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► **To cite this version:**

Théo Winterhalter, Matthieu Sozeau, Nicolas Tabareau. Eliminating Reflection from Type Theory. 2018. hal-01849166v1

HAL Id: hal-01849166

<https://hal.science/hal-01849166v1>

Preprint submitted on 25 Jul 2018 (v1), last revised 22 Nov 2018 (v3)

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Eliminating Reflection from Type Theory

To the Legacy of Martin Hofmann

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Type theories with equality reflection, such as extensional type theory (ETT), are convenient theories in which to formalise mathematics, as they make it possible to consider provably equal terms as convertible. Although type-checking is undecidable in this context, variants of ETT have been implemented, for example in NuPRL and more recently in Andromeda. The actual objects that can be checked are not proof-terms, but derivations of proof-terms. This suggests that any derivation of ETT can be translated into a typecheckable proof term of intensional type theory (ITT). However, this result, investigated categorically by Hofmann in 1995, and 10 years later more syntactically by Oury, has never given rise to an effective translation. In this paper, we provide the first syntactical translation from ETT to ITT with uniqueness of identity proofs and functional extensionality. This translation has been defined and proven correct in Coq and yields an executable plugin that translates a derivation in ETT into an actual Coq typing judgment. Additionally, we show how this result is extended in the context of homotopy to a two-level type theory.

CCS Concepts: • **Theory of computation** → **Logic; Proof theory; Constructive mathematics; Type theory**; *Logic and verification; Automated reasoning*;

Additional Key Words and Phrases: dependent types, translation, formalisation

ACM Reference Format:

Théo Winterhalter, Matthieu Sozeau, and Nicolas Tabareau. 2018. Eliminating Reflection from Type Theory: To the Legacy of Martin Hofmann. 1, 1 (July 2018), 26 pages. <https://doi.org/10.1145/nnnnnnn.nnnnnnn>

1 INTRODUCTION

Type theories with equality reflection, such as extensional type theory (ETT), are convenient theories in which to formalise mathematics, as they make it possible to consider provably equal terms as convertible, as expressed in the following typing rule:

$$\frac{\Gamma \vdash_x e : u =_A v}{\Gamma \vdash_x u \equiv v : A}$$

Here, the type $u =_A v$ is Martin-Löf's identity type with only one constructor $\text{refl } u : u =_A u$ which represents proofs of equality inside type theory, whereas $u \equiv v : A$ means that u and v are convertible in the theory—and can thus be silently replaced by one another in any term. Several variants of ETT have been considered and implemented, for example in NuPRL [Allen et al. 2000] and more recently in Andromeda [Bauer et al. 2016]. The prototypical example of the use of equality

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XXXX-XXXX/2018/7-ART

<https://doi.org/10.1145/nnnnnnn.nnnnnnn>

reflection is the definition of a coercion function between two types A and B that are equal (but not convertible) by taking a term of type A and simply returning it as a term of type B :

$$(\lambda (A B : \text{Type}) (e : A = B) (x : A). x) : \Pi(A B : \text{Type}). A = B \rightarrow A \rightarrow B.$$

In intensional type theory (ITT), this term does not type-check because x of type A can not be given the type B by conversion. In ETT, however, equality reflection can be used to turn the witness of equality into a proof of conversion and thus the type system validates the fact that x can be given the type B . This means that one needs to guess equality proofs during type-checking, because the witness of equality has been lost at the application of the reflection rule. Guessing it was not so hard in this example but is in general undecidable, as one can for instance encode the halting problem of any Turing machine as an equality in ETT. That is, the actual objects that can be checked in ETT are not terms, but instead derivations of terms. It thus seems natural to wonder whether any derivation of ETT can be translated into a typecheckable term of ITT. And indeed, it is well known that one can find a corresponding term of the same type in ITT by *explicitly* transporting the term x of type A using the elimination of internal equality on the witness of equality e , noted e_* :

$$\lambda (A B : \text{Type}) (e : A = B) (x : A) \Rightarrow e_* x : \Pi(A B : \text{Type}). A = B \rightarrow A \rightarrow B.$$

This can be seen as a way to make explicit the *silent* use of reflection. Furthermore, by making the use of transport as economic as possible, the corresponding ITT term can be seen as a compact witness of the derivation tree of the original ETT term.

This result has first been investigated categorically in the pioneering work of Hofmann [1995, 1997], by showing that the term model of ITT can be turned into a model of ETT by quotienting this model with propositional equality. However, it is not clear how to extend this categorical construction to an explicit and constructive translation from a derivation in ETT to a term of ITT. In 2005, this result has been investigated more syntactically by Oury [2005]. However, his presentation does not give rise to an effective translation and requires some additional axioms in ITT, including one that has no clear motivation. By an *effective* translation we mean that it is entirely constructive and can be used to deterministically *compute* the translation of a given ETT typing derivation.

Contributions. In this paper, we present the first syntactical translation from ETT to ITT (assuming uniqueness of identity proof (UIP) and function extensionality in ITT). By syntactical translation, we mean an explicit translation from a derivation $\Gamma \vdash_x t : T$ of ETT (the x index testifies that it is a derivation in ETT) to a context Γ' , term t' and type T' of ITT such that $\Gamma' \vdash t' : T'$ in ITT. This translation enjoys the additional property that if T can be typed in ITT, *i.e.*, $\Gamma \vdash T$, then $T' \equiv T$. This means in particular that a theorem proven in ETT but whose statement is also valid in ITT can be automatically transferred to a theorem of ITT. For instance, one could use a *local* extension of the Coq proof assistant with a reflection rule, without being forced to rely on the reflection in the entire development.

This translation can be seen as a way to build a syntactical model of ETT from a model of ITT as described more generally in Boulier et al. [2017] and has been entirely programmed and formalised¹ in Coq [Coq development team 2017]. For this, we rely on TemplateCoq² [Anand et al. 2018], which provides a reifier for Coq terms as represented in Coq's kernel as well as a formalisation of the type system of Coq. Thus, our formalisation of ETT is just given by adding the reflection rule to a subset of the original type system of Coq. This allows us to extract concrete Coq terms and types

¹<https://github.com/TheoWinterhalter/ett-to-itt>

²<https://template-coq.github.io/template-coq/>

from a closed derivation of ETT, using a little trick to incorporate Inductive types and induction. We do not treat cumulativity of universes which is an orthogonal feature of Coq's type theory.

Outline of the Paper. Before going into the technical development of the translation, we explain its main ingredients and differences with previous works. Then, in Section 2, we define the extensional and intensional type theories we consider. In Section 3, we define the main ingredient of the translation, which is a relation between terms of ETT and terms in ITT. Then, the translation is given in Section 4. Section 5 describes the Coq formalisation and Sections 6 and 7 discuss limitations and related work. The main proofs are given in detail in Appendices A and B.

The Coq formalisation is provided as an anonymous supplementary material.

On the Need for UIP and Functional Extensionality.

Our translation targets ITT plus UIP and functional extensionality, which correspond to the two following axioms

$$\begin{aligned} \text{UIP} & : \quad \Pi(A : \square_i). \Pi(x \ y : A). \Pi(e \ e' : x = y). e = e' \\ \text{FunExt} & : \quad \Pi(A \ B : \square_i). \Pi(f \ g : \Pi(x : A). B). (\Pi(x : A). f \ x = g \ x) \rightarrow f = g \end{aligned}$$

The first axiom says that any two proofs of the same equality are equal, and the other one says that two (dependent) functions are equal whenever they are pointwise equal³. These two axioms are perfectly valid statements of ITT and they can be proven in ETT. Indeed, UIP can be shown to be equivalent to the Streicher's axiom K

$$\text{K} : \quad \Pi(A : \square_i). \Pi(x : A). \Pi(e : x = x). e = \text{refl}_x$$

using the elimination on the identity type. But K is provable in ETT by considering the type

$$\Pi(A : \square_i). \Pi(x \ y : A). \Pi(e : x = y). e = \text{refl}_x$$

which is well typed (using the reflection rule to show that e has type $x = x$) and which can be inhabited by elimination of the identity type. In the same way, functional extensionality is provable in ETT because

$$\begin{aligned} \Pi(x : A). f \ x = g \ x & \rightarrow x : A \vdash f \ x \equiv g \ x && \text{by reflection} \\ & \rightarrow (\lambda(x : A). f \ x) \equiv (\lambda(x : A). g \ x) && \text{by congruence of } \equiv \\ & \rightarrow f \equiv g && \text{by } \eta\text{-law} \\ & \rightarrow f = g \end{aligned}$$

Therefore, applying our translation to the proofs of those theorems in ETT gives corresponding proofs of the same theorems in ITT. However, UIP is independent from ITT, as first shown by Hofmann and Streicher using the groupoid model [Hofmann and Streicher 1998], which has recently been extended in the setting of univalent type theory using the simplicial or cubical models [Bezem et al. 2013; Kapulkin and Lumsdaine 2012]. Similarly, Boulier *et al.* have shown that functional extensionality is independent from ITT using a simple syntactical translation [Boulier et al. 2017].

Therefore, our translation provides proofs of axioms independent from ITT, which means that the target of the translation already needs to have both UIP and functional extensionality. Part of our work is to show formally that they are the only axioms required.

³In Homotopy Type Theory (HoTT) [Univalent Foundations Program 2013], the functional extensionality axiom is stated in a more complete way, using the notion of adjoint equivalences, but this more complete way collapses to our simpler statement in presence of UIP.

Heterogeneous Equality and the Parametricity Translation.

The basic idea behind the translation from ETT to ITT is to interpret conversion using the internal notion of equality, *i.e.*, the identity type. But this means that two terms of two convertible types that were comparable in ETT become comparable in ITT only up-to the equality between the two types. One possible solution of this problem is to consider a native heterogeneous equality, such as *John Major equality* introduced by McBride [2000]. However, to avoid adding additional axioms to ITT like was done by Oury [2005], we prefer to encode this heterogeneous equality using the following dependent sums:

$$t \vDash_U u := \Sigma(p : T = U). p_* t = u.$$

During the translation, the same term occurring twice can be translated in two different manners, if the corresponding typing derivations are different. Even the types of the two different translations may be different. However, we have the strong property that any two translations of the same term only differ in places where transports of proof of equality have been injected. To keep track of this property, we introduce the relation $t \sim t'$ between two terms of ITT, of possibly different types. The crux of the proof of the translation is to guarantee that for every two terms t_1 and t_2 such that $\Gamma \vdash t_1 : T_1$, $\Gamma \vdash t_2 : T_2$ and $t_1 \sim t_2$, there exists p such that $\Gamma \vdash p : t_1 \vDash_{T_1} t_2$. However, during the proof, variables of different but (propositionally) equal types are introduced and the context cannot be maintained to be the same for both t_1 and t_2 . Therefore, the translation needs to keep track of this duplication of variables, plus a proof that they are heterogeneously equal. This mechanism is similar to what happens in the (relational) internal parametricity translation in ITT introduced by Bernardy et al. [2012] and recently rephrased in the setting of TemplateCoq [Anand et al. 2018]. Namely, a context is not translated as a telescope of variables, but as a telescope of triples consisting of two variables plus a witness that they are in the parametric relation. In our setting, this amounts to consider telescope of triples consisting of two variables plus a witness that they are heterogeneously equal. We can express this by considering the following dependent sums:

$$\text{Pack } A_1 A_2 := \Sigma(x : A_1). \Sigma(y : A_2). x \vDash_{A_1} y.$$

This presentation inspired by the parametricity translation is crucial in order to get an effective translation, because it is necessary to keep track of the evolution of contexts when doing the translation on open terms. This ingredient is missing in Oury's work [Oury 2005], which prevents him from deducing an effective (*i.e.*, constructive and computable) translation from his theorem.

2 DEFINITIONS OF EXTENSIONAL AND INTENSIONAL TYPE THEORIES

This section presents the common syntax, typing and main properties of ETT and ITT. Our type theories feature a universe hierarchy, dependent products and sums as well as Martin L of's identity types.

2.1 Syntax of ETT and ITT

The common syntax of ETT and ITT is given in Figure 1. It features: dependent products $\Pi(x : A). B$, with (annotated) λ -abstractions and (annotated) applications, negative dependent sums $\Sigma(x : A). B$ with (annotated) projections, sorts \square_i , identity types $u =_A v$ with reflection and elimination as well as terms realising UIP and functional extensionality. Annotating terms with otherwise computationally irrelevant typing information is a common practice when studying the syntax of type theory precisely (see [Streicher 1993] for a similar example). We will write $A \rightarrow B$ for $\Pi(_ : A). B$ the non-dependent product / function type.

We consider a fixed universe hierarchy without cumulativity, which ensures in particular uniqueness of typing (2.2) which is important for the translation.

s	$::= \square_i (i \in \mathbb{N})$	sorts (universes)
T, A, B, t, u, v	$::= x \mid \lambda(x : A).B.t \mid t @_{x:A.B} u$ $\mid \langle u; v \rangle_{x:A.B} \mid \pi_1^{x:A.B} p \mid \pi_2^{x:A.B} p$ $\mid \text{refl}_A u \mid J(A, u, x.e.P, w, v, p)$ $\mid \text{funext}(x : A, B, f, g, e) \mid \text{uip}(A, u, v, p, q)$ $\mid s \mid \Pi(x : A). B \mid \Sigma(x : A). B \mid u =_A v$	λ -terms pair terms equality terms equality axioms
Γ, Δ	$::= \bullet \mid \Gamma, x : A$	types contexts

Fig. 1. Syntax of ETT/ITT

About Annotations. Although it may look like a technical detail, the use of annotation is more fundamental in ETT than it is in ITT (where it is irrelevant and doesn't affect the theory). And this is actually one of the main differences between our work (and that of Martin Hofmann [1995] who has a similar presentation) and the work of Oury [2005].

Indeed, by using the standard model where types are interpreted as cardinals rather than sets, it is possible to see that the equality $\text{nat} \rightarrow \text{nat} = \text{nat} \rightarrow \text{bool}$ is independent from the theory, it is thus possible to assume it (as an axiom, or for those that would still not be convinced, simply under a λ that would introduce this equality). In that context, the identity map $\lambda(x : \text{nat}). x$ can be given the type $\text{nat} \rightarrow \text{bool}$ and we thus type $(\lambda(x : \text{nat}). x) 0 : \text{bool}$. Moreover, the β -reduction of the non-annotated system concludes that this expression reduces to 0 which cannot be given the type bool (as we said, the equality $\text{nat} \rightarrow \text{nat} = \text{nat} \rightarrow \text{bool}$ is independent from the theory, so the context is consistent), meaning we lack subject reduction in this case (or uniqueness of types, depending on how we see the issue). Our presentation has a blocked β -reduction limited to matching annotations: $(\lambda(x : A).B. t) @_{x:A.B} u = t[x \leftarrow u]$, from which subject reduction and uniqueness of types follow.

We believe that it is this difference, in essence, that is responsible for Oury's need for an extra axiom. Indeed, he needs to assume the congruence rule for heterogeneous equality of applications, which is not provable when formulated with John Major equality:

$$\text{JMAPP} \quad \frac{f_1 \ \forall(x:U_1).V_1 \cong \forall(x:U_2).V_2 \quad f_2 \quad u_1 \ U_1 \cong U_2 \ v_2}{f_1 \ u_1 \ V_1[x \leftarrow u_1] \cong V_2[x \leftarrow u_2] \quad u_2 \ v_2}$$

We can avoid using this congruence rule at the application case.

2.2 The Typing Systems

As usual in dependent type theory, we consider contexts which are telescopes whose declarations may depend on any variable already introduced. We note $\Gamma \vdash t : A$ to say that t has type A in context Γ . $\Gamma \vdash A$ shall stand for $\Gamma \vdash A : s$ for some sort s .

We use two relations $(s, s') \in \text{Ax}$ (written (s, s') for short) and $(s, s', s'') \in \text{R}$ (written (s, s', s'')) to constrain the sorts in the typing rules for universes, dependent products and dependent sums, as is done in any Pure Type System (PTS). In our case, because we do not have cumulativity, the rules are as follows:

$$(\square_i, \square_{i+1}) \in \text{Ax} \quad (\square_i, \square_j, \square_{\max(i,j)}) \in \text{R}$$

We give the typing rules of ITT in Figure 2. The rules are standard and we do not explain them. Let us just point out the conversion rule, which says that $u : A$ can be given the type $u : B$ when

Well-formedness of contexts.

$$\frac{}{\vdash \bullet} \qquad \frac{\vdash \Gamma \quad \Gamma \vdash A}{\vdash \Gamma, x : A} (x \notin \Gamma)$$

Types.

$$\frac{\vdash \Gamma}{\Gamma \vdash s : s'} (s, s') \qquad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : s'}{\Gamma \vdash \Pi(x : A). B : s''} (s, s', s'')$$

$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : s'}{\Gamma \vdash \Sigma(x : A). B : s''} (s, s', s'') \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash u : A \quad \Gamma \vdash v : A}{\Gamma \vdash u =_A v : s}$$

Structural rules.

$$\frac{\vdash \Gamma \quad (x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash u : A \quad \Gamma \vdash A \equiv B}{\Gamma \vdash u : B}$$

λ -calculus terms.

$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : s' \quad \Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda(x : A). B. t : \Pi(x : A). B}$$

$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : s' \quad \Gamma \vdash t : \Pi(x : A). B \quad \Gamma \vdash u : A}{\Gamma \vdash t @_{x:A.B} u : B[x \leftarrow u]}$$

$$\frac{\Gamma \vdash u : A \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : s' \quad \Gamma \vdash v : B[x \leftarrow u]}{\Gamma \vdash \langle u; v \rangle_{x:A.B} : \Sigma(x : A). B}$$

$$\frac{\Gamma \vdash p : \Sigma(x : A). B}{\Gamma \vdash \pi_1^{x:A.B} p : A} \qquad \frac{\Gamma \vdash p : \Sigma(x : A). B}{\Gamma \vdash \pi_2^{x:A.B} p : B[x \leftarrow \pi_1^{x:A.B} p]}$$

Equality terms.

$$\frac{\Gamma \vdash A : s \quad \Gamma \vdash u : A}{\Gamma \vdash \text{refl}_A u : u =_A u} \qquad \frac{\Gamma \vdash e_1, e_2 : u =_A v}{\Gamma \vdash \text{uip}(A, u, v, e_1, e_2) : e_1 = e_2}$$

$$\frac{\Gamma \vdash u, v : A \quad \Gamma, x : A, e : u =_A x \vdash P : s' \quad \Gamma \vdash A : s \quad \Gamma \vdash p : u =_A v \quad \Gamma \vdash w : P[x \leftarrow u, e \leftarrow \text{refl}_A u]}{\Gamma \vdash \text{J}(A, u, x.e.P, w, v, p) : P[x \leftarrow v, e \leftarrow p]}$$

$$\frac{\Gamma \vdash f, g : \Pi(x : A). B \quad \Gamma \vdash e : \Pi(x : A). f @_{x:A.B} x =_B g @_{x:A.B} x}{\Gamma \vdash \text{funext}(x : A, B, f, g, e) : f = g}$$

Fig. 2. Typing rules

$A \equiv B$, i.e., when A and B are convertible. As the notion of conversion is central in our work—the conversion of ETT being translated to an equality in ITT—we provide an exhaustive definition of it, with computational conversion rules (including β -conversion or reduction of the elimination principle of equality over reflexivity, see Figure 3) and congruence conversion rules (Figure 4). Note that we use Christine Paulin-Möhrling’s variant of the J rule rather than Martin-Löf’s original formulation. Although pretty straightforward, being precise here is very important, as for instance the congruence rule for λ -terms is the reason why functional extensionality is derivable in ETT. Congruence of equality terms is a standard extension of congruence to the new principles we add (UIP and functional extensionality).

ETT is thus simply an extension of ITT (we write \vdash_x for the associated typing judgment) with the reflection rule on equality, which axiomatises that propositionally equal terms are convertible:

$$\frac{\Gamma \vdash_x e : u =_A v}{\Gamma \vdash_x u \equiv v : A}$$

Note that, as already mentioned, in the presence of reflection and J, UIP is derivable so we could remove it from ETT, but keeping it allows us to share a common syntax which makes the statements of theorems simpler and does not affect the development.

Computation.

$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : s' \quad \Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda(x : A).B.t) @_{x:A.B} u \equiv t[x \leftarrow u] : B[x \leftarrow u]}$$

$$\frac{\Gamma \vdash A : s \quad \Gamma \vdash u : A \quad \Gamma, x : A, e : u =_A x \vdash P : s' \quad \Gamma \vdash w : P[x \leftarrow u, e \leftarrow \text{refl}_A u]}{\Gamma \vdash J(A, u, x.e.P, w, u, \text{refl}_A u) \equiv w : P[x \leftarrow u, e \leftarrow \text{refl}_A u]}$$

$$\frac{\Gamma \vdash A : s \quad \Gamma \vdash u : A \quad \Gamma, x : A \vdash B : s' \quad \Gamma \vdash v : B[x \leftarrow u]}{\Gamma \vdash \pi_1^{x:A.B} \langle u; v \rangle_{x:A.B} \equiv u : A}$$

$$\frac{\Gamma \vdash A : s \quad \Gamma \vdash u : A \quad \Gamma, x : A \vdash B : s' \quad \Gamma \vdash v : B[x \leftarrow u]}{\Gamma \vdash \pi_2^{x:A.B} \langle u; v \rangle_{x:A.B} \equiv v : B[x \leftarrow u]}$$

Equivalence relation.

$$\frac{\Gamma \vdash u : A}{\Gamma \vdash u \equiv u : A} \quad \frac{\Gamma \vdash u \equiv v : A \quad \Gamma \vdash v \equiv u : A}{\Gamma \vdash v \equiv u : A} \quad \frac{\Gamma \vdash u \equiv v : A \quad \Gamma \vdash v \equiv w : A}{\Gamma \vdash u \equiv w : A}$$

Conversion.

$$\frac{\Gamma \vdash t_1 \equiv t_2 : T_1 \quad \Gamma \vdash T_1 \equiv T_2}{\Gamma \vdash t_1 \equiv t_2 : T_2}$$

Fig. 3. Main conversion rules

2.3 General Properties of ITT and ETT

We now state the main properties of both ITT and ETT. We do not detail their proof as they are standard and can be found in the Coq formalisation.

First, although not explicit in the typing system, weakening is admissible in ETT and ITT.

Congruence of type constructors.

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s'}{\Gamma \vdash \Pi(x : A_1). B_1 \equiv \Pi(x : A_2). B_2 : s''} (s, s', s'')$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s'}{\Gamma \vdash \Sigma(x : A_1). B_1 \equiv \Sigma(x : A_2). B_2 : s''} (s, s', s'')$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma \vdash u_1 \equiv u_2 : A_1 \quad \Gamma \vdash v_1 \equiv v_2 : A_1}{\Gamma \vdash u_1 =_{A_1} v_1 \equiv u_2 =_{A_2} v_2 : s}$$

Congruence of λ -calculus terms.

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s' \quad \Gamma, x : A_1 \vdash t_1 \equiv t_2 : B_1}{\Gamma \vdash \lambda(x : A_1). B_1. t_1 \equiv \lambda(x : A_2). B_2. t_2 : \Pi(x : A_1). B_1}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s' \quad \Gamma \vdash t_1 \equiv t_2 : \Pi(x : A_1). B_1 \quad \Gamma \vdash u_1 \equiv u_2 : A_1}{\Gamma \vdash t_1 @_{x:A_1.B_1} u_1 \equiv t_2 @_{x:A_1.B_1} u_2 : B_1[x \leftarrow u_1]}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma \vdash u_1 \equiv u_2 : A_1 \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s' \quad \Gamma \vdash v_1 \equiv v_2 : B_1[x \leftarrow u_1]}{\Gamma \vdash \langle u_1; v_1 \rangle_{x:A_1.B_1} \equiv \langle u_2; v_2 \rangle_{x:A_2.B_2} : \Sigma(x : A_1). B_1}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s' \quad \Gamma \vdash p_1 \equiv p_2 : \Sigma(x : A_1). B_1}{\Gamma \vdash \pi_1^{x:A_1.B_1} p_1 \equiv \pi_1^{x:A_2.B_2} p_2 : A_1}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s' \quad \Gamma \vdash p_1 \equiv p_2 : \Sigma(x : A_1). B_1}{\Gamma \vdash \pi_2^{x:A_1.B_1} p_1 \equiv \pi_2^{x:A_2.B_2} p_2 : B_1[x \leftarrow \pi_1^{x:A_1.B_1} p_1]}$$

Congruence of equality terms.

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma \vdash u_1 \equiv u_2 : A}{\Gamma \vdash \text{refl}_{A_1} u_1 \equiv \text{refl}_{A_2} u_2 : u_1 =_{A_1} u_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma \vdash u_1 \equiv u_2 : A_1 \quad \Gamma \vdash v_1 \equiv v_2 : A_1 \quad \Gamma, x : A_1, e : u_1 =_{A_1} x \vdash P_1 \equiv P_2 : s' \quad \Gamma \vdash p_1 \equiv p_2 : u_1 =_{A_1} v_1 \quad \Gamma \vdash w_1 \equiv w_2 : P_1[x \leftarrow u_1, e \leftarrow \text{refl}_{A_1} u_1]}{\Gamma \vdash J(A_1, u_1, x.e.P_1, w_1, v_1, p_1) \equiv J(A_2, u_2, x.e.P_2, w_2, v_2, p_2) : P[x \leftarrow v_1, e \leftarrow p_1]}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash B_1 \equiv B_2 : s' \quad \Gamma \vdash f_1 \equiv f_2 : \Pi(x : A_1). B_1 \quad \Gamma \vdash g_1 \equiv g_2 : \Pi(x : A_1). B_1 \quad \Gamma \vdash e_1 \equiv e_2 : \Pi(x : A_1). f_1 @_{x:A_1.B_1} x =_{B_1} g_1 @_{x:A_1.B_1} x}{\Gamma \vdash \text{funext}(x : A_1, B_1, f_1, g_1, e_1) \equiv \text{funext}(x : A_2, B_2, f_2, g_2, e_2) : f_1 = g_1}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \quad \Gamma \vdash u_1 \equiv u_2 : A_1 \quad \Gamma \vdash v_1 \equiv v_2 : A_2 \quad \Gamma \vdash p_1 \equiv p_2 : u_1 =_{A_1} v_1 \quad \Gamma \vdash q_1 \equiv q_2 : u_1 =_{A_1} v_1}{\Gamma \vdash \text{uip}(A_1, u_1, v_1, p_1, q_1) \equiv \text{uip}(A_2, u_2, v_2, p_2, q_2) : p_1 = q_1}$$

Fig. 4. Congruence rules

LEMMA 2.1 (WEAKENING). *If $\Gamma \vdash \mathcal{J}$ and Δ extends Γ (possibly interleaving variables) then $\Delta \vdash \mathcal{J}$.*

Then, as mentioned above, the use of a non-cumulative hierarchy allows us to prove that a term t can be given at most one type in a context Γ , up-to conversion.

LEMMA 2.2 (UNIQUENESS OF TYPING). *If $\Gamma \vdash u : T_1$ and $\Gamma \vdash u : T_2$ then $\Gamma \vdash T_1 \equiv T_2$.*

Finally, an important property of the typing system (seen as a mutual inductive definition) is the possibility to deduce hypotheses from their conclusion, thanks to inversion of typing. Note that it is important here that our syntax is annotated for applications and projections as it provides a richer inversion principle.

LEMMA 2.3 (INVERSION OF TYPING).

- (1) *If $\Gamma \vdash x : T$ then $(x : A) \in \Gamma$ and $\Gamma \vdash A \equiv T$.*
- (2) *If $\Gamma \vdash \square_i : T$ then $\Gamma \vdash \square_{i+1} \equiv T$.*
- (3) *If $\Gamma \vdash \Pi(x : A). B : T$ then $\Gamma \vdash A : s$ and $\Gamma, x : A \vdash B : s'$ and $\Gamma \vdash s'' \equiv T$ for some (s, s', s'') .*
- (4) *If $\Gamma \vdash \lambda(x : A). B.t : T$ then $\Gamma \vdash A : s$ and $\Gamma, x : A \vdash B : s'$ and $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash \Pi(x : A). B \equiv T$.*
- (5) *If $\Gamma \vdash u @_{x:A.B} v : T$ then $\Gamma \vdash A : s$ and $\Gamma, x : A \vdash B : s'$ and $\Gamma \vdash u : \Pi(x : A). B$ and $\Gamma \vdash v : A$ and $\Gamma \vdash B[x \leftarrow u] \equiv T$.*
- (6) *... Analogous for the remaining term and type constructors.*

PROOF. Each case is proven by induction on the derivation (which corresponds to any number of applications of the conversion rule following one introduction rule). \square

3 RELATING TRANSLATED EXPRESSIONS

We want to define a relation on terms that equates two terms that are the same up to transport. This begs the question of what notion of transport is going to be used. Transport can be defined from elimination of equality as follows:

Definition 3.1 (Transport). Given $\Gamma \vdash p : T_1 =_s T_2$ and $\Gamma \vdash t : T_1$ we define the transport of t along p , written $p_* t$, as $J(s, T_1, X.e. T_1 \rightarrow X, \lambda(x : T_1). T_1.x, T_2, p) @_{T_1.T_2} t$ such that $\Gamma \vdash p_* t : T_2$.

However, in order not to confuse the transports added by the translation with the transports that were already present in the source, we consider p_* as part of the syntax in the reasoning. It will be unfolded to its definition only after the complete translation is performed. This idea is not novel as Hofmann already had a Subst operator that was part of his ITT (noted TT_1 in his paper [Hofmann 1995]).

We first define the (purely syntactic) relation \sqsubset between ETT terms and ITT terms in Figure 5 stating that the ITT term is simply a decoration of the first term by transports. Its purpose is to state how close to the original term its translation is. Then, in Figure 6 we extend this relation to \sim that is meant to deal with two ITT terms (basically $\sim = (\sqsubset \cdot \sqsubset)^*$). We remark easily that whenever $t \sqsubset \bar{t}$ we also have $t \sim \bar{t}$.

The goal is to prove that two terms in this relation, that are well-typed in the target type theory, are heterogeneously equal. As for this notion, we recall the definition we previously gave: $t \mathop{T} \cong_U u := \Sigma(p : T = U). p_* t = u$. This definition of heterogeneous equality can be shown to be reflexive, symmetric and transitive. Because of UIP, heterogeneous equality collapses to equality when taken on the same type.

LEMMA 3.2. *If $\Gamma \vdash e : u \mathop{A} \cong_A v$ then there exists p such that $\Gamma \vdash p : u \mathop{A} = v$.*

PROOF. This holds thanks to UIP on equality, which implies K, and so the proof of $A = A$ can be taken to be reflexivity. \square

$$\begin{array}{c}
\frac{t_1 \sqsubset t_2}{t_1 \sqsubset p_* t_2} \\
\\
\frac{}{x \sqsubset x} \quad \frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2}{\Pi(x : A_1). B_1 \sqsubset \Pi(x : A_2). B_2} \quad \frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2}{\Sigma(x : A_1). B_1 \sqsubset \Sigma(x : A_2). B_2} \\
\\
\frac{A_1 \sqsubset A_2 \quad u_1 \sqsubset u_2 \quad v_1 \sqsubset v_2}{u_1 =_{A_1} v_1 \sqsubset u_2 =_{A_2} v_2} \quad \frac{}{s \sqsubset s} \quad \frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2 \quad t_1 \sqsubset t_2}{\lambda(x : A_1). B_1. t_1 \sqsubset \lambda(x : A_2). B_2. t_2} \\
\\
\frac{t_1 \sqsubset t_2 \quad A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2 \quad u_1 \sqsubset u_2}{t_1 @_{x:A_1.B_1} u_1 \sqsubset t_2 @_{x:A_2.B_2} u_2} \quad \frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2 \quad t_1 \sqsubset t_2 \quad u_1 \sqsubset u_2}{\langle t_1; u_1 \rangle_{x:A_1.B_1} \sqsubset \langle t_2; u_2 \rangle_{x:A_2.B_2}} \\
\\
\frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2 \quad p_1 \sqsubset p_2}{\pi_1^{x:A_1.B_1} p_1 \sqsubset \pi_1^{x:A_2.B_1} p_2} \quad \frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2 \quad p_1 \sqsubset p_2}{\pi_2^{x:A_1.B_1} p_1 \sqsubset \pi_2^{x:A_2.B_2} p_2} \\
\\
\frac{A_1 \sqsubset A_2 \quad u_1 \sqsubset u_2}{\text{refl}_{A_1} u_1 \sqsubset \text{refl}_{A_2} u_2} \quad \frac{A_1 \sqsubset A_2 \quad B_1 \sqsubset B_2 \quad f_1 \sqsubset f_2 \quad g_1 \sqsubset g_2 \quad e_1 \sqsubset e_2}{\text{funext}(x : A_1, B_1, f_1, g_1, e_1) \sqsubset \text{funext}(x : A_2, B_2, f_2, g_2, e_2)} \\
\\
\frac{A_1 \sqsubset A_2 \quad u_1 \sqsubset u_2 \quad v_1 \sqsubset v_2 \quad p_1 \sqsubset p_2 \quad q_1 \sqsubset q_2}{\text{uip}(A_1, u_1, v_1, p_1, q_1) \sqsubset \text{uip}(A_2, u_2, v_2, p_2, q_2)} \\
\\
\frac{A_1 \sqsubset A_2 \quad u_1 \sqsubset u_2 \quad P_1 \sqsubset P_2 \quad w_1 \sqsubset w_2 \quad v_1 \sqsubset v_2 \quad p_1 \sqsubset p_2}{J(A_1, u_1, x.e.P_1, w_1, v_1, p_1) \sqsubset J(A_2, u_2, x.e.P_2, w_2, v_2, p_2)}
\end{array}$$

Fig. 5. Relation \sqsubset

NOTE. In particular, \cong on types corresponds to equality. This is not as trivial as it sounds, one might be concerned about what happens if we have $\Gamma \vdash e : A \cong_{s'} B$ with two distinct sorts s and s' . We would thus have $s = s'$, however, for this to be well-typed, we need to give a common type to s and s' , which can only be achieved if s and s' are actually the same sort.

Before we can prove the fundamental lemma stating that two terms in relation are heterogeneously equal, we need to consider another construction. As explained in the introduction, when proving the property by induction on terms, we introduce variables in the context that are equal only up-to heterogeneous equality. This phenomenon is similar to what happens in the parametricity translation [Bernardy et al. 2012]. Our fundamental lemma on the decoration relation \sim assumes two related terms of potentially different types T_1 and T_2 to produce an heterogeneous equality between them. For induction to go through under binders (e.g. for dependent products and abstractions), we hence need to consider the two terms under different, but heterogeneously equal contexts. Therefore, the context we produce will not only be a telescope of variables, but rather a telescope of triples consisting of two variables of possibly different types, and a witness that they are heterogeneously equal. To make this precise, we define the following macro:

$$\text{Pack } A_1 A_2 := \Sigma(x : A_1). \Sigma(y : A_2). x \cong y$$

$$\begin{array}{c}
\frac{}{x \sim x} \quad \frac{t_1 \sim t_2}{p_* t_1 \sim t_2} \quad \frac{t_1 \sim t_2}{t_1 \sim p_* t_2} \quad \frac{A_1 \sim A_2 \quad B_1 \sim B_2}{\Pi(x : A_1). B_1 \sim \Pi(x : A_2). B_2} \\
\\
\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{\Sigma(x : A_1). B_1 \sim \Sigma(x : A_2). B_2} \quad \frac{A_1 \sim A_2 \quad u_1 \sim u_2 \quad v_1 \sim v_2}{u_1 =_{A_1} v_1 \sim u_2 =_{A_2} v_2} \quad \frac{}{s \sim s} \\
\\
\frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad t_1 \sim t_2}{\lambda(x : A_1). B_1. t_1 \sim \lambda(x : A_2). B_2. t_2} \quad \frac{t_1 \sim t_2 \quad A_1 \sim A_2 \quad B_1 \sim B_2 \quad u_1 \sim u_2}{t_1 @_{x:A_1.B_1} u_1 \sim t_2 @_{x:A_2.B_2} u_2} \\
\\
\frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad t_1 \sim t_2 \quad u_1 \sim u_2}{\langle t_1; u_1 \rangle_{x:A_1.B_1} \sim \langle t_2; u_2 \rangle_{x:A_2.B_2}} \quad \frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad p_1 \sim p_2}{\pi_1^{x:A_1.B_1} p_1 \sim \pi_1^{x:A_2.B_1} p_2} \\
\\
\frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad p_1 \sim p_2}{\pi_2^{x:A_1.B_1} p_1 \sim \pi_2^{x:A_2.B_2} p_2} \quad \frac{A_1 \sim A_2 \quad u_1 \sim u_2}{\text{refl}_{A_1} u_1 \sim \text{refl}_{A_2} u_2} \\
\\
\frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad f_1 \sim f_2 \quad g_1 \sim g_2 \quad e_1 \sim e_2}{\text{funext}(x : A_1, B_1, f_1, g_1, e_1) \sim \text{funext}(x : A_2, B_2, f_2, g_2, e_2)} \\
\\
\frac{A_1 \sim A_2 \quad u_1 \sim u_2 \quad v_1 \sim v_2 \quad p_1 \sim p_2 \quad q_1 \sim q_2}{\text{uip}(A_1, u_1, v_1, p_1, q_1) \sim \text{uip}(A_2, u_2, v_2, p_2, q_2)} \\
\\
\frac{A_1 \sim A_2 \quad u_1 \sim u_2 \quad P_1 \sim P_2 \quad w_1 \sim w_2 \quad v_1 \sim v_2 \quad p_1 \sim p_2}{J(A_1, u_1, x.e.P_1, w_1, v_1, p_1) \sim J(A_2, u_2, x.e.P_2, w_2, v_2, p_2)}
\end{array}$$

Fig. 6. Relation \sim

together with its projections

$$\text{Proj}_1 p := \pi_1 p \quad \text{Proj}_2 p := \pi_1 \pi_2 p \quad \text{Proj}_e p := \pi_2 \pi_2 p.$$

We can then extend this notion canonically to contexts of the same length that are well formed using the same sorts:

$$\text{Pack}(\Gamma_1, x : A_1) (\Gamma_2, x : A_2) := (\text{Pack } \Gamma_1 \Gamma_2), x : \text{Pack}(A_1[\gamma_1]) (A_2[\gamma_2]) \quad \text{Pack } \bullet \bullet := \bullet.$$

When we pack contexts, we also need to apply the correct projections for the types in that context to still make sense. Assuming two contexts Γ_1 and Γ_2 of the same length, we can define left and right substitutions: $\gamma_1 := [x \leftarrow \text{Proj}_1 x \mid (x : _) \in \Gamma_1]$ and $\gamma_2 := [x \leftarrow \text{Proj}_2 x \mid (x : _) \in \Gamma_2]$. These substitutions implement lifting of terms to packed contexts: $\Gamma, \text{Pack } \Gamma_1 \Gamma_2 \vdash t[\gamma_1] : A[\gamma_1]$ whenever $\Gamma, \Gamma_1 \vdash t : A$ (resp. $\Gamma, \text{Pack } \Gamma_1 \Gamma_2 \vdash t[\gamma_2] : A[\gamma_2]$ whenever $\Gamma, \Gamma_2 \vdash t : A$).

For readability, when Γ_1 and Γ_2 are understood we will write Γ_p for $\text{Pack } \Gamma_1 \Gamma_2$.

Implicitly, whenever we use the notation $\text{Pack } \Gamma_1 \Gamma_2$ it means that the two contexts are of the same length and well-formed with the same sorts. We can now state the fundamental lemma.

LEMMA 3.3 (FUNDAMENTAL LEMMA). *Let t_1 and t_2 be two terms. If $\Gamma, \Gamma_1 \vdash t_1 : T_1$ and $\Gamma, \Gamma_2 \vdash t_2 : T_2$ and $t_1 \sim t_2$ then there exists p such that $\Gamma, \text{Pack } \Gamma_1 \Gamma_2 \vdash p : t_1[\gamma_1] \text{ }_{T_1[\gamma_1]} \cong_{T_2[\gamma_2]} t_2[\gamma_2]$.*

PROOF. The proof is by induction on the derivation of $t_1 \sim t_2$. It is given in Appendix A. \square

We can also prove that \sim preserves substitution and is an equivalence relation.

LEMMA 3.4. *If $t_1 \sim t_2$ and $u_1 \sim u_2$ then $t_1[x \leftarrow u_1] \sim t_2[x \leftarrow u_2]$.*

PROOF. We proceed by induction on the derivation of $t_1 \sim t_2$. \square

LEMMA 3.5 (\sim IS AN EQUIVALENCE RELATION). *\sim is reflexive, symmetric and transitive.*

PROOF. By induction on the derivation. \square

4 TRANSLATING ETT TO ITT

4.1 The Translation

We now define the translations (let us stress the plural here) of an extensional judgment. We extend \sqsubset canonically to contexts ($\Gamma \sqsubset \bar{\Gamma}$ when they bind the same variables and the types are in relation for \sqsubset).

Before defining the translation, we define a set $\llbracket \Gamma \vdash_x t : A \rrbracket$ of typing judgments in ITT associated to a typing judgment $\Gamma \vdash_x t : A$ in ETT. The idea is that this set describes all the possible translations that lead to the expected property. When $\bar{\Gamma} \vdash \bar{t} : \bar{A} \in \llbracket \Gamma \vdash_x t : A \rrbracket$, we say that $\bar{\Gamma} \vdash \bar{t} : \bar{A}$ realises $\Gamma \vdash_x t : A$. The translation will be given by showing that this set is inhabited by induction on the derivation.

Definition 4.1 (Characterisation of possible translations).

- For any $\vdash_x \Gamma$ we define $\llbracket \vdash_x \Gamma \rrbracket$ as a set of valid judgments (in ITT) such that $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ if and only if $\Gamma \sqsubset \bar{\Gamma}$.
- Similarly, $\bar{\Gamma} \vdash \bar{t} : \bar{A} \in \llbracket \Gamma \vdash_x t : A \rrbracket$ iff $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ and $A \sqsubset \bar{A}$ and $t \sqsubset \bar{t}$.

In order to better master the shape of the produced realiser, we state the following lemma which shows that it has the same head type constructor as the type it realises. This is important for instance for the case of an application, where we do not know a priori if the translated function has a dependent product type, which is required to be able to use the typing rule for application.

LEMMA 4.2. *We can always choose types \bar{T} that have the same head constructor as T .*

PROOF. Assume we have $\bar{\Gamma} \vdash \bar{t} : \bar{T} \in \llbracket \Gamma \vdash_x t : T \rrbracket$. By definition of \sqsubset , $T \sqsubset \bar{T}$ means that \bar{T} is shaped $p_* q_* \dots r_* \bar{T}'$ with \bar{T}' having the same head constructor as T . By inversion (2.3), the subterms are typable, including \bar{T}' . Actually, from inversion, we even get that the type of \bar{T}' is a universe. Then, using lemma 3.3 and lemma 3.2, we get $\bar{\Gamma} \vdash e : \bar{T} = \bar{T}'$. We conclude with $\bar{\Gamma} \vdash e_* \bar{t} : \bar{T}' \in \llbracket \Gamma \vdash_x t : T \rrbracket$. \square

Finally, in order for the induction to go through, we need to know that when we have a realiser of a derivation $\Gamma \vdash_x t : T$, we can pick an arbitrary other type realising $\Gamma \vdash_x T$ and still get a new derivation realising $\Gamma \vdash_x t : T$ with that type. This is important for instance for the case of an application, where the type of the domain of the translated function may differ from the type of the translated argument. So we need to be able to change it *a posteriori*.

LEMMA 4.3. *When we have $\bar{\Gamma} \vdash \bar{t} : \bar{T} \in \llbracket \Gamma \vdash_x t : T \rrbracket$ and $\bar{\Gamma} \vdash \bar{T}' \in \llbracket \Gamma \vdash_x T \rrbracket$ then we also have $\bar{\Gamma} \vdash \bar{t}' : \bar{T}' \in \llbracket \Gamma \vdash_x t : T \rrbracket$ for some \bar{t}' .*

PROOF. By definition we have $T \sqsubset \bar{T}$ and $T \sqsubset \bar{T}'$ and thus $T \sim \bar{T}$ and $T \sim \bar{T}'$, implying $\bar{T} \sim \bar{T}'$ by transitivity (3.5). By lemma 3.3 (in the case $\Gamma_1 \equiv \Gamma_2 \equiv \bullet$) we get $\bar{\Gamma} \vdash p : \bar{T} \cong \bar{T}'$ for some p . By

lemma 3.2 (and lemma 4.2 to give universes as types to \bar{T} and \bar{T}') we can assume $\bar{\Gamma} \vdash p : \bar{T} = \bar{T}'$. Then $\bar{\Gamma} \vdash p_* \bar{t} : \bar{T}'$ is still a translation since \sqsubset ignores transports. \square

We can now define the translation. This is done by mutual induction on context well-formedness, typing and conversion derivations. Indeed, in order to be able to produce a realiser by induction, we need to show that every conversion in ETT is translated as an heterogeneous equality in ITT.

THEOREM 4.4 (TRANSLATION).

- If $\vdash_x \Gamma$ then there exists $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$,
- If $\Gamma \vdash_x t : T$ then for any $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ there exist \bar{t} and \bar{T} such that $\vdash \bar{t} : \bar{T} \in \llbracket \Gamma \vdash_x t : T \rrbracket$,
- If $\Gamma \vdash_x u \equiv v : A$ then for any $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ there exist $A \sqsubset \bar{A}, A \sqsubset \bar{A}', u \sqsubset \bar{u}, v \sqsubset \bar{v}$ and \bar{e} such that $\vdash \bar{e} : \bar{u} \bar{A} \cong_{\bar{A}} \bar{v}$.

PROOF. We prove the theorem by induction on the derivation in the extensional type theory. To keep the flow of the presentation, the proof is given in Appendix B. \square

4.2 Meta-theoretical Consequences

We can check that all ETT theorems whose type are typable in ITT have proofs in ITT as well:

COROLLARY 4.5 (PRESERVATION OF ITT). *If $\vdash_x t : T$ and $\vdash T$ then there exist \bar{t} such that $\vdash \bar{t} : T \in \llbracket \vdash_x t : T \rrbracket$.*

PROOF. Since $\vdash \bullet \in \llbracket \vdash_x \bullet \rrbracket$, by Theorem (4.4), there exists \bar{t} and \bar{T} such that $\vdash \bar{t} : \bar{T} \in \llbracket \vdash_x t : T \rrbracket$. But as $\vdash T$, we have $\vdash T \in \llbracket \vdash_x T \rrbracket$, and, using Lemma 4.3, we obtain $\vdash \bar{t} : T \in \llbracket \vdash_x t : T \rrbracket$. \square

COROLLARY 4.6 (RELATIVE CONSISTENCY). *Assuming ITT is consistent, there is no term t such that $\vdash_x t : \Pi(A : \square_0). A$.*

PROOF. Assume such a t exists. By the Corollary 4.5, because $\vdash \Pi(A : \square_0). A$, there exists \bar{t} such that $\vdash \bar{t} : \Pi(A : \square_0). A$ which contradicts the assumed consistency of ITT. \square

4.3 Optimisations

Up until now, we remained silent about one thing: the size of the translated terms. Indeed, the translated term is a decoration of the initial one by transports which appear in many locations. For example, at each application we use a transport by lemma 4.2 to ensure that the term in function position is given a function type. In most cases—in particular when translating ITT terms—this produces unnecessary transports (often by reflexivity) that we wish to avoid.

In order to limit the size explosion, in the above we use a different version of transport, namely transport' such that

$$\begin{aligned} \text{transport}'_{A_1, A_2}(p, t) &= t && \text{when } A_1 =_\alpha A_2 \\ &= p_* t && \text{otherwise.} \end{aligned}$$

The idea is that we avoid *trivially* unnecessary transports (we do not deal with β -conversion for instance). We extend this technique to the different constructors of equality (symmetry, transitivity, ...) so that they reduce to reflexivity whenever possible. Take transitivity for instance:

$$\begin{aligned} \text{transitivity}'(\text{refl } u, q) &= q \\ \text{transitivity}'(p, \text{refl } u) &= p \\ \text{transitivity}'(p, q) &= \text{transitivity}(p, q). \end{aligned}$$

We show these *defined terms* enjoy the same typing rules as their counterparts and use them instead. In practice it is enough to recover the exact same term when it is typed in ITT.

5 FORMALISATION WITH TEMPLATE-COQ

We have formalised the translation (provided as an anonymous supplementary material) in the setting of TemplateCoq [Anand et al. 2018] in order to have a more precise proof, but also to evidence the fact that the translation is indeed constructive and can be used to perform computations.

TemplateCoq is a Coq library that has a representation of Coq terms as they are in Coq kernel (in particular using de Bruijn indices for variables), a (partial) implementation of the type checking algorithm and so on. It comes with a Coq plugin that permits to quote Coq terms into their representations, and to produce Coq terms from their representation (if they indeed denote well-typed terms). We have integrated our formalisation within that framework in order to ensure our presentations of ETT and ITT are close to Coq, but also to take advantage of the quoting mechanism to produce terms using interactive mode (in particular we get to use tactics). Note that we also rely on Mangin and Sozeau’s Equations [Sozeau 2010] plugin to derive nice dependent induction principles.

Our formalisation takes full advantage of its easy interfacing with TemplateCoq: we define two theories, namely ETT and ITT, but ITT enjoys a lot of syntactic sugar by having things such as transport, heterogeneous equality and packing as part of the syntax. The operations regarding these constructors—in particular the tedious ones—are written in Coq and then quoted to finally be *realised* in the translation from ITT to TemplateCoq.

Interoperability with TemplateCoq. The translation we define from ITT to TemplateCoq is not proven correct, but it is not really important as it can just be seen as a feature to observe the produced terms in a nicer setting. In any case, TemplateCoq does not yet provide a complete formalisation of CIC rules, as guard checking of recursive definitions and strict positivity of inductive type declarations are not formalised yet.

We also provide a translation from TemplateCoq to ETT that we will describe more extensively with the examples (5.4).

5.1 Quick Overview of the Formalisation

The file SAst.v contains the definition of the (common) abstract syntax of ETT and ITT in the form of an inductive definition with de Bruijn indices for variables (like in TemplateCoq). Sorts are defined separately in Sorts.v and we will address them later in Section 5.3.

```

Inductive sterm : Type :=
| sRel (n : nat)
| sSort (s : sort)
| sProd (nx : name) (A B : sterm)
| sLambda (nx : name) (A B t : sterm)
| sApp (u : sterm) (nx : name) (A B v : sterm)
| sEq (A u v : sterm)
| sRef1 (A u : sterm)
| (* ... *) .

```

The files ITyping.v and XTyping.v define respectively the typing judgments for ITT and ETT, using mutual inductive types. Then, most of the files are focused on the meta-theory of ITT and can be ignored by readers who don’t need to see yet another proof of subject reduction.

The most interesting files are obviously those where the fundamental lemma and the translation are formalised: FundamentalLemma.v and Translation.v. For instance, here is the main theorem, as stated in our formalisation:

```

Theorem complete_translation  $\Sigma$  :
  type_glob  $\Sigma$  ->

```

```

(forall  $\Gamma$  (h : XTyping.wf  $\Sigma$   $\Gamma$ ),  $\Sigma$   $\Gamma'$ ,  $\Sigma$  |--i  $\Gamma'$  #  $\llbracket \Gamma \rrbracket$ ) *
(forall  $\Gamma$  t A (h :  $\Sigma$  ;;;  $\Gamma$  |-x t : A)  $\Gamma'$  (h $\Gamma$  :  $\Sigma$  |--i  $\Gamma'$  #  $\llbracket \Gamma \rrbracket$ ),
   $\Sigma$  A' t',  $\Sigma$  ;;;;  $\Gamma'$  |--- [t'] : A' #  $\llbracket \Gamma$  |--- [t] : A)) *
(forall  $\Gamma$  u v A (h :  $\Sigma$  ;;;  $\Gamma$  |-x u = v : A)  $\Gamma'$  (h $\Gamma$  :  $\Sigma$  |--i  $\Gamma'$  #  $\llbracket \Gamma \rrbracket$ ),
   $\Sigma$  A' A'' u' v' p', eqtrans  $\Sigma$   $\Gamma$  A u v  $\Gamma'$  A' A'' u' v' p').

```

The fact that the theorem holds in Coq ensures we can actually compute a translated term and type out of a derivation in ETT.

5.2 Inductive Types and Recursion

In the proof of Section 4, we didn't mention anything about inductive types, pattern-matching or recursion as it is a bit technical on paper. In the formalisation, we offer a way to still be able to use them, and we will even show how it works in practice with the examples (5.4).

The main guiding principle is that inductive types and induction are orthogonal to the translation, they should more or less be translated to themselves. To realise that easily, we just treat an inductive definition as a way to introduce new constants in the theory, one for the type, one for each constructor, one for its elimination principle, and one equality per computation rule. For instance, the natural numbers can be represented by having the following constants in the context:

```

nat : Type
0 : nat
S : nat → nat
natrec :  $\forall P, P\ 0 \rightarrow (\forall m, P\ m \rightarrow P\ (S\ m)) \rightarrow \forall n, P\ n$ 
natrec0 :  $\forall P\ P_z\ P_s, \text{natrec}\ P\ P_z\ P_s\ 0 = P_z$ 
natrecS :  $\forall P\ P_z\ P_s\ n, \text{natrec}\ P\ P_z\ P_s\ (S\ n) = P_s\ n\ (\text{natrec}\ P\ P_z\ P_s\ n)$ 

```

Here we rely on the reflection rule to obtain the computational behavior of the eliminator natrec.

This means for instance that we do not consider inductive types that would only make sense in ETT, but we deem this not to be a restriction and to the best of our knowledge isn't something that is usually considered in the literature. With that in mind, our translation features a global context of typed constants with the restriction that the types of those constants should be well-formed in ITT. Those constants are thus used as black boxes inside ETT.

With this we are able to recover what we were missing from Coq, without having to deal with the trouble of proving that the translation doesn't break the guard condition of fixed points, and we are instead relying on a more type-based approach.

5.3 About Universes and Homotopy

The experienced reader might have noticed that our treatment of universes (except perhaps for the absence of cumulativity) was really superficial and the notion of sorts used is rather orthogonal to our main development. This is even more apparent in the formalisation. Indeed, we didn't fix a specific universe hierarchy, but instead specify what properties it should have, in what is reminiscent to a (functional⁴) PTS formulation.

```

Class Sorts.notion := {
  sort : Type ;
  succ : sort -> sort ;
  prod_sort : sort -> sort -> sort ;
  sum_sort : sort -> sort -> sort ;

```

⁴Meaning the sort of a sort, and the sort of a product are functions, necessary to the uniqueness of types (2.2).


```

eq_sort : sort -> sort ;
eq_dec : forall s z : sort, {s = z} + {s <> z} ;
succ_inj : forall s z, succ s = succ z -> s = z
}.

```

From the notion of sorts, we require functions to get the sort of a sort, the sort of a product from the sorts of its arguments, and (crucially) the sort of an identity type. We also require some measure of decidable equality and injectivity on those.

This allows us to instantiate this by a lot of different notions including the one presented earlier in the paper or even its extension with a universe Prop of propositions (like CIC [Bertot and Castéran 2004]). We present here two instances that have their own interest.

Type in Type. One of the instances we provide is one with only one universe Type, with the inconsistent typing rule $\text{Type} : \text{Type}$. Although inconsistent, this allows us to interface with TemplateCoq, without the—for the time being—very time-consuming universe constraint checking.

Homotopy Type System and Two-Level Type Theory. Another interesting application (or rather instance) of our formalisation is a translation from Homotopy Type System (HTS) [Voevodsky 2013] to Two-Level Type Theory (2TT) [Altenkirch et al. 2016; Annenkov et al. 2017].

HTS and 2TT arise from the incompatibility between UIP—recall it is provable in ETT—and univalence. The idea is to have two distinct notions of equality in the theory, a *strict* one satisfying UIP, and a *fibrant* one corresponding to the homotopy type theory equality, possibly satisfying univalence. This actually induces a separation in the types of the theory: some of them are called *fibrant* and the fibrant or homotopic equality can only be eliminated on those. HTS can be seen as an extension of 2TT with reflection on the strict equality just like ETT is an extension of ITT.

We can recover HTS and 2TT in our setting by taking F_i and U_i as respectively the fibrant and strict universes of those theories (for $i \in \mathbb{N}$), along with the following PTS rules:

$$\begin{aligned}
(F_i, F_{i+1}) &\in \text{Ax} \\
(U_i, U_{i+1}) &\in \text{Ax} \\
(F_i, F_j, F_{\max(i,j)}) &\in \text{R} \\
(F_i, U_j, U_{\max(i,j)}) &\in \text{R} \\
(U_i, F_j, U_{\max(i,j)}) &\in \text{R} \\
(U_i, U_j, U_{\max(i,j)}) &\in \text{R}
\end{aligned}$$

and the fact that the sort of the (strict) identity type on $A : s$ is the *strictified* version of s , i.e., U_i for $s = U_i$ or $s = F_i$. In order to have the fibrant equality, one simply needs to do as in Section 5.2.

In short, the translation from HTS to 2TT is basically the same as the one from ETT to ITT we presented in this paper, and this fact is factorised through our formalisation.

5.4 Examples

In this section, we show our translation in action with some small examples (the main problem usually being the generation of the tedious ETT typing derivation). This, we believe, already illustrates how it can be used in practice. The examples can be found in Example.v.

Identity with a Coercion. We will detail extensively a first, self-contained example without any inductive types or recursion, illustrating a very simple case of reflection. The term we would like to type is the following:

$$\lambda A B e x \Rightarrow x : \forall (A B : \text{Type}), A = B \rightarrow A \rightarrow B$$

This is, in some sense the identity, relying on the equality $e : A = B$ to convert $x : A$ to $x : B$. Of course, this definition isn't accepted in Coq because the conversion doesn't hold in ITT.

Fail **Definition** pseudoid (A B : **Type**) (e : A = B) (x : A) : B := x.

However, we still want to be able to write it *in some way*, in order to avoid manipulating de Bruijn indices directly. For this, we use a little trick by first defining a Coq axiom to represent an ill-typed term:

Axiom candidate : forall A B (t : A), B.

candidate A B t is a candidate t of type A to inhabit type B. We complete this by adding a notation that is reminiscent to Agda's [Norell 2007] hole mechanism.

Notation "'{! t !}'" := (candidate _ _ t).

We can now write the ETT function within Coq.

Definition pseudoid (A B : **Type**) (e : A = B) (x : A) : B := {! x !}.

We can then quote the term and its type to TemplateCoq thanks to the Quote **Definition** command provided by the plugin.

Quote **Definition** pseudoid_term :=
ltac:(let t := eval compute in pseudoid in exact t).

Quote **Definition** pseudoid_type :=
ltac:(let T := type of pseudoid in exact T).

The terms that we get are now TemplateCoq terms, representing Coq syntax. We need to put them in ETT, meaning adding the annotations, and also removing the candidate axiom. This is the purpose of the fullquote function that we provide in our formalisation.

Definition pretm_pseudoid :=
Eval lazy in fullquote (2 ^ 18) Σ [] pseudoid_term empty empty nomap.

Definition tm_pseudoid :=
Eval lazy in
match pretm_pseudoid with
| Success t => t
| Error _ => sRel 0
end.

Definition prety_pseudoid :=
Eval lazy in fullquote (2 ^ 18) Σ [] pseudoid_type empty empty nomap.

Definition ty_pseudoid :=
Eval lazy in
match prety_pseudoid with
| Success t => t
| Error _ => sRel 0
end.

tm_pseudoid and ty_pseudoid correspond respectively to the ETT representation of pseudoid and its type. We then produce, using our home-brewed Ltac type-checking tactic, the corresponding ETT typing derivation (notice the use of reflection to typecheck).

Lemma type_pseudoid : $\Sigma i ; ; ; [] \dashv x$ tm_pseudoid : ty_pseudoid.

Proof.
unfold tm_pseudoid, ty_pseudoid.
ettcheck. cbn.
eapply reflection with (e := sRel 1).
ettcheck.

Defined.

We can then translate this derivation, obtain the translated term and then convert it to TemplateCoq.

```
Definition itt_pseudoid : sterm :=
  Eval lazy in
  let '(_ ; t ; _) := type_translation type_pseudoid istrans_nil in t.
```

```
Definition tc_pseudoid : tsl_result term :=
  Eval lazy in
  tsl_rec (2 ^ 18)  $\Sigma$  [] itt_pseudoid empty.
```

Once we have it, we *unquote* the term to obtain a Coq term (notice that the only use of reflection has been replaced by a transport).

```
fun (A B : Type) (e : A = B) (x : A) => transport e x
  : forall A B : Type, A = B -> A -> B
```

Towards a Plugin with TemplateCoq. All of this work is pretty systematic and could be automated. Fortunately for us, TemplateCoq also features a really useful monad to reify Coq commands. All this can be condensed in the following TemplateCoq command:

```
Definition Translate ident : TemplateMonad () :=
  entry <- tmQuoteConstant ident false ;;
  match entry with
  | DefinitionEntry { | definition_entry_body := tm ; definition_entry_type := ty | } =>
    pretm <- tmEval lazy (fullquote (2 ^ 18)  $\Sigma$  [] tm empty empty nomap) ;;
    prety <- tmEval lazy (fullquote (2 ^ 18)  $\Sigma$  [] ty empty empty nomap) ;;
    match pretm, prety with
    | Success tm, Success ty =>
      name <- tmEval all (ident ++ "_der") ;;
      name <- tmFreshName name ;;
      der <- tmLemma name ( $\Sigma$ i ;; [] |-x tm : ty) ;;
      let '(_ ; itt_tm ; _) := type_translation der istrans_nil in
      t <- tmEval lazy (tsl_rec (2 ^ 18)  $\Sigma$  [] itt_tm empty) ;;
      match t with
      | FinalTranslation.Success _ t =>
        t' <- tmUnquote t ;;
        t' <- tmEval Ast.hnf (my_projT2 t') ;;
        tmPrint t'
      | _ => tmFail "Cannot translate from ITT to TemplateCoq"
      end
    | _,_ => tmFail "Cannot translate from TemplateCoq to ETT"
      end
    | _ => tmFail "Expected a constant definition"
  end.
```

This command basically does everything we showed above. It takes an identifier corresponding to a Coq definition, takes its type and body and applies the translation. Of course, this cannot guess the ETT derivation, this is handled by the `tmLemma` command which generates an obligation corresponding to it.

The command is then executed as follows.

```
Run TemplateProgram (Translate "pseudoid").
Next Obligation.
  pose proof xh $\Sigma$ i.
  ettcheck. cbn. eapply reflection with (e := sRel 1). ettcheck.
Defined.
```

As you can see, an obligation corresponding to the derivation is generated and proven as above. Once we hit **Defined**, the rest is executed and the command prints the expected translation.

About inductive types. As we promised, our translation is able to handle inductive types. For this consider the inductive type of vectors (or length-indexed lists) below, together with a simple definition (we will remain in ITT for simplicity).

```
Inductive vec A : nat -> Type :=
| vnil : vec A 0
| vcons : A -> forall n, vec A n -> vec A (S n).
```

```
Arguments vnil {_}.
Arguments vcons {_} _ _ _.
```

```
Definition vv : vec nat 1 := vcons 2 _ vnil.
```

Like before we quote this definition to TemplateCoq and then to ETT by removing the axiom. We also do something more here: we give an interpretation to the constants that we encounter in the term, like **nat**, **vec**, **S** or **vcons**. In the ETT term they get replaced by things like `sAx "nat"`, `sAx "vec"`, etc. In fact we preemptively put in the global context the constants and their types (and we showed they were indeed types).

Similarly, once we have the derivation, we need to provide an interpretation to the `sAx`, this is done by a correspondence table saying which Coq term realises which constant. Unsurprisingly, the translated Coq term is the same, namely `vcons 2 0 vnil`.

5.5 Towards an Interfacing between Andromeda and Coq

Andromeda [Bauer et al. 2016] is a proof assistant implementing ETT in a sense that is really close to our formalisation. Aside from a concise nucleus with a basic type theory, most things happen with the declaration of constants with given types, including equalities to define the computational behaviour of eliminators for instance. This is essentially what we do in our formalisation. Furthermore, their theory relies on `Type : Type`, meaning, our modular handling of universes can accommodate for this as well.

All in all, it should be possible in the near future to use our translation to produce Coq terms out of Andromeda developments. Note that this would not suffer from the difficulties in generating typing derivations since Andromeda does it for you.

5.6 Composition with other Translations

This translation also enables the formalisation of translations that target ETT rather than ITT and still get mechanised proofs of (relative) consistency by composition with this ETT to ITT translation. This could also be used to implement plugins based on the composition of translations. In particular, supposing we have a theory which forms a subset of ETT and whose conversion is decidable. Using this translation, we could formalise it as an embedded domain-specific type theory and provide an automatic translation of well-typed terms into witnesses in Coq. This would make it possible to extend conversion with the theory of lists for example.

This would provide a simple way to justify the consistency of CoqMT [Jouannaud and Strub 2017] for example, seeing it as an extensional type theory where reflection is restricted to equalities on a specific domain whose theory is decidable.

6 LIMITATIONS AND AXIOMS

Currently, the representation of terms and derivations and the computational content of the proof only allow us to deal with the translation of relatively small terms but we hope to improve that in the future. As we have seen, the actual translation involves the computational content of lemmata of inversion, substitution, weakening and equational reasoning and thus cannot be presented as a simple recursive definition on derivations.

As we already mentioned, the axioms K and functional extensionality are both necessary in ITT if we want the translation to be conservative as they are provable in ETT [Hofmann 1995]. However, one might still be concerned about having axioms as they can for instance hinder canonicity of the system. In that respect, K isn't really a restriction since it preserves canonicity. The best proof of that is probably Agda itself which natively features K—in fact, one needs to explicitly deactivate it with a flag if they wish to work without.

The case of functional extensionality is trickier. It is still possible to realise the axiom by composing our translation with a setoid interpretation [Altenkirch 1999] which validates it, or by going into a system featuring it, for instance by implementing Observational Type Theory [Altenkirch et al. 2007] like EPIGRAM [McBride 2004] if it still existed.

7 RELATED WORKS AND CONCLUSION

The seminal works on the precise connection between ETT and ITT go back to Streicher [1993] and Hofmann [1995, 1997]. In particular, the work of Hofmann provides a categorical answer to the question of consistency and conservativity of ETT over ITT with UIP and functional extensionality. Ten years later, Oury [2005, 2006] provided a translation from ETT to ITT with UIP and functional extensionality and other axioms (mainly due to technical difficulties). Although a first step towards a move from categorical semantics to a syntactic translation, his work does not stress any constructive aspect of the proof and shows that there merely exist translations in ITT to a typed term in ETT.

van Doorn et al. [2013] have later proposed and formalised a similar translation between a PTS with and without explicit conversion. This does not entail anything about ETT to ITT but we can find similarities in that there is a witness of conversion between any term and itself under an explicit conversion, which internalises irrelevance of explicit conversions. This morally corresponds to a Uniqueness of Conversions principle.

In this paper we provide the first effective translation from ETT to ITT with UIP and functional extensionality. The translation has been formalised in Coq using TemplateCoq, a meta-programming plugin of Coq. This translation is also effective in the sense that we can produce in the end a Coq term using the TemplateCoq denotation machinery. With ongoing work to extend the translation to the inductive fragment of Coq, we are paving the way to an extensional version of the Coq proof assistant which could be translated back to its intensional version, allowing the user to navigate between the two modes, and in the end produce a proof term checkable in the intensional fragment.

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A PROOF OF THE FUNDAMENTAL LEMMA

LEMMA A.1 (FUNDAMENTAL LEMMA). *Let t_1 and t_2 be two terms. If $\Gamma, \Gamma_1 \vdash t_1 : T_1$ and $\Gamma, \Gamma_2 \vdash t_2 : T_2$ and $t_1 \sim t_2$ then there exists p such that $\Gamma, \text{Pack } \Gamma_1 \Gamma_2 \vdash p : t_1[\gamma_1] \ T_1[\gamma_1] \cong_{T_2[\gamma_2]} \ t_2[\gamma_2]$.*

For readability we will abbreviate the left and right substitutions $_[\gamma_1]$ and $_[\gamma_2]$ by \uparrow and \downarrow respectively.

PROOF. We prove it by induction on the derivation of $t_1 \sim t_2$.

- VAR

$$\frac{}{x \sim x}$$

If x belongs to Γ , we apply reflexivity—together with uniqueness of typing (2.2)—to conclude. Otherwise, $\text{Proj}_e x$ has the expected type (since $x[\gamma_1] \equiv \text{Proj}_1 x$ and $x[\gamma_2] \equiv \text{Proj}_2 x$).

- TRANSPORTLEFT

$$\frac{t_1 \sim t_2}{p_* t_1 \sim t_2}$$

We have $\Gamma, \Gamma_1 \vdash p_* t_1 : T_1$ and $\Gamma, \Gamma_2 \vdash t_2 : T_2$. By inversion (2.3) we have $\Gamma, \Gamma_1 \vdash p : T'_1 = T_1$ and $\Gamma, \Gamma_1 \vdash t_1 : T'_1$. Then by induction hypothesis we have e such that $\Gamma, \Gamma_p \vdash e : t_1 \uparrow \cong t_2 \downarrow$. From transitivity and symmetry we only need to provide a proof of $t_1 \uparrow \cong p \uparrow_* t_1 \downarrow$ which is inhabited by $\langle p \uparrow; \text{refl } (p \uparrow_* t_1 \downarrow) \rangle_{-..}$.

- TRANSPORTRIGHT

$$\frac{t_1 \sim t_2}{t_1 \sim p_* t_2}$$

Similarly.

- PRODUCT

$$\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{\Pi(x : A_1). B_1 \sim \Pi(x : A_2). B_2}$$

We have $\Gamma, \Gamma_1 \vdash \Pi(x : A_1). B_1 : T_1$ and $\Gamma, \Gamma_2 \vdash \Pi(x : A_2). B_2 : T_2$ so by inversion (2.3) we have $\Gamma, \Gamma_1 \vdash A_1 : s_1$ and $\Gamma, \Gamma_1, x : A_1 \vdash B_1 : s'_1$ and $\Gamma, \Gamma_1 \vdash s''_1 \equiv T_1$ for $(s_1, s'_1, s''_1) \in \mathbb{R}$ (and similarly with 2s). By induction hypothesis we have $\Gamma, \Gamma_p \vdash p_A : A_1 \uparrow \cong A_2 \downarrow$ and $\Gamma, \Gamma_p, x : \text{Pack } A_1 A_2 \vdash p_B : B_1 \uparrow \cong B_2 \downarrow$ hence the result (using UIP and functional extensionality, refer to the formalisation and especially to the file `Quotes.v` for more details on how to realise this equality).

- EQUALITY

$$\frac{A_1 \sim A_2 \quad u_1 \sim u_2 \quad v_1 \sim v_2}{u_1 =_{A_1} v_1 \sim u_2 =_{A_2} v_2}$$

We have $\Gamma, \Gamma_1 \vdash u_1 =_{A_1} v_1 : T_1$ and $\Gamma, \Gamma_2 \vdash u_2 =_{A_2} v_2 : T_2$ so, by inversion (2.3), we have $\Gamma, \Gamma_1 \vdash A_1 : s_1$ and $\Gamma, \Gamma_1 \vdash u_1 : A_1$ and $\Gamma, \Gamma_1 \vdash v_1 : A_1$ as well as $\Gamma, \Gamma_1 \vdash s_1 \equiv T_1$ (and the same with 2s). By induction hypothesis we thus have $\Gamma, \Gamma_p \vdash p_A : A_1 \cong A_2$ and $\Gamma, \Gamma_p \vdash p_u : u_1 \cong u_2$ and $\Gamma, \Gamma_p \vdash p_v : v_1 \cong v_2$. We can thus conclude.

- REFLEXIVITY

$$\frac{}{s \sim s}$$

This one holds by reflexivity and uniqueness of typing (2.2) (indeed, $s \uparrow \equiv s$ and $s \downarrow \equiv s$).

- LAMBDA

$$\frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad t_1 \sim t_2}{\lambda(x : A_1). B_1.t_1 \sim \lambda(x : A_2). B_2.t_2}$$

We have $\Gamma, \Gamma_1 \vdash \lambda(x : A_1). B_1.t_1 : T_1$ and $\Gamma, \Gamma_2 \vdash \lambda(x : A_2). B_2.t_2 : T_2$, thus, by inversion 2.3 the subterms are well-typed and we can apply induction hypothesis. The conclusion follows similarly to the Π case.

- APPLICATION

$$\frac{t_1 \sim t_2 \quad A_1 \sim A_2 \quad B_1 \sim B_2 \quad u_1 \sim u_2}{t_1 @_{x:A_1.B_1} u_1 \sim t_2 @_{x:A_2.B_2} u_2}$$

We have $\Gamma, \Gamma_1 \vdash t_1 @_{x:A_1.B_1} u_1 : T_1$ and $\Gamma, \Gamma_2 \vdash t_2 @_{x:A_2.B_2} u_2 : T_2$ which means by inversion (2.3) that the subterms are well-typed. We apply the induction hypothesis and then conclude.

- REFLEXIVITY

$$\frac{A_1 \sim A_2 \quad u_1 \sim u_2}{\text{refl}_{A_1} u_1 \sim \text{refl}_{A_2} u_2}$$

We have $\Gamma, \Gamma_1 \vdash \text{refl}_{A_1} u_1 : T_1$ and $\Gamma, \Gamma_2 \vdash \text{refl}_{A_2} u_2 : T_2$ so by inversion (2.3) we have $\Gamma, \Gamma_1 \vdash A_1 : s_1$ and $\Gamma, \Gamma_1 \vdash u_1 : A_1$ (same with 2s). By IH we have $A_1 \upharpoonright \cong A_2 \upharpoonright$ and $u_1 \upharpoonright_{A_1 \upharpoonright \cong A_2 \upharpoonright} u_2 \upharpoonright$. The proof follows easily.

- FUNEXT

$$\frac{A_1 \sim A_2 \quad B_1 \sim B_2 \quad f_1 \sim f_2 \quad g_1 \sim g_2 \quad e_1 \sim e_2}{\text{funext}(x : A_1, B_1, f_1, g_1, e_1) \sim \text{funext}(x : A_2, B_2, f_2, g_2, e_2)}$$

Similar.

- UIP

$$\frac{A_1 \sim A_2 \quad u_1 \sim u_2 \quad v_1 \sim v_2 \quad p_1 \sim p_2 \quad q_1 \sim q_2}{\text{uip}(A_1, u_1, v_1, p_1, q_1) \sim \text{uip}(A_2, u_2, v_2, p_2, q_2)}$$

Similar.

- J

$$\frac{A_1 \sim A_2 \quad u_1 \sim u_2 \quad P_1 \sim P_2 \quad w_1 \sim w_2 \quad v_1 \sim v_2 \quad p_1 \sim p_2}{J(A_1, u_1, x.e.P_1, w_1, v_1, p_1) \sim J(A_2, u_2, x.e.P_2, w_2, v_2, p_2)}$$

Similar.

□

B CORRECTNESS OF THE TRANSLATION

THEOREM B.1 (TRANSLATION).

- If $\vdash_x \Gamma$ then there exists $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$,
- If $\Gamma \vdash_x t : T$ then for any $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ there exist \bar{t} and \bar{T} such that $\bar{\Gamma} \vdash \bar{t} : \bar{T} \in \llbracket \Gamma \vdash_x t : T \rrbracket$,
- If $\Gamma \vdash_x u \equiv v : A$ then for any $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ there exist $A \sqsubset \bar{A}, A \sqsubset \bar{A}', u \sqsubset \bar{u}, v \sqsubset \bar{v}$ and \bar{e} such that $\bar{\Gamma} \vdash \bar{e} : \bar{u} \bar{A} \cong_{\bar{A}'} \bar{v}$.

PROOF. We prove the theorem by induction on the derivation in the extensional type theory. In most cases we need to assume some $\bar{\Gamma}$, translation of the context, we will implicitly refer to $\bar{\Gamma}$ in such cases as the one given as hypothesis.

- EMPTY

$$\frac{}{\vdash_x \bullet}$$

We have $\vdash \bullet \in \llbracket \vdash_x \bullet \rrbracket$.

- EXTEND

$$\frac{\vdash_x \Gamma \quad \Gamma \vdash_x A}{\vdash_x \Gamma, x : A} (x \notin \Gamma)$$

By IH we have $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ and, using $\bar{\Gamma}$ as well as lemma 4.2, $\bar{\Gamma} \vdash \bar{A} : s \in \llbracket \Gamma \vdash_x A : s \rrbracket$. Thus $\vdash \bar{\Gamma}, x : \bar{A} \in \llbracket \vdash_x \Gamma, x : A \rrbracket$.

- SORT

$$\frac{\vdash_x \Gamma}{\Gamma \vdash_x s : s'}(s, s')$$

We have $\bar{\Gamma} \vdash s : s' \in \llbracket \Gamma \vdash_x s : s' \rrbracket$.

- PRODUCT

$$\frac{\Gamma \vdash_x A : s \quad \Gamma, x : A \vdash_x B : s'}{\Gamma \vdash_x \Pi(x : A). B : s''}(s, s', s'')$$

By IH and lemma 4.2 we have $\bar{\Gamma} \vdash \bar{A} : s$, meaning $\vdash \bar{\Gamma}, x : \bar{A} \in \llbracket \vdash_x \Gamma, x : A \rrbracket$, and then $\bar{\Gamma}, x : \bar{A} \vdash \bar{B} : s'$. We thus conclude $\bar{\Gamma} \vdash \Pi(x : \bar{A}). \bar{B} : s'' \in \llbracket \Gamma \vdash_x \Pi(x : A). B : s'' \rrbracket$.

- SIGMA

$$\frac{\Gamma \vdash_x A : s \quad \Gamma, x : A \vdash_x B : s'}{\Gamma \vdash_x \Sigma(x : A). B : s''}(s, s', s'')$$

Similar.

- EQUALITY

$$\frac{\Gamma \vdash_x A : s \quad \Gamma \vdash_x u : A \quad \Gamma \vdash_x v : A}{\Gamma \vdash_x u =_A v : s}$$

By IH and lemma 4.2 we have $\bar{\Gamma} \vdash \bar{A} : s$, and—using lemma 4.3—we also have $\bar{\Gamma} \vdash \bar{u} : \bar{A}$ and $\bar{\Gamma} \vdash \bar{v} : \bar{A}$. Then $\bar{\Gamma} \vdash \bar{u} =_{\bar{A}} \bar{v} : s \in \llbracket \Gamma \vdash_x u =_A v : s \rrbracket$.

- VARIABLE

$$\frac{\vdash_x \Gamma \quad (x : A) \in \Gamma}{\Gamma \vdash_x x : A}$$

We have $\vdash \bar{\Gamma} \in \llbracket \vdash_x \Gamma \rrbracket$ (as we assumed, this is not an instance of the induction hypothesis) and $(x : A) \in \Gamma$. By definition of $\Gamma \sqsubset \bar{\Gamma}$ we also have some $(x : \bar{A}) \in \bar{\Gamma}$ with $A \sqsubset \bar{A}$, thus $\bar{\Gamma} \vdash x : \bar{A} \in \llbracket \Gamma \vdash_x x : A \rrbracket$.

- CONVERSION

$$\frac{\Gamma \vdash_x u : A \quad \Gamma \vdash_x A \equiv B}{\Gamma \vdash_x u : B}$$

By IH and lemma 3.2 we have $\bar{\Gamma} \vdash \bar{e} : \bar{A} = \bar{B}$ which implies $\bar{\Gamma} \vdash \bar{A} \in \llbracket \Gamma \vdash_x A \rrbracket$ by inversion (2.3), thus, from lemma 4.3 and IH we get $\bar{\Gamma} \vdash \bar{u} : \bar{A}$, yielding $\bar{\Gamma} \vdash \bar{e}_* \bar{u} : \bar{B} \in \llbracket \Gamma \vdash_x u : B \rrbracket$.

- LAMBDA

$$\frac{\Gamma \vdash_x A : s \quad \Gamma, x : A \vdash_x B : s' \quad \Gamma, x : A \vdash_x t : B}{\Gamma \vdash_x \lambda(x : A). B.t : \Pi(x : A). B}$$

By IH and lemma 4.2 we have $\bar{\Gamma} \vdash \bar{A} : s$ and thus $\vdash \bar{\Gamma}, x : \bar{A} \in \llbracket \vdash_x \Gamma, x : A \rrbracket$, meaning we can apply IH and lemma 4.2 to the second hypothesis to get $\bar{\Gamma}, x : \bar{A} \vdash \bar{B} : s' \in \llbracket \Gamma, x : A \vdash_x B : s' \rrbracket$ and then IH and lemma 4.3 to get $\bar{\Gamma}, x : \bar{A} \vdash \bar{t} : \bar{B} \in \llbracket \Gamma, x : A \vdash_x t : B \rrbracket$. All of this yields $\bar{\Gamma} \vdash \lambda(x : \bar{A}). \bar{B}. \bar{t} : \Pi(x : \bar{A}). \bar{B} \in \llbracket \Gamma \vdash_x \lambda(x : A). B.t : \Pi(x : A). B \rrbracket$.

- APPLICATION

$$\frac{\Gamma \vdash_x A : s \quad \Gamma, x : A \vdash_x B : s' \quad \Gamma \vdash_x t : \Pi(x : A). B \quad \Gamma \vdash_x u : A}{\Gamma \vdash_x t @_{x:A.B} u : B[x \leftarrow u]}$$

Using IH together with lemmata 4.2 and 4.3 we get $\bar{\Gamma} \vdash \bar{A} : s$ and $\bar{\Gamma}, x : \bar{A} \vdash \bar{B} : s'$ and $\bar{\Gamma} \vdash \bar{t} : \Pi(x : \bar{A}). \bar{B}$ and $\bar{\Gamma} \vdash \bar{u} : \bar{A}$ meaning we can conclude $\bar{\Gamma} \vdash \bar{t} @_{x:\bar{A}. \bar{B}} \bar{u} : \bar{B}[x \leftarrow \bar{u}] \in \llbracket \Gamma \vdash_x t @_{x:A.B} u : B[x \leftarrow u] \rrbracket$.

- PAIR

$$\frac{\Gamma \vdash_x u : A \quad \Gamma \vdash_x A : s \quad \Gamma, x : A \vdash_x B : s' \quad \Gamma \vdash_x v : B[x \leftarrow u]}{\Gamma \vdash_x \langle u; v \rangle_{x:A.B} : \Sigma(x : A). B}$$

Using IH with lemmata 4.2 and 4.3 we translate all the hypotheses to conclude $\bar{\Gamma} \vdash \langle \bar{u}; \bar{v} \rangle_{x:\bar{A}.\bar{B}} : \Sigma(x : \bar{A}). \bar{B} \in \llbracket \Gamma \vdash_x \langle u; v \rangle_{x:A.B} : \Sigma(x : A). B \rrbracket$.

- PROJ₁

$$\frac{\Gamma \vdash_x p : \Sigma(x : A). B}{\Gamma \vdash_x \pi_1^{x:A.B} p : A}$$

Similar.

- PROJ₂

$$\frac{\Gamma \vdash_x p : \Sigma(x : A). B}{\Gamma \vdash_x \pi_2^{x:A.B} p : B[x \leftarrow \pi_1^{x:A.B} p]}$$

Similar.

- REFLEXIVITY

$$\frac{\Gamma \vdash_x A : s \quad \Gamma \vdash_x u : A}{\Gamma \vdash_x \text{refl}_A u : u =_A u}$$

By IH we have $\bar{\Gamma} \vdash \bar{u} : \bar{A}$ and thus $\bar{\Gamma} \vdash \text{refl}_{\bar{A}} \bar{u} : \bar{u} =_{\bar{A}} \bar{u} \in \llbracket \Gamma \vdash_x \text{refl}_A u : u =_A u \rrbracket$.

- J

$$\frac{\Gamma \vdash_x A : s \quad \Gamma \vdash_x u, v : A \quad \Gamma, x : A, e : u =_A x \vdash_x P : s' \quad \Gamma \vdash_x p : u =_A v \quad \Gamma \vdash_x w : P[x \leftarrow u, e \leftarrow \text{refl}_A u]}{\Gamma \vdash_x J(A, u, x.e.P, w, v, p) : P[x \leftarrow v, e \leftarrow p]}$$

By IH and lemma 4.2 we have $\bar{\Gamma} \vdash \bar{A} : s$. From this and IH and lemma 4.3 we have $\bar{\Gamma} \vdash \bar{u}, \bar{v} : \bar{A}$. We can thus deduce $\vdash \bar{\Gamma}, x : \bar{A}, e : \bar{u} =_{\bar{A}} x \in \llbracket \Gamma, x : A, e : u =_A x \rrbracket$ which in turn gives us $\bar{\Gamma}, x : \bar{A}, e : \bar{u} =_{\bar{A}} x \vdash \bar{P} : s'$. Similarly we also get $\bar{\Gamma} \vdash \bar{p} : \bar{u} =_{\bar{A}} \bar{v}$ and $\bar{\Gamma} \vdash \bar{w} : \bar{P}[x \leftarrow \bar{u}, e \leftarrow \text{refl}_{\bar{A}} \bar{u}]$. All of this allows us to conclude $\bar{\Gamma} \vdash J(\bar{A}, \bar{u}, x.e.\bar{P}, \bar{w}, \bar{v}, \bar{p}) : \bar{P}[x \leftarrow \bar{v}, e \leftarrow \bar{p}] \in \llbracket \Gamma \vdash_x J(A, u, x.e.P, w, v, p) : P[x \leftarrow v, e \leftarrow p] \rrbracket$.

- FUNEXT

$$\frac{\Gamma \vdash f, g : \Pi(x : A). B \quad \Gamma \vdash e : \Pi(x : A). f @_{x:A.B} x =_B g @_{x:A.B} x}{\Gamma \vdash \text{funext}(x : A, B, f, g, e) : f = g}$$

Similar.

- UIP

$$\frac{\Gamma \vdash_x e_1, e_2 : u =_A v}{\Gamma \vdash_x \text{uip}(A, u, v, e_1, e_2) : e_1 = e_2}$$

Similar.

- BETA

$$\frac{\Gamma \vdash_x A : s \quad \Gamma, x : A \vdash_x B : s' \quad \Gamma, x : A \vdash_x t : B \quad \Gamma \vdash_x u : A}{\Gamma \vdash_x (\lambda(x : A). B.t) @_{x:A.B} u \equiv t[x \leftarrow u] : B[x \leftarrow u]}$$

From IH and the lemmata, we even get the conversion, we conclude using reflexivity.

- PROJ₁-RED

$$\frac{\Gamma \vdash_x A : s \quad \Gamma \vdash_x u : A \quad \Gamma, x : A \vdash_x B : s' \quad \Gamma \vdash_x v : B[x \leftarrow u]}{\Gamma \vdash_x \pi_1^{x:A.B} \langle u; v \rangle_{x:A.B} \equiv u : A}$$

Likewise.

- PROJ₂-RED

$$\frac{\Gamma \vdash_x A : s \quad \Gamma \vdash_x u : A \quad \Gamma, x : A \vdash_x B : s' \quad \Gamma \vdash_x v : B[x \leftarrow u]}{\Gamma \vdash_x \pi_2^{x:A.B} \langle u; v \rangle_{x:A.B} \equiv v : B[x \leftarrow u]}$$

Likewise.

- J-RED

$$\frac{\Gamma \vdash_x u : A \quad \Gamma, x : A, e : u =_A x \vdash_x P : U_j \quad \Gamma \vdash_x w : P[x \leftarrow u, e \leftarrow \text{refl}_A u]}{\Gamma \vdash_x J(A, u, x.e.P, w, u, \text{refl}_A u) \equiv w : P[x \leftarrow u, e \leftarrow \text{refl}_A u]}$$

Likewise.

- CONV-REFL

$$\frac{\Gamma \vdash_x u : A}{\Gamma \vdash_x u \equiv u : A}$$

We conclude from IH and reflexivity of \cong .

- CONV-SYM

$$\frac{\Gamma \vdash_x u \equiv v : A}{\Gamma \vdash_x v \equiv u : A}$$

We conclude from IH and symmetry of \cong .

- CONV-TRANS

$$\frac{\Gamma \vdash_x u \equiv v : A \quad \Gamma \vdash_x v \equiv w : A}{\Gamma \vdash_x u \equiv w : A}$$

We conclude from IH and transitivity of \cong .

- CONV-CONV

$$\frac{\Gamma \vdash_x t_1 \equiv t_2 : T_1 \quad \Gamma \vdash_x T_1 \equiv T_2}{\Gamma \vdash_x t_1 \equiv t_2 : T_2}$$

By IH (and lemma 3.2) we have $\bar{\Gamma} \vdash \bar{e} : \bar{t}_1 \bar{T}_1 \cong_{\bar{T}_1} \bar{t}_2$ and $\bar{\Gamma} \vdash p : \bar{T}_1'' = \bar{T}_2$. Also from lemmata 3.3 and 3.2 we have $\bar{T}_1' = \bar{T}_1''$ and $\bar{T}_1 = \bar{T}_1''$, meaning we get $\bar{T}_1' = \bar{T}_2$ and $\bar{T}_1 = \bar{T}_2$. This allows us to conclude by transporting along the aforementioned equalities.

- CONV-PROD

$$\frac{\Gamma \vdash_x A_1 \equiv A_2 : s \quad \Gamma, x : A_1 \vdash_x B_1 \equiv B_2 : s'}{\Gamma \vdash_x \Pi(x : A_1). B_1 \equiv \Pi(x : A_2). B_2 : s''} (s, s', s'')$$

We conclude exactly like we did in the proof of lemma 3.3.

- All congruences hold like in proof of lemma 3.3.
- CONV-EQ

$$\frac{\Gamma \vdash_x e : u =_A v}{\Gamma \vdash_x u \equiv v : A}$$

By IH and lemma 4.2 we have $\bar{\Gamma} \vdash \bar{e} : \bar{u} =_{\bar{A}} \bar{v} \in \llbracket \Gamma \vdash_x e : u =_A v \rrbracket$ which yields the conclusion we wanted. \square