_1 a\epsilon \log n + j\psi(-r_1)} \le C e^{-\epsilon \log n (r_1 a - \psi(-r_1))}\end{aligned}$$

for some constant $C > 0$. Then choosing a large enough so that $r_1 a > \psi(-r_1)$ and taking $\epsilon = \delta/a$, we obtain the bound for $\max V$. For $\min V$, the proof is the same except we work with t_0 instead of r_1:

$$\begin{aligned}\mathbb{P}\Big(\min_{|u| \le \epsilon \log n} V(u) \le -a\epsilon \log n \Big) &\le \sum_{j \le \epsilon \log n} \mathbb{E}\Big[\sum_{|v|=j} \mathbf{1}_{V(v) \le -a\epsilon \log n} \Big] \\ \le \sum_{j \le \epsilon \log n} \mathbb{E}\Big[\sum_{|v|=j} e^{-t_0 V(v) - t_0 a\epsilon \log n} \Big] &\le \sum_{j \le \epsilon \log n} e^{-t_0 a\epsilon \log n} \le C n^{-a\epsilon t_0} \log n.\end{aligned}$$

□

The above lemma is used in the following section to obtain independence between the different branches of the tree. It will be used together with the following concentration lemma.

Lemma 2.4. *Consider two integers $\ell \le L$ and $L - \ell + 1$ measurable sets $A_\ell, \dots, A_L$. Consider, for any $u \in \mathbb{T}$ such that $|u| = \ell$, the variable*

$$Z_u := \sum_{u \le x, |x| \le L} e^{-t_0 V_u(x)} \mathbf{1}_{\left(V_u(y),\ u \le y \le x \right) \in A_{|x|}} \ .$$

There exist two constants $C > 0$ and $a > 1$ depending only on the reproduction distribution ν such that

$$\mathbb{P}\Big(\sum_{|u|=\ell} Z_u < \mathbb{E}[Z] \Big) \le \frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} C a^{-\ell} \ .$$

where

$$Z = \sum_{|x| \le L - \ell} e^{-t_0 V(x)} \mathbf{1}_{\left(V(y),\ y \le x \right) \in A_{|x|}} \ .$$

Proof. We first work with the conditional probability $\mathbb{P}(\cdot | \mathbb{T}_\ell)$. The $(Z_u^n, |u| = \ell)$ are hence independent and identically distributed as Z under $\mathbb{P}$. Denote by D_ℓ the number of vertices at generation ℓ; Tchebychev's inequality gives:

$$\mathbb{P}\Big(\Big| \sum_{|u|=\ell} Z_u - D_\ell \mathbb{E}[Z] \Big| > \frac{D_\ell}{2} \mathbb{E}[Z] \Big| \mathbb{T}_\ell \Big) \le \frac{4}{D_\ell} \frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \ .$$

Now, Theorem A.3 in [AD14a] tells us that for some $a, b > 1$, $\forall \ell \ge 1$, $\mathbb{P}\big(D_\ell \le a^\ell \big) \le b^{-\ell}$ which is enough to conclude the proof. □

2.2. Number of vertices with low potential

Fix some constant $c > 0$. In this subsection we are interested in a lower bound for the random variable

$$\sum_{|x|\le(\log n)^3} \mathbf{1}_{\overline{V}(x)\le c\log n}.$$

It counts the number of sites x with generation smaller than $(\log n)^3$ such that the potential in the path from the root to x remains below $c\log n$. We will see in the next section that this random variable is naturally related to the heavy range.

The proof we propose here is based on Lemma 2.3 which implies that the very first generations of the tree have no important impact on the value of $\overline{V}$. This point can be used to obtain independence and to apply Lemma 2.4.

Proposition 2.5. *Assume A1. Let $c > 0$ and $0 < \delta < 1 \wedge c$ and $t_0 := \inf\{s > 0, \psi(s) = 0\}$. There exists $a > 0$ such that for large n,*

$$\mathbb{P}\Big(\sum_{|x|\le(\log n)^3} \mathbf{1}_{\overline{V}(x)\le c\log n} \ge n^{t_0(c-\delta)}\Big) \le n^{-a\delta} .$$

Proof. We first use Lemma 2.3 to control $\overline{V}(u)$ on the very first generations of the tree: there exist two positive constants b and ϵ such that

$$\mathbb{P}\Big(\max_{|u|=\epsilon_n} \overline{V}(u) \ge \frac{\delta}{4}\log n\Big) \le n^{-b} \quad \text{where} \quad \epsilon_n = \lfloor \epsilon \log n \rfloor . \tag{13}$$

The strategy is then similar for the different cases (depending on ψ), the only difference is the generation we have to work with. Let us take some constant $B \ge 0$ and a sequence of integers $(\ell_n)_{n\in\mathbb{N}}$ such that for $n$large enough, ℓ_n is smaller than $(\log n)^3$; the exact choice of the sequence depends on ψ and will be explicitly given later. Consider now the collection of random variables:

$$\forall n \ge 1,\ \forall u \in \mathbb{T}, \quad Z_u^n := \sum_{\substack{x>u \\ |x|=\ell_n+\epsilon_n}} e^{-t_0 V_u(x)} \mathbf{1}_{(c-\delta/2)\log n \le V_u(x), \overline{V}_u(x)\le (c-\delta/4)\log n,\ \underline{V}_u(x)\ge -B} .$$

For n large enough, on the event $\mathcal{A}_n := \{\max_{|u|=\epsilon_n} \overline{V}(u) < \frac{\delta}{4}\log n\}$,

$$\sum_{|x|\le(\log n)^3} \mathbf{1}_{\overline{V}(x)\le c\log n} \ge \sum_{|u|=\epsilon_n} \sum_{x\ge u} \mathbf{1}_{\overline{V}_u(x)\le (c-\delta/4)\log n} \ge n^{t_0(c-\delta/2)} \sum_{|u|=\epsilon_n} Z_u^n.$$

Hence, concentration Lemma 2.4 implies that there exists a constant ϵ_1 such that for n large enough,

$$\mathbb{P}\left(\sum_{|x|\le(\log n)^3} \mathbf{1}_{\overline{V}(x)\le c\log n} \le n^{t_0(c-\delta/2)}\mathbb{E}[Z^n]\right) \le \frac{\mathbb{E}\Big[(Z^n)^2\Big]}{\mathbb{E}[Z^n]^2} n^{-\epsilon_1} + \mathbb{P}\big(\overline{\mathcal{A}}_n\big) . \tag{14}$$

where

$$\forall n \ge 1, \quad Z^n := \sum_{|x|=\ell_n} e^{-t_0 V(x)} \mathbf{1}_{(c-\delta/2)\log n \le V(x), \overline{V}(x)\le (c-\delta/4)\log n,\ \underline{V}(x)\ge -B} .$$

So we only have to control $\mathbb{E}[Z^n]$ and $\mathbb{E}\Big[(Z^n)^2\Big]$ to obtain the bound of the proposition.

• First for the mean $\mathbb{E}[Z^n]$ the proof is different depending on the value of $\psi'(t_0)$:

Case 1: $\psi'(t_0) < 0$**:** recall that in this case either $t_0 = 1$ and $\psi(1) = \inf_{s\in[0,1]} \psi(s) = 0$, or $t_0 < 1$ and $\inf_{s\in[0,1]} \psi(s) < 0$. Moreover, we choose $\ell_n = \lfloor \ell \log n \rfloor$ with $\ell = (c - 3\delta/8)/|\psi'(t_0)|$. Using Many-to-one formula of Lemma 2.1 with $t = t_0$, gives

$$\begin{aligned}\mathbb{E}[\,Z^n\,] &= \mathbb{P}\big(\,(c-\delta/2)\log n \le S_{\ell_n},\ \overline{S}_{\ell_n} \le (c-\delta/4)\log n,\ \underline{S}_{\ell_n} \ge -B\,\big)\\ &\ge 1 - \mathbb{P}(\,(c-\delta/2)\log n > S_{\ell_n}\,) - \mathbb{P}\big(\overline{S}_{\ell_n} > (c-\delta/4)\log n\big) - \mathbb{P}\big(\underline{S}_{\ell_n} < -B\big)\,.\end{aligned}$$

For any $B \ge 0$ and $\lambda > 0$ such that $\psi(t_0 + \lambda) < 0$, Markov inequality yields

$$\mathbb{P}\big(\underline{S}_{\ell_n} < -B\big) \le \sum_{k\le \ell_n} \mathbb{P}(\,S_k < -B\,) \le \sum_{k\le \ell_n} e^{-\lambda B} e^{\psi(t_0+\lambda)k} \le \frac{e^{-\lambda B}}{1 - e^{\psi(t_0+\lambda)}}\ ,$$

so taking B large enough, for any n, $\mathbb{P}(\underline{S}_{\ell_n} < -B) \le C < 1$. And, as the process S is the sum of i.i.d. random variables with mean $|\psi'(t_0)|$ and $c - \delta/2 < |\psi'(t_0)|\ell < c - \delta/4$, using exponential Markov inequality we can also prove that for some constant $a > 0$,

$$\mathbb{P}(\,(c-\delta/2)\log n > S_{\ell_n}\,) \le n^{-a} \quad \text{and} \quad \mathbb{P}\big(\overline{S}_{\ell_n} > (c-\delta/4)\log n\big) \le n^{-a}\ .$$

Combining these bounds, we obtain that for any $n \ge 1$,

$$\mathbb{E}[\,Z^n\,] \ge C > 0\ . \tag{15}$$

Case 2: $\psi'(t_0) = 0$: in this case either $t_0 < 1$ or $\psi(1) = \psi'(1) = 0$. We can basically use the same method as in the case $\psi'(t_0) < 0$ except that, for this case, important generations (see [AD14b], [AC18]) are the ones of the order $(\log n)^2$, so essentially we take this time $\ell_n = \lfloor (\ell \log n)^2 \rfloor$ for some $\ell > 0$, and we can choose $B = 0$. Many-to-one Lemma yields like above

$$\mathbb{E}[\,Z^n\,] = \mathbb{P}\big(\,(c-\delta/2)\log n \le S_{\ell_n},\ \overline{S}_{\ell_n} \le (c-\delta/4)\log n | \underline{S}_{\ell_n} \ge 0\,\big)\,\mathbb{P}\big(\underline{S}_{\ell_n} \ge 0\big)\,.$$

We know (see [AS14] equation (2.8)) that the limit, $\lim_{n\to+\infty} {\ell_n}^{1/2}\mathbb{P}(\underline{S}_{\ell_n} \ge 0) = d > 0$ exists, also by invariance principle (see [Bol76]),

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\big(\,(c-\delta/2)\log n \le S_{\ell_n},\ \overline{S}_{\ell_n} \le (c-\delta/4)\log n \ \big|\ \underline{S}_{\ell_n} \ge 0\,\big)\\ &= \lim_{n\to\infty} \mathbb{P}\Big(\,(c-\delta/2)(\sigma\ell)^{-1} \le S_{\ell_n}/\sigma\sqrt{\ell_n},\ \overline{S}_{\ell_n}/\sigma\sqrt{\ell_n} \le (c-\delta/4)(\sigma\ell)^{-1}\ \Big|\ \underline{S}_{\ell_n} \ge 0\Big)\\ &= \mathbb{P}\big(\,(c-\delta/2)(\sigma\ell)^{-1} \le \mathcal{M},\ \overline{\mathcal{M}} \le (c-\delta/4)(\sigma\ell)^{-1}\,\big) \ge C > 0.\end{aligned}$$

where $\mathcal{M}$ is the Brownian meander and $\sigma^2 := \mathbb{V}\mathrm{ar}(S_1) = \mathbb{E}(\sum_{|x|=1} V^2(x)e^{-V(x)}) < +\infty$ by (1). So finally we obtain in this case: for n large enough,

$$\mathbb{E}[\,Z^n\,] \ge C(\log n)^{-1} \tag{16}$$

for some constant $C > 0$.

• For $\mathbb{E}\big[\,(Z^n)^2\,\big]$, we first introduce the sequence of variables

$$\forall k \in \mathbb{N},\ M_k := \sum_{|x|=|y|=k} e^{-t_0V(x)-t_0V(y)} \mathbf{1}_{\underline{V}(x)\wedge\underline{V}(y)\ge -B}\ .$$

For any $k \in \mathbb{N}$,

$$\begin{aligned} M_{k+1} &\leq \sum_{|u|=|v|=k} e^{-t_0 V(u)-t_0 V(v)} \mathbf{1}_{\underline{V}(u)\wedge \underline{V}(v)\geq -B} \sum_{\substack{x \text{ s.t. } x^*=u \\ y \text{ s.t.} y^*=v}} e^{-t_0 V_u(x)-t_0 V_v(y)} \\ &= \sum_{\substack{u\neq v \\ |u|=|v|=k}} e^{-t_0 V(u)-t_0 V(v)} \mathbf{1}_{\underline{V}(u)\wedge \underline{V}(v)\geq -B} \sum_{\substack{x \text{ s.t. } x^*=u \\ y \text{ s.t.} y^*=v}} e^{-t_0 V_u(x)-t_0 V_v(y)} \\ &\quad + \sum_{|u|=k} e^{-2t_0 V(u)} \mathbf{1}_{\underline{V}(u)\geq -B} \sum_{\substack{x,y \text{ s.t.} \\ x^*=y^*=u}} e^{-t_0 V_u(x)-t_0 V_u(y)} \, . \end{aligned}$$

Then taking conditional expectation with respect to $\mathcal{E}_k := \sigma\left(\mathbb{T}_k,\ (V(x), |x| \leq k)\right)$, the above expression gives

$$\begin{aligned} \mathbb{E}[M_{k+1}|\mathcal{E}_k] &\leq M_k \mathbb{E}\left[\sum_{|x|=1} e^{-t_0 V(x)}\right]^2 + \sum_{|u|=k} e^{-2t_0 V(u)} \mathbf{1}_{\underline{V}(u)\geq -B} \mathbb{E}\left[\sum_{|x|=|y|=1} e^{-t_0 V(x)-t_0 V(y)}\right] \\ &\leq M_k + C \sum_{|u|=k} e^{-t_0 V(u)} \, . \end{aligned}$$

Note that $\mathbb{E}\Big[\sum_{|x|=|y|=1} e^{-t_0 V(x)-t_0 V(y)}\Big] = \mathbb{E}\Big[\big(\sum_{|x|=1} e^{-t_0 V(x)}\big)^2\Big]$ is finite thanks to (4). And we get recursively, as $M_0 = 1$,

$$\mathbb{E}[M_{k+1}] \leq 1 + C\mathbb{E}\left[\sum_{|u|\leq k} e^{-t_0 V(u)}\right] = 1 + C(k+1).$$

This gives the bound for the second moment: $\mathbb{E}\big[(Z^n)^2\big] \leq \mathbb{E}[M_{\ell_n}] \leq C\ell_n$. Collecting this bound together with the one for $\mathbb{E}[Z^n]$ in (14), this concludes the proof. □

3. Lower and Upper bound for the heavy range of recurrent walks

In this section, we prove Theorem 1.1 by considering separately the lower and upper bounds. The upper bounds are easier to obtain than the lower ones so they are treated for every cases at the beginning of this section in Proposition 3.1. For the lower bounds we treat separately the contribution coming from vertices with low potential in Proposition 3.3 and the contribution coming from vertices with high potential in Proposition 3.4. It turns out that for slow random walks, that is to say random walks with logarithmic behavior, the contribution coming from vertices with low potential is sufficient to obtain the asymptotics for the log-heavy range. Conversely, for fast but sub-diffusive cases, that is to say when $1 < \kappa < 2$, then only vertices with high potential contribute. Finally for diffusive cases, that is to say for $\kappa \geq 2$ then either vertices with low or high potential contribute depending on the value of θ.

Remind that $\mathbb{P}_x^{\mathcal{E}}$ stands for the probability where the environment is fixed and the index x stands for the starting point of the random walk. To obtain the bounds, an environment-related variable is essential:

$$\forall x \in \mathbb{T}, \quad H_x := \sum_{y\leq x} e^{V(y)-V(x)} \, . \tag{17}$$

Indeed some important probabilities, for a fixed environment, are related to H_x. For any $x \in \mathbb{T}$, let $T_x := \inf\{k > 0,\ X_k = x\}$ be the hitting time of vertex x. As the walk $(X_n)_{n\in\mathbb{N}}$ is recurrent, the expressions of the probability of hitting x before e^* starting from the root and the probability of hitting x before the root starting from x^* are the same as for a one-dimensional walk: the restriction of $(X_n)_{n\in\mathbb{N}}$ to the path $[\![e, x]\!]$. So a standard result for one-dimensional random walks in random environment (see [Gol84]) yields:

$$a_x := \mathbb{P}_e^{\mathcal{E}}(T_x < T_{e^*} = 1/\sum_{z\in[\![e,x]\!]} e^{V(z)} = \frac{e^{-V(x)}}{H_x} \tag{18}$$

and

$$b_x := \mathbb{P}_{x^*}^{\mathcal{E}}(T_x < T_{e^*}) = \sum_{z\in[\![e,x^*]\!]} e^{V(z)}/\sum_{z\in[\![e,x]\!]} e^{V(z)} = 1 - \frac{1}{H_x}\ . \tag{19}$$

3.1. Upper bounds

The main results of this section is the following proposition, which gives the upper bounds for $R_{n^\theta}^{(n)}$ depending on the value of θ. The proof is quite short compared to the lower bound and gives a good idea of what can be expected for the lower bounds.

Proposition 3.1. *Assume A1 and fix $\theta \in [0, 1)$. For any $\delta > 0$, there is a constant $\epsilon > 0$ such that, for n large enough,*

- *in all cases but $\psi(1) = 0$ and $\psi'(1) < 0$,*

$$\mathbb{P}^*\Big(R_{n^\theta}^{(n)} > n^{t_0(1-\theta)+\delta}\Big) \le n^{-\epsilon}\ ,$$

- *if instead $\psi(1) = 0$ and $\psi'(1) < 0$,*
  - *if $1 < \kappa \le 2$,* $\mathbb{P}^*\Big(R_{n^\theta}^{(n)} > n^{\kappa(1-\theta)+\delta}\Big) \le n^{-\epsilon}\ ,$
  - *if $\kappa \in (2, \infty]$,* $\mathbb{P}^*\Big(R_{n^\theta}^{(n)} > n^{(2-\kappa\theta)\vee(1-\theta)+\delta}\Big) \le n^{-\epsilon}\ .$

To prove the proposition, we have to control the largest generation visited by the walk before n returns to e^*: $X^*_{T^{(n)}} = \max_{0\le k\le T^{(n)}} |X_k|$. To study the different behaviors of this random variable depending on ψ was the purpose of [HS07a], [HS07b] and [FHS11]. In the following lemma we present a simpler version of their results adapted to our purpose:

Lemma 3.2. *If $\inf_{s\in[0,1]} \psi(s) = 0$ and $\psi'(1) \ge 0$ there exists $B, b > 0$ such that for n large enough,*

$$\mathbb{P}^*\big(X^*_{T^{(n)}} \ge L_n\big) \le n^{-b}\ , \text{with } L_n = B(\log n)^3,$$

otherwise if $\psi'(1) < 0$ for any $\epsilon > 0$, there exists $b > 0$ such that, for n large enough,

$$\mathbb{P}^*\big(X^*_{T^{(n)}} \ge L_n\big) \le n^{-b} \text{ with } L_n = n^{\min(\kappa-1,1)+\epsilon}.$$

Finally if $\inf_{s\in[0,1]} \psi(s) < 0$ there exists $B > 0$ and $b > 0$ such that for n large enough,

$$\mathbb{P}^*\big(X^*_{T^{(n)}} \ge L_n\big) \le n^{-b},\ \text{with } L_n = B\log n\ .$$

Proof. By the strong Markov property, $\mathbb{P}^{\mathcal{E}}\big(X^*_{T^{(n)}} \geq L_n\big) = 1 - \big(1 - \mathbb{P}^{\mathcal{E}}\big(T_{|L_n|} < T_{e^*}\big)\big)^n$ where $T_{|L_n|}$ is the hitting time of generation L_n:

$$T_{|L_n|} = \inf\{t \geq 0 \ / \ \exists x \in \mathbb{T}, X_t = x \text{ and } |x| = L_n\} \ .$$

As for any $x \in [0,1]$, $1-(1-x)^n \leq nx$, integrating the previous equality with respect to the distribution of $\mathcal{E}$ gives

$$\mathbb{P}^*\big(X^*_{T^{(n)}} > L_n\big) \leq n\mathbb{P}^*\big(T_{|L_n|} < T_{e^*}\big) .$$

This probability has been intensively studied in [HS07a], [HS07b], [FHS11] and [Hu17] (replacing the return time in e^* by the return time in e but this does not change the normalization rate). More precisely, when $\inf_{s\in[0,1]} \psi(s) = 0$: if $\psi'(1) \geq 0$ then Equations (5.4) in [FHS11] gives the existence, for B large enough, of a constant $c_1 > 1$ such that

$$\limsup_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}^*\big(T_{|L_n|} < T_{e^*}\big) \leq -c_1, \tag{20}$$

If instead $\psi'(1) < 0$ then Proposition 4.2 (ii and iii) in [HS07b] implies

$$\limsup_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}^*\big(T_{|L_n|} < T_{e^*}\big) \leq -1 - \epsilon/2, \tag{21}$$

(note that Proposition 4.2 is written for regular trees, but the proof also works in our case). Finally, if $\inf_{s\in[0,1]} \psi(s) < 0$, according to the proof of Theorem 1.1 in [HS07b], for B large enough there exists $c_2 > 1$ such that,

$$\limsup_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}^*\big(T_{|L_n|} < T_{e^*}\big) \leq -c_2. \tag{22}$$

The different results of the lemma are now direct consequences of (20), (21) and (22). □

We are now ready to prove the proposition.

Proof of Proposition 3.1. We first restrict the sum over the whole tree to the first L_n generations where L_n is the sequence introduced in Lemma 3.2 corresponding to the assumptions about ψ:

$$\mathbb{P}^*\Big(R^{(n)}_{n^\theta} > n^{t_0(1-\theta)+\delta}\Big) \leq \mathbb{P}^*\big(X^*_{T^{(n)}} \geq L_n\big) + n^{-t_0(1-\theta)+\delta}\mathbb{E}^* \sum_{|x|\leq L_n} \mathbb{P}^{\mathcal{E}}\Big(N^{(n)}_x \geq n^\theta\Big) \tag{23}$$

where the second term in the above upper bound comes from Markov inequality. Thanks to Lemma 3.2, we only have to bound the last expectation $\mathbb{E}^*\Big[\sum_{|x|\leq L_n} \mathbb{P}^{\mathcal{E}}(N^{(n)}_x \geq n^\theta)\Big]$. The proof is different whether L_n is a power of n or a power of $\log n$.

Case 1: *L_n is of the order $(\log n)^p$, $p > 0$.*

We split the sum into two terms depending on the value of $V(x)$: define

$$\Sigma_1 := \mathbb{E}^*\Big[\sum_{|x|\leq L_n} \mathbb{P}^{\mathcal{E}}(N^{(n)}_x \geq n^\theta)\mathbf{1}_{V(x)\leq(1-\theta)\log n}\Big]$$

and

$$\Sigma_2 := \mathbb{E}^*\Big[\sum_{|x|\leq L_n} \mathbb{P}^{\mathcal{E}}(N^{(n)}_x \geq n^\theta)\mathbf{1}_{V(x)>(1-\theta)\log n}\Big].$$

For Σ_1, we just say that $\mathbb{P}^{\mathcal{E}}(N_x^{(n)} \geq n^\theta) \leq 1$ and use the Many-to-one Formula, Lemma 2.1, with $t = t_0$:

$$\Sigma_1 \leq \mathbb{E}^*\Big[\sum_{|x|\leq L_n} \mathbf{1}_{V(x)\leq(1-\theta)\log n}\Big] \leq C\mathbb{E}\Big[\sum_{|x|\leq L_n} e^{t_0((1-\theta)\log n - V(x))}\Big]$$
$$= Cn^{t_0(1-\theta)}\sum_{i=0}^{L_n}\mathbb{E}\Big[e^{t_0(S_i-S_i)}\Big] = Cn^{t_0(1-\theta)}(L_n+1).$$

For Σ_2, we first compute the expectation of $N_x^{(n)}$ at fixed environment:

$$\mathbb{E}^{\mathcal{E}}\Big[N_x^{(n)}\Big] = \sum_{i=1}^{n}\mathbb{E}^{\mathcal{E}}\Big[N_x^{(i)} - N_x^{(i-1)}\Big] = n\mathbb{E}^{\mathcal{E}}\Big[N_x^{(1)}\Big] = n\frac{a_x}{1-b_x} = ne^{-V(x)} \tag{24}$$

where a_x and b_x have been defined in (18) and (19). Now we can apply Markov inequality to $\mathbb{P}^{\mathcal{E}}(N_x^{(n)} \geq n^\theta)$ and use Many-to-one formula with $t = t_0 \leq 1$:

$$\Sigma_2 \leq Cn^{(1-\theta)}\mathbb{E}\Big[\sum_{|x|\leq L_n} e^{-V(x)}\mathbf{1}_{V(x)>(1-\theta)\log n}\Big]$$
$$\leq Cn^{(1-\theta)}e^{(t_0-1)(1-\theta)\log n}\mathbb{E}\Big[\sum_{|x|\leq L_n} e^{-t_0V(x)}\mathbf{1}_{V(x)>(1-\theta)\log n}\Big] \leq Cn^{t_0(1-\theta)}(L_n+1).$$

Inserting the upper bounds of Σ_1 and Σ_2 in (23) concludes the proof in this first case.

Case 2: *L_n is of the order n^p, $p > 0$.*
Note that in this case, ψ satisfies $\inf_{s\in[0,1]}\psi(s) = 0$ with $\psi'(1) < 0$, so $t_0 = 1$. We assume in the definition of L_n that $\epsilon = \delta/2$ and take some real $d > 0$ which value will be fixed later. First we split the expectation $\mathbb{E}^*\Big[\sum_{|x|\leq L_n}\mathbf{1}_{N_x^{(n)}\geq n^\theta}\Big]$ into two parts Σ_1 and Σ_2, depending on the values of $\overline{V}(x)$:

$$\Sigma_1 := \mathbb{E}^*\Big[\sum_{|x|\leq L_n}\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)} \geq n^\theta\Big)\mathbf{1}_{\overline{V}(x)\leq d\log n}\Big]$$

and

$$\Sigma_2 := \mathbb{E}^*\Big[\sum_{|x|\leq L_n}\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)} \geq n^\theta\Big)\mathbf{1}_{\overline{V}(x)>d\log n}\Big].$$

We first deal with Σ_1. Equality (24), Markov inequality and Many-to-one Formula, Lemma 2.1 with $t = 1$ yield

$$\Sigma_1 \leq C\mathbb{E}\Big[\sum_{|x|\leq L_n} ne^{-V(x)}n^{-\theta}\mathbf{1}_{\overline{V}(x)\leq d\log n}\Big] = Cn^{1-\theta}\sum_{j=1}^{L_n}\mathbb{P}\Big(\overline{S}_j \leq d\log n\Big).$$

As $\psi'(1) < 0$ we can choose a $r > 0$ in such way that $\psi(1+r) < 0$. Let $A > 0$. When L_n is larger

than $A\log n$,

$$\begin{aligned}\sum_{j=1}^{L_n}\mathbb{P}\big(\overline{S}_j\le d\log n\big) &\le \sum_{j=1}^{A\log n}\mathbb{P}\big(\overline{S}_j\le d\log n\big)+\sum_{j=A\log n}^{L_n}\mathbb{P}\big(\overline{S}_j\le d\log n\big)\\ &\le A\log n+\sum_{j=A\log n}^{L_n}\mathbb{P}\Big(e^{-r\overline{S}_j}\ge e^{-rd\log n}\Big)\\ &\le A\log n+\sum_{j=A\log n}^{L_n}e^{rd\log n}e^{\psi(1+r)j},\end{aligned}\tag{25}$$

where we have used the equality $\mathbb{E}\big[e^{-rS_1}\big]=e^{\psi(1+r)}$ obtained from Many-to-one Formula. So the above sum is smaller than $Ce^{rd\log n}e^{A\psi(1+r)\log n}$, and choosing A large enough, this converges to 0. Therefore when $\psi'(1)<0$, the following bound holds

$$\Sigma_1\le Cn^{1-\theta}\log n\ .\tag{26}$$

We now have to deal with Σ_2. For any $x\in\mathbb{T}$, consider $E_x^{(n)}:=\sum_{i=1}^n\mathbf{1}_{\{\exists k\in[T^{i-1},T^i),\ X_k=x\}}$ the number of excursions between two returns to e^*. We split once again Σ_2 into two other sums depending on the value of $E_x^{(n)}$:

$$\Sigma_{2,1}=\mathbb{E}^*\Big[\sum_{|x|\le L_n}\mathbf{1}_{N_x^{(n)}\ge n^\theta}\mathbf{1}_{\overline{V}(x)>d\log n}\mathbf{1}_{E_x^{(n)}\ge 2}\Big]$$

and

$$\Sigma_{2,2}=\mathbb{E}^*\Big[\sum_{|x|\le L_n}\mathbf{1}_{N_x^{(n)}\ge n^\theta}\mathbf{1}_{\overline{V}(x)>d\log n}\mathbf{1}_{E_x^{(n)}=1}\Big]\ .$$

We first prove that, for d large enough, the walk will be able to reach a vertex x satisfying $\overline{V}(x)>d\log n$ during a unique excursion $[T^{i-1},T^i)$. Under $\mathbb{P}^{\mathcal{E}}$, $E_x^{(n)}$ follows the binomial distribution $\mathcal{B}(n,a_x)$, thus

$$\mathbb{P}^{\mathcal{E}}\Big(E_x^{(n)}\ge 2\Big)\le\mathbb{E}^{\mathcal{E}}\Big[E_x^{(n)}\Big]-\mathbb{P}^{\mathcal{E}}\Big(E_x^{(n)}=1\Big)=na_x(1-(1-a_x)^{n-1})\le n^2a_x^2\le n^2e^{-2\overline{V}(x)}$$

and Many-to-one Formula yields

$$\Sigma_{2,1}\le Cn^2\sum_{j=1}^{L_n}\mathbb{E}\Big[e^{S_j-2\overline{S}_j}\mathbf{1}_{\overline{S}_j>d\log n}\Big]\le CL_nn^{2-d}\le 1\tag{27}$$

for d large enough. For $\Sigma_{2,2}$, we first notice that

$$\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)}\ge n^\theta,E_x^{(n)}=1\Big)=\sum_{i=1}^n\mathbb{P}^{\mathcal{E}}(N_x^{(i)}-N_x^{(i-1)}\ge n^\theta,\ \forall j\ne i,\ N_x^{(j)}-N_x^{(j-1)}=0).$$

In particular, $\mathbb{P}^{\mathcal{E}}(N_x^{(n)}\ge n^\theta,E_x^{(n)}=1)\le na_x(b_x)^{n^\theta}$ and thanks to Many-to-one Formula, we get

$$\Sigma_{2,2}\le Cn\mathbb{E}\Big[\sum_{|x|\le L_n}a_x(b_x)^{n^\theta}\mathbf{1}_{\overline{V}(x)>d\log n}\Big]\le Cn\sum_{j=1}^{L_n}\mathbb{E}\left[\frac{1}{H_j^{\overline{S}}}\left(1-\frac{1}{H_j^{\overline{S}}}\right)^{n^\theta}\right]$$

where $H_j^S = \sum_{m=1}^{j} e^{S_m - S_j}$ is the random variable defined in Lemma 2.2. Now remark that if $H_j^S \leq n^\theta/(3\log n)$, then $\left(1-1/H_j^S\right)^{n^\theta} \leq 1/n^3$ and so

$$\sum_{j=1}^{L_n} \mathbb{E}\left[\frac{1}{H_j^S}\left(1-\frac{1}{H_j^S}\right)^{n^\theta} \mathbf{1}_{H_j \leq n^\theta/(3\log n)}\right] \leq L_n/n^3$$

and according to Lemma 2.2,

$$\mathbb{E}\left[\frac{1}{H_j^S}\left(1-\frac{1}{H_j^S}\right)^{n^\theta} \mathbf{1}_{H_j > n^\theta/3\log n)}\right] \leq \frac{3\log n}{n^\theta}\mathbb{P}\left(H_j^S > \frac{n^\theta}{3\log n}\right) \leq C\frac{(\log n)^\kappa}{n^{\kappa\theta}}.$$

Finally,

$$\Sigma_{2,2} \leq C(\log n)^\kappa L_n n^{1-\kappa\theta}. \tag{28}$$

The bounds obtained for Σ_1, $\Sigma_{2,1}$ and $\Sigma_{2,2}$ in (26), (27), (28) and Markov inequality give the result in this second case. □

3.2. Lower bounds

In this section, we prove two propositions: the first one gives a lower bound for the heavy range $R_{n^\theta}^{(n)}$ for any cases but is only optimal for the slowest cases, that is to say for random walks with logarithmic behavior. This first proposition is obtained by studying only vertices with low potential (see definition of set A_n below). The second one, Proposition 3.4, which is more technical to obtain, deals only with the fast cases, that is to say when $\psi(1) = 0$ and $\psi'(1) < 0$ and focuses on vertices with high potential.

Proposition 3.3. *Assume A1 and fix $\theta \in [0,1)$. For any $0 < \delta < t_0(1-\theta)$, there exists a constant $\epsilon > 0$ such that for n large enough,*

$$\mathbb{P}^*\Big(\sum_{|x|\leq(\log n)^3} \mathbf{1}_{N_x^{(n)} \geq n^\theta} < n^{t_0(1-\theta)-\delta}\Big) \leq n^{-\epsilon}\ .$$

Remark that the upper bound for $R_{n^\theta}^{(n)}$ given in Proposition 3.1 for the case $\inf_{s\in[0,1]}\psi(s) = 0$ and $\psi'(1) < 0$ can be larger than the lower bound obtained in the above proposition. This means that we ignore too much vertices with high potential that are of great importance. They are treated in Proposition 3.4.

Proof. Consider the subset

$$\forall n \geq 1, \quad A_n := \left\{ x \in \mathbb{T},\ \overline{V}(x) \leq (1-\theta-\delta)\log n, \ \text{ and } \ |x| \leq (\log n)^3 \right\}.$$

We will prove that for n large enough,

$$\mathbb{P}^*\Big(\sum_{x\in A_n} \mathbf{1}_{\ N_x^{(n)} < n^\theta} \geq 1\Big) \leq e^{-n^\theta}\ . \tag{29}$$

Accepting this bound, we have

$$\mathbb{P}^*\Big(\sum_{|x|\leq(\log n)^3} \mathbf{1}_{N_x^{(n)} \geq n^\theta} < n^{t_0(1-\theta)-\delta}\Big) \leq \mathbb{P}^*\Big(|A_n| < n^{t_0(1-\theta)-\delta}\Big) + e^{-n^\theta}$$

and the control on the cardinal of A_n given in Proposition 2.5 taking $c = 1-\theta-\delta$ concludes the proof.

Let us now prove (29). Fix an environment $\mathcal{E} = (\mathbb{T}, \boldsymbol{\omega})$ and consider a vertex $x \in \mathbb{T}$. By the strong Markov property, the sequence $(N_x^{(i)} - N_x^{(i-1)}, i \geq 1)$ is, under $\mathbb{P}^{\mathcal{E}} = \mathbb{P}_e^{\mathcal{E}}$, an i.i.d. sequence with law given by $N_x^{(1)}$ (remark that $N_x^{(0)} = 0$). Applying exponential Markov inequality, we have for any $\eta > 0$ and $\alpha > 0$,

$$\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)} \leq \alpha\Big) = \mathbb{P}\Big(\sum_{i=1}^{n}[N_x^{(i)} - N_x^{(i-1)}] < \alpha\Big) = \mathbb{P}^{\mathcal{E}}\Big(e^{-\eta\sum_{i=1}^{n}[N_x^{(i)}-N_x^{(i-1)}]} > e^{-\eta\alpha}\Big)$$
$$\leq \exp\Big(\eta\alpha + n\log\mathbb{E}^{\mathcal{E}}\Big[e^{-\eta N_x^{(1)})}\Big]\Big). \tag{30}$$

It is easy to see, by the strong Markov property, that the distribution of $N_x^{(1)}$ under $\mathbb{P}_x^{\mathcal{E}}$ is geometrical with parameter $1-b_x$, so that

$$\begin{aligned}\log\mathbb{E}^{\mathcal{E}}\Big[\exp(-\eta N_x^{(1)})\Big] &= \log\Big((1-a_x) + a_x\mathbb{E}_x^{\mathcal{E}}\Big[\exp(-\eta N_x^{(1)})\Big]\Big)\\ &= \log\big(1 - a_x + a_x(1-b_x)/(e^{\eta} - b_x)\big)\\ &\leq -a_x + a_x(1-b_x)/(1+\eta-b_x)\ .\end{aligned}$$

Then coming back to (30) with $\eta = (1-b_x)$, we obtain

$$\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)} \leq \alpha\Big) \leq \exp\Big(-\frac{na_x}{2} + (1-b_x)\alpha\Big). \tag{31}$$

The equations of a_x and b_x given in (18) and (19) imply that $a_x/(1-b_x) = e^{-V(x)}$. Together with (31), this gives

$$\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)} \leq \alpha\Big) \leq e^{-\frac{a_x}{2}\left(n - e^{V(x)}\alpha\right)}\ .$$

As for any $x \in A_n$, $e^{-V(x)} \geq n^{-(1-\theta-\delta)}$ and $a_x \geq 1/(n^{1-\theta-\delta}(\log n)^3)$, the above inequality implies that

$$\mathbb{P}^{\mathcal{E}}\Big(\sum_{x\in A_n}\mathbf{1}_{\,N_x^{(n)}<n^{\theta}} \geq 1\Big) \leq \sum_{x\in A_n}\mathbb{P}^{\mathcal{E}}\Big(N_x^{(n)} \leq n^{1-\delta}e^{-V(x)}\Big) \leq |A_n|e^{-n^{\theta+\delta}(1-n^{-\delta})/(\log n)^3}\ .$$

Integrating with respect to the distribution of $\mathcal{E}$, we obtain for n large enough:

$$\mathbb{P}^{*}\Big(\sum_{x\in A_n}\mathbf{1}_{\,N_x^{(n)}<n^{\theta}} \geq 1\Big) \leq C\mathbb{E}[\,|A_n|\,]\,e^{-n^{\theta+\delta/2}}\ .$$

Then we only need an appropriate upper bound for $\mathbb{E}[\,|A_n|\,]$ to complete the proof. This is a direct consequence of Many-to-one Formula. Indeed, taking $t = t_0$ in Lemma 2.1, we get

$$\begin{aligned}\mathbb{E}[\,|A_n|\,] &= \mathbb{E}\Big[\sum_{|x|\leq(\log n)^3}\mathbf{1}_{\overline{V}(x)\leq(1-\theta-\delta)\log n}\Big]\\ &= \sum_{k=1}^{\lfloor(\log n)^3\rfloor}\mathbb{E}\Big[e^{t_0 S_k + k\psi(t_0)}\mathbf{1}_{\overline{S}_k\leq(1-\theta-\delta)\log n}\Big] \leq (\log n)^3 n^{t_0(1-\theta-\delta)}\ .\end{aligned}$$

□

The following proposition deals with the rapid cases ($\psi(1) = 0, \psi'(1) < 0$): we treat the vertices with large potential which were left aside in the previous proposition. Quite technical, the proof is decomposed in essentially four Lemmata.

Proposition 3.4. *Assume A1 and suppose we are in the case*

$$\inf_{s\in[0,1]} \psi(s) = \psi(1) = 0 \ , \ \psi'(1) < 0 \text{ and } \kappa = \inf\{ s > 1,\ \psi(s) = 0\} < \infty \ .$$

Define $\zeta := (\kappa-1)^{-1} \wedge 1$ *and consider a real* $\theta \in [0,\zeta)$. *For any constant* $0 < \delta < (\kappa-1)(\zeta-\theta)$, *there exists a positive number* ϵ *such that for* n *large enough,*

$$\mathbb{P}^*\Big(R^{(n)}_{n^\theta} \le n^{2\wedge\kappa-\kappa\theta-\delta}\Big) \le n^{-\epsilon}.$$

In the sequel it is enough to consider generations slightly smaller than typical visited generations for this cases (that is $n^{(\kappa-1)\wedge 1}$, as recalled in Lemma 3.2), so we define

$$L^\delta_n := n^{(\kappa-1)\wedge 1-\delta/2} = n^{(\kappa-1)\zeta-\delta/2} \ .$$

First let us introduce the set $\Gamma_n = \Gamma^1_n \cap \Gamma^2_n$ where

$$\Gamma^1_n = \Big\{ x\in\mathbb{T} \ / \ n^\theta \le H_x \le n^{\theta+\delta/16},\ V(x) \ge 4\log n \Big\} \text{ and}$$
$$\Gamma^2_n = \Big\{ x\in\mathbb{T} \ / \ \sum_{z\le x} H_z \le n^{1-7\delta/16} \text{ and } \forall z\le x,\ H_z \le n^\zeta \Big\}$$

(recall the definition of H_x in (17)). The set Γ_n contains the vertices which are the main contributors to the heavy range. Remark that, when $\kappa \le 2$, the condition $\{\forall z\le x,\ H_z \le n^\zeta\}$ is implied by the condition on the sum $\sum_{z\le x} H_z$ and is therefore useless in this case.

The set Γ_n is such that the walk visits most of these vertices more than n^θ times with a large probability. Then, as

$$R^{(n)}_{n^\theta} \ge \sum_{|x|\le L^\delta_n} \mathbf{1}_{N^{(n)}_x \ge n^\theta} \mathbf{1}_{x\in\Gamma_n},$$

we only have to obtain a lower bound for the above last sum. We first prove in the following lemma that, with a large probability, the walk reaches a given vertex of Γ_n during a single excursion $[T^{i-1}, T^i)$

Lemma 3.5. *For any* $x\in\mathbb{T}$, *consider* $E^{(n)}_x := \sum_{i=1}^n \mathbf{1}_{\{\exists k\in[T^{i-1},T^i),\ X_k = x\}}$ *the number of excursions where the walk hits vertex* x. *Then, for* n *large enough,*

$$\mathbb{P}^*\Big(\sum_{|x|\le L^\delta_n} \mathbf{1}_{E^{(n)}_x \ge 2} \mathbf{1}_{x\in\Gamma_n} \ge 1\Big) \le n^{-1} \ .$$

Proof. Under $\mathbb{P}^{\mathcal{E}}$, $E^{(n)}_x$ follows the binomial distribution $\mathcal{B}(n,a_x) = \mathcal{B}(n, e^{-V(x)}/H_x)$ so

$$\mathbb{P}^{\mathcal{E}}\Big(E^{(n)}_x \ge 2\Big) \le \mathbb{E}^{\mathcal{E}}\Big[E^{(n)}_x\Big] - \mathbb{P}^{\mathcal{E}}\Big(E^{(n)}_x = 1\Big) = n a_x (1-(1-a_x)^{n-1}) \le n^2 a_x^2 \ .$$

Moreover, for $x\in\Gamma_n$, $n^2 a_x^2 = n^2 e^{-2V(x)} H_x^{-2} \le n^{-2} e^{-V(x)}$ and

$$\mathbb{E}^*\Big[\sum_{|x|\le L^\delta_n} \mathbb{P}^{\mathcal{E}}\Big(E^{(n)}_x \ge 2\Big) \mathbf{1}_{x\in\Gamma_n}\Big] \le C n^{-2} \mathbb{E}\Big[\sum_{|x|\le L^\delta_n} e^{-V(x)}\Big] \le C n^{-2} L^\delta_n \le n^{-1} \ .$$

And Markov inequality gives the result of the lemma. □

The previous lemma shows that there is independence between the contributions of the different excursions. The following one shows that the sum of these contributions is close to the sum of the quenched mean of one contribution.

Lemma 3.6. *There is a constant C such that for n large enough,*

$$\mathbb{P}^{\mathcal{E}}\left(\sum_{i=1}^{n}\sum_{|x|\le L_n^\delta}\mathbf{1}_{N_x^{(i)}-N_x^{(i-1)}\ge n^\theta}\mathbf{1}_{x\in\Gamma_n}\le n^{1-\theta-\delta/8}\sum_{|x|\le L_n^\delta}e^{-V(x)}\mathbf{1}_{x\in\Gamma_n}\right)$$
$$\le\frac{C}{n^{1-\delta/8}}\frac{\sum_{|x|,|y|\le L_n^\delta}H_{x\wedge y}e^{-V(x)-V(y)+V(x\wedge y)}\mathbf{1}_{x,y\in\Gamma_n}}{\left(\sum_{|x|\le L_n^\delta}e^{-V(x)}\mathbf{1}_{x\in\Gamma_n}\right)^2},\quad \mathbb{P}^*\text{-a.s.}$$

Proof. As the excursions are i.i.d. under $\mathbb{P}^{\mathcal{E}}$, we first use Tchebytchev inequality with probability measure $\mathbb{P}^{\mathcal{E}}$ to obtain:

$$\mathbb{P}^{\mathcal{E}}\left(\sum_{i=1}^{n}\sum_{|x|\le L_n^\delta}\mathbf{1}_{N_x^{(i)}-N_x^{(i-1)}\ge n^\theta}\mathbf{1}_{x\in\Gamma_n}\le\frac{n}{2}\sum_{|x|\le L_n^\delta}\mathbb{P}^{\mathcal{E}}\left(N_x^{(1)}\ge n^\theta\right)\mathbf{1}_{x\in\Gamma_n}\right)$$
$$\le\frac{4}{n}\frac{\sum_{|x|,|y|\le L_n^\delta}\mathbb{P}^{\mathcal{E}}\left(N_x^{(1)}\wedge N_y^{(1)}\ge n^\theta\right)\mathbf{1}_{x,y\in\Gamma_n}}{\left(\sum_{|x|\le L_n^\delta}\mathbb{P}^{\mathcal{E}}\left(N_x^{(1)}\ge n^\theta\right)\mathbf{1}_{x\in\Gamma_n}\right)^2}\,. \tag{32}$$

Remark now that for any $x\in\mathbb{T}$, $\mathbb{P}^{\mathcal{E}}(N_x^{(1)}\ge n^\theta)=a_x(b_x)^{\lceil n^\theta\rceil}$, with a_x and b_x given in respectively (18) and (19) (and $\lceil n^\theta\rceil$ stands for the smallest integer larger than n^θ). As for $x\in\Gamma_n$, $H_x\in[n^\theta,n^{\theta+\delta/16}]$, there is a constant C such that for any $n\ge1$,

$$\sum_{|x|\le L_n^\delta}\mathbb{P}^{\mathcal{E}}(N_x^{(1)}\ge n^\theta)\mathbf{1}_{x\in\Gamma_n}\ge\frac{C}{n^{\theta+\delta/16}}\sum_{|x|\le L_n^\delta}e^{-V(x)}\mathbf{1}_{x\in\Gamma_n}. \tag{33}$$

To bound the numerator in (32), we use Markov-type inequality:

$$\sum_{|x|,|y|\le L_n^\delta}\mathbb{P}^{\mathcal{E}}\left(N_x^{(1)}\wedge N_y^{(1)}\ge n^\theta\right)\mathbf{1}_{x,y\in\Gamma_n}\le\frac{1}{n^{2\theta}}\sum_{x,y\le L_n^\delta}\mathbb{E}^{\mathcal{E}}\left[N_x^{(1)}N_y^{(1)}\right]\mathbf{1}_{x,y\in\Gamma_n}$$
$$\le\frac{2}{n^{2\theta}}\sum_{|x|,|y|\le L_n^\delta}e^{-V(x)-V(y)+V(x\wedge y)}H_{x\wedge y}\mathbf{1}_{x,y\in\Gamma_n}.$$

The last bound of the previous equation is obtained by the control of the covariances $\mathbb{E}^{\mathcal{E}}\left[N_x^{(1)}N_y^{(1)}\right]$. This result is similar to the ones presented in Lemma 5.2 in [HS16], we give here a short proof for completeness.

Lemma 3.7. *For any $x,y\in\mathbb{T}$,*

$$\text{if } x\le y,\quad \mathbb{E}^{\mathcal{E}}\left[N_x^{(1)}N_y^{(1)}\right]=e^{-V(y)}\left(2H_x-1\right) \tag{34}$$
$$\text{else}\quad \mathbb{E}^{\mathcal{E}}\left[N_x^{(1)}N_y^{(1)}\right]=2e^{-V(x)-V(y)+V(x\wedge y)}H_{x\wedge y}. \tag{35}$$

Proof. A direct calculation leads to (34) when $x=y$. We proceed by induction on the generations for the general case. The result is obvious for $x=y=e$. Suppose now that Equations (34) and (35) are true for any $|x|\vee|y|\le m$ for some integer m and consider two vertices x and y such that $|x|\vee|y|\le m+1$. If $|x|\vee|y|\le m$, the result is direct. If $|x|\le m$ and $|y|=m+1$, we consider the

σ-algebra $\mathcal{G}_m = \sigma\left(\mathcal{E}, (N_x^{(1)}, |x| \leq m)\right)$. Given $\mathcal{G}_m$, $N_y^{(1)}$ follows the negative binomial distribution $BN(N_{y^*}^{(1)}, 1/(1+e^{-(V(y)-V(y^*))}))$. Then,

$$\mathbb{E}^{\mathcal{E}}\Big[N_x^{(1)}N_y^{(1)}\big|\mathcal{F}_m\Big] = N_x^{(1)}N_{y^*}^{(1)}e^{-(V(y)-V(y^*))}.$$

Taking the expectation leads to (34). Finally, if $|x| = |y| = m+1$, same kinds of arguments show that

$$\mathbb{E}^{\mathcal{E}}\Big[N_x^{(1)}N_y^{(1)}\big|\mathcal{F}_m\Big] = N_{x^*}^{(1)}N_{y^*}^{(1)}e^{-(V(x)-V(x^*))-(V(y)-V(y^*))},$$

if $x^* \neq y^*$ and otherwise

$$\mathbb{E}^{\mathcal{E}}\Big[N_x^{(1)}N_y^{(1)}\big|\mathcal{F}_m\Big] = N_{x^*}^{(1)}(N_{x^*}^{(1)}+1)e^{-(V(x)-V(x^*))-(V(y)-V(y^*))}\ .$$

The result is now easily obtained considering (34). □

□

The following two lemmata control the numerator and the denominator of the bound obtained in Lemma 3.6.

Lemma 3.8. *For n large enough,*

$$G_n := \mathbb{E}\Big[\sum_{|x|,|y|\leq L_n^\delta} H_{x\wedge y}e^{-V(x)-V(y)+V(x\wedge y)}\mathbf{1}_{x,y\in\Gamma_n}\Big] \leq Cn^{1+2(\kappa-1)(\zeta-\theta)-23\delta/16}.$$

Proof. Remark first that G_n can be written in the following way

$$G_n = \sum_{i=1}^{L_n^\delta}\mathbb{E}\Big[\sum_{|u|=i}e^{-V(u)}H_u\Big(\mathbf{1}_{u\in\Gamma_n} + \sum_{\substack{v\neq\tilde{v}\\ v^*=\tilde{v}^*=u}}e^{-V_u(v)-V_u(\tilde{v})}\sum_{x\geq v}\sum_{y\geq\tilde{v}}e^{-V_v(x)}e^{-V_{\tilde{v}}(y)}\mathbf{1}_{x,y\in\Gamma_n}\Big)\Big]$$

where we recall that for any $z \geq w$, $V_w(z) := V(z) - V(w)$ is the potential centered at w. Furthermore, for any $x \in \Gamma_n$ and any ancestor $v \leq x$, the variable H_x can be decomposed as follows

$$H_x = e^{-V_v(x)}H_v + H_x^{(v)} \leq e^{-V_v(x)}n^\zeta + H_x^{(v)}$$

where $H_x^{(v)} = \sum_{z\in]\!]v,x]\!]}e^{V_v(z)-V_v(x)}$. Now remark that

$$\{x, y \in \Gamma_n\} \subset \Big\{n^\theta \leq e^{-V_v(x)}n^\zeta + H_x^{(v)},\ n^\theta \leq e^{-V_{\tilde{v}}(y)}n^\zeta + H_y^{(\tilde{v})},\ \sum_{z\leq u}H_z \leq n^{1-7\delta/16}\Big\}$$

$$\text{and } \{u \in \Gamma_n\} \subset \Big\{\sum_{z\leq u}H_z \leq n^{1-7\delta/16}\Big\}.$$

The new variable $H_x^{(v)}$ depends only on the values of the potential centered at v and are therefore independent of the other branches and of the potential of the vertices before v. The same is true for $H_y^{(\tilde{v})}$. Therefore, conditioning to variables up to generation i and using Many-to-one Formula of Lemma 2.1 with $t = 1$, we obtain

$$G_n \leq \sum_{i=1}^{L_n^\delta}\mathbb{E}\Big[H_i^S\mathbf{1}_{\sum_{j\leq i}H_j^S\leq n^{1-7\delta/16}}\Big]\left(1+\mathbb{E}\Big[\sum_{\substack{v\neq\tilde{v}\\ v^*=\tilde{v}^*=e}}e^{-V(v)-V(\tilde{v})}\Big]\Big(\sum_{k=0}^{L_n^\delta-i-1}\phi_{k,\zeta}\Big)^2\right) \tag{36}$$

where for any $1 \geq b > \theta$, $\phi_{k,b} := \mathbb{P}\big(e^{-S_k} n^b + H_k^S > n^\theta\big)$. The upper bound of $\phi_{k,b}$ will be use several times in the sequel so we start by giving a general estimation. First, $\phi_{k,b} \leq \mathbb{P}\big(e^{-S_k} n^b > n^\theta/2\big) + \mathbb{P}\big(H_k^S > n^\theta/2\big)$. According to Lemma 2.2, there is a constant $C > 0$ such that for any n large enough, $\mathbb{P}\big(H_k^S > n^\theta/2\big) \leq C n^{-(\kappa-1)\theta}$. And, by Markov inequality, for any $0 < \delta < (\kappa-1)(b-\theta)$,

$$\mathbb{P}\left(e^{-S_k} > \frac{1}{2n^{b-\theta}}\right) \leq C n^{(\kappa-1)(b-\theta)-\delta/2} \mathbb{E}\Big[e^{-(\kappa-1-\frac{\delta}{2(b-\theta)})S_k}\Big] = C e^{k\psi(\kappa-\frac{\delta}{2(b-\theta)})} n^{(\kappa-1)(b-\theta)-\delta/2}.$$

Therefore, as $\psi\big(\kappa - \frac{\delta}{2(b-\theta)}\big) < 0$, there exists $d > 0$ such that

$$\phi_{k,b} \leq C(n^{-(\kappa-1)\theta} + e^{-dk} n^{(\kappa-1)(b-\theta)-\delta/2}). \tag{37}$$

Then applying this inequality in our case that is to say when $b = \zeta$, this gives, as $L_n^\delta = n^{(\kappa-1)\zeta-\delta/2}$, for n large enough,

$$\forall i < L_n^\delta, \quad \sum_{k=0}^{L_n^\delta - i - 1} \phi_{k,\zeta} \leq C n^{(\kappa-1)(\zeta-\theta)-\delta/2}.$$

Moreover, by Assumption (2), $\mathbb{E}\Big[\sum_{v\neq\tilde{v}, |v^*|=|\tilde{v}^*|=1} e^{-V(v)-V(\tilde{v})}\Big]$ is finite. Hence, to conclude the proof of the lemma, we only have to remark that $\sum_{i=1}^{L_n^\delta} H_i^S \mathbf{1}_{\sum_{j\leq i} H_j^S \leq n^{1-7\delta/16}} \leq n^{1-7\delta/16}$. Indeed we add terms to the sum only while the sum stays smaller than $n^{1-7\delta/16}$ so the final result has to be smaller too. □

We now give a lower bound for the denominator.

Lemma 3.9. *There is a constant $\epsilon > 0$, such that for n large enough,*

$$\mathbb{P}\Big(\sum_{|x|\leq L_n^\delta} e^{-V(x)} \mathbf{1}_{x\in\Gamma_n} < n^{(\kappa-1)(\zeta-\theta)-5\delta/8}\Big) \leq n^{-\epsilon} .$$

Proof. We use the same strategy as in the proof of Proposition 2.5: we cut the tree at an early generation to obtain independence and then use concentration Lemma 2.4. We first use Lemma 2.3 to control $\overline{V}(u)$ on the very first generations of the tree: there exist $b, \epsilon > 0$ such that $\mathbb{P}\big(\overline{\mathcal{A}}_n\big) \leq n^{-b}$ where $\mathcal{A}_n = \big\{\forall u \in \mathbb{T}_{\epsilon_n}, |V(u)| \leq \frac{\delta}{8}\log n\big\}$ and $\epsilon_n = \lfloor \epsilon \log n \rfloor$. So for n large enough, on the event $\mathcal{A}_n$,

$$\forall u \in \mathbb{T}_{\epsilon_n},\ H_u \leq n^{2\delta/8}\epsilon_n \leq n^{3\delta/8} \ \text{ and } \ \forall y \geq u,\ H_y^{(u)} \leq H_y \leq H_y^{(u)} + e^{-V_u(y)} n^{3\delta/8} \tag{38}$$

Recall that for any $u \leq x$, $V_u(x) = V(x) - V(u)$ and $H_x^{(u)} = \sum_{u\leq y\leq x} e^{V_u(y)-V_u(x)}$. Fix now some constant $B > 0$, we can consider the collection of random variables :

$$\forall n \geq 1,\ \forall u \in \mathbb{T}, \quad Z_u^n = \sum_{\substack{x>u \\ |x|\leq L_n^\delta}} e^{-V_u(x)} \mathbf{1}_{x\in\Gamma_{n,u}}, \text{ with } \Gamma_{n,u} = \Gamma_{n,u}^1 \cap \Gamma_{n,u}^2$$

where the $\Gamma_{n,u}^i$ are the following new sets of constraints on the environment

$$\Gamma_{n,u}^1 = \Big\{x \in \mathbb{T} \ / \ n^\theta \leq H_x^u \leq n^{\theta+\delta/16}/2,\ V_u(x) \geq 2\log n\Big\} \text{ and}$$
$$\Gamma_{n,u}^2 = \Big\{x \in \mathbb{T} \ / \ \sum_{z\leq x} H_z^u \leq n^{1-3\delta/8}/2 \text{ and } \forall u \leq z \leq x,\ V_u(z) \geq -B \text{ and } H_z^u \leq n^\zeta/2\Big\}.$$

The definitions of Γ_n and $\Gamma_{n,u}$ and inequalities in (38) imply that on $\mathcal{A}_n$,

$$\sum_{|x|\le L_n^\delta} e^{-V(x)}\mathbf{1}_{x\in\Gamma_n} \ge n^{-\delta/8}\sum_{|u|=\epsilon_n} Z_u^n \ ,$$

so concentration Lemma 2.4 shows that, for some $\epsilon>0$, for n large enough

$$\mathbb{P}\Big(\sum_{|x|\le L_n^\delta} e^{-V(x)}\mathbf{1}_{x\in\Gamma_n} \le n^{-\delta/8}\mathbb{E}[\,Z^n\,]\Big) \le \frac{\mathbb{E}\Big[\,(Z^n)^2\,\Big]}{\mathbb{E}[\,Z^n\,]^2}n^{-\epsilon} + \mathbb{P}\big(\overline{\mathcal{A}}_n\big) \ , \tag{39}$$

where $Z^n=\sum_{|x|\le L_n^\delta-\epsilon_n} e^{-V(x)}\mathbf{1}_{x\in\Gamma_{n,e}}$. The next step is to control $\mathbb{E}[\,Z^n\,]$ and $\mathbb{E}\Big[\,(Z^n)^2\,\Big]$.

• *A lower bound for the mean* $\mathbb{E}[\,Z^n\,]$. Here Many-to-one Formula of Lemma 2.1 with $t=1$, gives :

$$\begin{aligned}
&\mathbb{E}[\,Z^n\,]\\
&=\sum_{i=0}^{L_n^\delta-\epsilon_n}\mathbb{P}\Big(n^\theta\le H_i^S\le n^{\theta+\delta/8}/2, S_i\ge 2\log n,\ \sum_{j\le i}H_j^S\le n^{1-3\delta/8}/2,\ \overline{H}_i^S\le n^\zeta/2, \underline{S}_i\ge -B\Big)\\
&\ge\sum_{i=d_n}^{L_n^\delta-\epsilon_n}\Big(\mathbb{P}\big(n^\theta\le H_i^S\big)-\sum_{m=1}^{5}p_{m,i}\Big),
\end{aligned}$$

where $\overline{r}_n:=\max_{j\le n}r_j$ and $\underline{r}_n:=\min_{j\le n}r_j$ for any sequence $(r_n)_{n\in\mathbb{N}}$ and $d_n=\lceil d\log n\rceil$ with d a constant which can be chosen as large as needed. Finally, the $p_{m,i}$ are defined by

$$\begin{aligned}
&p_{1,i}:=\mathbb{P}(\,S_i<2\log n\,),\ p_{2,i}:=\mathbb{P}\Big(n^{\theta+\delta/16}/2\le H_i^S\Big),\ p_{3,i}:=\mathbb{P}\big(n^\theta\le H_i^S,\underline{S}_i<-B\big)\\
&p_{4,i}:=\mathbb{P}\Big(n^\theta\le H_i^S\le n^{\theta+\delta},\overline{H}_i^S>n^\zeta/2\Big)\ \text{and}\ p_{5,i}:=\mathbb{P}\Big(n^\theta\le H_i^S,\ \sum_{j\le i}H_j^S\ge n^{1-3\delta/8}/2\Big).
\end{aligned}$$

By Lemma 2.2, for n large enough, $\mathbb{P}\big(n^\theta\le H_i^S\big)\ge cn^{-(\kappa-1)\theta}$, so we only have to prove that the other terms are small compared to $n^{-(\kappa-1)\theta}$. The probability $p_{1,i}$ has already been bounded in this paper, see (25): we get as $i\ge d_n$, $p_{1,i}\le n^{-D}$ with D as large as needed if d is large enough. By Lemma 2.2, $p_{2,i}\le Cn^{-(\kappa-1)(\theta+\delta/16)}$. The case of the other $p_{.,i}$ are a bit more delicate. At the end, the calculus for $p_{3,i}$, $p_{4,i}$ and $p_{5,i}$ are very similar so we detail the one for $p_{5,i}$ and then sketch the proofs for $p_{3,i}$ and $p_{4,i}$.

Fix some $A>0$ and let $i_n=\max(0,\lceil i-A\log n\rceil)$. Then $H_i^S=H_{i_n}^S e^{-S_{i_n,i}}+H_{i_n,i}^S$ where for $k\le\ell$,

$$S_{k,\ell}=S_\ell-S_k \text{ and } H_{k,\ell}^S:=\sum_{j=k+1}^{\ell}e^{S_{k,j}-S_{k,\ell}}\ .$$

Note that $H_{i_n}^S$ is independent of $(S_{i_n,i},H_{i_n,i}^S)$. First, we have that $p_{5,i}$ is smaller than

$$\mathbb{P}\Big(\widetilde{H}_{n,i}\ge n^\theta,\ \sum_{j\le i_n}H_j^S\ge n^{1-3\delta/8}/4, H_{i_n}^S\le n^2\Big)+\mathbb{P}\big(H_{i_n}^S\ge n^2\big)+\mathbb{P}\Big(\sum_{i_n<j\le i}H_j^S\ge n^{1-3\delta/8}/4\Big)$$

with $\widetilde{H}_{n,i}:=n^2e^{-S_{i_n,i}}+H_{i_n,i}^S$. By independence, the first term above is smaller than

$$\mathbb{P}\Big(\widetilde{H}_{n,i}\ge n^\theta\Big)\,\mathbb{P}\Big(\sum_{j\le i_n}H_j^S\ge n^{1-3\delta/8}/4\Big)\ .$$

As $\widetilde{H}_{n,i}$ has the same distribution as the random variable $n^2 e^{-S_{i-i_n}} + H^S_{i-i_n}$, then $\mathbb{P}\Big(\widetilde{H}_{n,i} \geq n^\theta\Big) = \phi_{i-i_n,2}$ where $\phi_{k,b}$ was defined previously and estimated in (37). So for A large enough

$$\mathbb{P}\Big(\widetilde{H}_{n,i} \geq n^\theta\Big) \leq Cn^{-\theta(\kappa-1)}.$$

Moreover, as $\zeta \leq 1$, $\mathbb{E}\Big[\big(H^S_j\big)^{(\kappa-1)\zeta-\delta/16}\Big]$ is bounded and as $(\kappa-1)\zeta \leq 1$,

$$\mathbb{E}\Bigg[\Big(\sum_{j\leq i_n} H^S_j\Big)^{(\kappa-1)\zeta-\delta/16}\Bigg] \leq \sum_{j\leq i_n} \mathbb{E}\Big[\big(H^S_j\big)^{(\kappa-1)\zeta-\delta/16}\Big] \leq CL_n^\delta\ .$$

And Markov inequality yields, for any $i \leq L_n^\delta$,

$$\mathbb{P}\Big(\sum_{j\leq i_n} H^S_j \geq n^{1-3\delta/8}/4\Big) \leq CL_n^\delta n^{-((\kappa-1)\zeta-\delta/16)(1-3\delta/8)} \leq Cn^{-\delta/16}\ .$$

Similarly $\mathbb{P}\big(\sum_{i_n<j\leq i} H^S_j \geq n^{1-3\delta/8}/4\big) \leq Cn^{-((\kappa-1)\zeta-\delta/16)(1-3\delta/8)}\log n$ and finally by Lemma 2.2, $\mathbb{P}\big(H^S_{i_n} \geq n^2\big) \leq Cn^{-2(\kappa-1)}$. As $\zeta > \theta$, for δ small enough, this yields

$$p_{5,i} \leq Cn^{-(\kappa-1)\theta-\delta/16}\ .$$

Same kind of arguments give the same bound for $p_{4,i}$. And finally, for $p_{3,i}$, with the same decomposition as above we easily get that

$$p_{3,i} \leq \mathbb{P}\Big(\widetilde{H}_{n,i} \geq n^\theta, \underline{S}_{i_n} < -B\Big) + \mathbb{P}\big(H^S_{i_n} \geq n^2\big) + \mathbb{P}\big(\min_{i_n\leq k\leq i} S_k \leq -B\big)$$

and by independence, $\mathbb{P}\Big(\widetilde{H}_{n,i} \geq n^\theta, \underline{S}_{i_n} < -B\Big) = \mathbb{P}\Big(\widetilde{H}_{n,i} \geq n^\theta\Big)\,\mathbb{P}\big(\underline{S}_{i_n} < -B\big)$. Let us now denote $\lambda = (\kappa-1)\zeta - \delta$. As $\psi(1+\lambda) < 0$, for any $0 \leq l < i$,

$$\mathbb{P}\Big(\min_{l\leq k\leq i} S_k \leq -B\Big) \leq e^{-\lambda B}\frac{e^{\psi(1+\lambda)l}}{1-e^{\psi(1+\lambda)}}$$

and then

$$p_{3,i} \leq C\left(e^{-\lambda B}n^{-(\kappa-1)\theta} + n^{-2\lambda} + e^{\psi(1+\lambda)A\log n}\right) \leq Ce^{-\lambda B}n^{-(\kappa-1)\theta}\ .$$

Collecting the different upper bounds of the $p_{\cdot,i}$, we obtain for large n,

$$\sum_{m=1}^{5} p^{m,i} \leq Cn^{-(\kappa-1)\theta}\left(n^{-\delta/16} + e^{-\lambda B}\right).$$

Therefore for B and n large enough,

$$\mathbb{E}[\,Z^n\,] \geq \sum_{i=d_n}^{L_n^\delta-\epsilon_n}\Big(\mathbb{P}\big(n^\theta \leq H^S_i\big) - \sum_{m=1}^{5} p_{m,i}\Big) \geq CL_n^\delta n^{-(\kappa-1)\theta} = Cn^{(\kappa-1)(\zeta-\theta)-\delta/2}.$$

• *An upper bound for* $\mathbb{E}\big[(Z^n)^2\big]$. We barely use the same arguments as in the proof of Lemma 3.8: first $\mathbb{E}\big[(Z^n)^2\big]$ can be written as

$$\sum_{i=1}^{L_n^\delta}\mathbb{E}\Big[\sum_{|u|=i} e^{-2V(u)}\Big(\mathbf{1}_{u\in\Gamma_n} + \sum_{\substack{v\neq\tilde{v}\\ v^*=\tilde{v}^*=u}} e^{-V_u(v)-V_u(\tilde{v})}\sum_{x\geq v}\sum_{y\geq\tilde{v}} e^{-V_v(x)}e^{-V_{\tilde{v}}(y)}\mathbf{1}_{x,y\in\Gamma_n}\Big)\Big]\ .$$

Then in the same way as we have obtained (36) we get

$$\mathbb{E}\big[(Z^n)^2\big] \le \sum_{i=1}^{L_n^\delta} \mathbb{E}\big[e^{-S_i}\mathbf{1}_{S_i\ge -B}\big]\left(1+\mathbb{E}\Big[\sum_{\substack{v\neq\tilde v\\ v^*=\tilde v^*=e}} e^{-V(v)-V(\tilde v)}\Big]\Big(\sum_{k=0}^{L_n^\delta-i-1}\phi_{k,\zeta}\Big)^2\right)$$

and same arguments as in the proof of Proposition 3.8 (see Equation (37) and below) show that

$$\mathbb{E}\big[(Z^n)^2\big] \le Cn^{2(\kappa-1)(\zeta-\theta)-\delta}\sum_{i=1}^{L_n^\delta}\mathbb{E}\big[e^{-S_i}\mathbf{1}_{S_i\ge -B}\big]\,.$$

So all that is left to do is to control the first part of the sum, again by a Markov inequality: as $\psi(2\wedge\kappa-\delta)<0$,

$$\sum_{i=1}^{L_n^\delta}\mathbb{E}\big[e^{-S_i}\mathbf{1}_{S_i\ge -B}\big] \le e^B\sum_{i=1}^{L_n^\delta}\mathbb{E}\Big[e^{-(1\wedge(\kappa-1)-\delta)S_i}\Big] \le e^B\sum_{i=1}^{\infty}e^{i\psi(2\wedge\kappa-\delta))}<\infty$$

and finally $\mathbb{E}\big[(Z^n)^2\big]\le Cn^{2(\kappa-1)(\zeta-\theta)-\delta}$. The bounds obtained for $\mathbb{E}[Z^n]$, $\mathbb{E}\big[(Z^n)^2\big]$ and Inequality (39) conclude the proof. □

Proof of Proposition 3.4. We only have to collect all the previous results. Thanks to Lemma 3.5 we can separate the excursions and, as $\kappa\wedge 2-\kappa\theta = 1-\theta+(\kappa-1)(\zeta-\theta)$, thanks to Lemma 3.9 we can introduce the variable $\widetilde\Sigma_n := \sum_{|x|\le L_n^\delta} e^{-V(x)}\mathbf{1}_{x\in\Gamma_n}$. So these two Lemmata imply

$$\begin{aligned}&\mathbb{P}\Big(R_{n^\theta}^{(n)}\le n^{\kappa\wedge 2-\kappa\theta-\delta}\Big)\le n^{-1}+n^{-\epsilon}\\ &+\mathbb{P}\Big(\sum_{i=1}^{n}\sum_{|x|\le L_n^\delta}\mathbf{1}_{N_x^{(i)}-N_x^{(i-1)}\ge n^\theta}\mathbf{1}_{x\in\Gamma_n}\le n^{1-\theta-3\delta/8}\widetilde\Sigma_n\ ,\ \widetilde\Sigma_n\ge n^{(\kappa-1)(\zeta-\theta)-5\delta/8}\Big)\end{aligned}$$

for some constant $\epsilon>0$. Using now the quenched concentration Lemma 3.6, the probability of the right-hand side of the previous equation can be bounded by

$$\begin{aligned}&\frac{C}{n^{1-\delta/8}}\mathbb{E}\left[\Big(\widetilde\Sigma_n\Big)^{-2}\sum_{|x|,|y|\le L_n^\delta}H_{x\wedge y}e^{-V(x)-V(y)+V(x\wedge y)}\mathbf{1}_{x,y\in\Gamma_n}\mathbf{1}_{\widetilde\Sigma_n\ge n^{(\kappa-1)(\zeta-\theta)-5\delta/8}}\right]\\ \le& Cn^{-1-2(\kappa-1)(\zeta-\theta)-11\delta/8}G_n\ .\end{aligned}$$

And finally, the bound of G_n given in Lemma 3.8 shows that

$$\mathbb{P}\Big(R_{n^\theta}^{(n)}\le n^{\kappa\wedge 2-\kappa\theta-\delta}\Big)\le\frac{1}{n}+\frac{1}{n^\epsilon}+\frac{1}{n^{\delta/16}}\ .$$

This concludes the proof of the proposition. □

4. Estimation of the c.d.f.

In this section we mainly prove Theorem 1.7. For that purpose, we use the same global strategy as in [DL17], that is to say we begin by estimating the moments of ρ (defined in (10)) and we then use these estimators to build the family of estimators $\widehat F_n^\alpha$ of the c.d.f. The important difference comparing to [DL17] is that in our case the state space is now a Galton-Watson tree instead of $\mathbb{Z}$. Recall that in this part, we assume not only A1 but also A2:

- the reproduction law of the Galton-Watson is bounded: $\exists K > 0, \quad \mathbb{P}(\nu \leq K) = 1$.
- given the tree up to generation n and the number of children ν_x of some $x \in \mathbb{T}$ such that $|x| = n$, the variables $(\omega_{x_i})_{1 \leq i \leq \nu_x}$ are i.i.d. with the same distribution as some variable ω.

4.1. Estimation of the moments of ρ

First remark that the marked tree $(x \in \mathbb{T},\ N_x^{(n)})$ is a sufficient statistic for the trajectory $(X_t)_{t \in [0, T^{(n)}]}$. Indeed for any admissible sequence $(a_k)_{k \in \{0, \dots, t^{(n)}\}}$, the likelihood is

$$\mathbb{P}\big(X_0 = a_0, \dots, X_{T^{(n)}} = a_{t^{(n)}} \big| \mathbb{T}\big) = \prod_{x \in \mathbb{T}} \mathbb{E}\left[\left(\frac{1}{1 + \sum_{i=1}^{\nu_x} e^{-\omega_{x_i}}} \right)^{n_x^{(n)}} \prod_{i=1}^{\nu_x} \left(\frac{e^{-\omega_{x_i}}}{1 + \sum_{i=1}^{\nu_x} e^{-\omega_{x_i}}} \right)^{n_{x_i}^{(n)}} \Big| \mathbb{T} \right]$$

where $n_x^{(n)} := \sum_{k=1}^{t^{(n)}} \mathbf{1}_{\{a_{k-1} = x^*,\ a_k = x\}}$. It is then natural to construct our estimator using these random variables. A second important point is that, for fixed environment and tree, that is to say under $\mathbb{P}^{\mathcal{E}}$, for any $x \in \mathbb{T}$,

$$\mathbb{P}^{\mathcal{E}}\left(N_x^{(n)} = j \Big| \left(N_y^{(n)} \right)_{|y| < |x|} \right) = \binom{N_{x^*}^{(n)} + j - 1}{j} \rho_x^i (1 - \rho_x)^j$$

where $\rho_x = (1 + e^{-\omega_x})^{-1}$. So the moments of ρ are convenient quantities to estimate and we first focus on them. The estimator of $m^{\alpha,\beta} = \mathbb{E}\big[\rho^\alpha (1-\rho)^\beta\big]$ is constructed following the same strategy as in [DL17]. Let us first introduce the function $\Phi_{\alpha,\beta}$,

$$\forall \alpha, \beta, i, j \geq 0, \quad \Phi_{\alpha,\beta}(i,j) = \mathbf{1}_{i \geq \alpha+1, j \geq \beta} \frac{\binom{i+j-(\alpha+1+\beta)}{i-(\alpha+1)}}{\binom{i+j-1}{j}} \ . \tag{40}$$

To estimate the moments $m^{\alpha,\beta}$ of ρ, we use the following random variable:

$$\widehat{m}_n^{\alpha,\beta} = \frac{1}{\mathbb{E}[\nu]\, R_{\alpha+1}^{(n)}} \sum_{x \in \mathbb{T}} \sum_{\substack{y, s.t. \\ y^* = x}} \Phi_{\alpha,\beta}(N_x^{(n)}, N_y^{(n)})$$

where we use the convention: $0/0 = 0$. Remark that the series here are in fact simple sums as only a finite number of $N_x^{(n)}$ are non zero. These estimators satisfy a concentration property:

Proposition 4.1. *Assume A1 and A2 and fix $\alpha, \beta \in \mathbb{Z}_+$. There is a positive constant C, such that for any integer $n \geq 0$ and any real number $z > 0$,*

$$\mathbb{P}\left(\big|\widehat{m}_n^{\alpha,\beta} - m^{\alpha,\beta}\big| \geq \frac{\alpha!\beta!}{(\alpha+\beta)!} \frac{K}{\mathbb{E}[\nu]} \sqrt{\frac{z + 2\log R_{\alpha+1}^{(n)}}{2R_{\alpha+1}^{(n)}}} \right) \leq C e^{-z} \ .$$

Proof. We first introduce the sequence $(y_k)_{k \in \mathbb{N}}$ which designates the vertices of the tree $\mathbb{T} \cup \{e^*\}$ in level order, generation by generation: we start with $y_0 = e^*$, $y_1 = e$ and then we move to the first generation, the first vertex (on the left) is called y_2, the second one y_3 and so on. Once no more vertices are present at this generation, we move to the next one and so on (see Figure 3). Using the sequence $(y_k)_{k \in \mathbb{N}}$, the estimators $\widehat{m}_n^{\alpha,\beta}$ have the following expression

$$\widehat{m}_n^{\alpha,\beta} = \frac{1}{\mathbb{E}[\nu]\, R_{\alpha+1}^{(n)}} \sum_{k=1}^{\infty} \sum_{\substack{y, s.t. \\ y^* = y_k}} \Phi_{\alpha,\beta}(N_{y_k}^{(n)}, N_y^{(n)}) \ .$$

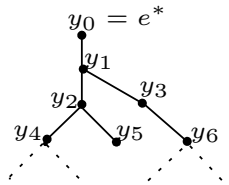

Figure 3: Vertices numbering

Denote by $(\mathcal{F}_k)_{k\in\mathbb{N}}$ the following filtration: $\mathcal{F}_0 = \sigma\left(N_e^{(n)}\right)$ and for $k \geq 0$,

$$\mathcal{F}_{k+1} = \sigma\left(\mathcal{F}_k,\ \sigma\left(\nu_{y_{k+1}},\ (N_y^{(n)}, y^* = y_{k+1})\right)\right)\ . \tag{41}$$

In the following lemma we introduce a martingale which the main tool to obtain Proposition 4.1.

Lemma 4.2. *Let $\alpha, \beta \in \mathbb{Z}_+$. For probability measure $\mathbb{P}$, the sequence $\left(Z_k^{\alpha,\beta}\right)_{k\geq 0}$ defined, for all integer $k \geq 0$, by*

$$Z_0^{\alpha,\beta} = 0 \quad \textit{and} \quad \forall k \geq 1,\ Z_k^{\alpha,\beta} = \mathbf{1}_{N_{y_k}^{(n)} \geq \alpha+1}\Big(\sum_{y, y^* = y_k} \Phi_{\alpha,\beta}\left(N_{y_k}^{(n)}, N_y^{(n)}\right) - \mathbb{E}[\nu]\, m^{\alpha,\beta}\Big)$$

are martingale difference sequence with respect to $(\mathcal{F}_k)_{k\in\mathbb{N}}$.

Proof. First recall the following result which is a slight variation of Lemma 3.1 in [AdR17], which can be proved using the same arguments: for any $n \geq 1$, under $\mathbb{P}$ the marked tree $\left(x \in \mathbb{T},\ N_x^{(n)}\right)$ is a multi-type Galton-Watson tree. Its initial type is n and its mean matrix is given by:

$$\begin{aligned} \forall i, j \geq 0, \quad m_{i,j} = \mathbb{E}\Big[\sum_{|x|=1} \mathbf{1}_{N_x^{(i)}=j}\Big] &= \binom{i-1+j}{j}\, \mathbb{E}\Big[\sum_{|x|=1} \frac{e^{-j\omega_x}}{(1+e^{-\omega_x})^{i+j}}\Big] \\ &= \binom{i-1+j}{j}\, \mathbb{E}[\nu]\, \mathbb{E}\Big[(1-\rho)^j\, \rho^i\Big] \end{aligned} \tag{42}$$

Now, $Z_k^{\alpha,\beta}$ is obviously an integrable and $\mathcal{F}_k$-measurable random variable. Moreover, according to (42), for any $i \geq \alpha+1$ and any $k \geq 1$,

$$\begin{aligned} &\mathbb{E}\Big[\mathbf{1}_{N_{y_k}^{(n)}=i} \sum_{x, x^*=y_k} \Phi_{\alpha,\beta}\left(N_{y_k}^{(n)}, N_x^{(n)}\right) \Big| \mathcal{F}_{k-1}\Big] \\ &= \mathbf{1}_{N_{y_k}^{(n)}=i}\, \mathbb{E}\Big[\sum_{|x|=1} \Phi_{\alpha,\beta}\left(i, N_x^{(i)}\right)\Big] = \mathbf{1}_{N_{y_k}^{(n)}=i} \sum_{j=0}^{\infty} \Phi_{\alpha,\beta}(i,j)\, \mathbb{E}\Big[\sum_{|x|=1} \mathbf{1}_{N_x^{(i)}=j}\Big] \\ &= \mathbf{1}_{N_{y_k}^{(n)}=i} \sum_{j=\beta}^{\infty} \binom{i-(\alpha+1)+j-\beta}{i-(\alpha+1)} \mathbb{E}[\nu]\, \mathbb{E}\Big[(1-\rho)^j\, \rho^i\Big] = \mathbf{1}_{N_{y_k}^{(n)}=i} \mathbb{E}[\nu]\, m^{\alpha,\beta}\ . \end{aligned}$$

From this last equality we easily obtain $\mathbb{E}\left[Z_k^{\alpha,\beta} \middle| \mathcal{F}_{k-1}\right] = 0$ which completes the proof of the lemma. □

We can now prove Proposition 4.1. We only have to show that:

$$\forall z>0,\ \mathbb{P}\left(\left|\sum_{k=1}^{\infty} Z_k^{\alpha,\beta}\right| \geq \frac{\alpha!\beta!}{(\alpha+\beta)!)} K\sqrt{R_{\alpha+1}^{(n)}\left(z+2\log R_{\alpha+1}^{(n)}\right)}\right) \leq Ce^{-z}\ . \tag{43}$$

We know, according to Lemma 4.2 that the process $(M_k)_{k\in\mathbb{N}}$ defined by

$$M_0=0 \ \text{ and } \ \forall k\leq 1,\ M_k=\sum_{i=1}^{k} Z_i^{\alpha,\beta}$$

is a martingale with respect to $(\mathcal{F}_k)_{k\in\mathbb{N}}$. We could directly apply a concentration inequality but to obtain a better bound we first remark that some of the increments $Z_k^{\alpha,\beta}$ are zero so we consider the sequence of stopping times (with respect to the filtration $(\mathcal{F}_k)_{k\in\mathbb{N}}$):

$$\tau_0=0 \text{ and } \forall m\geq 1,\ \tau_{m+1}=\inf\left\{k>\tau_m,\ N_{y_l}^{(n)}\geq \alpha+1\right\} \tag{44}$$

where $\inf\varnothing=\infty$. As the other variables$Z_k^{\alpha,\beta}$ are null, we have that $M_{\tau_m}=\sum_{i=1}^{m} Z_{\tau_i}^{\alpha,\beta}$ and $R_{\alpha+1}^{(n)}=\sum_{m=1}^{\infty}\mathbf{1}_{\tau_m<\infty}$. Now an elementary combinatoric argument shows that, for $i\geq\alpha+1$ and $j\geq\beta$,

$$\binom{i-1+j-\alpha-\beta}{i-1-\alpha}\binom{\alpha+\beta}{\alpha}\leq\binom{i-1+j}{i-1}\ .$$

Thus, for any $i,j\geq 0$,

$$0\leq\Phi_{\alpha,\beta}(i,j)\leq\binom{\alpha+\beta}{\alpha}^{-1}=\Phi_{\alpha,\beta}(\alpha+1,\beta)\leq 1$$

and for any $m\geq 0$, $|Z_m^{\alpha,\beta}|\leq K$ where K is the upper bound for the support of ν as described in Theorem 1.7. Moreover, for any $m\in\mathbb{N}$, the stopped process $(M_{\tau_m\wedge l})_{l\geq 0}$ is still a martingale. For any $m\in\mathbb{N}$, $|M_{\tau_m\wedge l}|\leq mK$. Therefore the stopped martingale $(M_{\tau_m\wedge l})_{l\geq 0}$ is uniformly integrable and Doob's optional sampling theorem implies that $(M_{\tau_m})_{m\in\mathbb{N}}$ is a martingale. As for any $m\geq 0$,

$$M_{\tau_{m-1}}-\mathbb{E}[\nu]\,m^{\alpha,\beta}\leq M_{\tau_m}\leq M_{\tau_{m-1}}-\mathbb{E}[\nu]\,m^{\alpha,\beta}+K,$$

Mc Diarmid's inequality (see Theorem 6.7 in [McD89]) leads now to the following concentration result: for any integer $m\geq 0$ and any real $z>0$,

$$\mathbb{P}\Big(|M_{\tau_m}|\geq\binom{\alpha+\beta}{\alpha}^{-1}K\sqrt{mz/2}\Big)\leq 2e^{-z}\ .$$

And a union bound concludes the proof:

$$\begin{aligned}
&\mathbb{P}\left(\Big|\sum_{m=1}^{\infty} Z_m^{\alpha,\beta}\Big|\geq\frac{\alpha!\beta!}{(\alpha+\beta)!)}\sqrt{R_{\alpha+1}^{(n)}\left(z+2\log R_{\alpha+1}^{(n)}\right)}\right)\\
=&\mathbb{P}\left(\bigcup_{m=1}^{\infty}\left\{R_{\alpha+1}^{(n)}=m\ ;\ |M_{\tau_m}|\geq\binom{\alpha+\beta}{\alpha}^{-1}K\sqrt{m\frac{z+2\log m}{2}}\right\}\right)\\
\leq&\sum_{m=1}^{\infty}\mathbb{P}\left(|M_{\tau_m}|\geq\binom{\alpha+\beta}{\alpha}^{-1}K\sqrt{m\frac{z+2\log m}{2}}\right)\leq 2e^{-z}\sum_{m\geq 1}e^{-2\log m}=\frac{\pi^2}{3}e^{-z}\ .
\end{aligned}$$

□

4.2. Estimation of the cumulative distribution function

In this section we prove Theorem 1.7. For that purpose, we use the estimation of the moments $m^{\alpha,\beta}$ to approximate the cumulative distribution function F of ρ. It is a straightforward consequence of Lemma 4.3 and Lemma 4.4.

Define for any $u \in [0,1]$ and any $\alpha \in \mathbb{N}^*$,

$$F^\alpha(u) = \sum_{k=0}^{\lfloor \alpha u \rfloor - 1} \binom{\alpha-1}{k} m^{k,\alpha-1-k} \ ,$$

where $x \to \lfloor x \rfloor$ is the floor function and $\sum_{k=0}^{-1} = 0$. When the function F is regular enough, it is a classical result that it is well approximate by F^α. A precise statement is given in Lemma 6 of [DL17] which we recall here.

Lemma 4.3. *Suppose that the function $F \in \mathcal{C}^\gamma$ for some $\gamma \in (0,2]$. For any integer $\alpha \geq 1$,*

$$\max_{0 \leq \ell \leq \alpha} |F(\ell/\alpha) - F^\alpha(\ell/\alpha)| \leq \frac{\|F\|_\gamma}{2^\gamma(\alpha+1)^{\gamma/2}} \ .$$

According to the results of the previous section, it is then natural to use the estimator defined in (11):

$$\widehat{F}_n^\alpha(u) = \frac{1}{R_\alpha^{(n)} \mathbb{E}[\nu]} \sum_{x \in \mathbb{T}} \psi_\alpha^{\lfloor \alpha u \rfloor}\left(N_{x^*}^{(n)}, N_x^{(n)}\right) \text{ with } \psi_\alpha^l(i,j) = \frac{\mathbf{1}_{i \geq \alpha}}{\binom{i-1+j}{\alpha-1}} \sum_{k=0}^{l-1} \binom{i-1}{k}\binom{j}{\alpha-1-k}.$$

Indeed, for any $i \geq \alpha$, $\psi_\alpha^l(i,j) = \sum_{k=0}^{l-1} \binom{\alpha}{k} \Phi_{k,\alpha-1-k}(i,j)$, where $\Phi_{k,\alpha-1-k}$ is defined in (40). Thus, this estimator is essentially the one obtained from the moment estimators of the previous subsection, but using only the sites x satisfying $N_x^{(n)} \geq \alpha$. Notice that for any $1 \leq l \leq \alpha$ and $i \geq \alpha, j \geq 0$,

$$\sum_{k=0}^{l-1} \binom{i}{k}\binom{j}{\alpha-1-k} \leq \sum_{k=0}^{\alpha-1} \binom{i-1}{k}\binom{j}{\alpha-1-k} = \binom{i-1+j}{\alpha-1} \ ,$$

thus $\psi_\alpha^l \in [0,1]$. Moreover, Vandermonde's identity:

$$\forall i,j \geq 0, \quad \sum_{k=0}^{\alpha-1} \binom{i-1}{k}\binom{j}{\alpha-1-k} = \binom{i-1+j}{\alpha-1}$$

shows that any $\widehat{F}_n^\alpha$ is a (random) c.d.f. We now have to prove that $\widehat{F}_n^\alpha$ estimates correctly F^α. This is done in the following lemma.

Lemma 4.4. *For any integers $\alpha, n \geq 1$ and any real $z > 0$,*

$$\mathbb{P}\left(\max_{0 \leq \ell \leq \alpha} \left| \widehat{F}_n^\alpha(\ell/\alpha) - F^\alpha(\ell/\alpha) \right| \geq \frac{K}{\mathbb{E}[\nu]} \sqrt{\frac{z + \log \alpha + 2 \log R_\alpha^{(n)}}{2 R_\alpha^{(n)}}} \right) \leq C e^{-z} \ .$$

Proof. The proof follows the same lines as the one of Proposition 4.1. We introduce the sequence:

$$\forall 0 \leq \ell \leq \alpha,\ \forall k \in \mathbb{N}, \quad Y_k^{\alpha,\ell} = \mathbf{1}_{N_{y_k}^{(n)} \geq \alpha} \left(\sum_{y, y^* = y_k} \psi_\alpha^\ell \left(N_{y_k}^{(n)}, N_y^{(n)} \right) - \mathbb{E}[\nu]\, F^\alpha\left(\ell/\alpha\right) \right) .$$

Using same kind of calculus as in the proof of Lemma 4.2, we can show that $\left(Y_k^{\alpha,\ell}\right)_{k\in\mathbb{N}}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_k)_{k\in\mathbb{N}}$ defined in (41). Lemma 4.4 is equivalent to

$$\forall z > 0,\ \mathbb{P}\left(\max_{1\leq l\leq \alpha} \left| \sum_{k\in\mathbb{N}} Y_k^{\alpha,\ell} \right| \geq K \sqrt{\frac{R_\alpha^{(n)}}{2}(z + \log\alpha + 2\log R_\alpha^{(n)})} \right) \leq C e^{-z} .$$

So we now consider the martingale $\left(M_k^{\alpha,l}\right)_{k\in\mathbb{N}}$ defined by $M_k^{\alpha,l} := \sum_{j\leq k} Y_j^{\alpha,l}$ and the same sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$ as the one defined in (44) except that $\alpha+1$ is replaced by α:

$$\tau_0 = 0 \text{ and } \forall m \geq 1,\ \tau_{m+1} = \inf\left\{ l > \tau_m,\ N_{y_l}^{(n)} \geq \alpha \right\} .$$

The process $\left(M_{\tau_m}^{\alpha,l}\right)_{m\in\mathbb{N}}$ is still a martingale and for any $m \geq 1$,

$$M_{\tau_{m-1}}^{\alpha,l} - \mathbb{E}[\nu]\, F^\alpha\left(\ell/\alpha\right) \leq M_{\tau_m}^{\alpha,l} \leq M_{\tau_{m-1}}^{\alpha,l} - \mathbb{E}[\nu]\, F^\alpha\left(\ell/\alpha\right) + K .$$

Then Mc Diarmid's inequality (Theorem 6.7 in [McD89]) yields, for any $1 \leq \ell \leq \alpha$,

$$\forall k \in \mathbb{N},\ \forall z > 0, \quad \mathbb{P}\left(\left| M_k^{\alpha,\ell} \right| \geq K\sqrt{\frac{kz}{2}} \right) \leq 2e^{-z}$$

and a union bound gives the result:

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq \ell\leq \alpha} \left| \sum_{k\in\mathbb{N}} Y_k^{\alpha,\ell} \right| \geq K\sqrt{\frac{R_\alpha^{(n)}}{2}(z + \log\alpha + 2\log R_\alpha^{(n)})} \right) \\
=&\mathbb{P}\left(\bigcup_{k=1}^{\infty} \left\{ R_\alpha^{(n)} = k\ ;\ \max_{1\leq \ell\leq \alpha} \left| M_k^{\alpha,\ell} \right| \geq K\sqrt{\frac{k}{2}(z + \log\alpha + 2\log k)} \right\} \right) \\
\leq &\sum_{k=1}^{\infty} \sum_{\ell=1}^{\alpha} \mathbb{P}\left(\left| M_k^{\alpha,\ell} \right| \geq K\sqrt{\frac{k}{2}(z + \log\alpha + 2\log k)} \right) \leq 2e^{-z} \sum_{k\geq 1} \sum_{\ell=1}^{\alpha} \frac{e^{-2\log k}}{\alpha} = \frac{\pi^2}{3} e^{-z} .
\end{aligned}$$

□

The inequality of Theorem 1.7 follows now immediately from Lemma 4.3, Lemma 4.4 and the fact that F is $\gamma \wedge 1$-Hölder.

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