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# EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS TO A FRACTIONAL PARABOLIC PROBLEM WITH SINGULAR WEIGHT AT THE BOUNDARY

BOUMEDIENE ABDELLAOUI\*, KHEIREDDINE BIROUD, EL-HAJ LAAMRI

ABSTRACT. In this work, we consider a nonlocal semilinear parabolic problem related to a fractional Hardy inequality with singular weight at the boundary. More precisely, we consider the problem

$$(P) \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u^p}{d^{2s}} & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \end{cases}$$

where  $0 < s < 1$ ,  $\Omega \subset \mathbb{R}^N$  is a bounded regular domain,  $d(x) = d(x, \partial\Omega)$ ,  $p > 0$  and  $\lambda > 0$  is a positive constant. The initial data  $u_0 \geq 0$  is a nonnegative function in a suitable Lebesgue space that we make precise later.

The main goal of this work is to analyze the interaction between the parameters  $s$ ,  $p$  and  $\lambda$  in order to show the existence or the nonexistence of solution to problem (P) in a suitable sense. We will show that our results have a significative difference with respect to the local case  $s = 1$ .

## 1. INTRODUCTION AND MAIN RESULTS

Recently, much attention has been devoted to the fractional Laplacian  $(-\Delta)^s$ , which for  $0 < s < 1$  is defined on smooth functions as follows

$$(1.1) \quad (-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where P.V stands for the Cauchy principal value and  $a_{N,s}$  is a dimensional constant given by (see e.g [21, section 2]),

$$a_{N,s} := \frac{s 2^{2s} \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)}$$

where  $\Gamma$  denotes the Gamma function. The choice of the constant  $a_{N,s}$  is motivated, among others, by the following proposition (see [21, Proposition 4.4]) :

**Proposition 1.1.** *For all  $\phi \in C_0^\infty(\mathbb{R}^N)$ , the following statements hold :*

$$(1.2) \quad \lim_{s \rightarrow 0^+} (-\Delta)^s \phi = \phi \text{ and } \lim_{s \rightarrow 1^-} (-\Delta)^s \phi = -\Delta \phi$$

where  $\Delta$  is the classical Laplacian.

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Of course, there are other ways to define the fractional Laplacian. For this, we refer the interested reader to [21, 10, 16] and the references therein.

Note that the fractional Laplacian is a nonlocal operator and that this raises several technical difficulties. However, great attention has recently been focused on the study of nonlocal operators because they play a crucial role in describing several phenomena, as, for instance, the thin obstacle problem, anomalous diffusion, quasi-geostrophic flows, ...

In spirit of this nonlocal setting, the aim of this paper is the study of the following nonlocal parabolic problem :

$$(1.3) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u^p}{d^{2s}} & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \end{cases}$$

where  $\Omega$  is a bounded regular domain of  $\mathbb{R}^N$ ,  $d(x) = \text{dist}(x, \partial\Omega)$ ,  $p > 0$ ,  $\lambda > 0$ ,  $u_0 \geq 0$  is a nonnegative measurable function,  $0 < s < 1$  and  $(-\Delta)^s$  is the fractional Laplacian .

More precisely, our goal here is to get *natural conditions on the parameters  $s, p, \lambda$*  and initial data in order to prove the existence of positive solutions to the problem (1.3). By solution, we mean energy solution or weak solution (for more specifics, see Definition 2.4 and Definition 2.5).

In order to give precise statements of the results obtained, let us introduce first the following Hardy's inequality (for a proof, see [22] and [23, Theorem 5]) under the form that we will use.

**Theorem 1.2.** *Assume that  $s \in (0, 1)$ ,  $N \geq 2s$  and let  $\Omega$  be a bounded regular domain ( $C^{1,1}$  regularity is sufficient). Then, there exists a positive constant  $\Lambda_s(\Omega)$  such that for all  $\phi \in C_0^\infty(\Omega)$*

$$(1.4) \quad \Lambda_s(\Omega) \int_{\Omega} \frac{\phi^2(x)}{d^{2s}(x)} dx \leq \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy,$$

where  $D_\Omega := \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^C \times \Omega^C)$ .

If  $s \in [\frac{1}{2}, 1)$  and  $\Omega$  is a convex domain, then the constant  $\Lambda_s(\Omega)$  is independent of  $\Omega$  and it is given by

$$(1.5) \quad \Lambda_s(\Omega) = \Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}.$$

**Remark 1.3.**

- In all cases, we always have  $\Lambda_s(\Omega) \leq \Lambda_s^*$ , see [23, Theorem 5].
- Let us point out that  $\Lambda_s(\Omega)$  is achieved if and only if  $\Lambda_s(\Omega) < \Lambda_s^*$ . We refer to [14] for more details.

Before going further, let us mention some previous works concerning the local case, i.e.  $s = 1$  in order to compare it with the nonlocal case.

First, Problem (1.3) has been recently treated in [4] (see also [3] for the elliptic case). In their work, the authors have proved a strong nonexistence result if  $p < 1$ . However, for  $p > 1$  and for adequate assumptions on  $u_0$ , they proved the existence of a positive solution using suitable sub and supersolution and iteration arguments.

If  $p = 1$  and  $\Omega$  is a bounded convex domain, then the authors proved in [15] a strong nonexistence result for  $\lambda > \Lambda_1^*$ . As a consequence, a complete and instantaneous blow-up result was proved for approximating problems.

Second, in the case where the weight  $d^{-2}$  is replaced by  $|x|^{-2}$  with  $0 \in \Omega$ , the best constant in the Hardy's inequality is  $\Lambda^{**} = \left(\frac{N-2}{2}\right)^2$ , see for instance [24] and the references therein. Moreover, if  $p = 1$ , the authors proved in [8] that the existence of solutions holds if and only if  $\lambda \leq \Lambda^{**}$ . For  $p > 1$ , a strong nonexistence result was obtained in [11].

In the nonlocal case (i.e.  $s < 1$ ), a different phenomenon appears in both cases  $p < 1$  and  $p = 1$ .

The main results of this paper can be summarized in the following points.

- If  $p < 1$ , we will show that Problem (1.3) has a nonnegative solution, in a suitable sense, for all  $u_0 \in L^1(\Omega)$ . This makes a significative difference in comparing with the local case  $s = 1$ , where a strong nonexistence result is proved in [4].

Here we will prove the following result.

**Theorem 1.4.** *Assume  $0 < s < 1$  and  $0 < p < 1$ . Then, for all  $0 \leq u_0 \in L^1(\Omega)$ , Problem (1.3) has a positive distributional solution.*

*Moreover, if  $s > \frac{1}{2}$  and  $0 < p < 2s - 1$ , then  $u$  is a weak solution in the sense of Definition 2.5 given below.*

- In the linear case  $p = 1$ , we will show that the range of the parameter  $s$ , together with the validity of the Hardy's inequality with optimal constant stated in Theorem 1.2 and Remark 1.3, will be decisive for existence and nonexistence of positive solutions for  $\lambda$  large. The main result in this case is the following.

**Theorem 1.5.** *Assume that  $s \in (0, \frac{1}{2})$ . Then, for all  $\lambda > 0$ , Problem (1.3) has a positive weak solution. If  $s \in [\frac{1}{2}, 1)$  and  $\Lambda_s(\Omega) = \Lambda_s^*$  (that includes the case of convex domain), then the existence of a nonnegative solution holds if and only if  $0 < \lambda \leq \Lambda_s^*$ .*

- Finally, in the superlinear case  $p > 1$ , we are able to show the existence of a nonnegative solution for a suitable  $u_0 \in L^\infty(\Omega)$ . More precisely, we will prove the next theorem.

**Theorem 1.6.** *Assume that  $s \in (0, 1)$  and  $p > 1$ . Then, there exists  $u_0 \in L^\infty(\Omega)$  such that Problem (1.3) has a positive bounded solution.*

As far as we know, all the results presented here are new.

For the sake of completeness and for the reader's convenience, we have tried to write this paper in almost self contained form. Moreover, we give precise references for all points that are not detailed in this work.

To end this section, our paper is organized as follows. In Section 2, we give some auxiliary results related to fractional Sobolev spaces and some functional inequalities. We also define the two notions of solution that we will use systematically along the paper : energy solutions and weak solutions.

The case  $0 < p < 1$  is treated in Section 3. Using monotone arguments and suitable test functions, we are able to prove the existence and uniqueness of a global weak solution that is in a suitable Sobolev space. This seems to be surprising since in the local case a strong nonexistence result holds with complete blow-up for the

sequence of solutions to the approximating problems.

In section 4, we treat the linear case  $p = 1$ . According to the value of  $s$ , we will show the existence and the nonexistence of solution for  $\lambda$  large. In particular, if  $s \in (0, \frac{1}{2})$ , we are able to prove the existence and the uniqueness of a positive weak solution for all  $\lambda > 0$ . However, if  $s \in [\frac{1}{2}, 1)$  and  $\Lambda_s^*(\Omega) = \Lambda_s^*$  (that holds for example if  $\Omega$  is convex), then the existence of solution holds if and only if  $\lambda \leq \Lambda_s^*$ .

The superlinear case  $p > 1$  is treated in Section 5. Under suitable assumptions on the initial data and using monotone arguments, we are able to get the existence of a bounded positive solution.

Finally, in the last section we give some extension to the case of the spectral fractional Laplacian.  $\blacksquare$

## 2. THE FUNCTIONAL SETTING AND TOOLS

**2.1. Functional framework.** As already announced in the introduction, we are working in a bounded domain  $\Omega$  with homogeneous Dirichlet boundary conditions  $u = 0$  in  $\mathbb{R} \setminus \Omega$  and not simply  $u = 0$  in  $\partial\Omega$ . This makes the classical fractional Sobolev space approach not sufficient for studying our problem. Then, we need to set a natural functional framework that is inspired by (but not equivalent to) the fractional Sobolev spaces which will allow us to deal with the Dirichlet boundary condition.

Of course, this subsection can be skipped by readers already familiar with them.

Let  $\Omega \subset \mathbb{R}^N$  and  $s \in (0, 1)$ . In the sequel, we denote  $D_\Omega = \mathbb{R}^{2N} \setminus (\Omega^C \times \Omega^C)$ . In the following, we denote by  $H^s(\Omega)$  the *classical* fractional Sobolev space defined as

$$H^s(\Omega) =: \left\{ u \in L^2(\Omega) ; \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < +\infty \right\},$$

endowed with the norm

$$(2.1) \quad \|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left( \frac{a_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

Now, we introduce the space  $\mathbb{H}_0^s(\Omega)$  as the completion of  $\mathcal{C}_0^\infty(\Omega)$  with respect to the norm of  $H^s(\mathbb{R}^N)$ . Notice that if  $u \in \mathbb{H}_0^s(\Omega)$ , we have  $u = 0$  a.e. in  $\mathbb{R}^N \setminus \Omega$ . It is clear that for  $u \in \mathbb{H}_0^s(\Omega)$ , we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy = \iint_{D_\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

If  $\Omega$  is a bounded domain, then using Poincaré's inequality, we can endowed  $\mathbb{H}_0^s(\Omega)$  with the equivalent norm

$$\|u\|_{\mathbb{H}_0^s(\Omega)}^2 = \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

The pair  $(\mathbb{H}_0^s(\Omega), \|\cdot\|_{\mathbb{H}_0^s(\Omega)})$  is a Hilbert space (see for instance [31, Lemma 7] or [10, Lemma 1.29]). The dual space of  $\mathbb{H}_0^s(\Omega)$  will be denoted by  $\mathbb{H}^{-s}(\Omega)$ . For more properties of the previous spaces, we refer to [21], [6] and [10, subsection 1.5].

**2.2. Some useful functional inequalities.** In this subsection, we collect and present some functional inequalities under the form in which they will be exploited.

• The following Sobolev's inequality is proved in [21] (see also [30] for a simple proof).

**Theorem 2.1.** *Assume that  $0 < s < 1$  with  $2s < N$ . There exists a positive constant  $S = S(N, s)$  such that for all  $u \in \mathbb{H}_0^s(\Omega)$ , we have*

$$S \left( \int_{\Omega} |u(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

where  $2_s^* := \frac{2N}{N - 2s}$  is the so-called critical Sobolev exponent.

• In order to treat the case  $\lambda = \Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}$ , we need the following improved Hardy-Sobolev inequality obtained in [23]. For this purpose, let us first state the definition of inner radius of a domain  $\Omega$  :

the inner radius of a domain  $\Omega$  is defined as  $R_{in} := \sup_{x \in \Omega} d(x)$ . We say that the domain  $\Omega$  has a finite inner radius whenever  $R_{in} < \infty$ .

**Theorem 2.2.** (Improved Hardy-Sobolev inequality) *Assume that  $\frac{1}{2} \leq s < 1$  and  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) is a uniformly Lipschitz and convex domain with finite inner radius. Then, there exists a positive constant  $C \equiv C(\Omega, s)$  such that*

$$(2.2) \quad C \left( \int_{\Omega} |u(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \Lambda_s^* \int_{\Omega} \frac{u^2}{d^{2s}} dx$$

where  $2_s^* = \frac{2N}{N - 2s}$ .

• The Kato type inequality below will be useful in this paper.

**Theorem 2.3.** *Let  $\Phi \in C^2(\mathbb{R})$  be a convex function. Assume that  $u$  and  $\Phi(u)$  are such that  $(-\Delta)^s u$  and  $(-\Delta)^s \Phi(u)$  exist. Then*

$$(2.3) \quad (-\Delta)^s \Phi(u) \leq \Phi'(u) (-\Delta)^s u.$$

**2.3. Notion of solutions.** Now, let us begin by making precise the sense that solutions are defined.

**Definition 2.4.** *Let  $(g, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$ . We say that  $u$  is an energy solution to the problem*

$$(2.4) \quad \begin{cases} u_t + (-\Delta)^s u = g(x, t) & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

if  $u \in L^2((0, T), \mathbb{H}_0^s(\Omega)) \cap C([0, T], L^2(\Omega))$  with  $u_t \in L^2((0, T), L^2(\Omega))$  and for all  $w \in L^2((0, T), \mathbb{H}_0^s(\Omega))$ , we have

$$(2.5) \quad \int_0^T \int_{\Omega} u_t w dx dt + \int_0^T \iint_{D_{\Omega}} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy dt = \int_0^T \int_{\Omega} g w dx dt,$$

for all  $w \in L^2((0, T), \mathbb{H}_0^s(\Omega))$ .

Since we will consider problems with general datum in  $L^1$ , we have to specify the meaning of weak solutions. We first need to define the space of test functions

$$P(\Omega_T) := \{\phi : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}, \text{ s.t. } -\phi_t + (-\Delta)^s \phi = \varphi, \varphi \in L^\infty(\Omega_T) \\ \phi(x, t) = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0, T], \phi(x, T) = 0 \text{ in } \Omega\}.$$

Notice that if  $\phi \in P(\Omega_T)$ , then  $\phi \in L^\infty(\Omega_T)$  (see for instance [26]). Then we have the subsequent definition.

**Definition 2.5.** Let  $(g, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$ . We say that  $u \in \mathcal{C}([0, T], L^1(\Omega))$  is a weak solution to

$$(2.6) \quad \begin{cases} u_t + (-\Delta)^s u = g(x, t) & \text{in } \Omega_T \\ u(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

if for all  $\phi \in P(\Omega_T)$ , we have

$$\int_{\Omega_T} u(-\phi_t + (-\Delta)^s \phi) dx dt = \int_{\Omega_T} u \varphi dx dt = \int_{\Omega_T} g \phi dx dt + \int_{\Omega} u_0 \phi(x, 0) dx.$$

The following existence and regularity result is obtained in [26].

**Theorem 2.6** ( See [26, Theorem 28]). Assume that  $(g, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$ . Then, Problem (2.6) has a unique weak solution  $u$ , that is obtained as a limit of approximations, such that  $\forall k \geq 0$ ,  $T_k(u) \in L^2((0, T), \mathbb{H}_0^s(\Omega))$ ,  $u \in L^q(\Omega_T)$  for all  $q \in [1, \frac{N+2s}{N})$  and  $|(-\Delta)^{\frac{s}{2}} u| \in L^r(\Omega_T)$  for all  $r \in [1, \frac{N+2s}{N+s})$ .

Here,  $T_k(\sigma) := \max\{-k, \min\{k, \sigma\}\}$  for  $k > 0$ , Taking into consideration the singular weight  $d^{-2s}$  at the boundary of  $\Omega$  and in order to prove a priori estimates for approximate problems, we will use this existence result obtained in [1].

**Theorem 2.7** (See [1, Proposition 1.10]). *theorem 2.7* Assume that  $s \in (0, 1)$  and  $\beta \in (0, 2s)$ . Then, the following problem

$$(2.7) \quad \begin{cases} (-\Delta)^s \xi = \frac{1}{d^\beta(x)} & \text{in } \Omega, \\ \phi = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique bounded distributional solution such that

- (i) if  $\beta < s$ , then  $\xi \simeq d^s$ ,
- (ii) if  $\beta = s$ , then  $\xi \simeq d^s \log(\frac{D}{d(x)})$  where  $D \gg \text{diam}(\Omega)$ ,
- (iii) if  $\beta \in (s, 2s)$ , then  $\xi \simeq d^{2s-\beta}$ .

Note that, if  $f$  and  $g$  are nonnegative functions, then by  $f \simeq g$ , we mean that there exist two positive constants  $C_1$  and  $C_2$  such that  $C_1 f \leq g \leq C_2 f$ .

Finally, we conclude this section with the next Picone's inequality that the proof can be found in [26] and [5].

**Proposition 2.8.** Consider  $u \in \mathbb{H}_0^s(\Omega)$  such that  $u > 0$  in  $\Omega$ . Then, for all  $\phi \in \mathcal{C}_0^\infty(\Omega)$ , we have

$$(2.8) \quad \int_{\Omega} \frac{(-\Delta)^s u}{u} \phi^2 dx \leq \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy.$$

■

3. THE SUBLINEAR CASE  $p < 1$  : EXISTENCE RESULTS.

In this section, we are interested in the sub-linear case, namely  $0 < p < 1$ . Contrary to the local case studied in [4], we are able here to show the existence of a solution, at least, in a weak sense.

In order to make easier the calculations and using the homogeneity, we can assume that  $\lambda = 1$ .

The main existence result of this section is the following.

**Theorem 3.1.** *Assume that  $0 < s < 1$  and  $0 < p < 1$ . Then, for all  $0 \lesssim u_0 \in L^1(\Omega)$ , Problem (1.3) has a positive distributional solution. Moreover, if  $s > \frac{1}{2}$  and  $0 < p < 2s - 1$ , we get the existence of a weak solution in the sense of Definition 2.5.*

*Proof.* : We proceed by iterations. Let  $n \geq 1$  and define  $u_n$  to be the unique positive solution to

$$(3.1) \quad \begin{cases} u_{nt} + (-\Delta)^s u_n &= \frac{u_{n-1}^p}{(d(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T \\ u_n(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) &= u_{0n}(x) & \text{in } \Omega, \end{cases}$$

where  $u_{0n} := \min(n, u_0)$  and  $u_{00} := 0$ . The existence and the uniqueness of  $u_n$  follow using classical arguments for monotone operators. Moreover  $u_n \geq 0$ . Now, taking into consideration that  $u_{0n} \uparrow u_0$  as  $n \rightarrow \infty$ , then by the comparison principle it holds that  $u_n \leq u_{n+1}$  for all  $n$ .

Let  $\xi$  be the solution to Problem (2.7) with  $\beta \in (s, \min\{2s, 1\})$ , then  $\beta < 1$ . By Theorem 2.7, it follows that  $\xi \simeq d^{2s-\beta}$ .

Using  $\xi$  as a test function in (3.1) and taking into consideration the estimate on  $\xi$  given in Theorem 2.7, it holds

$$\frac{d}{dt} \int_{\Omega} u_n \xi \, dx + \int_{\Omega} \frac{u_n}{d^{\beta}} \, dx = \int_{\Omega} \frac{u_{n-1}^p \xi}{(d(x) + \frac{1}{n})^{2s}} \, dx \leq C \int_{\Omega} \frac{u_n^p}{d^{\beta}} \, dx.$$

Since  $p < 1$ , by Young's inequality, we obtain

$$(3.2) \quad \frac{d}{dt} \int_{\Omega} u_n \xi \, dx + (1 - \varepsilon) \int_{\Omega} \frac{u_n}{d^{\beta}} \, dx \leq C(\varepsilon) \int_{\Omega} \frac{1}{d^{\beta}} \, dx.$$

Since  $\beta < 1$ , then  $\int_{\Omega} \frac{1}{d^{\beta}} \, dx < \infty$ . Choosing  $\varepsilon$  small enough and integrating (3.2) in time, we get

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \xi \, dx + (1 - \varepsilon) \int_0^T \int_{\Omega} \frac{u_n}{d^{\beta}} \, dx \, dt \leq TCC(\varepsilon) + \int_{\Omega} u_0 \xi \, dx < \infty.$$

Thus, there exists a measurable function  $u \in L^1(\Omega_T) \cap L^{\infty}((0, T), L^1_{loc}(\Omega))$  such that  $u_n \uparrow u$  strongly in  $L^1(\Omega_T)$  and in  $L^{\infty}((0, T), L^1_{loc}(\Omega))$  as  $n \rightarrow \infty$ . Moreover

$$\frac{u_n^p}{(d(x) + \frac{1}{n})^{2s}} \uparrow \frac{u^p}{d^{2s}} \quad \text{strongly in } L^1_{loc}(\Omega_T).$$

Hence  $u$  is, at least, a distributional solution to problem (1.3) with  $\frac{u}{d^{\beta}} \in L^1(\Omega_T)$ .



Let us assume now that  $s > \frac{1}{2}$  and that  $p < 2s - 1$ . Taking into consideration that  $u \in L^1(\Omega_T)$  and using Hölder's inequality, we can show that  $\frac{u^p}{d^{2s}} \in L^1(\Omega_T)$ . Finally, by Theorem 2.6, we easily obtain the desired result.  $\square$

**Remark 3.2.**

- In the local case (i.e  $s = 1$ ), the authors proved in [4] a strong nonexistence result to problem (1.3) for all  $p < 1$ . As a consequence, they get a complete and instantaneous blow-up for the approximating problems. Hence, our existence result in Theorem 3.1 shows a significative difference between the local and the nonlocal cases.
- In the linear case, as it was established in [19] a closely phenomenon occurs. More precisely, the authors proved that all functions are locally  $s$ -harmonic up to a small error. This produces more solutions in the fractional case than the local case (that disappear when letting  $s \rightarrow 1$ ).

■

#### 4. THE LINEAR CASE : EXISTENCE AND NONEXISTENCE RESULTS.

In this section, we consider the linear problem

$$(4.1) \quad \begin{cases} u_t + (-\Delta)^s u = \lambda \frac{u}{d^{2s}} & \text{in } \Omega_T, \\ u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where  $\lambda > 0$  and  $u_0$  is a measurable positive function satisfying some assumptions that we will specify later.

We will show that the rang of the parameter  $s$  will be the key point in order to show the existence or the nonexistence of solution for large value of  $\lambda$ .

**4.1. The case  $s \in (0, \frac{1}{2})$  : existence result.** The main existence result of this subsection is the following.

**Theorem 4.1.** *Assume that  $\Omega$  is a bounded regular domain of  $\mathbb{R}^N$  with  $N > 2s$ ,  $s \in (0, \frac{1}{2})$  and  $0 \not\leq u_0 \in L^1(\Omega)$ . Then, for all  $\lambda > 0$ , Problem (4.1) has a weak solution  $u \in \mathcal{C}((0, T), L^1(\Omega))$  such that*

- 1)  $\forall k \geq 0, T_k(u) \in L^2((0, T), \mathbb{H}_0^s(\Omega))$  ;
- 2)  $u \in L^q(\Omega_T)$  for all  $q \in [1, \frac{N+2s}{N}]$  ;
- 3)  $|(-\Delta)^{\frac{s}{2}} u| \in L^r(\Omega_T)$  for all  $r \in [1, \frac{N+2s}{N+s}]$ .

**Remark 4.2.**

- The above existence result holds for all bounded domains and in particular for convex domains.
- The existence result in Theorem 4.1 (at least for convex domains) is another case where local and nonlocal problems have significant difference. In fact, if  $s = 1$  and if  $\Omega$  is a convex domain, existence holds **only** if  $\lambda \leq \Lambda_1^*$ .

Before starting the proof of Theorem 4.1, we shall recall the following existence result obtained in [5, Theorem 3.1] which will have a key role in our demonstration.

**Theorem 4.3.** *Assume that  $0 < s < \frac{1}{2}$  and fix  $\alpha > 0$  be such that  $2s + 2\alpha < 1$ . Then, the problem*

$$(4.2) \quad \begin{cases} (-\Delta)^s w = \frac{1}{d^{2s+\alpha}} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique weak solution  $w$  obtained as a limit of approximate problems with  $w \in \mathbb{H}_0^s(\Omega)$ . Moreover

$$(4.3) \quad w(x) \geq C d^s(x) \quad \text{in } \Omega.$$

*Proof.* Let  $0 < s < \frac{1}{2}$  and choosing  $0 < \alpha < \frac{1-2s}{2}$ , then  $d^{-2(s+\alpha)} \in L^1(\Omega)$ . Therefore defining  $w_n$  as the unique solution to the problem

$$(4.4) \quad \begin{cases} (-\Delta)^s w_n = \frac{1}{d^{2s+\alpha} + \frac{1}{n}} & \text{in } \Omega, \\ w_n > 0 & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

Thanks to Theorem 2.6,  $w_n \uparrow w$  strongly in  $L^1(\Omega)$  where  $w$  is the unique weak solution to Problem (4.3). Now, using  $w_n$  as a test function in (4.4) and applying Hölder's inequality, we get

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy \leq C(\Omega) \left( \int_\Omega \frac{w_n^2(x)}{d^{2s}(x)} dx \right)^{\frac{1}{2}}.$$

Hence by the Hardy-Sobolev inequality (1.4), we conclude that

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy \leq C(\Omega) \text{ for all } n.$$

Thus  $w \in \mathbb{H}_0^s(\Omega)$  and the result follows.  $\square$

Now we are ready to show Theorem 4.1.

**Proof of Theorem 4.1.**

We proceed by approximation. Let  $n \in \mathbb{N}^*$  and  $u_n \in L^2((0, T), \mathbb{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$  be the positive solution to the approximate problem

$$(4.5) \quad \begin{cases} u_{nt} + (-\Delta)^s u_n = \lambda \frac{u_{n-1}}{(d(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T \\ u_n(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) = u_{0n}(x) & \text{in } \Omega, \end{cases}$$

where  $u_{0n} := T_n(u_0)$  and  $u_{00} := 0$ .

In what follows, we denote by  $C$  any positive constant that can change from one line to the other and it is independent of  $n$ .

Fix  $\alpha \in (0, \frac{1-2s}{2})$  and consider  $w \in \mathbb{H}_0^s(\Omega)$  the unique weak solution to Problem (4.2) given in Theorem 4.3.

We claim that,

$$(4.6) \quad \forall \varepsilon > 0, \forall \lambda > 0, \exists C = C(\lambda, \varepsilon) > 0 \text{ such that } \lambda < C d^{2s}(x) + \frac{\varepsilon}{d^\alpha(x)} \quad \forall x \in \Omega.$$

It is clear that (4.6) holds trivially in a neighborhood of the boundary. Now, far from the boundary we use the fact that for every compact set  $K \subset\subset \Omega$ ,  $d^{2s} > C(K)$ . Thus, by choosing  $C$  large in (4.6), the claim follows.

Taking  $w$  (the solution to Problem (4.2)) as test function in (4.5), we get

$$\frac{d}{dt} \int_{\Omega} u_n w \, dx + \int_{\Omega} w (-\Delta)^s u_n \, dx \leq \lambda \int_{\Omega} \frac{u_n w}{d^{2s}} \, dx.$$

Thus

$$(4.7) \quad \frac{d}{dt} \int_{\Omega} u_n w \, dx + \int_{\Omega} \frac{u_n}{d^{2s+\alpha}} \, dx \leq \lambda \int_{\Omega} \frac{u_n w}{d^{2s}} \, dx.$$

For  $k > 0$ , we consider the set  $A_k := \{x \in \Omega ; w(x) \geq k\}$ . Then, we get the existence of  $\hat{k} > 0$  such that  $A_k \subset B_{\hat{k}} := \{x \in \Omega ; d(x) \geq \hat{k}\}$ .

Hence, by using (4.6) we have

$$\begin{aligned} \lambda \int_{\Omega} \frac{u_n w}{d^{2s}} \, dx &= \lambda \int_{A_k} \frac{u_n w}{d^{2s}} \, dx + \lambda \int_{\Omega \setminus A_k} \frac{u_n w}{d^{2s}} \, dx \\ &\leq \lambda \int_{B_{\hat{k}}} \frac{u_n w}{d^{2s}} \, dx + \varepsilon \int_{\Omega \setminus A_k} \frac{u_n w}{d^{2s+\alpha}} \, dx + C(\varepsilon) \int_{\Omega \setminus A_k} u_n w \, dx \\ &\leq (\lambda + \hat{k}^{-2s} + C(\varepsilon)) \int_{\Omega} u_n w \, dx + \varepsilon k \int_{\Omega \setminus A_k} \frac{u_n}{d^{2s+\alpha}} \, dx. \end{aligned}$$

Going back to (4.7) and choosing  $\varepsilon$  such that  $\varepsilon k \ll 1$ , we obtain

$$(4.8) \quad \frac{d}{dt} \int_{\Omega} u_n w \, dx + (1 - \varepsilon \hat{k}) \int_{\Omega} \frac{u_n}{d^{2s+\alpha}} \, dx < C \int_{\Omega} u_n w \, dx.$$

Thus

$$\frac{d}{dt} \int_{\Omega} u_n w \, dx \leq C \int_{\Omega} u_n w \, dx.$$

By Gronwall's Lemma and integrating in time, it holds that

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) w \, dx + (1 - \varepsilon) \int_0^T \int_{\Omega} \frac{u_n}{d^{2s+\alpha}} \, dx \, dt \leq C(T) \int_{\Omega} u_0 w \, dx.$$

Thus

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) w \, dx \leq C(T) \quad \text{and} \quad \int_0^T \int_{\Omega} \frac{u_n}{d^{2s}} \, dx \, dt \leq C(T).$$

Taking into consideration that the sequence  $\{u_n\}_n$  is increasing in  $n$ , we get the existence of a measurable function  $u$  such that

- (1)  $u_n \rightarrow u$  strongly in  $L^1_{loc}(\Omega_T)$ ,
- (2)  $\frac{u_n}{d^{2s}} \rightarrow \frac{u}{d^{2s}}$  strongly in  $L^1(\Omega_T)$ .

Moreover, thanks to the above estimates, we can show that  $u \in \mathcal{C}([0, T], L^1(\Omega))$ .

Thus,  $u$  is a weak solution to problem (4.1) in the sense of Definition 2.5.

Since  $u$  is obtained as a limit of approximate problems with *monotone behavior*, then by a simple comparison principle, it follows that  $u$  is the minimal solution to Problem (4.1).

To prove the uniqueness of the solution of Problem (4.1), we argue by contradiction. Assume that  $v$  is another weak solution to Problem (4.1). It is clear that  $v \geq u$ . Define  $\mathcal{U} = v - u$ , then  $\frac{\mathcal{U}}{d^{2s}} \in L^1(\Omega_T)$  and  $\mathcal{U}$  satisfies

$$(4.9) \quad \begin{cases} \mathcal{U}_t + (-\Delta)^s \mathcal{U} &= \lambda \frac{\mathcal{U}}{d^{2s}} & \text{in } \Omega_T, \\ \mathcal{U} &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \mathcal{U}(x, 0) &= 0 & \text{in } \Omega. \end{cases}$$

Taking  $w$  (the solution to Problem (4.2)) as a test function in (4.9), then thanks to (4.6), we obtain

$$\frac{d}{dt} \int_{\Omega} \mathcal{U} w \, dx + (1 - \varepsilon) \int_{\Omega} \frac{\mathcal{U}}{d^{2s}} \, dx < C \int_{\Omega} \mathcal{U} w \, dx,$$

where  $\varepsilon \ll 1$ . Moreover  $\mathcal{U}, w \geq 0$  in  $\Omega_T$ , then

$$\frac{d}{dt} \int_{\Omega} \mathcal{U} w \, dx \leq C \int_{\Omega} \mathcal{U} w \, dx.$$

Using Gronwall's lemma, we reach that  $\mathcal{U}(t, x)w(x) = 0$  a.e. in  $\Omega$  for all  $t > 0$ . Thus  $\mathcal{U}(t, x) \equiv 0$  for all  $t \geq 0$  and the uniqueness follows.  $\square$

In the case where the data  $u_0$  has more regularity, we can improve the regularity of the solution  $u$ . More precisely, we have :

**Theorem 4.4.** *Assume that  $\Omega$  is a bounded regular domain with  $N > 2s$ ,  $s \in (0, \frac{1}{2})$  and  $\lambda > 0$ . Assume that  $0 \not\leq u_0 \in L^\sigma(\Omega, w \, dx)$  with  $\sigma > 1$  where  $w$  is the solution to Problem (4.2). Then, the unique solution  $u$  to Problem (4.1) satisfies  $u \in L^\infty((0, T), L^\sigma(\Omega, w \, dx))$ .*

*Proof.* As in the proof of Theorem 4.1, let us consider  $u_n$  the unique solution to the approximate Problem (4.5). Since  $\sigma > 1$ , then we use the Kato type inequality (??) to obtain

$$(4.10) \quad u_{nt}^\sigma + (-\Delta)^s u_n^\sigma \leq \sigma u_n^{\sigma-1} u_{nt} + \sigma u_n^{\sigma-1} (-\Delta)^s u_n.$$

By using  $u_n^{\sigma-1} w$  as a test function in (4.5) and taking into consideration estimates (4.6) and (4.10), we get

$$\begin{aligned} \frac{1}{\sigma} \frac{d}{dt} \int_{\Omega} u_n^\sigma w \, dx + \frac{1}{\sigma} \int_{\Omega} \frac{u_n^\sigma}{d^{2s+\alpha}} \, dx &\leq \frac{d}{dt} \int_{\Omega} u_n^{\sigma-1} u_{nt} w \, dx + \int_{\Omega} w u_n^{\sigma-1} (-\Delta)^s u_n \, dx \\ &\leq \lambda \int_{\Omega} \frac{u_n^\sigma w}{d^{2s}} \\ &\leq C \int_{\Omega} u_n^\sigma w \, dx + \varepsilon \int_{\Omega} \frac{u_n^\sigma}{d^{2s+\alpha}} \, dx. \end{aligned}$$

Therefore

$$\frac{1}{\sigma} \frac{d}{dt} \int_{\Omega} u_n^\sigma w \, dx + \left( \frac{1}{\sigma} - \varepsilon \right) \int_{\Omega} \frac{u_n^\sigma}{d^{2s+\alpha}} \, dx \leq C \int_{\Omega} u_n^\sigma w \, dx.$$

Hence, by choosing  $\varepsilon \ll \frac{1}{\sigma}$ , we reach that

$$\frac{d}{dt} \int_{\Omega} u_n^\sigma w \, dx \leq C \int_{\Omega} u_n^\sigma w \, dx.$$

Gronwall's inequality allows us to conclude that

$$\int_{\Omega} u_n^\sigma(x, t) w \, dx \leq \left( \int_{\Omega} u_0^\sigma w \, dx \right) e^{Ct}, \quad \forall t \geq 0.$$

Using the monotone convergence Theorem, we get the existence of a solution  $u$  such that  $u \in L^\sigma(\Omega_T, w \, dx \, dt)$  and

$$\int_0^T \int_{\Omega} \frac{u^\sigma}{d^{2s+\alpha}} \, dx \, dt \leq C(T).$$

□

In the case where  $\lambda \leq \Lambda_s(\Omega)$ , then according to the attainability of  $\Lambda_s(\Omega)$ , we are able to improve the regularity of the solution. More precisely, we have :

**Theorem 4.5.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded regular domain with  $N \geq 2s$  and  $s \in (\frac{1}{2}, 1)$ . Assume that  $0 \leq u_0 \in L^2(\Omega)$ .*

1) *If  $\Lambda_s(\Omega)$  is attained, then for all  $\lambda \leq \Lambda_s^*(\Omega)$ , Problem (4.1) has a unique positive  $u$  such that  $u \in L^2(0, T, \mathbb{H}_0^s(\Omega))$  if  $\lambda < \Lambda_s(\Omega)$  and  $u \in L^2(\Omega_T)$  if  $\lambda = \Lambda_s(\Omega)$ .*

2) *If  $\Lambda_s(\Omega) = \Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}$  (recall that in this case,  $\Lambda_s(\Omega)$  is not attained), then :*

- *if  $\lambda < \Lambda_s^*$ , then Problem (4.1) has a unique positive finite energy solution  $u \in L^2((0, T), \mathbb{H}_0^s(\Omega))$  ;*
- *if  $\lambda = \Lambda_s^*$ , then problem (4.1) has a distributional solution  $u \in L^2(\Omega_T)$ .*

*Proof.* We proceed by approximation. Let  $n \in \mathbb{N}^*$  and define  $u_n$  to be the unique nonnegative solution to the following approximate problem,

$$(4.11) \quad \begin{cases} u_{nt} + (-\Delta)^s u_n = \lambda \frac{u_{n-1}}{(d(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T, \\ u_n(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) = u_{0n}(x) & \text{in } \Omega, \end{cases}$$

where  $u_{0n} := T_n(u_0)$  and  $u_{00} := 0$ . The existence of  $u_n$  follows using classical arguments for monotone operators. It is clear that  $\{u_n\}_n$  is monotone in  $n$ .

We have to distinguish two cases :

**First case**  $\Lambda_s(\Omega) < \Lambda_s^*$ . In this case,  $\Lambda_s(\Omega)$  is attained. Using  $u_n$  as a test function in (4.11) and by Hardy's inequality, we get,

$$\begin{aligned} \frac{1}{2} \int_{\Omega} u_n^2(x, T) \, dx &+ \frac{a_{N,s}}{2} \left( 1 - \frac{\lambda}{\Lambda_s(\Omega)} \right) \int_0^T \iint_{D_\Omega} \frac{|u_n(x, t) - u_n(y, t)|^2}{|x - y|^{N+2s}} \, dx \, dy \, dt \\ &\leq \frac{1}{2} \int_{\Omega} u_0^2 \, dx. \end{aligned}$$

- If  $\lambda < \Lambda_s(\Omega)$ , we deduce the existence of a measurable function  $u \in L^2((0, T), \mathbb{H}_0^s(\Omega))$  such that  $u_n \rightharpoonup u$  weakly in  $L^2((0, T), \mathbb{H}_0^s(\Omega))$ ,  $u_n \rightarrow u$  a.e in  $\Omega_T$ , and  $\frac{u_n^2}{d^{2s}} \rightarrow \frac{u^2}{d^{2s}}$  strongly in  $L^1(\Omega_T)$ . It is not difficult to show that  $u_n \rightarrow u$  strongly in  $L^2((0, T), \mathbb{H}_0^s(\Omega))$ . Thus  $u$  is an energy solution to Problem (4.1).
- If  $\lambda = \Lambda_s(\Omega)$ , we have

$$\max_{t \in [0, T]} \int_{\Omega} u_n^2(x, t) dx \leq C.$$

Then we conclude that the sequence  $\{u_n\}_n$  is bounded in  $L^2(\Omega_T)$  and by monotonicity of  $u_n$  we get  $u_n \rightarrow u$  in  $L^2(\Omega_T)$  and  $u$  is a distributional solution to (4.1). **Second case**  $\Lambda_s(\Omega) = \Lambda_s^*$ .

If  $\lambda < \Lambda_s^*$ , then as in the first case, the existence of an energy solution follows directly by using the same a priori estimates.

Now, let us deal with the case  $\lambda = \Lambda_s^*$ . As above, using  $u_n$  as test function in (4.11), we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega} u_n^2(x, t) dx + \frac{a_{N,s}}{2} \int_0^t \int \int_{D_{\Omega}} \frac{|u_n(x, \sigma) - u_n(y, \sigma)|^2}{|x - y|^{N+2s}} dx dy d\sigma \\ \leq C_s \int_0^t \int_{\Omega} \frac{u_n^2}{d^{2s}} dx d\sigma + \frac{1}{2} \int_{\Omega} u_0^2 dx. \end{aligned}$$

Now using the improved Hardy-Sobolev inequality (2.2) (see Theorem 2.2), it holds that

$$\frac{1}{2} \int_{\Omega} u_n^2(x, t) dx + k(N, s) \int_0^t \left( \int_{\Omega} u_n^{2^*}(x, \sigma) dx d\sigma \right)^{\frac{2}{2^*}} \leq \frac{1}{2} \int_{\Omega} u_0^2 dx.$$

By Sobolev's inequality, we get

$$\frac{1}{2} \int_{\Omega} u_n^2(x, t) dx + C \int_0^t \int_{\Omega} u_n^2(x, \sigma) dx d\sigma \leq \frac{1}{2} \int_{\Omega} u_0^2 dx.$$

Hence

$$\sup_{t \in [0, T]} \int_{\Omega} u_n^2(x, t) dx \leq \int_{\Omega} u_0^2 dx.$$

Therefore, we conclude that the sequence  $\{u_n\}_n$  is bounded in  $L^2(\Omega_T)$ . As above by monotonicity of  $u_n$  we get  $u_n \rightarrow u$  in  $L^2(\Omega_T)$  and  $u$  is a distributional solution of (4.1).  $\square$

Related to the asymptotic behavior of the above solution, we have the following proposition.

**Proposition 4.6.** *Assume that  $\lambda \in (0, \Lambda_s(\Omega))$  if  $\Lambda_s(\Omega) < \Lambda_s^*$  or  $\lambda \in (0, \Lambda_s^*]$  if  $\Lambda_s(\Omega) = \Lambda_s^*$ . If  $u$  is the solution to problem (4.1) obtained above, then*

$$u(x, t) \rightarrow 0 \text{ in } L^2(\Omega) \text{ as } t \rightarrow +\infty.$$

*Proof.* The first case follows easily by using Gronwall's lemma.

Suppose that  $\lambda = \Lambda_s^*$ , then

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx \leq -\frac{a_{N,s}}{2} \int \int_{D_{\Omega}} \frac{|u(x, t) - u(y, t)|^2}{|x - y|^{N+2s}} dx dy + \Lambda_s^* \int_{\Omega} \frac{u^2}{d^{2s}} dx.$$

Thus, by improved Hardy-Sobolev inequality (2.2), we obtain,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx \leq -C \left( \int_{\Omega} u^{2^*_s} dx \right)^{\frac{2}{2^*_s}}.$$

Thus, Hölder's inequality allows us to get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx \leq -C_0 \int_{\Omega} u^2 dx.$$

Thanks to Gronwall's lemma, we obtain

$$\|u(t)\|_{L^2(\Omega)} \leq e^{-C_0 t} \|u_0\|_{L^2(\Omega)}.$$

This ends up the proof.  $\square$

**4.2. The case  $s \in [\frac{1}{2}, 1)$  : Nonexistence result.** Let us recall that, in this case and if  $\Omega$  is a regular convex domain, then Hardy-Sobolev inequality in Theorem 1.2 holds with the optimal constant  $\Lambda_s(\Omega) = \Lambda_s^*$ .

As in [15], if  $\Omega$  is a convex domain, we are able to show the next nonexistence result.

**Theorem 4.7.** *Assume that  $\frac{1}{2} \leq s < 1$ ,  $N \geq 2$ ,  $\Omega \subset \mathbb{R}^N$  is a bounded regular domain such  $\Lambda_s(\Omega) = \Lambda_s^*$  (that includes the case of convex domains). If  $\lambda > \Lambda_s^*$ , then Problem (4.1) does not have any weak positive solution.*

In order to prove Theorem 4.7, we need to introduce two useful propositions. The first one is the next version of the maximum principle.

**Proposition 4.8.** *Assume that  $v_0 \in L^1(\Omega)$  and consider  $v$  the unique solution to the problem*

$$(4.12) \quad \begin{cases} v_t + (-\Delta)^s v = 0 & \text{in } \Omega_T, \\ v(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases}$$

*Then, for all  $t > 0$ , there exist two positive constants  $C_1(t)$  and  $C_2(t)$  such that for all  $x \in \Omega$*

$$(4.13) \quad C_1(t)d^s(x) \leq v(x, t) \leq C_2(t)d^s(x).$$

*Proof.* The proof follows closely the argument used in [28] for the local case. However, for the reader's convenience, we include here some details.

Define  $\varrho$  to be the unique solution to the problem

$$(4.14) \quad \begin{cases} (-\Delta)^s \varrho = 1 & \text{in } \Omega, \\ \varrho = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then by [18], we have  $c_1 d^s(x) \leq \varrho(x) \leq c_2 d^s(x)$  with  $c_1, c_2 > 0$ . From [9], we reach that, if  $t > 0$ , then  $v(x, t) \geq C(t)\varrho$ . Hence, the left hand side inequality in (4.13) follows.

Let us now prove the right hand side inequality in (4.13). For this, we will use the regularizing effect of the fractional heat semi-group.

Fix  $t > 0$ , then

$$|v(x, t)| \leq \frac{C}{t^{\frac{N}{2s}}} \|v_0\|_{L^1(\Omega)}.$$

Consider  $\mathcal{V}$  to be the unique solution to this problem

$$(4.15) \quad \begin{cases} \mathcal{V}_t + (-\Delta)^s \mathcal{V} = 0 & \text{in } \Omega_T, \\ \mathcal{V}(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \mathcal{V}(x, 0) = 1 & \text{in } \Omega. \end{cases}$$

By using Hopf's lemma (see [20]), it follows that for all  $t > 0$

$$\left\| \frac{\mathcal{V}(\cdot, t)}{d^s} \right\|_{L^\infty(\Omega)} \leq C(t).$$

Denoting by  $\mathcal{S}$  the corresponding semi-group, then for simplicity of typing we set  $\mathcal{V}(\cdot, t) = \mathcal{S}(t)v_0(\cdot)$ .

Thanks to the semi-group properties, if  $v$  solves (4.12), then

$$\|d^{-s}v(\cdot, t)\|_{L^\infty(\Omega)} = \left\| d^{-s}\mathcal{S}\left(\frac{t}{3}\right) \left[ \mathcal{S}\left(\frac{2t}{3}\right)v_0(\cdot) \right] \right\|_{L^\infty(\Omega)} \leq C(t) \|\mathcal{S}\left(\frac{2t}{3}\right)v_0(\cdot)\|_{L^\infty(\Omega)},$$

and

$$\left\| \mathcal{S}\left(\frac{2t}{3}\right)v_0(\cdot) \right\|_{L^\infty(\Omega)} \leq C_2(t) \|\mathcal{S}\left(\frac{t}{3}\right)v_0(\cdot)\|_{L^1(\Omega)}.$$

Now, taking into consideration that

$$\begin{aligned} \left\| \mathcal{S}\left(\frac{t}{3}\right)v_0(\cdot) \right\|_{L^1(\Omega)} &= \int_{\Omega} v_0(x) d^s(x) \frac{\mathcal{S}\left(\frac{t}{3}\right)\chi_{\Omega}}{d^s(x)} dx \\ &\leq C(\Omega) \int_{\Omega} v_0(x) d^s(x) \frac{\mathcal{V}\left(x, \frac{t}{3}\right)}{d^s(x)} dx \leq C(t) \|d^s u_0\|_{L^1(\Omega)}, \end{aligned}$$

and combining with the above estimates, it follows that

$$d^{-s}v(\cdot, t) \leq \bar{C}(t) \|d^s v_0\|_{L^1(\Omega)}.$$

Hence we conclude.  $\square$

We need also the next result that extends the one obtained in [12] in the local case.

**Proposition 4.9.** *Let  $\Omega$  be a bounded regular domain such that  $\Lambda_s(\Omega) = \Lambda_s^*$ . Define*

$$\hat{\Lambda} = \inf_{\phi \in C_0^\infty(\Omega), \phi \neq 0} \frac{\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - C \int_{\Omega} \frac{\phi^2}{d^{2s}} dx}{\int_{\Omega} \phi^2 dx},$$

where  $C > \Lambda_s^*$ , then  $\hat{\Lambda} = -\infty$ .

*Proof.* We argue by contradiction. Assume that  $|\hat{\Lambda}| < \infty$ , then for all  $\phi \in C_0^\infty(\Omega)$ , we have

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - C \int_{\Omega} \frac{\phi^2}{d^{2s}} dx \geq \hat{\Lambda} \int_{\Omega} \phi^2 dx.$$



Thus  
(4.16)

$$J(\hat{\Lambda}) := \inf_{\phi \in \mathcal{C}_0^\infty(\Omega), \phi \neq 0} \frac{\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy + |\hat{\Lambda}| \int_\Omega \phi^2 dx}{\int_\Omega \frac{\phi^2}{d^{2s}} dx} \geq C > \Lambda_s^*.$$

For  $\beta > 0$ , we define the set

$$\Omega_\beta := \left\{ x \in \Omega ; \text{dist}(x, \partial\Omega) < \beta \right\},$$

then

$$J(\hat{\Lambda}) \leq \inf_{\phi \in \mathcal{C}_0^\infty(\Omega_\beta), \phi \neq 0} \frac{\frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy + |\hat{\Lambda}| \int_{\Omega_\beta} \phi^2 dx}{\int_{\Omega_\beta} \frac{\phi^2}{d^{2s}} dx}.$$

By using Hardy's inequality (1.4), we obtain

$$\begin{aligned} \int_{\Omega_\beta} \phi^2 dx &= \int_{\Omega_\beta} \frac{\phi^2}{d^{2s}} d^{2s} dx \\ &\leq \beta^{2s} \int_{\Omega_\beta} \frac{\phi^2}{d^{2s}} dx \\ &\leq \frac{\beta^{2s} a_{N,s}}{\Lambda_s^*} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy. \end{aligned}$$

Hence, for  $\phi \in \mathcal{C}_0^\infty(\Omega_\beta)$ ,

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy + |\hat{\Lambda}| \int_{\Omega_\beta} \phi^2 dx &\leq \\ \frac{a_{N,s}}{2} \left( 1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy. & \end{aligned}$$

Therefore we conclude that

$$\begin{aligned} J(\hat{\Lambda}) &\leq \left( 1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \inf_{\phi \in \mathcal{C}_0^\infty(\Omega_\beta)} \frac{\frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy}{\int_{\Omega_\beta} \frac{\phi^2}{d^{2s}} dx} \\ &\leq \left( 1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \Lambda_s^*(\Omega_\beta). \end{aligned}$$

Recall that  $\Lambda_s^*(\Omega) = \Lambda_s^*$ , then since  $\Lambda_s^*(\Omega) \leq \Lambda_s^*(\Omega_\beta) \leq \Lambda_s^*$ , it follows that  $\Lambda_s^*(\Omega_\beta) = \Lambda_s^*$ . Now, using the fact that  $C > \Lambda_s^*$ , we can choose  $\beta$  small enough such that

$$\left( 1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \Lambda_s^* < C,$$

wich gives a contradiction with (4.16). Thus  $\hat{\Lambda} = -\infty$ .  $\square$

We are now ready to prove our main nonexistence result.

**Proof of Theorem 4.7.**

Without loss of generality, we suppose that  $0 \not\leq u_0 \in L^\infty(\Omega)$ . We argue by contradiction and suppose that problem (4.1) has a positive weak solution  $u$ . Using an approximating argument, we get the existence of a minimal solution to Problem (4.1) obtained as a limit of approximating problems. We denote this minimal solution by  $u$ . Thus  $u = \lim_{n \rightarrow +\infty} u_n$  where  $u_n$  is the unique positive solution to

$$(4.17) \quad \begin{cases} u_{nt} + (-\Delta)^s u_n = \lambda a_n(x) u_n & \text{in } \Omega_T, \\ u_n(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

with  $a_n(x) = \min\{n, \frac{1}{d^{2s}}\}$ ,  $\lambda > \Lambda_s^*$ . It is clear that  $\{u_n\}_n$  is increasing in  $n$  and  $u_n \rightarrow u$  strongly in  $\mathcal{C}([0, T], L^1(\Omega))$ .

Notice that  $u(x, t) > 0$  for all  $(x, t) \in \Omega \times (0, T)$ . By applying Proposition 4.8, it follows that for all  $(x, t) \in \Omega \times [t_1, t_2]$  with  $t_1 > 0$ , we have

$$u(x, t) \geq C(t_1, t_2, \Omega) d^s(x).$$

Let  $\phi \in \mathcal{C}_0^\infty(\Omega)$ , using  $\frac{\phi^2}{u_n}$  as a test function in (4.17), we obtain

$$(4.18) \quad \int_{\Omega} \frac{u_{nt}}{u_n} \phi^2 dx + \int_{\Omega} \frac{\phi^2}{u_n} (-\Delta)^s u_n dx = \lambda \int_{\Omega} a_n(x) \phi^2 dx.$$

Using Picone's inequality stated in Proposition 2.8, it holds

$$(4.19) \quad \int_{\Omega} \frac{\phi^2}{u_n} (-\Delta)^s u_n dx \leq \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy,$$

and integrating in  $[t_1, t_2]$  with  $t_1 > 0$ , we obtain

$$\lambda \int_{\Omega} a_n(x) \phi^2 dx \leq \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{t_2 - t_1} \int_{\Omega} \ln \left( \frac{u_n(t_2)}{u_n(t_1)} \right) \phi^2 dx.$$

Notice that  $\ln \left( \frac{u_n(\cdot, t_2)}{u_n(\cdot, t_1)} \right) = \ln \left( \frac{u_n(\cdot, t_2) d(\cdot)}{u_n(\cdot, t_1) d(\cdot)} \right)$ .

Since  $u_n \uparrow u$  strongly in  $L^1(\Omega_T)$  and using the fact that  $u_n(\cdot, t) \geq C d(\cdot)$  for all  $n \geq 1$  and uniformly for  $t \in [t_1, t_2]$ , it holds that  $\left\| \ln(u_n(\cdot, t_2) d(\cdot)) \right\|_{L^\sigma(\Omega)} \leq C$  for all  $\sigma > 1$ . Thus

$$\left\| \ln \left( \frac{u_n(\cdot, t_2)}{u_n(\cdot, t_1)} \right) \right\|_{L^\sigma(\Omega)} \leq \left\| \ln(u_n(\cdot, t_2) d(\cdot)) \right\|_{L^\sigma(\Omega)} + \left\| \ln(u_n(\cdot, t_1) d(\cdot)) \right\|_{L^\sigma(\Omega)} \leq C.$$

Hence for all  $\varepsilon > 0$ , we get the existence of  $C(\varepsilon)$  such that

$$\frac{1}{t_2 - t_1} \int_{\Omega} \ln \left( \frac{u_n(t_2)}{u_n(t_1)} \right) \phi^2 dx \leq \varepsilon \frac{a_{N,s}}{2} \iint_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + C(\varepsilon) \int_{\Omega} \phi^2 dx.$$

Letting  $n \rightarrow \infty$ , it follows that for all  $\phi \in C_0^\infty(\Omega)$  with  $\phi \neq 0$ , we have

$$\frac{(1 + \varepsilon)^{\frac{a_{N,s}}{2}} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - \lambda \int_{\Omega} \frac{\phi^2}{d^{2s}} dx}{\int_{\Omega} \phi^2 dx} \geq -C(\varepsilon).$$

Since  $\lambda > \Lambda_s^*$ , we can choose  $\varepsilon$  small enough such that  $\frac{\lambda}{1 + \varepsilon} > \Lambda_s^*$ . Then, we obtain

$$\inf_{\phi \in C_0^\infty(\Omega)} \frac{\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{\lambda}{1 + \varepsilon} \int_{\Omega} \frac{\phi^2}{d^{2s}} dx}{\int_{\Omega} \phi^2 dx} \geq -\frac{C(\varepsilon)}{1 + \varepsilon} > -\infty,$$

Hence, we reach a contradiction with the result of Proposition 4.9. This ends up the proof.  $\blacksquare$

## 5. THE SUPERLINEAR CASE $p > 1$ : EXISTENCE RESULTS.

In this section, we are interested in the super-linear case, namely  $p > 1$ . The main idea to get the existence result is to find a suitable supersolution and then we proceed by iteration. The main existence result of this section is the following.

**Theorem 5.1.** *Assume that  $\Omega$  is a bounded regular domain of  $\mathbb{R}^N$  with  $N > 2s$  and  $p > 1$ . Then, there exists  $u_0 \in L^\infty(\Omega)$  such that the problem*

$$(5.1) \quad \begin{cases} u_t + (-\Delta)^s u = \frac{u^p}{d^{2s}} & \text{in } \Omega_T, \\ u(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

has a minimal positive solution  $u \in L^2((0, T), \mathbb{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ .

*Proof.* In order to prove Theorem 5.1, we use a monotony argument. Fix  $\beta > 0$  be such that  $\beta \in (0, s)$  if  $p \geq 2$  and  $\beta < (2 - p)s$  if  $p \in (1, 2)$ . Let  $\xi$  be the solution to Problem (2.7), then by Theorem 2.7, it holds that  $\xi \simeq d^s$ .

Since  $p > 1$ , by setting  $\bar{\xi} = C\xi$ , we can choose  $C$  small enough such that

$$\bar{\xi}_t + (-\Delta)^s \bar{\xi} \geq \frac{\bar{\xi}^p}{d^{2s}}.$$

It is clear that if  $u_0 \leq \bar{\xi}$ , then  $\bar{\xi}$  is supersolution to (5.1). Consider now  $\underline{\xi}$ , the unique solution to the following problem

$$(5.2) \quad \begin{cases} v_t + (-\Delta)^s v = 0 & \text{in } \Omega_T, \\ v = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ v(x, 0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Then  $\underline{\xi} \in L^2(0, T, \mathbb{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$  and  $\underline{\xi}$  is a subsolution to (5.1) with  $\underline{\xi} \leq \bar{\xi}$ .

We define now the sequence  $\{u^k\}_k$  by  $u^0 := \underline{\xi}$ , and for  $k \geq 1$ ,  $u^k$  is the unique positive solution to the problem

$$(5.3) \quad \begin{cases} (u^k)_t + (-\Delta)^s u^k &= \frac{(u^{k-1})^p}{d^{2s} + \frac{1}{k}} & \text{in } \Omega_T, \\ u^k &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u^k(x, 0) &= u_0(x) & \text{in } \Omega. \end{cases}$$

Applying the comparison principle to obtain  $\underline{\xi} \leq u^k \leq u^{k+1} \leq \bar{\xi}$  for all  $k \geq 1$ .

Thus, we get the existence of  $u \leq \bar{\xi}$  such that  $u^k \uparrow u$  strongly in  $L^\sigma(\Omega_T)$  for all  $\sigma \geq 1$  and  $u^k \rightarrow u$  strongly in  $L^2((0, T), \mathbb{H}_0^s(\Omega))$ . Then  $u$  is a bounded solution to Problem (5.1) with  $\frac{u^p}{d^{2s}} \in L^1(\Omega_T)$ . This concludes the proof.  $\square$

**Remark 5.2.** *It seems to be interesting to find **natural** class of initial data  $u_0$  such that Problem (5.1) has a weak solution in the sense that  $\frac{u^p}{d^{2s}} \in L^1(\Omega_T)$ .*

## 6. EXTENSION AND OPEN PROBLEMS.

In this section, we state some extension of the previous results to the case of the spectral Laplacian.

First let us define the space

$$(6.1) \quad \widetilde{H}^s(\Omega) := \left\{ u = \sum_{i=1}^{+\infty} a_i \phi_i \in L^2(\Omega) \text{ with } \sum_{i=1}^{\infty} \lambda_i^s a_i^2 < +\infty \right\},$$

where  $(\lambda_i, \phi_i)$  are the eigenvalues and the eigenfunctions of  $(-\Delta)$  with Dirichlet boundary conditions. The space  $\widetilde{H}^s(\Omega)$  endowed with the norm

$$\|u\|_{\widetilde{H}^s(\Omega)}^2 = \sum_{i=1}^{+\infty} \lambda_i^s a_i^2$$

is a Hilbert space.

In addition, Observe that  $\widetilde{H}^s(\Omega)$  is the interpolation space  $(\mathbb{H}_0^s(\Omega), L^2(\Omega))_{[1-s]}$ , see [6, 10, 27]. Therefore, we obtain

$$\widetilde{H}^s(\Omega) = \begin{cases} H^s(\Omega) & \text{if } 0 < s < \frac{1}{2}, \\ H_{00}^{\frac{1}{2}}(\Omega) & \text{if } s = \frac{1}{2}, \\ \mathbb{H}_0^s(\Omega) & \text{if } \frac{1}{2} < s < 1, \end{cases}$$

where

$$H_{00}^{\frac{1}{2}}(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) ; \int_{\Omega} \frac{u^2}{d(x)} < +\infty \right\}.$$

The spectral fractional Laplacian  $\mathcal{A}^s$  is defined by

$$(6.2) \quad \mathcal{A}^s(u) = \sum_{i=1}^{+\infty} \lambda_i^s a_i \phi_i.$$

As it was proved by Caffarelli-Silvestre in [16] by adding a new variable, it is possible to consider the spectral fractional Laplacian operator in a bounded domain. This extension allows us to write nonlocal problems in a local way.

More precisely, consider the problem

$$(6.3) \quad \begin{cases} \mathcal{A}^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, define  $C_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}$  and denote by  $(x, y)$  a point in  $C_\Omega$ . For  $u \in \widetilde{H}^s(\Omega)$ , we define the  $s$ -harmonic extension  $w = E_s(u)$  in  $C_\Omega$  as the solution to the problem

$$(6.4) \quad \begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } C_\Omega, \\ w = 0 & \text{in } \partial_L\Omega, \\ w = u & \text{on } \Omega \times \{0\}, \end{cases}$$

with  $\partial_L\Omega = \partial\Omega \times (0, \infty)$ . It is obvious that  $w$  belongs to the following space

$$X_0^s(C_\Omega) := \overline{C_0^\infty(C_\Omega)}^{|\cdot|_{X_0^s(C_\Omega)}} \quad \text{with } \|w\|_{X_0^s(C_\Omega)}^2 = k_s \int_{C_\Omega} y^{1-2s} |\nabla w|^2 dx dy,$$

where  $k_s$  is a normalization constant. As consequence, for all  $u \in \widetilde{H}^s(\Omega)$ , we have

$$(6.5) \quad \|E_s(u)\|_{X_0^s(C_\Omega)} = \|u\|_{\widetilde{H}^s(\Omega)}.$$

Going back to problem (6.4), we get

$$(6.6) \quad \frac{\partial w(x, y)}{\partial \nu^s} \equiv -\frac{1}{k_s} \lim_{y \rightarrow 0^+} \frac{\partial w(x, y)}{\partial y} = \mathcal{A}^s w(x, 0) = \mathcal{A}^s u(x).$$

Now, let us recall the next regularity result obtained in [18, Lemma 2.9].

**Theorem 6.1.** *Assume that  $s \in (0, 1)$  and consider  $\zeta$  to be the unique solution to problem*

$$(6.7) \quad \begin{cases} \mathcal{A}^s \zeta = 1 & \text{in } \Omega, \\ \zeta = 0 & \text{in } \partial\Omega. \end{cases}$$

*Then,  $\zeta \in C^\alpha(\bar{\Omega})$  for all  $\alpha \in (0, \min\{2s, 1\})$ . Moreover, there exists a positive constant  $C = C(\Omega, s)$  such that*

- (i)  $\zeta \leq Cd^{2s}$  if  $s < \frac{1}{2}$ ,
- (ii)  $\zeta \leq Cd^s$  if  $\frac{1}{2} \leq s < 1$ .

Set  $\psi := \zeta^a$  where  $a \in (0, 1)$ . By the Kato type inequality 2.3, we get

$$\mathcal{A}^s \psi \geq \frac{C}{\zeta^{1-a}}.$$

As in Theorem 3.1, we have the next existence result where an additional hypothesis is needed in the case where  $s < \frac{1}{2}$ .

**Theorem 6.2.** *Let  $0 < s < 1$  and consider the problem*

$$(6.8) \quad \begin{cases} u_t + \mathcal{A}^s(u) = \lambda \frac{u^p}{d^{2s}} & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \end{cases}$$

where  $0 \leq u_0 \in L^1(\Omega)$ . Then

- (1) if  $s \in (0, \frac{1}{2})$ , Problem (6.8) has a positive distributional solution for all  $p \in (0, 1)$ ;
- (2) if  $s \in [\frac{1}{2}, 1)$ , Problem (6.8) has a positive distributional solution for all  $p \in (0, 1 - s)$ .

*Proof.* As in the proof of Theorem 3.1, we proceed by approximation. Define  $u_n$  to be the unique solution to the problem

$$(6.9) \quad \begin{cases} u_{nt} + \mathcal{A}^s u_n &= \frac{u_{n-1}^p}{(d(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T \\ u_n(x, t) &= 0 & \text{in } \partial\Omega \times (0, T), \\ u_n(x, 0) &= u_{0n}(x) & \text{in } \Omega, \end{cases}$$

with  $u_{0n} := \min(n, u_0)$  and  $u_{00} := 0$ . It is clear that  $u_n \geq 0$  and  $u_n \leq u_{n+1}$  for all  $n$ .

• Let us begin by the case  $s \in (0, \frac{1}{2})$ . Define  $\psi = \zeta^a$  with  $a \in (0, 1)$  where  $\zeta$  is the unique solution to Problem (6.7). Using  $\psi$  as a test function in (6.9), we reach that

$$\frac{d}{dt} \int_{\Omega} u_n \psi \, dx + \int_{\Omega} \frac{u_n}{d^{(1-a)2s}} \, dx \leq \int_{\Omega} \frac{u_n^p \psi}{(d(x) + \frac{1}{n})^{2s}} \, dx \leq C \int_{\Omega} \frac{u_n^p}{d^{2s(1-a)}} \, dx.$$

Since  $p < 1$ , by Young's inequality, we obtain

$$(6.10) \quad \frac{d}{dt} \int_{\Omega} u_n \psi \, dx + C \int_{\Omega} \frac{u_n}{d^{2s(1-a)}} \, dx \leq \bar{C} \int_{\Omega} \frac{1}{d^{2s(1-a)}} \, dx.$$

Taking into consideration that  $2s(1-a) < 1$ , it follows that  $\int_{\Omega} \frac{1}{d^{2s(1-a)}} \, dx < +\infty$ . Thus

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \psi \, dx + C \int_0^T \int_{\Omega} \frac{u_n}{d^{2s(1-a)}} \, dx \, dt \leq CT + \int_{\Omega} u_0 \psi \, dx < +\infty.$$

Hence the rest of the proof follows as in the proof of Theorem 3.1.

• We deal now with the case  $s \in [\frac{1}{2}, 1)$ . In this case,  $\phi \leq Cd^s$ . Since  $p < 1 - s$ , we can choose  $a \in (0, 1)$  such that  $p < (1 - \frac{s}{1-s(1-a)})$ , then as above we have

$$(6.11) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} u_n \psi \, dx + C \int_{\Omega} \frac{u_n}{d^{s(1-a)}} \, dx \\ & \leq \bar{C} \int_{\Omega} \frac{u_n^p}{d^{2s-a}} \, dx \leq \varepsilon \int_{\Omega} \frac{u_n}{d^{s(1-a)}} \, dx + C(\varepsilon) \int_{\Omega} \frac{1}{d^{s(\frac{1}{p-1} + 1 - a)}} \, dx. \end{aligned}$$

But,  $p < (1 - \frac{s}{1-s(1-a)})$  implies  $s(\frac{1}{p-1} + 1 - a) < 1$ , then  $\int_{\Omega} \frac{1}{d^{s(\frac{1}{p-1} + 1 - a)}} \, dx < +\infty$ .

Hence

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \psi \, dx + C \int_0^T \int_{\Omega} \frac{u_n}{d^{s(1-a)}} \, dx \, dt \leq CT + \int_{\Omega} u_0 \psi \, dx < +\infty.$$

Then we conclude.  $\square$

**Remark 6.3.** In the case  $s > \frac{1}{2}$ , we believe that the condition  $p < 1 - s$  is technical and general existence result holds for all  $p < 1$ .

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