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► **To cite this version:**

Alain Finkel, Jérôme Leroux, Grégoire Sutre. Reachability for Two-Counter Machines with One Test and One Reset. FSTTCS 2018 - 38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, Dec 2018, Ahmedabad, India. pp.31:1-31:14, 10.4230/LIPIcs.FSTTCS.2018.31 . hal-01848554v2

**HAL Id: hal-01848554**

**<https://hal.science/hal-01848554v2>**

Submitted on 20 Dec 2018

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# 1 Reachability for Two-Counter Machines with One 2 Test and One Reset

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## 12 — Abstract —

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13 We prove that the reachability relation of two-counter machines with one zero-test and one reset  
14 is Presburger-definable and effectively computable. Our proof is based on the introduction of two  
15 classes of Presburger-definable relations effectively stable by transitive closure. This approach  
16 generalizes and simplifies the existing different proofs and it solves an open problem introduced  
17 by Finkel and Sutre in 2000.

18 **2012 ACM Subject Classification** Theory of computation → Logic → Logic and verification

19 **Keywords and phrases** Counter machine, Vector addition system, Reachability problem, Formal  
20 verification, Presburger arithmetic, Infinite-state system

21 **Digital Object Identifier** 10.4230/LIPIcs.FSTTCS.2018.31

22 **Funding** This work was supported by the grant ANR-17-CE40-0028 of the French National  
23 Research Agency ANR (project BRAVAS).

24 **Acknowledgements** The work reported was carried out in the framework of ReLaX, UMI2000  
25 (ENS Paris-Saclay, CNRS, Univ. Bordeaux, CMI, IMSc).

## 26 **1** Introduction

27 **Context** Vector addition systems with states (VASS) are equivalent to Petri nets and  
28 to counter machines without the ability to test counters for zero. Although VASS have  
29 been studied since the 1970's, they remain fascinating since there are still some important  
30 open problems like the complexity of reachability (known between EXPSpace and cubic-  
31 Ackermannian) or even an efficient (in practice) algorithm to solve reachability. In 1979,  
32 Hopcroft and Pansiot [13] gave an algorithm that computes the Presburger-definable reach-  
33 ability set of a 2-dim VASS, hence VASS in dimension 2 are more easy to verify and they  
34 enjoy interesting properties like reachability and equivalence of reachability sets, for instance,  
35 are both decidable. Unfortunately, these results do not extend in dimension 3 or for 2-dim  
36 VASS with zero-tests on the two counters: the reachability set (hence also the reachability  
37 relation) is not Presburger-definable for 3-dim VASS [13]; reachability, and all non-trivial  
38 problems, are undecidable for 2-dim VASS extended with zero-tests on the two counters.

39 In 2004, Leroux and Sutre proved that the reachability relation of a 2-dim VASS is  
40 also effectively Presburger-definable [17] and this is not a consequence of the Presburger-



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38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science  
(FSTTCS 2018).

Editors: Sumit Ganguly and Paritosh Pandya; Article No. 31; pp. 31:1–31:14



Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

<i>Class</i>	<i>Post*</i>	<i>Pre*</i>	$\overset{*}{\rightarrow}$
$\mathbf{T}_1\mathbf{Tr}_2 \simeq \mathbf{T}_{1,2} \simeq \mathbf{T}_{1,2}\mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Not Recursive	Not Recursive	Not Recursive
$\mathbf{T}_1\mathbf{R}_2 \simeq \mathbf{T}_1\mathbf{R}_{1,2}\mathbf{Tr}_1$	<b>Eff.</b> Presburger	Eff. Presburger	<b>Eff. Presburger</b>
$\mathbf{R}_{1,2}\mathbf{Tr}_1 \simeq \mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Eff. Presburger	Eff. Presburger	<b>Eff. Presburger</b>
$\mathbf{T}_1 \simeq \mathbf{T}_1\mathbf{R}_1\mathbf{Tr}_1$	Eff. Presburger	Eff. Presburger	<b>Eff. Presburger</b>
2-dim VASS	Eff. Presburger	Eff. Presburger	Eff. Presburger

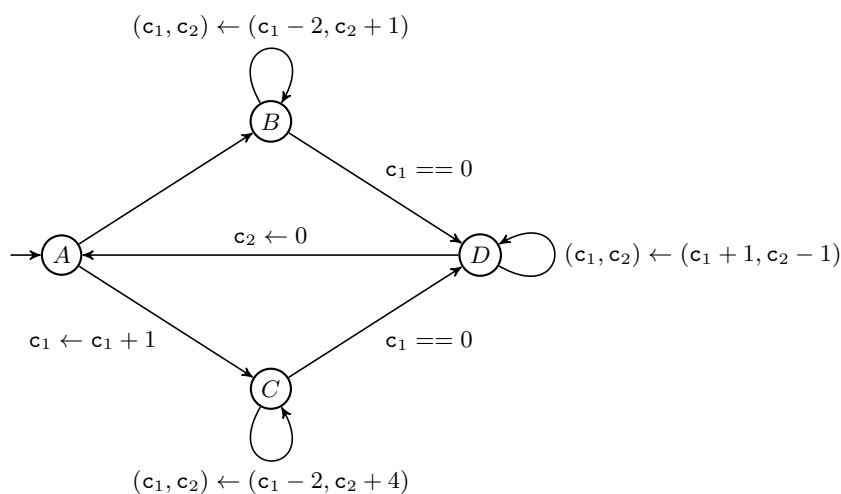
■ **Figure 1** Reachability sets ( $post^*$  and  $pre^*$ ) and reachability relation ( $\overset{*}{\rightarrow}$ ) for extensions of 2-dimensional VASS. We let  $\simeq$  denote the existence of mutual reductions between two classes of machines that preserve the effective Presburger-definability of the reachability sets and relation. The contributions of this paper are indicated in boldface.

41 definability of the reachability set. As a matter of fact, there exist counter machines (even  
42 3-dim VASS) with a Presburger-definable reachability set but with a non Presburger-definable  
43 reachability relation [13, 17]. But, for all recursive 2-dim extended VASS, the reachability  
44 sets are Presburger-definable [11, 10]. More precisely, let us denote by  $\mathbf{T}_I\mathbf{R}_J\mathbf{Tr}_K$ , with  
45  $I, J, K \subseteq \{1, 2\}$ , the class of 2-dim VASS extended with zero-tests on the  $I$ -counters, resets  
46 on the  $J$ -counters and transfers from the  $K$ -counters. For instance,  $\mathbf{T}_{\{1\}}\mathbf{R}_{\{1,2\}}\mathbf{Tr}_\emptyset$ , also  
47 written  $\mathbf{T}_1\mathbf{R}_{1,2}$  for short, is the class of 2-dim VASS extended with zero-tests on the first  
48 counter, resets on both counters, and no transfer. The relations between classes from [11] are  
49 recalled in Figure 1 and the class  $\mathbf{T}_1\mathbf{R}_2$  has been shown to be the “maximal” class having  
50 Presburger-definable  $post^*$  and  $pre^*$  reachability sets [11]. However, it was unknown whether  
51 the Presburger-definable reachability set  $post^*$  can be effectively computed or not. In fact,  
52 even the boundedness problem (is the reachability set  $post^*$  finite?) was open for this class.

53 **Contributions** Our main contribution is a proof that the reachability relation of counter  
54 machines in  $\mathbf{T}_1\mathbf{R}_2$  is effectively Presburger-definable. Our proof relies on the effective  
55 Presburger-definability of the reachability relation for 2-dim VASS [17]. The impact of our  
56 result is threefold.

- 57 ■ We solve the main *open problem* in [11] which was the question of the existence of  
58 an algorithm that computes the Presburger-definable reachability set for two-counter  
59 machines in  $\mathbf{T}_1\mathbf{R}_2$ .
- 60 ■ In fact, we prove a stronger result, namely that the *reachability relation* of counter machines  
61 in  $\mathbf{T}_1\mathbf{R}_2$  is Presburger-definable and computable. This completes the decidability picture  
62 of 2-dim extended VASS.
- 63 ■ We provide a *simple* proof of the effective Presburger-definability of the reachability  
64 relation in  $\mathbf{T}_1\mathbf{R}_2$ . As an immediate consequence, one may deduce all existing results [11,  
65 10] for 2-dim extended VASS and our proof unifies all different existing proofs on 2-dim  
66 extended VASS, including the proof in [6] that the boundedness problem is decidable for  
67 the class  $\mathbf{R}_{1,2}$  of 2-dim VASS extended with resets on both counters.

68 **Related work** VASS have been extended with resets, transfers and zero-tests. Extended  
69 VASS with resets and transfers are well structured transition systems [9] hence termination  
70 and coverability are decidable; but reachability and boundedness are undecidable (except  
71 boundedness which is decidable for extended VASS with transfers) [5, 6]. The reachability and  
72 place-boundedness problems are decidable for extended VASS with *one* zero-test [19, 3, 8, 4].  
73 Recently, Akshay et al. studied extended Petri nets with a hierarchy on places and with  
74 resets, transfers and zero-tests [1]. As a counter is a particular case of a stack, it is natural



■ **Figure 2** A 2-dimensional VASS extended with zero-tests on the first counter and resets on the second counter (shortly called TRVASS).

75 to study counter machines with one stack. Termination and boundedness are decidable for  
 76 VASS with one stack [16] but surprisingly, the decidability status of the reachability problem  
 77 is open for VASS with one stack, both in arbitrary dimension and in dimension 1. We only  
 78 know that reachability and coverability for VASS with one stack are TOWER-hard [14, 15].

79 **Outline** We present in Section 2 an example of 2-dim extended VASS in  $\mathbf{T}_1\mathbf{R}_2$ . This  
 80 example motivates the study of two classes of binary relations on natural numbers, namely  
 81 diagonal relations in Section 3 and horizontal relations in Section 4. These two classes of  
 82 relations are combined in Section 5 into a new class of one counter automata with effectively  
 83 Presburger-definable reachability relations. These automata are used in Section 6 to compute  
 84 the reachability relations of 2-dim extended VASS in  $\mathbf{T}_1\mathbf{R}_2$ .

85 For the remainder of the paper, 2-dim extended VASS in  $\mathbf{T}_1\mathbf{R}_2$  are shortly called TRVASS.

## 86 2 Motivating Example

87 Figure 2 depicts an example of a TRVASS. There are four *states*  $A$ ,  $B$ ,  $C$  and  $D$ , and two  
 88 *counters*  $c_1$  and  $c_2$ . Following the standard semantics of vector addition systems, these  
 89 counters range over natural numbers. The operations labeling the three loops and the edge  
 90 from  $A$  to  $C$  are classical addition instructions of vector addition systems. In dimension 2,  
 91 these addition instructions are always of the form  $(c_1, c_2) \leftarrow (c_1 + a_1, c_2 + a_2)$  where  $a_1$  and  
 92  $a_2$  are integer constants. For instance, the instruction  $(c_1, c_2) \leftarrow (c_1 - 2, c_2 + 1)$  labeling the  
 93 loop on  $B$  means that  $c_1$  is decremented by 2 and at the same time  $c_2$  is incremented by 1.  
 94 As the counters must remain nonnegative, this instruction may be executed (i.e., the loop on  
 95  $B$  may be taken) only if  $c_1 \geq 2$ . In addition to classical addition instructions, TRVASS may  
 96 test the first counter for zero, written  $c_1 == 0$ , and reset the second counter to zero, written  
 97  $c_2 \leftarrow 0$ .

98 The operational semantics of a TRVASS is given, as for vector addition systems, by an  
 99 infinite directed graph whose nodes are called *configurations* and whose edges are called *steps*.  
 100 Formal definitions will be given in Section 6. For the TRVASS of Figure 2, configurations

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101 are triples  $q(x_1, x_2)$  where  $q \in \{A, B, C, D\}$  is a state and  $x_1, x_2 \in \mathbb{N}$  are values of the  
 102 counters  $c_1$  and  $c_2$ , respectively. It is understood that  $\mathbb{N}$  denotes the set of natural numbers  
 103  $\{0, 1, 2, \dots\}$ . There is a step from a configuration  $p(x_1, x_2)$  to a configuration  $q(y_1, y_2)$ ,  
 104 written  $p(x_1, x_2) \rightarrow q(y_1, y_2)$ , if there is an edge from  $p$  to  $q$  labeled by an operation (1) that  
 105 can be executed from the counter values  $(x_1, x_2)$  and (2) whose execution changes the counter  
 106 values from  $(x_1, x_2)$  to  $(y_1, y_2)$ . Here, we have the steps  $B(5, 1) \rightarrow B(3, 2)$ ,  $C(0, 2) \rightarrow D(0, 2)$   
 107 and  $D(7, 3) \rightarrow A(7, 0)$ . But there is no step from  $C(1, 2)$  and there is no step to  $A(7, 1)$ .

The *reachability relation* of a TRVASS, written  $\xrightarrow{*}$ , is the reflexive-transitive closure of the step relation  $\rightarrow$ . The reachability relation is one of the main objects of interest for verification purposes. Coming back to our example of Figure 2, we have  $A(1, 0) \xrightarrow{*} A(2, 0)$  since we have the following contiguous sequence of steps:

$$A(1, 0) \rightarrow C(2, 0) \rightarrow C(0, 4) \rightarrow D(0, 4) \rightarrow D(1, 3) \rightarrow D(2, 2) \rightarrow A(2, 0)$$

By removing the steps  $\rightarrow D(1, 3) \rightarrow D(2, 2)$ , we also get that  $A(1, 0) \xrightarrow{*} A(0, 0)$ . In fact, it can be shown that  $A(1, 0) \xrightarrow{*} A(y, 0)$  for every  $y \in \mathbb{N}$ , thanks to the following pattern, where  $k$  denotes an odd natural number and  $i \in \{1, 2\}$ :

$$A(k, 0) \rightarrow C(k+1, 0) \xrightarrow{*} D(2k+2, 0) \xrightarrow{*} D(k+i, k+2-i) \xrightarrow{*} A(k+i, 0)$$

108 One may wonder whether it also holds that  $A(x, 0) \xrightarrow{*} A(y, 0)$  for every  $x, y \in \mathbb{N}$ . A  
 109 consequence of our main result (see Theorem 14) is that we can do even better: we can  
 110 compute the set of pairs  $(x, y) \in \mathbb{N} \times \mathbb{N}$  such that  $A(x, 0) \xrightarrow{*} A(y, 0)$ , as a formula in Presburger  
 111 arithmetic<sup>1</sup>.

112 ► **Remark.** It is well-known that zero-tests are more expressive than resets. Indeed, a reset  
 113  $c_1 \leftarrow 0$  can be simulated by a loop  $c_1 \leftarrow c_1 - 1$  followed by a zero-test  $c_1 == 0$ . A crucial  
 114 difference between resets and zero-tests is *monotony*. In a 2-dimensional VASS extended with  
 115 resets on both counters (shortly called RRVASS), larger counter values are always better,  
 116 in the sense that every behavior from a configuration  $q(x_1, x_2)$  can be reproduced from a  
 117 configuration  $q(x'_1, x'_2)$  with  $x'_1 \geq x_1$  and  $x'_2 \geq x_2$ . This is not true anymore in presence  
 118 of zero-tests. This difference makes the analysis of TRVASS more complex than that of  
 119 RRVASS, as illustrated in the following example. ◀

► **Example 1.** Consider the RRVASS obtained from the TRVASS of Figure 2 by replacing the two zero-tests (from  $B$  to  $D$  and from  $C$  to  $D$ ) with resets  $c_1 \leftarrow 0$ . Suppose that we want to show that  $c_1$  is unbounded in state  $A$  from  $A(1, 0)$ , i.e.,  $A(1, 0) \xrightarrow{*} A(y, 0)$  for infinitely many  $y \in \mathbb{N}$ . A natural strategy is, starting from  $A(x, 0)$  with  $x \geq 1$ , to reach  $D(0, y)$  with  $y$  as large as possible (without visiting  $A$  on the way), and then to reach  $A(y, 0)$  by taking the “transfer” loop on  $D$  as much as possible. By iterating this strategy, we get

$$A(1, 0) \xrightarrow{*} D(0, 4) \xrightarrow{*} A(4, 0) \xrightarrow{*} D(0, 8) \xrightarrow{*} A(8, 0) \xrightarrow{*} D(0, 16) \xrightarrow{*} A(16, 0) \dots$$

This witnesses that  $c_1$  is unbounded in state  $A$  from  $A(1, 0)$ . In comparison, this strategy does not work for the original TRVASS of Figure 2. Indeed, we get

$$A(1, 0) \xrightarrow{*} D(0, 4) \xrightarrow{*} A(4, 0) \xrightarrow{*} D(0, 2) \xrightarrow{*} A(2, 0) \xrightarrow{*} D(0, 1) \xrightarrow{*} A(1, 0)$$

120 by following this strategy. This is because the only way to reach  $D$  from a configuration  
 121  $A(x, 0)$  with  $x$  even is via  $B$ . ◀

<sup>1</sup> Recall that Presburger arithmetic [18] is the first-order theory of the natural numbers with addition.

122 The rest of the paper is devoted to the proof that the reachability relation of a TRVASS  
 123 is *effectively Presburger-definable*, i.e., there is an algorithm that, given a TRVASS and two  
 124 states  $p$  and  $q$ , computes a formula  $\varphi(x_1, x_2, y_1, y_2)$  in Presburger arithmetic whose models  
 125 are precisely the quadruples  $(x_1, x_2, y_1, y_2)$  of natural numbers such that  $p(x_1, x_2) \xrightarrow{*} q(y_1, y_2)$ .  
 126 It is already known that the reachability relation is effectively Presburger-definable in the  
 127 absence of zero-tests and resets [17]. Obviously, the counter  $c_1$  is zero after a zero-test  
 128  $c_1 == 0$  and, similarly, the counter  $c_2$  is zero after a reset  $c_2 \leftarrow 0$ . So we focus on the  
 129 reachability subrelations between configurations where at least one of the counters is zero,  
 130 for instance,  $\{(x, 0, 0, y) \mid p(x, 0) \xrightarrow{*} q(0, y)\}$ . Such a subrelation can be seen as a (binary)  
 131 relation on  $\mathbb{N}$ . This motivates our study in Sections 3 and 4 of two classes of relations on  $\mathbb{N}$   
 132 that naturally stem from the operational semantics of TRVASS.

### 133 3 Diagonal Relations

134 We call a relation  $R \subseteq \mathbb{N} \times \mathbb{N}$  *diagonal* when  $(x, y) \in R$  implies  $(x + c, y + c) \in R$  for every  
 135  $c \in \mathbb{N}$ . For instance, the *identity relation* on  $\mathbb{N}$ , namely  $\{(x, x) \mid x \in \mathbb{N}\}$ , is a diagonal relation.  
 136 The usual order  $\leq$  on natural numbers is also a diagonal relation. It is readily seen that the  
 137 class of diagonal relations is closed under union, intersection, composition, and transitive  
 138 closure. In this section, we show that the transitive closure of a diagonal Presburger-definable  
 139 relation is effectively Presburger-definable. Our study of diagonal relations is motivated by  
 140 the following observation.

141 ► **Remark.** The reachability subrelations  $\{(x, y) \mid p(0, x) \xrightarrow{*} q(0, y)\}$ , where  $p$  and  $q$  are  
 142 states, are diagonal in a TRVASS with no reset. Analogously, the reachability subrelations  
 143  $\{(x, y) \mid p(x, 0) \xrightarrow{*} q(y, 0)\}$  are diagonal in a TRVASS with no zero-test. ◀

144 ► **Example 2.** Let us consider the diagonal relation  $R \subseteq \mathbb{N} \times \mathbb{N}$  defined by  $(x, y) \in R$  if,  
 145 and only if, the Presburger formula  $x \leq y \wedge y \leq 2x$  holds. It is routinely checked that  
 146 the transitive closure  $R^+$  of  $R$  satisfies  $(x, y) \in R^+$  if, and only if, the Presburger formula  
 147  $(x = 0 \Leftrightarrow y = 0) \wedge x \leq y$  holds. ◀

We fix, for the remainder of this section, a diagonal relation  $R \subseteq \mathbb{N} \times \mathbb{N}$ . Consider the  
 subsets  $I_R$  and  $D_R$  of  $\mathbb{N}$  defined by

$$I_R \stackrel{\text{def}}{=} \{x \mid \exists y : (x, y) \in R \wedge x < y\} \quad D_R \stackrel{\text{def}}{=} \{y \mid \exists x : (x, y) \in R \wedge x > y\}$$

148 Since  $R$  is diagonal, the sets  $I_R$  and  $D_R$  are upward-closed, meaning that  $x \in I_R$  implies  
 149  $x' \in I_R$  for every  $x' \geq x$  (and similarly for  $D_R$ ). If  $x \in I_R$  then  $(x, x + \delta) \in R$  for some  
 150 positive integer  $\delta > 0$ . Since  $R$  is diagonal,  $(x', x' + \delta) \in R$  for every  $x' \geq x$ . So the pair  
 151  $(x, x + \delta)$  can be viewed as an “increasing loop” that applies to every  $x' \geq x$ . Similarly, if  
 152  $y \in D_R$  then there is a “decreasing loop”  $(y + \delta, y) \in R$  that applies to every  $y' \geq y$ . We are  
 153 mostly interested in increasing and decreasing loops that apply to every element of  $I_R$  and  
 154  $D_R$ , respectively. This leads us to the following definitions:

$$155 \quad \alpha \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\} & \text{if } I_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$156 \quad \beta \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\} & \text{if } D_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

157  
 158 Let us explain why the natural numbers  $\alpha$  and  $\beta$  are well-defined. If  $I_R \neq \emptyset$  then there  
 159 exists  $\delta > 0$  such that  $(m, m + \delta) \in R$  where  $m = \min I_R$ . It follows from diagonality

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of  $R$  that  $(x, x + \delta) \in R$  for every  $x \geq m$ , hence, for every  $x \in I_R$ . Therefore the set  
 $\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\}$  is non-empty, and so it has a minimum. A similar  
argument shows that  $\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\}$  is non-empty when  $D_R \neq \emptyset$ .

We are now almost ready to provide a characterization of the transitive closure of  $R^+$ .  
To do so, we introduce the relations  $\text{Inc}_R$  and  $\text{Dec}_R$  on  $\mathbb{N}$  defined by

$$\text{Inc}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (x \in I_R \wedge \exists h \in \mathbb{N} : y = x + h\alpha)$$

$$\text{Dec}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (y \in D_R \wedge \exists k \in \mathbb{N} : x = y + k\beta)$$

We let  $\circledast$  denote *relational composition* ( $S \circledast R \stackrel{\text{def}}{=} \{(x, z) \mid \exists y : x S y R z\}$ ). The *powers* of a  
relation  $R$  are inductively defined by  $R^1 \stackrel{\text{def}}{=} R$  and  $R^{n+1} \stackrel{\text{def}}{=} R \circledast R^n$ .

► **Lemma 3.** *It holds that  $R^+ = \text{Inc}_R \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R$ .*

**Proof.** We introduce the relation  $C = \text{Inc}_R \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R$ , so as to reduce  
clutter. To prove that  $C \subseteq R^+$ , we show that  $\text{Inc}_R$  and  $\text{Dec}_R$  are both contained in  $R^*$ . Let  
 $(x, y) \in \text{Inc}_R$ . If  $x = y$  then  $(x, y) \in R^*$ . Otherwise,  $x \in I_R$  and there exists  $h \in \mathbb{N}$  such that  
 $y = x + h\alpha$ . Moreover,  $h$  and  $\alpha$  are positive as  $x \neq y$ . It follows from  $x \in I_R$  and  $\alpha > 0$  that  
 $(x, x + \alpha) \in R$ . Since  $R$  is diagonal, we derive that  $(x, x + \alpha), \dots, (x + (h-1)\alpha, x + h\alpha)$  are all  
in  $R$ . Hence,  $(x, y) \in R^+$ . We have shown that  $\text{Inc}_R \subseteq R^*$ . Now let  $(x, y) \in \text{Dec}_R$ . If  $x = y$   
then  $(x, y) \in R^*$ . Otherwise,  $y \in D_R$  and there exists  $k \in \mathbb{N}$  such that  $x = y + k\beta$ . Moreover,  
 $k$  and  $\beta$  are positive as  $x \neq y$ . It follows from  $y \in D_R$  and  $\beta > 0$  that  $(y + \beta, y) \in R$ .  
Since  $R$  is diagonal, we derive that  $(y + k\beta, y + (k-1)\beta), \dots, (y + \beta, y)$  are all in  $R$ . Hence,  
 $(x, y) \in R^+$ . We have shown that  $\text{Dec}_R \subseteq R^*$ . We derive from  $\text{Inc}_R \subseteq R^*$  and  $\text{Dec}_R \subseteq R^*$   
that  $C \subseteq R^+$ .

Let us now prove the converse inclusion  $R^+ \subseteq C$ . We first observe that  $\text{Inc}_R = \text{Inc}_R^*$   
and  $\text{Dec}_R = \text{Dec}_R^*$ . These equalities easily follow from the definitions of  $\text{Inc}_R$  and  $\text{Dec}_R$ . As  
a consequence, we get that

$$C = \text{Inc}_R^* \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R^* \tag{3}$$

Let us prove by induction on  $n$  that  $R^n \subseteq C$  for all  $n \geq 1$ . The base cases  $n = 1, \dots, \alpha + \beta + 1$   
are trivial. Assume that  $R^m \subseteq C$  for all  $1 \leq m < n$ , where  $n \geq \alpha + \beta + 2$ , and let us show  
that this inclusion also holds for  $m = n$ . Let  $(x, y) \in R^n$ . There exists  $x_0, \dots, x_n$  such that  
 $x = x_0 R x_1 R \dots R x_n = y$ . We start by showing the two following properties, as they will  
be crucial for the rest of the proof.

$$x \notin I_R \implies x_0 \geq x_1 \geq \dots \geq x_n \quad \text{and} \quad y \notin D_R \implies x_0 \leq x_1 \leq \dots \leq x_n$$

We prove these properties by contraposition. If  $x_i < x_{i+1}$  for some  $0 \leq i < n$ , then we may,  
w.l.o.g., choose the first such  $i$ . This entails that  $x_0 \geq \dots \geq x_i$ . Moreover,  $x_i \in I_R$  since  
 $x_i < x_{i+1}$  and  $x_i R x_{i+1}$ . It follows that  $x = x_0 \in I_R$  as  $I_R$  is upward-closed. Similarly, if  
 $x_{i-1} > x_i$  for some  $0 < i \leq n$ , then we may, w.l.o.g., choose the last such  $i$ . This entails  
that  $x_i \leq \dots \leq x_n$ . Moreover,  $x_i \in D_R$  since  $x_{i-1} > x_i$  and  $x_{i-1} R x_i$ . It follows that  
 $y = x_n \in D_R$  as  $D_R$  is upward-closed.

To prove that  $(x, y) \in C$ , we consider four cases, depending on the membership of  $x$  in  
 $I_R$  and on the membership of  $y$  in  $D_R$ .

If  $x \notin I_R$  and  $y \notin D_R$  then  $x_0 = x_1 = \dots = x_n$ . This means in particular that  $x_0 R x_n$ ,  
hence,  $x = x_0 C x_n = y$ .

196 If  $x \notin I_R$  and  $y \in D_R$  then  $x_0 \geq x_1 \geq \dots \geq x_n$ . Note that  $\beta > 0$  since  $D_R$  is non-empty.  
 197 Since  $n \geq \beta$ , there exists  $0 \leq i < j \leq n$  such that  $x_i \equiv x_j \pmod{\beta}$ , hence,  $x_i = x_j + k\beta$  for  
 198 some  $k \in \mathbb{N}$ . Recall that  $x = x_0 R^i x_i$  and  $x_j R^{n-j} x_n = y$ . As  $R$  is diagonal, we derive that  
 199  $x_i R^{n-j} y'$  where  $y' = y + k\beta$ . We obtain that  $x R^{n+i-j} y'$ . It follows from the induction  
 200 hypothesis that  $x C y'$ . Moreover, we have  $(y', y) \in \text{Dec}_R$  since  $y \in D_R$  and  $y' = y + k\beta$ .  
 201 Hence,  $x (C \circ \text{Dec}_R) y$  and we derive from Equation 3 that  $x C y$ .

202 If  $x \in I_R$  and  $y \notin D_R$  then  $x_0 \leq x_1 \leq \dots \leq x_n$ . Note that  $\alpha > 0$  since  $I_R$  is non-empty.  
 203 Since  $n \geq \alpha$ , there exists  $0 \leq i < j \leq n$  such that  $x_i \equiv x_j \pmod{\alpha}$ , hence,  $x_j = x_i + h\alpha$   
 204 for some  $h \in \mathbb{N}$ . Recall that  $x = x_0 R^i x_i$  and  $x_j R^{n-j} x_n = y$ . As  $R$  is diagonal, we derive  
 205 that  $x' R^i x_j$  where  $x' = x + h\alpha$ . We obtain that  $x' R^{n+i-j} y$ . It follows from the induction  
 206 hypothesis that  $x' C y$ . Moreover, we have  $(x, x') \in \text{Inc}_R$  since  $x \in I_R$  and  $x' = x + h\alpha$ .  
 207 Hence,  $x (\text{Inc}_R \circ C) y$  and we derive from Equation 3 that  $x C y$ .

208 If  $x \in I_R$  and  $y \in D_R$  then both  $\alpha$  and  $\beta$  are positive. Since  $n \geq \alpha$ , there exists  
 209  $0 \leq i < j \leq n$  such that  $x_i \equiv x_j \pmod{\alpha}$ . If  $x_i \leq x_j$  then  $x_j = x_i + h\alpha$  for some  $h \in \mathbb{N}$  and  
 210 we may proceed as in the case  $x \in I_R \wedge y \notin D_R$  to show that  $x C y$ . Otherwise,  $x_i = x_j + k\alpha$   
 211 for some  $k \in \mathbb{N}$ . Recall that  $x = x_0 R^i x_i$  and  $x_j R^{n-j} x_n = y$ . As  $R$  is diagonal, we  
 212 derive that  $x' R^i z' R^{n-j} y'$  where  $x' = x + k\alpha(\beta - 1)$ ,  $z' = x_i + k\alpha(\beta - 1) = x_j + k\alpha\beta$  and  
 213  $y' = y + k\alpha\beta$ . We obtain that  $x' R^{n+i-j} y'$ . It follows from the induction hypothesis that  
 214  $x' C y'$ . Moreover, we have  $(x, x') \in \text{Inc}_R$  since  $x \in I_R$  and  $x' = x + k\alpha(\beta - 1)$ , and we also  
 215 have  $(y', y) \in \text{Dec}_R$  since  $y \in D_R$  and  $y' = y + k\alpha\beta$ . Hence,  $x (\text{Inc}_R \circ C \circ \text{Dec}_R) y$  and we  
 216 derive from Equation 3 that  $x C y$ . ◀

217 We derive the following theorem.

218 ▶ **Theorem 4.** *The transitive closure of a diagonal Presburger-definable relation is effectively*  
 219 *Presburger-definable.*

220 **Proof.** Assume that  $\varphi_R(x, y)$  is a Presburger formula denoting a diagonal relation  $R$ . The  
 221 sets  $I_R$  and  $D_R$  are defined by the Presburger formulas  $\exists y : \varphi_R(x, y) \wedge x < y$  and  $\exists x :$   
 222  $\varphi_R(x, y) \wedge x > y$ , respectively. The natural numbers  $\alpha$  and  $\beta$  defined in Equations 1 and 2  
 223 are obviously computable from  $\varphi_R$ . So the characterization given in Lemma 3 immediately  
 224 provides a computable Presburger formula denoting  $R^+$ . ◀

## 225 4 Horizontal Relations

226 A relation  $R \subseteq \mathbb{N} \times \mathbb{N}$  is said to be *horizontal* if  $(x, y) \in R$  implies  $(x + c, y) \in R$  for every  
 227  $c \in \mathbb{N}$ . The class of horizontal relations is clearly stable by union, intersection, composition,  
 228 and transitive closure. In this section we prove that the transitive closure of a horizontal  
 229 Presburger-definable relation is effectively Presburger-definable. Our study of horizontal  
 230 relations is motivated by the following observation.

231 ▶ **Remark.** The reachability subrelations  $\{(x, y) \mid p(0, x) \xrightarrow{*} \xrightarrow{c_2 \leftarrow 0} \xrightarrow{*} q(y, 0)\}$ , where  $p$  and  $q$   
 232 are states, are horizontal in a TRVASS. ◀

▶ **Example 5.** Let us consider the following horizontal relation  $R$ :

$$R \stackrel{\text{def}}{=} \{(x, y) \mid 2y \leq x \vee (y \in 4\mathbb{N} \wedge y \leq 2x + 2)\}$$

233 We prove that  $R^+$  is equal to  $C \stackrel{\text{def}}{=} \{(x, y) \mid x = 0 \Rightarrow y = 0\}$  as follows. Since  $R \subseteq C$  and  
 234  $C$  is transitive, we get  $R^+ \subseteq C$ . Conversely, let  $(x, y) \in C$ . If  $x = 0$  then  $y = 0$  and from  
 235  $(0, 0) \in R$  we derive  $(x, y) \in R^+$ . So, we can assume that  $x \geq 1$ . In that case  $(x, 4) \in R$  and



236  $(4z, 4(z+1)) \in R$  for every  $z > 0$ . It follows that  $(x, n) \in R^+$  for every  $n \in 4 + 4\mathbb{N}$ . Moreover,  
 237 there exists such an  $n$  satisfying  $2y \leq n$ . For such an  $n$ , we have  $(x, n) \in R^+$  and  $(n, y) \in R$ .  
 238 We deduce that  $(x, y) \in R^+$ . It follows that  $R^+ = C$ . ◀

239 The effective Presburger-definability of the transitive closure comes from the following  
 240 characterization.

241 ▶ **Lemma 6.** *For every horizontal relation  $R$  we have:*

$$242 \quad R^+ = \{(x, y) \mid \exists z : (z, y) \in R \wedge \forall u : x \leq u < z \Rightarrow \exists v : (u, v) \in R \wedge u < v\} \quad (4)$$

243 **Proof.** Assume first that  $(x, y) \in R^+$ . There exists a sequence  $x_0, \dots, x_k$  such that  $x =$   
 244  $x_0 R x_1 \dots R x_k = y$  with  $k \geq 1$ . Let  $z = x_{k-1}$  and let us prove that for every  $u \in$   
 245  $\{x, \dots, z-1\}$  there exists  $v > u$  such that  $(u, v) \in R$ . If  $z \leq x$  we are done. So we can  
 246 assume that  $z > x$ . Since  $x_0 \leq u$ , there exists a maximal  $j \in \{1, \dots, k\}$  such that  $x_{j-1} \leq u$ .  
 247 Let  $v = x_j$  and observe that  $(u, v) \in R$ . Since  $x_{k-1} = z > u$ , it follows that  $j < k$  and by  
 248 maximality of  $j$  we deduce that  $x_j > u$ . Therefore  $v > u$ .

249 Conversely, let us consider  $(x, y) \in \mathbb{N} \times \mathbb{N}$  such that there exists  $z$  satisfying  $(z, y) \in R$   
 250 and such that for every  $u \in \{x, \dots, z-1\}$  there exists  $v > u$  such that  $(u, v) \in R$ . Notice  
 251 that there exists a sequence  $x_0 < \dots < x_k$  with  $k \geq 0$  such that  $x = x_0 R x_1 \dots R x_k \geq z$ . It  
 252 follows that  $(x, x_k) \in R^*$ . Moreover, since  $(z, y) \in R$ ,  $z \leq x_k$ , and  $R$  is horizontal we deduce  
 253 that  $(x_k, y) \in R$ . It follows that  $(x, y) \in R^+$ . ◀

254 The previous lemma shows that the transitive closure of a horizontal relation  $R$  denoted  
 255 by a Presburger formula  $\varphi_R$  is denoted by the Presburger formula obtained from (4) by  
 256 replacing  $(z, y) \in R$  and  $(u, v) \in R$  by  $\varphi_R(z, y)$  and  $\varphi_R(u, v)$  respectively. We have proved  
 257 the following theorem.

258 ▶ **Theorem 7.** *The transitive closure of a horizontal Presburger-definable relation is effectively*  
 259 *Presburger-definable.*

## 260 5 Presburger Automata

261 We exhibit in this section a general class of one counter automata with effectively Presburger-  
 262 definable reachability relations. These automata will be used in the next section to compute  
 263 the reachability relations of TRVASS.

264 A *Presburger automaton* is a pair  $\mathcal{P} = (Q, \Delta)$  where  $Q$  is a finite set of *states*, and  $\Delta$   
 265 is a finite set of *transitions*  $(p, R, q)$  where  $p, q \in Q$  and  $R \subseteq \mathbb{N} \times \mathbb{N}$  is a relation denoted  
 266 by a Presburger formula (which is left implicit). A *configuration* is a pair  $(q, x) \in Q \times \mathbb{N}$ ,  
 267 also written as  $q(x)$  in the sequel. The *one-step* relation  $\rightarrow_{\mathcal{P}}$  is the binary relation over  
 268 configurations defined by  $p(x) \rightarrow_{\mathcal{P}} q(y)$  if there exists  $(p, R, q) \in \Delta$  such that  $(x, y) \in R$ . The  
 269 *reachability relation*  $\xrightarrow{*}_{\mathcal{P}}$  is defined as the reflexive-transitive closure of  $\rightarrow_{\mathcal{P}}$ .

270 ▶ **Remark.** The reflexive-transitive closure  $R^*$  of a Presburger-definable relation  $R \subseteq \mathbb{N} \times \mathbb{N}$   
 271 need not be Presburger-definable, in general. For instance, if  $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = 2x\}$   
 272 then  $R^*$  is the relation  $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \exists k \in \mathbb{N} : y = 2^k x\}$ , which is not definable in  
 273 Presburger arithmetic. Worse, a simple reduction from the halting problem for Minsky  
 274 machines shows that membership of a pair  $(x, y)$  in  $R^*$  is undecidable (where  $R$  is a  
 275 Presburger-definable relation given as input along with  $x$  and  $y$ ). ◀

276 A consequence of the above remark is that the reachability problem for Presburger  
 277 automata is undecidable, even if we restrict ourselves to Presburger automata with a single  
 278 state and a single transition. This comes from the fact that transitions can use arbitrary  
 279 Presburger-definable relations. We will exhibit a subclass of Presburger automata with  
 280 effectively Presburger-definable reachability relations (hence, with a decidable reachability  
 281 problem) by limiting the expressive power of the transitions occurring on cycles. We say that  
 282 a transition  $(p, R, q)$  is *diagonal* if  $R$  is diagonal, *horizontal* if  $R$  is horizontal, and *ordinary*  
 283 if it is neither diagonal nor horizontal. Note that a relation on  $\mathbb{N}$  may be both diagonal  
 284 and horizontal, for instance  $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \leq 2x\}$ . A *cycle* is non-empty sequence of  
 285 transitions  $(p_1, R_1, q_1), \dots, (p_n, R_n, q_n)$  such that  $q_n = p_1$  and  $q_i = p_{i+1}$  for all  $1 \leq i < n$ .

286 ► **Lemma 8.** *Let  $\mathcal{P}$  be a Presburger automaton. If every cycle of  $\mathcal{P}$  contains only diagonal*  
 287 *transitions then  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable.*

288 **Proof.** We first observe that  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable when  $\mathcal{P} = (Q, \Delta)$  is  
 289 a Presburger automaton whose transitions are all diagonal. Indeed, we may view  $\mathcal{P}$  as a  
 290 finite-state automaton over the finite alphabet  $\{R \mid (p, R, q) \in \Delta\}$ . For every states  $p$  and  $q$ ,  
 291 we may compute a regular expression denoting the language accepted by  $\mathcal{P}$  with initial state  $p$   
 292 and final state  $q$ . The obvious evaluation of this regular expression (concatenation  $\cdot$  becomes  
 293 relational composition  $\circ$ , sum  $+$  becomes union  $\cup$ , and star  $*$  becomes reflexive-transitive  
 294 closure  $*$ ) yields the relation  $\{(x, y) \mid p(x) \xrightarrow{*}_{\mathcal{P}} q(y)\}$ . This evaluation is computable because  
 295 Presburger-definable diagonal relations are effectively closed under union, composition and  
 296 reflexive-transitive closure (as an immediate consequence of Theorem 4). We have shown  
 297 that  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable when all transitions of  $\mathcal{P}$  are diagonal.

We now prove the lemma. Let  $\mathcal{P} = (Q, \Delta)$  be a Presburger automaton such that every  
 cycle of  $\mathcal{P}$  contains only diagonal transitions. Let  $\mathcal{N}$  be the Presburger automaton obtained  
 from  $\mathcal{P}$  by keeping only diagonal transitions. Consider two configurations  $p(x)$  and  $q(y)$ . It  
 is readily seen that  $p(x) \xrightarrow{*}_{\mathcal{P}} q(y)$  if, and only if, there exists  $1 \leq k \leq |Q|$ ,  $s_1, \dots, s_k \in Q$  and  
 $x_1, y_1, \dots, x_k, y_k \in \mathbb{N}$  such that  $p(x) = s_1(x_1)$ ,  $s_k(y_k) = q(y)$  and

$$s_1(x_1) \xrightarrow{*}_{\mathcal{P}} s_1(y_1) \rightarrow_{\mathcal{P}} s_2(x_2) \xrightarrow{*}_{\mathcal{P}} s_2(y_2) \cdots s_{k-1}(y_{k-1}) \rightarrow_{\mathcal{P}} s_k(x_k) \xrightarrow{*}_{\mathcal{P}} s_k(y_k)$$

298 Observe that for every state  $s \in Q$  and for every  $x, y \in \mathbb{N}$ ,  $s(x) \xrightarrow{*}_{\mathcal{P}} s(y)$  if, and only if,  
 299  $s(x) \xrightarrow{*}_{\mathcal{N}} s(y)$ . Moreover,  $\xrightarrow{*}_{\mathcal{N}}$  is effectively Presburger-definable since all transitions of  $\mathcal{N}$   
 300 are diagonal. We derive from the above characterization of  $\xrightarrow{*}_{\mathcal{P}}$  that  $\xrightarrow{*}_{\mathcal{P}}$  is also effectively  
 301 Presburger-definable. ◀

302 We say that a Presburger automaton  $\mathcal{P}$  is *shallow* if every cycle that contains an ordinary  
 303 transition also contains a horizontal transition. Shallowness of Presburger automata is  
 304 decidable. This follows from two easy observations. Firstly, diagonality and horizontality of  
 305 Presburger-definable relations on  $\mathbb{N}$  are decidable, since these properties can be expressed in  
 306 Presburger arithmetic. Secondly, a Presburger automaton is shallow if, and only if, every  
 307 *simple* cycle containing an ordinary transition also contains a horizontal transition. We now  
 308 show the main result of this section.

309 ► **Theorem 9.** *The reachability relation of a shallow Presburger automaton is effectively*  
 310 *Presburger-definable.*

**Proof.** By induction on the number of horizontal transitions. The base case follows from  
 Lemma 8. Indeed, if  $\mathcal{P}$  is a shallow Presburger automaton with no horizontal transition then  
 every cycle of  $\mathcal{P}$  contains only diagonal transitions. Assume that the theorem holds for every

shallow Presburger automaton with  $n$  horizontal transitions, where  $n \in \mathbb{N}$ . Let  $\mathcal{P} = (Q, \Delta)$  be a Presburger automaton with  $n + 1$  horizontal transitions. Pick a horizontal transition  $(p, R, q) \in \Delta$  and let  $\mathcal{N}$  be the Presburger automaton obtained from  $\mathcal{P}$  by removing the transition  $(p, R, q)$ . Note that, by construction,  $\mathcal{N}$  is shallow since  $\mathcal{P}$  is shallow. Let  $S$  denote the reachability relation from  $q$  to  $p$  in  $\mathcal{N}$ , namely the relation  $S = \{(y, x) \mid q(y) \xrightarrow{*} \mathcal{N} p(x)\}$ . It is readily seen that, for every configurations  $s(x)$  and  $t(y)$  of  $\mathcal{P}$ ,  $s(x) \xrightarrow{*} \mathcal{P} t(y)$  if, and only if,  $s(x) \xrightarrow{*} \mathcal{N} t(y)$  or there exists  $x', y' \in \mathbb{N}$  such that

$$s(x) \xrightarrow{*} \mathcal{N} p(x') \quad \wedge \quad (x', y') \in ((R \circledast S)^* \circledast R) \quad \wedge \quad q(y') \xrightarrow{*} \mathcal{N} t(y)$$

311 By induction hypothesis, the relation  $\xrightarrow{*} \mathcal{N}$  is effectively Presburger-definable, and so is  $R \circledast S$ .  
 312 Moreover,  $R \circledast S$  is horizontal since  $R$  is horizontal. It follows from Theorem 7 that  $(R \circledast S)^*$   
 313 is effectively Presburger-definable. We derive from the above characterization of  $\xrightarrow{*} \mathcal{P}$  that  
 314  $\xrightarrow{*} \mathcal{P}$  is also effectively Presburger-definable.  $\blacktriangleleft$

315  $\blacktriangleright$  **Remark.** The notions of diagonal relations, horizontal relations and Presburger automata  
 316 are extended to larger dimensions in the obvious way. A relation  $R \subseteq \mathbb{N}^d \times \mathbb{N}^d$  is *diagonal*  
 317 (resp. *horizontal*) if  $(\mathbf{x}, \mathbf{y}) \in R$  implies  $(\mathbf{x} + \mathbf{c}, \mathbf{y} + \mathbf{c}) \in R$  (resp.  $(\mathbf{x} + \mathbf{c}, \mathbf{y}) \in R$ ) for every  
 318  $\mathbf{c} \in \mathbb{N}^d$ . But Theorem 9 does not extend to larger dimensions, even if we restrict ourselves  
 319 to Presburger automata with a single state and a single transition. In fact, the reflexive-  
 320 transitive closure of a Presburger-definable relation that is diagonal (resp. horizontal) need  
 321 not be Presburger-definable. Consider the relation  $R \subseteq \mathbb{N}^2 \times \mathbb{N}^2$  defined by  $(x_1, x_2) R (y_1, y_2)$   
 322 if, and only if, the Presburger formula  $y_1 \leq 2x_1 \wedge y_2 < x_2$  holds. The relation  $R$  is both  
 323 diagonal and horizontal. It is routinely checked that the reflexive-transitive closure  $R^*$  is the  
 324 set of pairs  $((x_1, x_2), (y_1, y_2)) \in \mathbb{N}^2 \times \mathbb{N}^2$  such that  $y_1 \leq 2^{x_2 - y_2} x_1$  and  $y_2 \leq x_2$ , which is not  
 325 definable in Presburger arithmetic.  $\blacktriangleleft$

## 326 6 Reachability Relations of TRVASS

A TRVASS is a 2-dimensional vector addition system with states (2-dim VASS) such that the first counter can be tested for zero and the second one can be reset to zero. Formally, a TRVASS is a triple  $\mathcal{V} = (Q, \Sigma, \Delta)$  where  $Q$  is a finite set of *states*,  $\Sigma \subseteq \mathbb{Z}^2 \cup \{T, R\}$  is a finite set of *actions*, and  $\Delta \subseteq Q \times \Sigma \times Q$  is a finite set of *transitions*. A *configuration* of  $\mathcal{V}$  is a triple  $(q, x_1, x_2) \in Q \times \mathbb{N} \times \mathbb{N}$  written as  $q(x_1, x_2)$  in the sequel. The operational semantics of  $\mathcal{V}$  is given by the binary relations  $\xrightarrow{a} \mathcal{V}$  over configurations, with  $a \in \Sigma$ , defined by  $p(x_1, x_2) \xrightarrow{a} \mathcal{V} q(y_1, y_2)$  if  $(p, a, q) \in \Delta$  and

$$\begin{aligned} (y_1, y_2) &= (x_1 + a_1, x_2 + a_2) && \text{if } a = (a_1, a_2) \in \mathbb{Z}^2 \\ (y_1, y_2) &= (0, x_2) \wedge x_1 = 0 && \text{if } a = T \\ (y_1, y_2) &= (x_1, 0) && \text{if } a = R \end{aligned}$$

327 Given a word  $w = a_1 \dots a_k$  of actions  $a_j \in \Sigma$ , we denote by  $\xrightarrow{w} \mathcal{V}$  the binary relation over  
 328 configurations defined as the relational composition  $\xrightarrow{a_1} \mathcal{V} \circledast \dots \circledast \xrightarrow{a_k} \mathcal{V}$ . The relation  $\xrightarrow{\varepsilon} \mathcal{V}$   
 329 denotes the identity relation on configurations. Given a subset  $W \subseteq \Sigma^*$ , we let  $\xrightarrow{W} \mathcal{V}$  denote  
 330 the union  $\bigcup_{w \in W} \xrightarrow{w} \mathcal{V}$ . The relation  $\xrightarrow{\Sigma^*} \mathcal{V}$ , also written  $\xrightarrow{*} \mathcal{V}$ , is called the *reachability relation*  
 331 of  $\mathcal{V}$ . Observe that  $\xrightarrow{*} \mathcal{V}$  is the reflexive-transitive closure of the *step* relation  $\rightarrow \mathcal{V} \stackrel{\text{def}}{=} \bigcup_{a \in \Sigma} \xrightarrow{a} \mathcal{V}$ .

332 The remainder of this section is devoted to the proof that TRVASS have effectively  
 333 Presburger-definable reachability relations. Let us fix a TRVASS  $\mathcal{V} = (Q, \Sigma, \Delta)$ . We let  $A$   
 334 denote the set  $\Sigma \cap \mathbb{Z}^2$  of *addition vectors*.

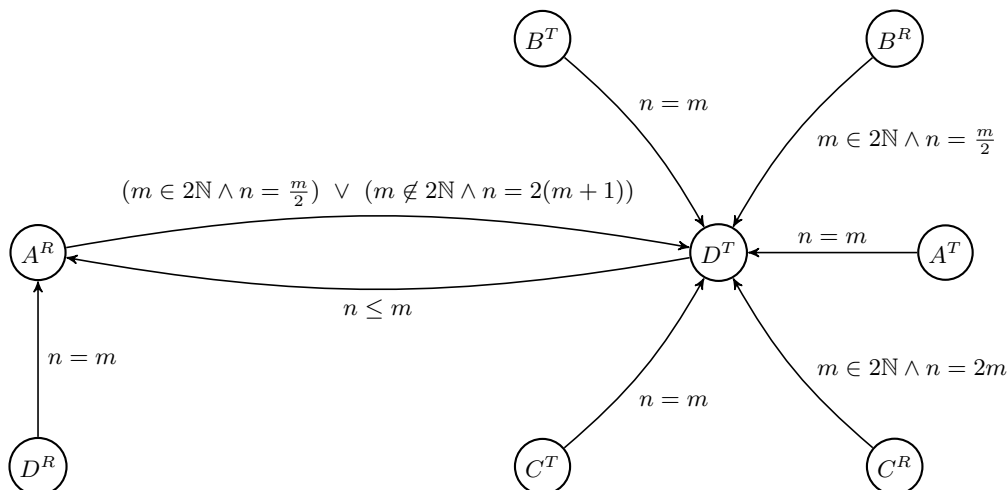
The reachability relation of  $\mathcal{V}$  can be expressed in terms of the reachability relation of a Presburger automaton by observing that configurations reachable just after a zero-test  $T$  or a reset  $R$  are restricted to  $q(0, n)$  or  $q(n, 0)$ , respectively, where  $q \in Q$  and  $n \in \mathbb{N}$ . Those configurations are parametrized by introducing the set  $S = \{q^T, q^R \mid q \in Q\}$  obtained as two disjoint copies of  $Q$ . Elements in  $\{q^T \mid q \in Q\}$  are called *test states*, and those in  $\{q^R \mid q \in Q\}$  are called *reset states*. Given  $s \in S$  and  $n \in \mathbb{N}$ , we introduce the configuration  $\llbracket s, n \rrbracket$  in  $Q \times \mathbb{N}^2$  defined as follows:

$$\llbracket s, n \rrbracket \stackrel{\text{def}}{=} \begin{cases} q(0, n) & \text{if } s = q^T \\ q(n, 0) & \text{if } s = q^R \end{cases}$$

We also introduce, for each pair  $(s, t) \in S \times S$ , the binary relation  $R_{s,t}$  defined by

$$R_{s,t} \stackrel{\text{def}}{=} \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid \llbracket s, m \rrbracket \xrightarrow{A^*X} \llbracket t, n \rrbracket\}$$

335 where  $X = T$  if  $t$  is a test state and  $X = R$  if  $t$  is a reset state. It is known that the  
 336 reachability relation of a 2-dim VASS is effectively Presburger-definable [17, 2]. This entails  
 337 that the relation  $\xrightarrow{A^*} \mathcal{V}$  is effectively Presburger-definable, and it follows that the relations  
 338  $R_{s,t}$  are also effectively Presburger-definable. We introduce the Presburger automaton  $\mathcal{P}$   
 339 with set of states  $S$  and set of transitions  $\{(s, R_{s,t}, t) \mid (s, t) \in S \times S\}$ . Note that  $\mathcal{P}$  is  
 340 computable from  $\mathcal{V}$ .



■ **Figure 3** The Presburger automaton  $\mathcal{P}$  associated to the TRVASS of Figure 2.

341 ► **Example 10.** Let us come back to the TRVASS of Figure 2. The relations  $R_{s,t}$  are all  
 342 empty except for  $R_{D^T, A^R}$ ,  $R_{D^R, A^R}$  and  $R_{s, D^T}$  with  $s \in \{A^T, A^R, B^T, B^R, C^T, C^R\}$ . The  
 343 corresponding automaton  $\mathcal{P}$  is depicted in Figure 3. Each transition  $(s, R_{s,t}, t)$  is depicted  
 344 by an edge from  $s$  to  $t$  labeled by a Presburger formula  $\varphi_{s,t}(m, n)$  denoting the relation  $R_{s,t}$ .  
 345 The empty relations (which are both diagonal and horizontal) are not depicted. Notice that  
 346 the transition from  $A^R$  to  $D^T$  is ordinary and the one from  $D^T$  to  $A^R$  is horizontal. It follows  
 347 that  $\mathcal{P}$  is shallow. We observe that the horizontal relation  $R$  defined as the composition  
 348  $R_{D^T, A^R} \circ R_{A^R, D^T}$  is the one introduced in Example 5. ◀

349 We first show that the Presburger automaton  $\mathcal{P}$  is shallow. By Theorem 9, this will entail  
 350 that its reachability relation  $\xrightarrow{*} \mathcal{P}$  is effectively Presburger-definable.

351 ► **Lemma 11.** *The Presburger automaton  $\mathcal{P}$  is shallow.*

352 **Proof.** It is readily seen that  $\mathcal{P}$  satisfies the following properties:

- 353 ■ Transitions from reset states to reset states are diagonal,
- 354 ■ Transitions from test states to reset states are horizontal,
- 355 ■ Transitions from test states to test states are diagonal.

356 It follows that an ordinary transition of  $\mathcal{P}$  is a transition from a reset state to a test state. If  
 357 a cycle contains such a transition then it must contain a transition from a test state to a  
 358 reset state as well. Since such a transition is horizontal, we obtain that  $\mathcal{P}$  is shallow. ◀

359 The two following lemmas show how to decompose the reachability relation of  $\mathcal{V}$  in terms  
 360 of the reachability relation of  $\mathcal{P}$ .

361 ► **Lemma 12.** *For every  $s, t \in S$  and  $m, n \in \mathbb{N}$ , if  $s(m) \xrightarrow{*}_{\mathcal{P}} t(n)$  then  $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$ .*

362 **Proof.** It is easily seen that  $s(m) \rightarrow_{\mathcal{P}} t(n)$  implies  $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$ , for every  $s, t \in S$  and  
 363  $m, n \in \mathbb{N}$ . We derive, by an immediate induction on  $k \geq 1$ , that  $s(m) (\rightarrow_{\mathcal{P}})^k t(n)$  implies  
 364  $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$ , for every  $s, t \in S$  and  $m, n \in \mathbb{N}$ . The lemma follows. ◀

► **Lemma 13.** *Consider two configurations  $p(x_1, x_2)$  and  $q(y_1, y_2)$  of  $\mathcal{V}$ . It holds that  
 $p(x_1, x_2) \xrightarrow{\Sigma^* \setminus A^*}_{\mathcal{V}} q(y_1, y_2)$  if, and only if, there exist  $s, t \in S$  and  $m, n \in \mathbb{N}$  such that:*

$$p(x_1, x_2) \xrightarrow{A^* \{T, R\}}_{\mathcal{V}} \llbracket s, m \rrbracket \wedge s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \wedge \llbracket t, n \rrbracket \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

**Proof.** Lemma 12 shows the “if” direction of the equivalence. For the other direction, let  
 $w \in \Sigma^* \setminus A^*$  such that  $p(x_1, x_2) \xrightarrow{w}_{\mathcal{V}} q(y_1, y_2)$ . By splitting  $w$  after each occurrence of an  
 action in  $\{T, R\}$ , we deduce that  $w = w_0 X_1 \dots w_{k-1} X_k w_k$  where  $k \geq 1$ , and  $w_0, \dots, w_k \in A^*$ .  
 Let us introduce the configurations  $c_1, \dots, c_k$  satisfying the following relations:

$$p(x_1, x_2) \xrightarrow{w_0 X_1}_{\mathcal{V}} c_1 \dots \xrightarrow{w_{k-1} X_k}_{\mathcal{V}} c_k \xrightarrow{w_k}_{\mathcal{V}} q(y_1, y_2)$$

365 Notice that  $c_j = \llbracket q_j^{X_j}, n_j \rrbracket$  for some  $q_j \in Q$  and some  $n_j \in \mathbb{N}$ . By definition of  $\mathcal{P}$ , we get  
 366  $q_{j-1}^{X_{j-1}}(n_{j-1}) \rightarrow_{\mathcal{P}} q_j^{X_j}(n_j)$  for every  $j \in \{1, \dots, k\}$ . We have proved the lemma. ◀

367 We deduce our main result.

368 ► **Theorem 14.** *The reachability relation of a TRVASS is effectively Presburger-definable.*

**Proof.** Lemma 13 shows that  $p(x_1, x_2) \xrightarrow{*}_{\mathcal{V}} q(y_1, y_2)$  if, and only if,  $p(x_1, x_2) \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$   
 or there exists  $s, t \in S$  and  $m, n \in \mathbb{N}$  such that:

$$p(x_1, x_2) \xrightarrow{A^* \{T, R\}}_{\mathcal{V}} \llbracket s, m \rrbracket \wedge s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \wedge \llbracket t, n \rrbracket \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

369 From [17, 2], the relation  $\xrightarrow{A^*}_{\mathcal{V}}$  is effectively Presburger-definable. From Lemma 11 and  
 370 Theorem 9, the relation  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable as well. ◀

371 Coming back to the classes of 2-dim extended VASS discussed in the introduction (see  
 372 Figure 1), Theorem 14 means that the reachability relation is effectively Presburger-definable  
 373 for the “maximal” class  $\mathbf{T}_1 \mathbf{R}_2$ . This result also applies to 2-dim VASS extended with resets  
 374 and transfers on both counters (i.e., the class  $\mathbf{R}_{1,2} \mathbf{Tr}_{1,2}$ ), since they can be simulated by  
 375 machines in  $\mathbf{T}_1 \mathbf{R}_2$ .

## 7 Conclusion and Open Problems

We have shown that the reachability relation of 2-dim VASS extended with tests on the first counter and resets on the second counter, is effectively Presburger-definable. This completes the decidability picture of 2-dim extended VASS initiated in [11]. Our proof techniques may also be used for other classes of counter machines where shallow Presburger automata would naturally appear. Many other problems on extensions of VASS are still interesting to solve.

- The reachability problem is NP-complete [12] for 1-dim VASS, PSPACE-complete [2] for 2-dim VASS, and NL-complete [7] for unary 2-dim VASS. But we do not know what are the complexities for the reachability problem, for the construction of the reachability set and for the reachability relation for all 2-dim extended VASS.
- The boundedness problem is undecidable for 3-dim VASS extended with resets on all counters [5] and it is decidable for arbitrary dimension VASS extended with resets on two counters [6]. Is boundedness decidable for arbitrary dimension TRVASS ?

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