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## Appendix

### Alzheimer's Disease Modelling and Staging through Independent Gaussian Process Analysis of Spatio-Temporal Brain Changes

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#### A. Lower bound derivation

In this section we detail the derivation of the lower bound :

$$\begin{aligned} \log(p(Y, C | \sigma, \lambda)) &= \log\left[\int_A \int_S \int_{S'} p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S, S'|\lambda)dAdSdS'\right] \\ &= \log\left[\int_A \int_S \int_{S'} p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S'|S, \lambda)p(S)dAdSdS'\right] \end{aligned} \quad (1)$$

If we know S this completely determines S', thus we have  $\int p(S'|S, \lambda)dS' = 1$  which gives us :

$$\begin{aligned} \log(p(Y, C | \sigma, \lambda)) &= \log\left[\int_A \int_S p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)dAdS\right] \\ &= \log\left[\int_A \int_S p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S) \frac{q_1(A)q_2(S)}{q_1(A)q_2(S)} dAdS\right] \\ &= \log\left[\mathbb{E}_{A \sim q_1, S \sim q_2} \left[\frac{p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)}{q_1(A)q_2(S)}\right]\right] \\ &\geq \mathbb{E}_{A \sim q_1, S \sim q_2} \left[\log\left[\frac{p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)}{q_1(A)q_2(S)}\right]\right] \end{aligned} \quad (2)$$

This is obtained thanks to Jensen's inequality. Finally this leads us to :

$$\begin{aligned} \mathbb{E}_{A \sim q_1, S \sim q_2} \left[\log\left[\frac{p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)}{q_1(A)q_2(S)}\right]\right] &= \mathbb{E}_{A \sim q_1, S \sim q_2} [\log[p(Y|A, S, \sigma)]] \\ &\quad + \mathbb{E}_{S \sim q_2} [\log(P(C|S', \lambda))] \\ &\quad - \mathcal{D}[q_1(A|Y)||p(A)] \\ &\quad - \mathcal{D}[q_2(S|Y)||p(S)] \end{aligned} \quad (3)$$

## Independent Gaussian Process Analysis

In the Method section we introduced the approximation  $q_1(A) = \prod_{n=1}^{Ns} \mathcal{N}(\mu_n, \Sigma(\alpha, \beta))$ .

The covariance matrix is shared by all the spatial processes which gives us the set of spatial parameters :

$$\psi = \{\mu_n, n \in [1, Ns], \alpha, \beta\} \quad (4)$$

Following [5] we introduce for each GP two vectors,  $\Omega_n$  with a prior  $p(\Omega_n) = \mathcal{N}(0, \frac{1}{l_n}I)$  for each element and  $W_n$  with a prior  $p(W_n) = \mathcal{N}(0, I)$ , such that  $S_n(t) = \Phi(t\Omega_n)W_n$ . Where  $\Phi$  is chosen to obtain a RBF kernel as explained in [5]. We define the approximated distributions  $q_3(W_n) = \prod_j \mathcal{N}(m_{n,j}, s_{n,j}^2)$  and  $q_4(\Omega_n) = \prod_j \mathcal{N}(\alpha_{n,j}, \beta_{n,j}^2)$  of  $p(W_n)$  and  $p(\Omega_n)$ . Using these approximations and following [5], we can derive a lower bound for S with the same technique than above. We have the set of temporal parameters :

$$\theta = \{m_n, s_n, \alpha_n, \beta_n, l_n, n \in [1, Ns]\} \quad (5)$$

Now we can obtain every term of (3). The Kullback-Leibler of a multivariate Gaussian has a closed-form :

$$\mathcal{D}[q_1(A|X)||p(A)] = \frac{1}{2} \sum_{n=1}^{Ns} Tr(\Sigma) + \mu_n^T \mu_n - F - \log[\det(\Sigma)] \quad (6)$$

Using the factorized form of  $q_2$  and the fact that the different Gaussian processes are independent from each other we can write :

$$\mathcal{D}[q_2(S|X)||p(S)] = \sum_{n=1}^{Ns} \mathcal{D}[q_3(W_n)|p(W_n)] + \mathcal{D}[q_4(\Omega_n)|p(\Omega_n)] \quad (7)$$

Since the approximations  $q_3$  and  $q_4$  and their respective priors are normally distributed we have an analytic formula for both Kullback-Leibler divergences.

$$\mathcal{D}[q_3(W_n)|p(W_n)] = \frac{1}{2} \sum_j s_{n,j}^2 + \mu_{n,j}^2 - 1 - \log(s_{n,j}^2) \quad (8)$$

$$\mathcal{D}[q_4(\Omega_n)|p(\Omega_n)] = \frac{1}{2} \sum_j \beta_{n,j}^2 l_n + \alpha_{n,j}^2 l_n - 1 - \log(\beta_{n,j}^2 l_n) \quad (9)$$

As in [10] we employ the reparameterization trick to have an efficient way of sampling the expectations of (3). Thus we have :

$$\begin{aligned} - W_{n,j} &= m_{n,j} + s_{n,j} * \epsilon_{n,j} \\ - \Omega_{n,j} &= \alpha_{n,j} + \beta_{n,j} * \zeta_{n,j} \\ - A_n &= \mu_n + \Sigma_n^{\frac{1}{2}} * \kappa_n \end{aligned}$$

Which gives us :

$$\mathbb{E}_{A \sim q_1, S \sim q_2}[\log(p(Y|A, S, \sigma))] = \mathbb{E}_{\epsilon, \zeta, \kappa}[\log(p(Y|m, s, \alpha, \beta, \mu, \Sigma, \sigma))] \quad (10)$$

$$\mathbb{E}_{S \sim q_2}[\log(p(C|S', \lambda))] = \mathbb{E}_{\epsilon, \zeta}[\log(p(C|m, s, \alpha, \beta, \lambda))] \quad (11)$$

Where  $\epsilon_{n,j} \sim \mathcal{N}(0, 1)$ ,  $\zeta_{n,j} \sim \mathcal{N}(0, 1)$  and  $\kappa_n \sim \mathcal{N}(0, I)$ .

**B. Kronecker factorization**

Here we detail how to split the covariance matrix in a Kronecker product of three matrices along each spatial dimensions. We have :

$$\Sigma_{i,j}(\alpha, \beta) = \alpha \exp\left(-\frac{\|u_i - u_j\|^2}{2\beta}\right) \quad (12)$$

We can use the separability properties of the exponential to decompose the covariance between two locations  $u_i = (x_i, y_i, z_i)$  and  $u_j = (x_j, y_j, z_j)$  :

$$\Sigma_{i,j}(\alpha, \beta) = \alpha \exp\left(-\frac{(x_i - x_j)^2}{2\beta}\right) \exp\left(-\frac{(y_i - y_j)^2}{2\beta}\right) \exp\left(-\frac{(z_i - z_j)^2}{2\beta}\right) \quad (13)$$

So  $\Sigma$  can be decomposed into the Kronecker product of 1D processes :

$$\Sigma = \Sigma_x \otimes \Sigma_y \otimes \Sigma_z \quad (14)$$

Allowing us to deal with large-size matrices.

### C. Comparison with ICA

We performed a comparison of our algorithm with ICA on a similar example than in 3.1. However the data was generated in a simplified setting since ICA can't be applied when the time associated to each image is unknown. To do so we assigned the ground truth parameter  $t_p$  beforehand. The goal was to compare the separation performances of both our algorithm and ICA, on data generated with three latent spatio-temporal processes. In Figure 1 we observe that the sources estimated by ICA are more noisy and uncertain than the ones estimated by our method, highlighting the performances of our algorithm in terms of sources separation.

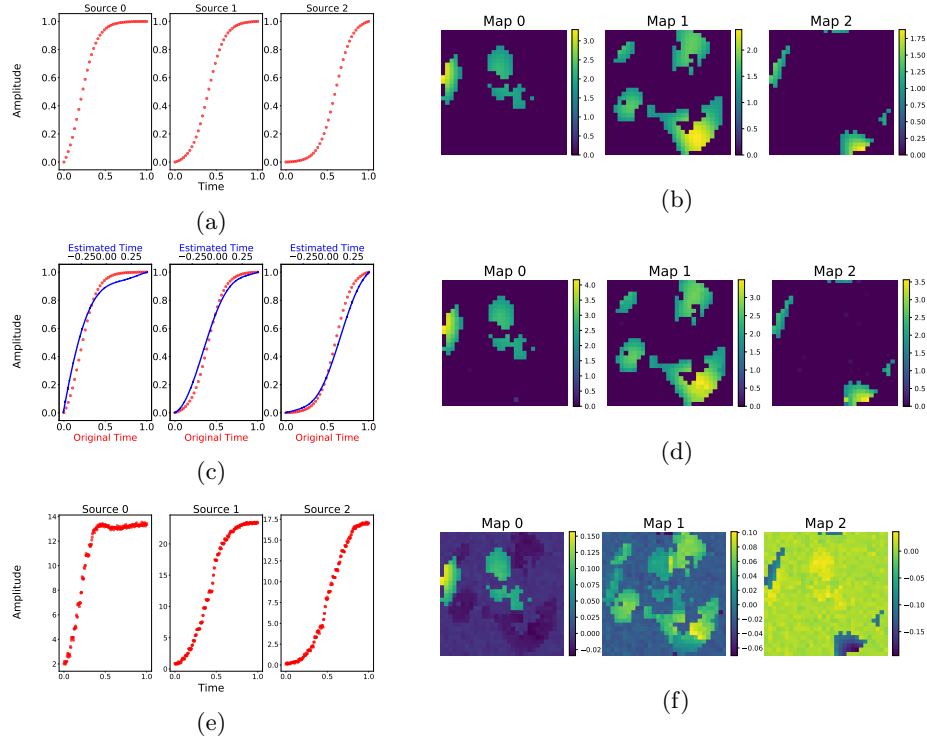


Fig. 1: First Row : Raw sources. Second Row : Sources estimated by our method. Third Row : Sources estimated by ICA.

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