



HAL
open science

Reachability for Two-Counter Machines with One Test and One Reset

Alain Finkel, Jérôme Leroux, Grégoire Sutre

► **To cite this version:**

Alain Finkel, Jérôme Leroux, Grégoire Sutre. Reachability for Two-Counter Machines with One Test and One Reset. FSTTCS 2018 - 38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, Dec 2018, Ahmedabad, India. pp.31:1-31:14, 10.4230/LIPIcs.FSTTCS.2018.31 . hal-01848554v1

HAL Id: hal-01848554

<https://hal.science/hal-01848554v1>

Submitted on 24 Jul 2018 (v1), last revised 20 Dec 2018 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 Reachability for Two-Counter Machines with One 2 Test and One Reset

3 **Alain Finkel**

4 LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay, France

5 alain.finkel@lsv.ens-paris-saclay.fr

6 **Jérôme Leroux**

7 LaBRI, Univ. Bordeaux, CNRS, Bordeaux-INP, Talence, France

8 jerome.leroux@labri.fr

9 **Grégoire Sutre**

10 LaBRI, Univ. Bordeaux, CNRS, Bordeaux-INP, Talence, France

11 gregoire.sutre@labri.fr

12 — Abstract —

13 We prove that the reachability relation of two-counter machines with one zero-test and one reset
14 is Presburger-definable and effectively computable. Our proof is based on the introduction of two
15 classes of Presburger-definable relations effectively stable by transitive closure. This approach
16 generalizes and simplifies the existing different proofs and it solves an open problem introduced
17 by Finkel and Sutre in 2000.

18 **2012 ACM Subject Classification** Theory of computation → Logic → Logic and verification

19 **Keywords and phrases** Counter machine, Vector addition system, Reachability problem, Formal
20 verification, Presburger arithmetic, Infinite-state system

21 **Digital Object Identifier** 10.4230/LIPIcs...

22 **Funding** This work was supported by the grant ANR-17-CE40-0028 of the French National
23 Research Agency ANR (project BRAVAS).

24 **1** Introduction

25 **Context** Vector addition systems with states (VASS) are equivalent to Petri nets and
26 to counter machines without the ability to test counters for zero. Although VASS have
27 been studied since the 1970's, they remain fascinating since there are still some important
28 open problems like the complexity of reachability (known between EXPSpace and cubic-
29 Ackermannian) or even an efficient (in practice) algorithm to solve reachability. In 1979,
30 Hopcroft and Pansiot [13] gave an algorithm that computes the Presburger-definable reach-
31 ability set of a 2-dim VASS, hence VASS in dimension 2 are more easy to verify and they
32 enjoy interesting properties like reachability and equivalence of reachability sets, for instance,
33 are both decidable. Unfortunately, these results do not extend in dimension 3 or for 2-dim
34 VASS with zero-tests on the two counters: the reachability set (hence also the reachability
35 relation) is not Presburger-definable for 3-dim VASS [13]; reachability, and all non-trivial
36 problems, are undecidable for 2-dim VASS extended with zero-tests on the two counters.

37 In 2004, Leroux and Sutre proved that the reachability relation of a 2-dim VASS is
38 also effectively Presburger-definable [17] and this is not a consequence of the Presburger-
39 definability of the reachability set. As a matter of fact, there exist counter machines with
40 a Presburger-definable reachability set but with a non Presburger-definable reachability



© Alain Finkel, Jérôme Leroux and Grégoire Sutre;
licensed under Creative Commons License CC-BY

Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

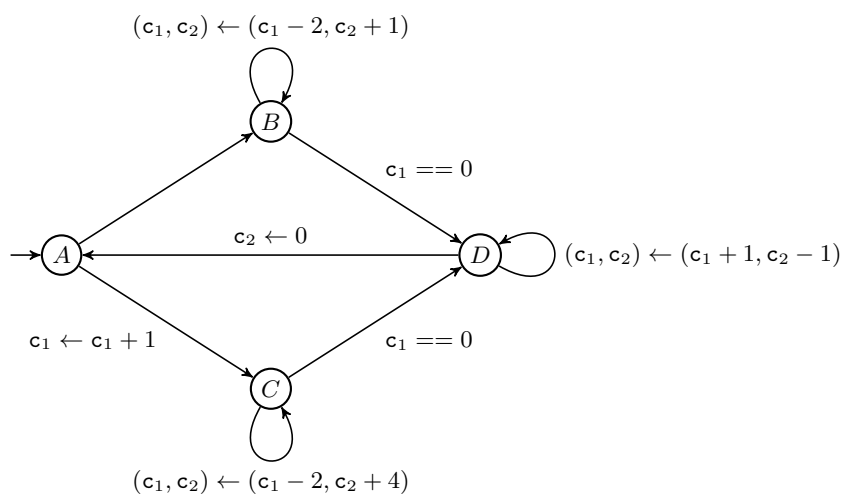
<i>Class</i>	<i>Post*</i>	<i>Pre*</i>	$\xrightarrow{*}$
$\mathbf{T}_1\mathbf{Tr}_2 \simeq \mathbf{T}_{1,2} \simeq \mathbf{T}_{1,2}\mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Not Recursive	Not Recursive	Not Recursive
$\mathbf{T}_1\mathbf{R}_2 \simeq \mathbf{T}_1\mathbf{R}_{1,2}\mathbf{Tr}_1$	Eff. Presburger	Eff. Presburger	Eff. Presburger
$\mathbf{R}_{1,2}\mathbf{Tr}_1 \simeq \mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Eff. Presburger	Eff. Presburger	Eff. Presburger
$\mathbf{T}_1 \simeq \mathbf{T}_1\mathbf{R}_1\mathbf{Tr}_1$	Eff. Presburger	Eff. Presburger	Eff. Presburger
2-dim VASS	Eff. Presburger	Eff. Presburger	Eff. Presburger

■ **Figure 1** Reachability sets ($Post^*$ and Pre^*) and reachability relation ($\xrightarrow{*}$) for extensions of 2-dimensional VASS. We let \simeq denote the existence of mutual reductions between two classes of machines that preserve the effective Presburger-definability of the reachability sets and relation. The contributions of this paper are indicated in boldface.

41 relation. But, for all recursive 2-dim extended VASSs, the reachability sets are Presburger-
42 definable [11, 10]. More precisely, let us denote by $\mathbf{T}_I\mathbf{R}_J\mathbf{Tr}_K$, with $I, J, K \subseteq \{1, 2\}$, the
43 class of 2-dim VASS extended with zero-tests on the I -counters, resets on the J -counters
44 and transfers from the K -counters. For instance, $\mathbf{T}_{\{1\}}\mathbf{R}_{\{1,2\}}\mathbf{Tr}_\emptyset$, also written $\mathbf{T}_1\mathbf{R}_{1,2}$ for
45 short, is the class of 2-dim VASS extended with zero-tests on the first counter, resets on both
46 counters, and no transfer. The relations between classes from [11] are recalled in Figure 1 and
47 the class $\mathbf{T}_1\mathbf{R}_2$ has been shown to be the “maximal” class having Presburger-definable $post^*$
48 and pre^* reachability sets [11]. However, it was unknown whether the Presburger-definable
49 reachability set $post^*$ can be effectively computed or not. In fact, even the boundedness
50 problem (is the reachability set $post^*$ finite?) was open for this class.

51 **Contributions** Our main contribution is a proof that the reachability relation of counter
52 machines in $\mathbf{T}_1\mathbf{R}_2$ is effectively Presburger-definable. The impact of our result is threefold.
53 ■ We solve the main *open problem* in [11] which was the question of the existence of
54 an algorithm that computes the Presburger-definable reachability set for two-counter
55 machines in $\mathbf{T}_1\mathbf{R}_2$.
56 ■ In fact, we prove a stronger result, namely that the *reachability relation* of counter machines
57 in $\mathbf{T}_1\mathbf{R}_2$ is Presburger-definable and computable. This completes the decidability picture
58 of 2-dim extended VASS.
59 ■ We provide a *simple* proof for the effective Presburger-definability of the reachability
60 relation in $\mathbf{T}_1\mathbf{R}_2$. As an immediate consequence, one may deduce all existing results [11,
61 10] for 2-dim extended VASS and our proof unifies all different existing proofs on 2-dim
62 extended VASS, including the proof in [6] that the boundedness problem is decidable for
63 the class $\mathbf{R}_{1,2}$ of 2-dim VASS extended with resets on both counters.

64 **Related work** VASS have been extended with resets, transfers and zero-tests. Extended
65 VASS with resets and transfers are well structured transition systems [9] hence termination
66 and coverability are decidable; but reachability and boundedness are undecidable (except
67 boundedness that is decidable for extended VASS with transfers) [5, 6]. Reachability and
68 place-boundedness problems are decidable for extended VASS with *one* zero-test [19, 3, 8, 4].
69 Recently, Akshay & al. studied extended Petri nets with a hierarchy on places and with
70 resets, transfers and zero-tests [1]. As a counter is a particular case of a stack, it is natural
71 to study counter machines with one stack. Termination and boundedness are decidable for
72 VASS with one stack [16] but surprisingly, the decidability status of the reachability problem
73 is open for VASS with one stack, both in arbitrary dimension and in dimension 1. We only
74 know that reachability and coverability for VASS with one stack are TOWER-hard [14, 15].



■ **Figure 2** A 2-dimensional VASS extended with zero-tests on the first counter and resets on the second counter (shortly called TRVASS).

75 **Outline** We present in Section 2 an example of 2-dim extended VASS in $\mathbf{T}_1\mathbf{R}_2$. This
 76 example motivates the study of two classes of binary relations on natural numbers, namely
 77 diagonal relations in Section 3 and horizontal relations in Section 4. These two classes of
 78 relations are combined in Section 5 into a new class of one counter automata with effectively
 79 Presburger-definable reachability relations. These automata are used in Section 6 to compute
 80 the reachability relations of 2-dim extended VASS in $\mathbf{T}_1\mathbf{R}_2$.

81 For the remainder of the paper, 2-dim extended VASS in $\mathbf{T}_1\mathbf{R}_2$ are shortly called TRVASS.

82 2 Motivating Example

83 Figure 2 depicts an example of a TRVASS. There are four *states* A, B, C and D , and two
 84 *counters* c_1 and c_2 . Following the standard semantics of vector addition systems, these
 85 counters range over natural numbers. The operations labeling the three loops and the edge
 86 from A to C are classical addition instructions of vector addition systems. In dimension 2,
 87 these addition instructions are always of the form $(c_1, c_2) \leftarrow (c_1 + a_1, c_2 + a_2)$ where a_1 and
 88 a_2 are integer constants. For instance, the instruction $(c_1, c_2) \leftarrow (c_1 - 2, c_2 + 1)$ labeling the
 89 loop on B means that c_1 is decremented by 2 and at the same time c_2 is incremented by 1.
 90 As the counters must remain nonnegative, this instruction may be executed (i.e., the loop on
 91 B may be taken) only if $c_1 \geq 2$. In addition to classical addition instructions, TRVASS may
 92 test the first counter for zero, written $c_1 == 0$, and reset the second counter to zero, written
 93 $c_2 \leftarrow 0$.

94 The operational semantics of a TRVASS is given, as for vector addition systems, by an
 95 infinite directed graph whose nodes are called *configurations* and whose edges are called *steps*.
 96 Formal definitions will be given in Section 6. For the TRVASS of Figure 2, configurations
 97 are triples $q(x_1, x_2)$ where $q \in \{A, B, C, D\}$ is a state and $x_1, x_2 \in \mathbb{N}$ are values of the
 98 counters c_1 and c_2 , respectively. It is understood that \mathbb{N} denotes the set of natural numbers
 99 $\{0, 1, 2, \dots\}$. There is a step from a configuration $p(x_1, x_2)$ to a configuration $q(y_1, y_2)$,
 100 written $p(x_1, x_2) \rightarrow q(y_1, y_2)$, if there is an edge from p to q labeled by an operation (1) that
 101 can be executed from the counter values (x_1, x_2) and (2) whose execution changes the counter

XX:4 Reachability for Test/Reset Two-Counter Machines

102 values from (x_1, x_2) to (y_1, y_2) . Here, we have the steps $B(5, 1) \rightarrow B(3, 2)$, $C(0, 2) \rightarrow D(0, 2)$
 103 and $D(7, 3) \rightarrow A(7, 0)$. But there is no step from $C(1, 2)$ and there is no step to $A(7, 1)$.

The *reachability relation* of a TRVASS, written $\overset{*}{\rightarrow}$, is the reflexive-transitive closure of the step relation \rightarrow . The reachability relation is one of the main objects of interest for verification purposes. Coming back to our example of Figure 2, we have $A(1, 0) \overset{*}{\rightarrow} A(2, 0)$ since we have the following contiguous sequence of steps:

$$A(1, 0) \rightarrow C(2, 0) \rightarrow C(0, 4) \rightarrow D(0, 4) \rightarrow D(1, 3) \rightarrow D(2, 2) \rightarrow A(2, 0)$$

By removing the steps $\rightarrow D(1, 3) \rightarrow D(2, 2)$, we also get that $A(1, 0) \overset{*}{\rightarrow} A(0, 0)$. In fact, it can be shown that $A(1, 0) \overset{*}{\rightarrow} A(y, 0)$ for every $y \in \mathbb{N}$, thanks to the following pattern, where k denotes an odd natural number and $i \in \{1, 2\}$:

$$A(k, 0) \rightarrow C(k + 1, 0) \overset{*}{\rightarrow} D(2k + 2, 0) \overset{*}{\rightarrow} D(k + i, k + 2 - i) \overset{*}{\rightarrow} A(k + i, 0)$$

104 One may wonder whether it also holds that $A(x, 0) \overset{*}{\rightarrow} A(y, 0)$ for every $x, y \in \mathbb{N}$. A
 105 consequence of our main result (see Theorem 14) is that we can do even better: we can
 106 compute the set of pairs $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $A(x, 0) \overset{*}{\rightarrow} A(y, 0)$, as a formula in Presburger
 107 arithmetic¹.

108 ► **Remark.** It is well-known that zero-tests are more expressive than resets. Indeed, a reset
 109 $c_1 \leftarrow 0$ can be simulated by a loop $c_1 \leftarrow c_1 - 1$ followed by a zero-test $c_1 == 0$. A crucial
 110 difference between resets and zero-tests is *monotony*. In a 2-dimensional VASS extended with
 111 resets on both counters (shortly called RRVASS), larger counter values are always better,
 112 in the sense that every behavior from a configuration $q(x_1, x_2)$ can be reproduced from a
 113 configuration $q(x'_1, x'_2)$ with $x'_1 \geq x_1$ and $x'_2 \geq x_2$. This is not true anymore in presence
 114 of zero-tests. This difference makes the analysis of TRVASS more complex than that of
 115 RRVASS, as illustrated in the following example. ◀

► **Example 1.** Consider the RRVASS obtained from the TRVASS of Figure 2 by replacing the two zero-tests (from B to D and from C to D) with resets $c_1 \leftarrow 0$. Suppose that we want to show that c_1 is unbounded in state A from $A(1, 0)$, i.e., $A(1, 0) \overset{*}{\rightarrow} A(y, 0)$ for infinitely many $y \in \mathbb{N}$. A natural strategy is, starting from $A(x, 0)$ with $x \geq 1$, to reach $D(0, y)$ with y as large as possible (without visiting A on the way), and then to reach $A(y, 0)$ by taking the “transfer” loop on D as much as possible. By iterating this strategy, we get

$$A(1, 0) \overset{*}{\rightarrow} D(0, 4) \overset{*}{\rightarrow} A(4, 0) \overset{*}{\rightarrow} D(0, 8) \overset{*}{\rightarrow} A(8, 0) \overset{*}{\rightarrow} D(0, 16) \overset{*}{\rightarrow} A(16, 0) \dots$$

This witnesses that c_1 is unbounded in state A from $A(1, 0)$. In comparison, this strategy does not work for the original TRVASS of Figure 2. Indeed, we get

$$A(1, 0) \overset{*}{\rightarrow} D(0, 4) \overset{*}{\rightarrow} A(4, 0) \overset{*}{\rightarrow} D(0, 2) \overset{*}{\rightarrow} A(2, 0) \overset{*}{\rightarrow} D(0, 1) \overset{*}{\rightarrow} A(1, 0)$$

116 by following this strategy. This is because the only way to reach D from a configuration
 117 $A(x, 0)$ with x even is via B . ◀

118 The rest of the paper is devoted to the proof that the reachability relation of a TRVASS
 119 is *effectively Presburger-definable*, i.e., there is an algorithm that, given a TRVASS and two
 120 states p and q , computes a formula $\varphi(x_1, x_2, y_1, y_2)$ in Presburger arithmetic whose models

¹ Recall that Presburger arithmetic [18] is the first-order theory of the natural numbers with addition.

121 are precisely the quadruples (x_1, x_2, y_1, y_2) of natural numbers such that $p(x_1, x_2) \xrightarrow{*} q(y_1, y_2)$.
 122 It is already known that the reachability relation is effectively Presburger-definable in absence
 123 of zero-tests and resets [17]. Obviously, the counter c_1 is zero after a zero-test $c_1 == 0$
 124 and, similarly, the counter c_2 is zero after a reset $c_2 \leftarrow 0$. So we focus on the reachability
 125 subrelations between configurations where at least one of the counters is zero, for instance,
 126 $\{(x, 0, 0, y) \mid p(x, 0) \xrightarrow{*} q(0, y)\}$. Such a subrelation can be seen as a (binary) relation on \mathbb{N} .
 127 This motivates our study in Sections 3 and 4 of two classes of relations on \mathbb{N} that naturally
 128 stem from the operational semantics of TRVASS.

129 3 Diagonal Relations

130 We call a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ *diagonal* when $(x, y) \in R$ implies $(x + c, y + c) \in R$ for every
 131 $c \in \mathbb{N}$. For instance, the *identity relation* on \mathbb{N} , namely $\{(x, x) \mid x \in \mathbb{N}\}$, is a diagonal relation.
 132 The usual order \leq on natural numbers is also a diagonal relation. It is readily seen that the
 133 class of diagonal relations is closed under union, intersection, composition, and transitive
 134 closure. In this section, we show that the transitive closure of a diagonal Presburger-definable
 135 relation is effectively Presburger-definable. Our study of diagonal relations is motivated by
 136 the following observation.

137 ► **Remark.** The reachability subrelations $\{(x, y) \mid p(0, x) \xrightarrow{*} q(0, y)\}$, where p and q are
 138 states, are diagonal in a TRVASS with no reset. Analogously, the reachability subrelations
 139 $\{(x, y) \mid p(x, 0) \xrightarrow{*} q(y, 0)\}$ are diagonal in a TRVASS with no zero-test. ◀

140 ► **Example 2.** Let us consider the diagonal relation $R \subseteq \mathbb{N} \times \mathbb{N}$ defined by $(x, y) \in R$ if,
 141 and only if, the Presburger formula $x \leq y \wedge y \leq 2x$ holds. It is routinely checked that
 142 the transitive closure R^+ of R satisfies $(x, y) \in R^+$ if, and only if, the Presburger formula
 143 $(x = 0 \Leftrightarrow y = 0) \wedge x \leq y$ holds. ◀

We fix, for the remainder of this section, a diagonal relation $R \subseteq \mathbb{N} \times \mathbb{N}$. Consider the
 subsets I_R and D_R of \mathbb{N} defined by

$$I_R \stackrel{\text{def}}{=} \{x \mid \exists y : (x, y) \in R \wedge x < y\} \quad D_R \stackrel{\text{def}}{=} \{y \mid \exists x : (x, y) \in R \wedge x > y\}$$

144 Since R is diagonal, the sets I_R and D_R are upward-closed, meaning that $x \in I_R$ implies
 145 $x' \in I_R$ for every $x' \geq x$ (and similarly for D_R). If $x \in I_R$ then $(x, x + \delta) \in R$ for some
 146 positive integer $\delta > 0$. Since R is diagonal, $(x', x' + \delta) \in R$ for every $x' \geq x$. So the pair
 147 $(x, x + \delta)$ can be viewed as an “increasing loop” that applies to every $x' \geq x$. Similarly, if
 148 $y \in D_R$ then there is a “decreasing loop” $(y + \delta, y) \in R$ that applies to every $y' \geq y$. We are
 149 mostly interested in increasing and decreasing loops that apply to every element of I_R and
 150 D_R , respectively. This leads us to the following definitions:

$$151 \quad \alpha \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\} & \text{if } I_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$152 \quad \beta \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\} & \text{if } D_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

154 Let us explain why the natural numbers α and β are well-defined. If $I_R \neq \emptyset$ then there
 155 exists $\delta > 0$ such that $(m, m + \delta) \in R$ where $m = \min I_R$. It follows from diagonality
 156 of R that $(x, x + \delta) \in R$ for every $x \geq m$, hence, for every $x \in I_R$. Therefore the set

XX:6 Reachability for Test/Reset Two-Counter Machines

157 $\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\}$ is non-empty, and so it has a minimum. A similar
 158 argument shows that $\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\}$ is non-empty when $D_R \neq \emptyset$.

159 We are now almost ready to provide a characterization of the transitive closure of R^+ .
 160 To do so, we introduce the relations Inc_R and Dec_R on \mathbb{N} defined by

$$161 \quad \text{Inc}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (x \in I_R \wedge \exists h \in \mathbb{N} : y = x + h\alpha)$$

$$162 \quad \text{Dec}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (y \in D_R \wedge \exists k \in \mathbb{N} : x = y + k\beta)$$

164 We let \circledast denote *relational composition* ($S \circledast R \stackrel{\text{def}}{=} \{(x, z) \mid \exists y : x S y R z\}$). The *powers* of a
 165 relation R are inductively defined by $R^1 \stackrel{\text{def}}{=} R$ and $R^{n+1} \stackrel{\text{def}}{=} R \circledast R^n$.

166 ► **Lemma 3.** *It holds that $R^+ = \text{Inc}_R \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R$.*

167 **Proof.** We introduce the relation $C = \text{Inc}_R \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R$, so as to reduce
 168 clutter. To prove that $C \subseteq R^+$, we show that Inc_R and Dec_R are both contained in R^* . Let
 169 $(x, y) \in \text{Inc}_R$. If $x = y$ then $(x, y) \in R^*$. Otherwise, $x \in I_R$ and there exists $h \in \mathbb{N}$ such that
 170 $y = x + h\alpha$. Moreover, h and α are positive as $x \neq y$. It follows from $x \in I_R$ and $\alpha > 0$ that
 171 $(x, x + \alpha) \in R$. Since R is diagonal, we derive that $(x, x + \alpha), \dots, (x + (h-1)\alpha, x + h\alpha)$ are all
 172 in R . Hence, $(x, y) \in R^+$. We have shown that $\text{Inc}_R \subseteq R^*$. Now let $(x, y) \in \text{Dec}_R$. If $x = y$
 173 then $(x, y) \in R^*$. Otherwise, $y \in D_R$ and there exists $k \in \mathbb{N}$ such that $x = y + k\beta$. Moreover,
 174 k and β are positive as $x \neq y$. It follows from $y \in D_R$ and $\beta > 0$ that $(y + \beta, y) \in R$.
 175 Since R is diagonal, we derive that $(y + k\beta, y + (k-1)\beta), \dots, (y + \beta, y)$ are all in R . Hence,
 176 $(x, y) \in R^+$. We have shown that $\text{Dec}_R \subseteq R^*$. We derive from $\text{Inc}_R \subseteq R^*$ and $\text{Dec}_R \subseteq R^*$
 177 that $C \subseteq R^+$.

178 Let us now prove the converse inclusion $R^+ \subseteq C$. We first observe that $\text{Inc}_R = \text{Inc}_R^*$
 179 and $\text{Dec}_R = \text{Dec}_R^*$. These equalities easily follow from the definitions of Inc_R and Dec_R . As
 180 a consequence, we get that

$$181 \quad C = \text{Inc}_R^* \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R^* \tag{3}$$

Let us prove by induction on n that $R^n \subseteq C$ for all $n \geq 1$. The base cases $n = 1, \dots, \alpha + \beta + 1$
 are trivial. Assume that $R^m \subseteq C$ for all $1 \leq m < n$, where $n \geq \alpha + \beta + 2$, and let us show
 that this inclusion also holds for $m = n$. Let $(x, y) \in R^n$. There exists x_0, \dots, x_n such that
 $x = x_0 R x_1 R \dots R x_n = y$. We start by showing the two following properties, as they will
 be crucial for the rest of the proof.

$$x \notin I_R \implies x_0 \geq x_1 \geq \dots \geq x_n \quad \text{and} \quad y \notin D_R \implies x_0 \leq x_1 \leq \dots \leq x_n$$

182 We prove these properties by contraposition. If $x_i < x_{i+1}$ for some $0 \leq i < n$, then we may,
 183 w.l.o.g., choose the first such i . This entails that $x_0 \geq \dots \geq x_i$. Moreover, $x_i \in I_R$ since
 184 $x_i < x_{i+1}$ and $x_i R x_{i+1}$. It follows that $x = x_0 \in I_R$ as I_R is upward-closed. Similarly, if
 185 $x_{i-1} > x_i$ for some $0 < i \leq n$, then we may, w.l.o.g., choose the last such i . This entails
 186 that $x_i \leq \dots \leq x_n$. Moreover, $x_i \in D_R$ since $x_{i-1} > x_i$ and $x_{i-1} R x_i$. It follows that
 187 $y = x_n \in D_R$ as D_R is upward-closed.

188 To prove that $(x, y) \in C$, we consider four cases, depending on the membership of x in
 189 I_R and on the membership of y in D_R .

190 If $x \notin I_R$ and $y \notin D_R$ then $x_0 = x_1 = \dots = x_n$. This means in particular that $x_0 R x_n$,
 191 hence, $x = x_0 C x_n = y$.

192 If $x \notin I_R$ and $y \in D_R$ then $x_0 \geq x_1 \geq \dots \geq x_n$. Note that $\beta > 0$ since D_R is non-empty.
 193 Since $n \geq \beta$, there exists $0 \leq i < j \leq n$ and $k \in \mathbb{N}$ such that $x_i = x_j + k\beta$. Recall

194 that $x = x_0 R^i x_i$ and $x_j R^{n-j} x_n = y$. As R is diagonal, we derive that $x_i R^{n-j} y'$ where
 195 $y' = y + k\beta$. We obtain that $x R^{n+i-j} y'$. It follows from the induction hypothesis that $x C y'$.
 196 Moreover, we have $(y', y) \in \text{Dec}_R$ since $y \in D_R$ and $y' = y + k\beta$. Hence, $x (C \circ \text{Dec}_R) y$ and
 197 we derive from Equation 3 that $x C y$.

198 If $x \in I_R$ and $y \notin D_R$ then $x_0 \leq x_1 \leq \dots \leq x_n$. Note that $\alpha > 0$ since I_R is non-empty.
 199 Since $n \geq \alpha$, there exists $0 \leq i < j \leq n$ and $h \in \mathbb{N}$ such that $x_j = x_i + h\alpha$. Recall that
 200 $x = x_0 R^i x_i$ and $x_j R^{n-j} x_n = y$. As R is diagonal, we derive that $x' R^i x_j$ where $x' = x + h\alpha$.
 201 We obtain that $x' R^{n+i-j} y$. It follows from the induction hypothesis that $x' C y$. Moreover,
 202 we have $(x, x') \in \text{Inc}_R$ since $x \in I_R$ and $x' = x + h\alpha$. Hence, $x (\text{Inc}_R \circ C) y$ and we derive
 203 from Equation 3 that $x C y$.

204 If $x \in I_R$ and $y \in D_R$ then both α and β are positive. Since $n \geq \alpha$, there exists
 205 $0 \leq i < j \leq n$ such that $x_i \equiv x_j \pmod{\alpha}$. If $x_i \leq x_j$ then $x_j = x_i + h\alpha$ for some $h \in \mathbb{N}$ and
 206 we may proceed as in the case $x \in I_R \wedge y \notin D_R$ to show that $x C y$. Otherwise, $x_i = x_j + k\alpha$
 207 for some $k \in \mathbb{N}$. Recall that $x = x_0 R^i x_i$ and $x_j R^{n-j} x_n = y$. As R is diagonal, we
 208 derive that $x' R^i z' R^{n-j} y'$ where $x' = x + k\alpha(\beta - 1)$, $z' = x_i + k\alpha(\beta - 1) = x_j + k\alpha\beta$ and
 209 $y' = y + k\alpha\beta$. We obtain that $x' R^{n+i-j} y'$. It follows from the induction hypothesis that
 210 $x' C y'$. Moreover, we have $(x, x') \in \text{Inc}_R$ since $x \in I_R$ and $x' = x + k\alpha(\beta - 1)$, and we also
 211 have $(y', y) \in \text{Dec}_R$ since $y \in D_R$ and $y' = y + k\alpha\beta$. Hence, $x (\text{Inc}_R \circ C \circ \text{Dec}_R) y$ and we
 212 derive from Equation 3 that $x C y$. ◀

213 We derive the following theorem.

214 ▶ **Theorem 4.** *The transitive closure of a diagonal Presburger-definable relation is effectively*
 215 *Presburger-definable.*

216 **Proof.** Assume that $\varphi_R(x, y)$ is a Presburger formula denoting a diagonal relation R . The
 217 sets I_R and D_R are defined by the Presburger formulas $\exists y : \varphi_R(x, y) \wedge x < y$ and $\exists x :$
 218 $\varphi_R(x, y) \wedge x > y$, respectively. The natural numbers α and β defined in Equations 1 and 2
 219 are obviously computable from φ_R . So the characterization given in Lemma 3 immediately
 220 provides a computable Presburger formula denoting R^+ . ◀

221 4 Horizontal Relations

222 A relation $R \subseteq \mathbb{N} \times \mathbb{N}$ is said to be *horizontal* if $(x, y) \in R$ implies $(x + c, y) \in R$ for every
 223 $c \in \mathbb{N}$. The class of horizontal relations is clearly stable by union, intersection, composition,
 224 and transitive closure. In this section we prove that the transitive closure of a horizontal
 225 Presburger-definable relation is effectively Presburger-definable. Our study of horizontal
 226 relations is motivated by the following observation.

227 ▶ **Remark.** The reachability subrelations $\{(x, y) \mid p(0, x) \xrightarrow{*} \xrightarrow{c_2 \leftarrow 0} \xrightarrow{*} q(y, 0)\}$, where p and q
 228 are states, are horizontal in a TRVASS. ◀

▶ **Example 5.** Let us consider the following horizontal relation R :

$$R \stackrel{\text{def}}{=} \{(x, y) \mid 2y \leq x \vee (y \in 4\mathbb{N} \wedge y \leq 2x + 2)\}$$

229 We prove that R^+ is equal to $C \stackrel{\text{def}}{=} \{(x, y) \mid x = 0 \Rightarrow y = 0\}$ as follows. Since $R \subseteq C$ and
 230 C is transitive, we get $R^+ \subseteq C$. Conversely, let $(x, y) \in C$. If $x = 0$ then $y = 0$ and from
 231 $(0, 0) \in R$ we derive $(x, y) \in R^+$. So, we can assume that $x \geq 1$. In that case $(x, 4) \in R$ and
 232 $(4z, 4(z+1)) \in R$ for every $z > 0$. It follows that $(x, n) \in R^+$ for every $n \in 4 + 4\mathbb{N}$. Moreover,
 233 there exists such an n satisfying $2y \leq n$. For such an n , we have $(x, n) \in R^+$ and $(n, y) \in R$.
 234 We deduce that $(x, y) \in R^+$. It follows that $R^+ = C$. ◀

XX:8 Reachability for Test/Reset Two-Counter Machines

235 The effective Presburger-definability of the transitive closure comes from the following
236 characterization.

237 ► **Lemma 6.** *For every horizontal relation R we have:*

$$238 \quad R^+ = \{(x, y) \mid \exists z : (z, y) \in R \wedge \forall u : x \leq u < z \Rightarrow \exists v : (u, v) \in R \wedge u < v\} \quad (4)$$

239 **Proof.** Assume first that $(x, y) \in R^+$. There exists a sequence x_0, \dots, x_k such that $x =$
240 $x_0 R x_1 \dots R x_k = y$ with $k \geq 1$. Let $z = x_{k-1}$ and let us prove that for every $u \in$
241 $\{x, \dots, z-1\}$ there exists $v > u$ such that $(u, v) \in R$. If $z \leq x$ we are done. So we can
242 assume that $z > x$. Since $x_0 \leq u$, there exists a maximal $j \in \{1, \dots, k\}$ such that $x_{j-1} \leq u$.
243 Let $v = x_j$ and observe that $(u, v) \in R$. Since $x_{k-1} = z > u$, it follows that $j < k$ and by
244 maximality of j we deduce that $x_j > u$. Therefore $v > u$.

245 Conversely, let us consider $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that there exists z satisfying $(z, y) \in R$
246 and such that for every $u \in \{x, \dots, z-1\}$ there exists $v > u$ such that $(u, v) \in R$. Notice
247 that there exists a sequence $x_0 < \dots < x_k$ with $k \geq 0$ such that $x = x_0 R x_1 \dots R x_k \geq z$. It
248 follows that $(x, x_k) \in R^*$. Moreover, since $(z, y) \in R$, $z \leq x_k$, and R is horizontal we deduce
249 that $(x_k, y) \in R$. It follows that $(x, y) \in R^+$. ◀

250 The previous lemma shows that the transitive closure of a horizontal relation R denoted
251 by a Presburger formula φ_R is denoted by the Presburger formula obtained from (4) by
252 replacing $(z, y) \in R$ and $(u, v) \in R$ by $\varphi_R(z, y)$ and $\varphi_R(u, v)$ respectively. We have proved
253 the following theorem.

254 ► **Theorem 7.** *The transitive closure of a horizontal Presburger-definable relation is effectively*
255 *Presburger-definable.*

256 5 Presburger Automata

257 We exhibit in this section a general class of one counter automata with effectively Presburger-
258 definable reachability relations. These automata will be used in the next section to compute
259 the reachability relations of TRVASS.

260 A *Presburger automaton* is a pair $\mathcal{P} = (Q, \Delta)$ where Q is a finite set of *states*, and Δ
261 is a finite set of *transitions* (p, R, q) where $p, q \in Q$ and $R \subseteq \mathbb{N} \times \mathbb{N}$ is a relation denoted
262 by a Presburger formula (which is left implicit). A *configuration* is a pair $(q, x) \in Q \times \mathbb{N}$,
263 also written as $q(x)$ in the sequel. The *one-step* relation $\rightarrow_{\mathcal{P}}$ is the binary relation over
264 configurations defined by $p(x) \rightarrow_{\mathcal{P}} q(y)$ if there exists $(p, R, q) \in \Delta$ such that $(x, y) \in R$. The
265 *reachability relation* $\xrightarrow{*}_{\mathcal{P}}$ is defined as the reflexive-transitive closure of $\rightarrow_{\mathcal{P}}$.

266 ► **Remark.** The reflexive-transitive closure R^* of a Presburger-definable relation $R \subseteq \mathbb{N} \times \mathbb{N}$
267 need not be Presburger-definable, in general. For instance, if $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = 2x\}$
268 then R^* is the relation $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \exists k \in \mathbb{N} : y = 2^k x\}$, which is not definable in
269 Presburger arithmetic. Worse, a simple reduction from the halting problem for Minsky
270 machines shows that membership of a pair (x, y) in R^* is undecidable (where R is a
271 Presburger-definable relation given as input along with x and y). ◀

272 A consequence of the above remark is that the reachability problem for Presburger
273 automata is undecidable, even if we restrict ourselves to Presburger automata with a single
274 state and a single transition. This comes from the fact that transitions can use arbitrary
275 Presburger-definable relations. We will exhibit a subclass of Presburger automata with
276 effectively Presburger-definable reachability relations (hence, with a decidable reachability

277 problem) by limiting the expressive power of the transitions occurring on cycles. We say that
 278 a transition (p, R, q) is *diagonal* if R is diagonal, *horizontal* if R is horizontal, and *ordinary*
 279 if it is neither diagonal nor horizontal. Note that a relation on \mathbb{N} may be both diagonal
 280 and horizontal, for instance $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \leq 2x\}$. A *cycle* is non-empty sequence of
 281 transitions $(p_1, R_1, q_1), \dots, (p_n, R_n, q_n)$ such that $q_n = p_1$ and $q_i = p_{i+1}$ for all $1 \leq i < n$.

282 ► **Lemma 8.** *Let \mathcal{P} be a Presburger automaton. If every cycle of \mathcal{P} contains only diagonal*
 283 *transitions then $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable.*

284 **Proof.** We first observe that $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable when $\mathcal{P} = (Q, \Delta)$ is
 285 a Presburger automaton whose transitions are all diagonal. Indeed, we may view \mathcal{P} as a
 286 finite-state automaton over the finite alphabet $\{R \mid (p, R, q) \in \Delta\}$. For every states p and q ,
 287 we may compute a regular expression denoting the language accepted by \mathcal{P} with initial state p
 288 and final state q . The obvious evaluation of this regular expression (concatenation \cdot becomes
 289 relational composition \circ , sum $+$ becomes union \cup , and star \star becomes reflexive-transitive
 290 closure $*$) yields the relation $\{(x, y) \mid p(x) \xrightarrow{*}_{\mathcal{P}} q(y)\}$. This evaluation is computable because
 291 Presburger-definable diagonal relations are effectively closed under union, composition and
 292 reflexive-transitive closure (as an immediate consequence of Theorem 4). We have shown
 293 that $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable when all transitions of \mathcal{P} are diagonal.

We now prove the lemma. Let $\mathcal{P} = (Q, \Delta)$ be a Presburger automaton such that every
 cycle of \mathcal{P} contains only diagonal transitions. Let \mathcal{N} be the Presburger automaton obtained
 from \mathcal{P} by keeping only diagonal transitions. Consider two configurations $p(x)$ and $q(y)$. It
 is readily seen that $p(x) \xrightarrow{*}_{\mathcal{P}} q(y)$ if, and only if, there exists $1 \leq k \leq |Q|$, $s_1, \dots, s_k \in Q$ and
 $x_1, y_1, \dots, x_k, y_k \in \mathbb{N}$ such that $p(x) = s_1(x_1)$, $s_k(y_k) = q(y)$ and

$$s_1(x_1) \xrightarrow{*}_{\mathcal{P}} s_1(y_1) \rightarrow_{\mathcal{P}} s_2(x_2) \xrightarrow{*}_{\mathcal{P}} s_2(y_2) \cdots s_{k-1}(y_{k-1}) \rightarrow_{\mathcal{P}} s_k(x_k) \xrightarrow{*}_{\mathcal{P}} s_k(y_k)$$

294 Observe that for every state $s \in Q$ and for every $x, y \in \mathbb{N}$, $s(x) \xrightarrow{*}_{\mathcal{P}} s(y)$ if, and only if,
 295 $s(x) \xrightarrow{*}_{\mathcal{N}} s(y)$. Moreover, $\xrightarrow{*}_{\mathcal{N}}$ is effectively Presburger-definable since all transitions of \mathcal{N}
 296 are diagonal. We derive from the above characterization of $\xrightarrow{*}_{\mathcal{P}}$ that $\xrightarrow{*}_{\mathcal{P}}$ is also effectively
 297 Presburger-definable. ◀

298 We say that a Presburger automaton \mathcal{P} is *shallow* if every cycle that contains an ordinary
 299 transition also contains a horizontal transition. Shallowness of Presburger automata is
 300 decidable. This follows from two easy observations. Firstly, diagonality and horizontality of
 301 Presburger-definable relations on \mathbb{N} are decidable, since these properties can be expressed in
 302 Presburger arithmetic. Secondly, a Presburger automaton is shallow if, and only if, every
 303 *simple* cycle containing an ordinary transition also contains a horizontal transition. We now
 304 show the main result of this section.

305 ► **Theorem 9.** *The reachability relation of a shallow Presburger automaton is effectively*
 306 *Presburger-definable.*

Proof. By induction on the number of horizontal transitions. The base case follows from
 Lemma 8. Indeed, if \mathcal{P} is a shallow Presburger automaton with no horizontal transition then
 every cycle of \mathcal{P} contains only diagonal transitions. Assume that the theorem holds for every
 shallow Presburger automaton with n horizontal transitions, where $n \in \mathbb{N}$. Let $\mathcal{P} = (Q, \Delta)$
 be a Presburger automaton with $n + 1$ horizontal transitions. Pick a horizontal transition
 $(p, R, q) \in \Delta$ and let \mathcal{N} be the Presburger automaton obtained from \mathcal{P} by removing the
 transition (p, R, q) . Let S denote the reachability relation from q to p in \mathcal{N} , namely the

XX:10 Reachability for Test/Reset Two-Counter Machines

relation $S = \{(y, x) \mid q(y) \xrightarrow{*} p(x)\}$. It is readily seen that, for every configurations $s(x)$ and $t(y)$ of \mathcal{P} , $s(x) \xrightarrow{*} t(y)$ if, and only if, $s(x) \xrightarrow{*} t(y)$ or there exists $x', y' \in \mathbb{N}$ such that

$$s(x) \xrightarrow{*} p(x') \quad \wedge \quad (x', y') \in ((R \circledast S)^* \circledast R) \quad \wedge \quad q(y') \xrightarrow{*} t(y)$$

307 By induction hypothesis, the relation $\xrightarrow{*}$ is effectively Presburger-definable, and so is $R \circledast S$.
 308 Moreover, $R \circledast S$ is horizontal since R is horizontal. It follows from Theorem 7 that $(R \circledast S)^*$
 309 is effectively Presburger-definable. We derive from the above characterization of $\xrightarrow{*}$ that
 310 $\xrightarrow{*}$ is also effectively Presburger-definable. ◀

311 ▶ **Remark.** The notions of diagonal relations, horizontal relations and Presburger automata
 312 are extended to larger dimensions in the obvious way. A relation $R \subseteq \mathbb{N}^d \times \mathbb{N}^d$ is *diagonal*
 313 (resp. *horizontal*) if $(\mathbf{x}, \mathbf{y}) \in R$ implies $(\mathbf{x} + \mathbf{c}, \mathbf{y} + \mathbf{c}) \in R$ (resp. $(\mathbf{x} + \mathbf{c}, \mathbf{y}) \in R$) for every
 314 $\mathbf{c} \in \mathbb{N}^d$. But Theorem 9 does not extend to larger dimensions, even if we restrict ourselves
 315 to Presburger automata with a single state and a single transition. In fact, the reflexive-
 316 transitive closure of a Presburger-definable relation that is diagonal (resp. horizontal) need
 317 not be Presburger-definable. Consider the relation $R \subseteq \mathbb{N}^2 \times \mathbb{N}^2$ defined by $(x_1, x_2) R (y_1, y_2)$
 318 if, and only if, the Presburger formula $y_1 \leq 2x_1 \wedge y_2 < x_2$ holds. The relation R is both
 319 diagonal and horizontal. It is routinely checked that the reflexive-transitive closure R^* is the
 320 set of pairs $((x_1, x_2), (y_1, y_2)) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $y_1 \leq 2^{x_2 - y_2} x_1$ and $y_2 \leq x_2$, which is not
 321 definable in Presburger arithmetic. ◀

6 Reachability Relations of TRVASS

322 A TRVASS is a 2-dimensional vector addition system with states (2-dim VASS) such that
 the first counter can be tested for zero and the second one can be reseted to zero. Formally,
 a *TRVASS* is a triple $\mathcal{V} = (Q, \Sigma, \Delta)$ where Q is a finite set of *states*, $\Sigma \subseteq \mathbb{Z}^2 \cup \{T, R\}$ is
 a finite set of *actions*, and $\Delta \subseteq Q \times \Sigma \times Q$ is a finite set of *transitions*. A *configuration*
 of \mathcal{V} is a triple $(q, x_1, x_2) \in Q \times \mathbb{N} \times \mathbb{N}$ written as $q(x_1, x_2)$ in the sequel. The operational
 semantics of \mathcal{V} is given by the binary relations \xrightarrow{a} over configurations, with $a \in \Sigma$, defined
 by $p(x_1, x_2) \xrightarrow{a} q(y_1, y_2)$ if $(p, a, q) \in \Delta$ and

$$\begin{aligned} (y_1, y_2) &= (x_1 + a_1, x_2 + a_2) & \text{if } a &= (a_1, a_2) \in \mathbb{Z}^2 \\ (y_1, y_2) &= (0, x_2) \wedge x_1 = 0 & \text{if } a &= T \\ (y_1, y_2) &= (x_1, 0) & \text{if } a &= R \end{aligned}$$

323 Given a word $w = a_1 \dots a_k$ of actions $a_j \in \Sigma$, we denote by \xrightarrow{w} the binary relation over
 324 configurations defined as the relational composition $\xrightarrow{a_1} \circledast \dots \circledast \xrightarrow{a_k}$. The relation $\xrightarrow{\varepsilon}$
 325 denotes the identity relation on configurations. Given a subset $W \subseteq \Sigma^*$, we let \xrightarrow{W} denote
 326 the union $\bigcup_{w \in W} \xrightarrow{w}$. The relation $\xrightarrow{\Sigma^*}$, also written $\xrightarrow{*}$, is called the *reachability relation*
 327 of \mathcal{V} . Observe that $\xrightarrow{*}$ is the reflexive-transitive closure of the *step* relation $\rightarrow \stackrel{\text{def}}{=} \bigcup_{a \in \Sigma} \xrightarrow{a}$.

328 The remainder of this section is devoted to the proof that TRVASS have effectively
 329 Presburger-definable reachability relations. Let us fix a TRVASS $\mathcal{V} = (Q, \Sigma, \Delta)$. We let A
 330 denote the set $\Sigma \cap \mathbb{Z}^2$ of *addition vectors*.

The reachability relation of \mathcal{V} can be expressed in terms of the reachability relation of
 a Presburger automaton by observing that configurations reachable just after a zero-test
 T or a reset R are restricted to $q(0, n)$ or $q(n, 0)$, respectively, where $q \in Q$ and $n \in \mathbb{N}$.
 Those configurations are parametrized by introducing the set $S = \{q^T, q^R \mid q \in Q\}$ obtained
 as two disjoint copies of Q . Elements in $\{q^T \mid q \in Q\}$ are called *test states*, and those in

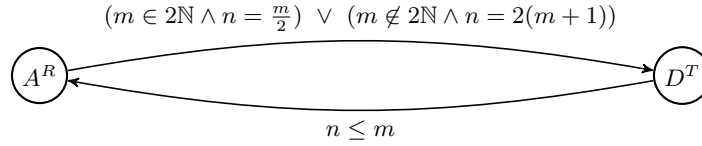
$\{q^R \mid q \in Q\}$ are called *reset states*. Given $s \in S$ and $n \in \mathbb{N}$, we introduce the configuration $\llbracket s, n \rrbracket$ in $Q \times \mathbb{N}^2$ defined as follows:

$$\llbracket s, n \rrbracket \stackrel{\text{def}}{=} \begin{cases} q(0, n) & \text{if } s = q^T \\ q(n, 0) & \text{if } s = q^R \end{cases}$$

We also introduce, for each pair $(s, t) \in S \times S$, the binary relation $R_{s,t}$ defined by

$$R_{s,t} \stackrel{\text{def}}{=} \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid \llbracket s, m \rrbracket \xrightarrow{A^*X} \llbracket t, n \rrbracket\}$$

331 where $X = T$ if t is a test state and $X = R$ if t is a reset state. It is known that the
 332 reachability relation of a 2-dim VASS is effectively Presburger-definable [17, 2]. This entails
 333 that the relation $\xrightarrow{A^*}$ is effectively Presburger-definable, and it follows that the relations
 334 $R_{s,t}$ are also effectively Presburger-definable. We introduce the Presburger automaton \mathcal{P}
 335 with set of states S and set of transitions $\{(s, R_{s,t}, t) \mid (s, t) \in S \times S\}$. Note that \mathcal{P} is
 336 computable from \mathcal{V} .



■ **Figure 3** The Presburger automaton \mathcal{P} associated to the TRVASS of Figure 2.

337 ► **Example 10.** Let us come back to the TRVASS of Figure 2. The relations $R_{s,t}$ are all empty
 338 except for R_{A^R, D^T} and R_{D^T, A^R} . The corresponding automaton \mathcal{P} is depicted in Figure 3.
 339 Each transition $(s, R_{s,t}, t)$ is depicted by an edge from s to t labeled by a Presburger formula
 340 $\varphi_{s,t}(m, n)$ denoting the relation $R_{s,t}$. The empty relations (which are both diagonal and
 341 horizontal) and the states not in $\{A^R, D^T\}$ are not depicted. Notice that the transition from
 342 A^R to D^T is ordinary and the one from D^T to A^R is horizontal. It follows that \mathcal{P} is shallow.
 343 We observe that the horizontal relation R defined as the composition $R_{D^T, A^R} \circ R_{A^R, D^T}$ is
 344 the one introduced in Example 5. ◀

345 We first show that the Presburger automaton \mathcal{P} is shallow. By Theorem 9, this will entail
 346 that its reachability relation $\xrightarrow{*}$ is effectively Presburger-definable.

347 ► **Lemma 11.** *The Presburger automaton \mathcal{P} is shallow.*

348 **Proof.** It is readily seen that \mathcal{P} satisfies the following properties:

- 349 ■ Transitions from reset states to reset states are diagonal,
- 350 ■ Transitions from test states to reset states are horizontal,
- 351 ■ Transitions from test states to test states are diagonal.

352 It follows that an ordinary transition of \mathcal{P} is a transition from a reset state to a test state. If
 353 a cycle contains such a transition then it must contain a transition from a test state to a
 354 reset state as well. Since such a transition is horizontal, we obtain that \mathcal{P} is shallow. ◀

355 The two following lemmas show how to decompose the reachability relation of \mathcal{V} in terms
 356 of the reachability relation of \mathcal{P} .

357 ► **Lemma 12.** *For every $s, t \in S$ and $m, n \in \mathbb{N}$, if $s(m) \xrightarrow{*} t(n)$ then $\llbracket s, m \rrbracket \xrightarrow{*} \llbracket t, n \rrbracket$.*

XX:12 Reachability for Test/Reset Two-Counter Machines

358 **Proof.** It is easily seen that $s(m) \rightarrow_{\mathcal{P}} t(n)$ implies $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$, for every $s, t \in S$ and
 359 $m, n \in \mathbb{N}$. We derive, by an immediate induction on $k \geq 1$, that $s(m) (\rightarrow_{\mathcal{P}})^k t(n)$ implies
 360 $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$, for every $s, t \in S$ and $m, n \in \mathbb{N}$. The lemma follows. ◀

▶ **Lemma 13.** *Consider two configurations $p(x_1, x_2)$ and $q(y_1, y_2)$ of \mathcal{V} . It holds that $p(x_1, x_2) \xrightarrow{\Sigma^* \setminus A^*}_{\mathcal{V}} q(y_1, y_2)$ if, and only if, there exist $s, t \in S$ and $m, n \in \mathbb{N}$ such that:*

$$p(x_1, x_2) \xrightarrow{A^* \{T, R\}}_{\mathcal{V}} \llbracket s, m \rrbracket \wedge s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \wedge \llbracket t, n \rrbracket \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

Proof. Lemma 12 shows the “if” direction of the equivalence. For the other direction, let $w \in \Sigma^* \setminus A^*$ such that $p(x_1, x_2) \xrightarrow{w}_{\mathcal{V}} q(y_1, y_2)$. By splitting w after each occurrence of an action in $\{T, R\}$, we deduce that $w = w_0 X_1 \dots w_{k-1} X_k w_k$ where $k \geq 1$, and $w_0, \dots, w_k \in A^*$. Let us introduce the configurations c_1, \dots, c_k satisfying the following relations:

$$p(x_1, x_2) \xrightarrow{w_0 X_1}_{\mathcal{V}} c_1 \cdots \xrightarrow{w_{k-1} X_k}_{\mathcal{V}} c_k \xrightarrow{w_k}_{\mathcal{V}} q(y_1, y_2)$$

361 Notice that $c_j = \llbracket q_j^{X_j}, n_j \rrbracket$ for some $q_j \in Q$ and some $n_j \in \mathbb{N}$. By definition of \mathcal{P} , we get
 362 $q_{j-1}^{X_{j-1}}(n_{j-1}) \rightarrow_{\mathcal{P}} q_j^{X_j}(n_j)$ for every $j \in \{1, \dots, k\}$. We have proved the lemma. ◀

363 We deduce our main result.

364 ▶ **Theorem 14.** *The reachability relation of a TRVASS is effectively Presburger-definable.*

Proof. Lemma 13 shows that $p(x_1, x_2) \xrightarrow{*}_{\mathcal{V}} q(y_1, y_2)$ if, and only if, $p(x_1, x_2) \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$ or there exists $s, t \in S$ and $m, n \in \mathbb{N}$ such that:

$$p(x_1, x_2) \xrightarrow{A^* \{T, R\}}_{\mathcal{V}} \llbracket s, m \rrbracket \wedge s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \wedge \llbracket t, n \rrbracket \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

365 From [17, 2], the relation $\xrightarrow{A^*}_{\mathcal{V}}$ is effectively Presburger-definable. From Lemma 11 and
 366 Theorem 9, the relation $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable as well. ◀

367 Coming back to the classes of 2-dim extended VASS discussed in the introduction (see
 368 Figure 1), Theorem 14 means that the reachability relation is effectively Presburger-definable
 369 for the “maximal” class $\mathbf{T}_1 \mathbf{R}_2$. This result also applies to 2-dim VASS extended with resets
 370 and transfers on both counters (i.e., the class $\mathbf{R}_{1,2} \mathbf{Tr}_{1,2}$), since they can be simulated by
 371 machines in $\mathbf{T}_1 \mathbf{R}_2$.

372 7 Conclusion and open problems

373 We have shown that the reachability relation of 2-dim VASS extended with tests on the first
 374 counter and resets on the second counter, is effectively Presburger-definable. This completes
 375 the decidability picture of 2-dim extended VASS initiated in [11]. Our proof techniques may
 376 also be used for other classes of counter machines where shallow Presburger automata would
 377 naturally appear. Many other problems on extensions of VASS are still interesting to solve.

378 ■ The reachability problem is NP-complete [12] for 1-dim VASS, PSPACE-complete [2] for
 379 2-dim VASS, and NL-complete [7] for unary 2-dim VASS. But we do not know what are
 380 the complexities for the reachability problem, for the construction of the reachability set
 381 and for the reachability relation for all 2-dim extended VASS.

382 ■ The boundedness problem is undecidable for 3-dim VASS extended with resets on all
 383 counters [5] and it is decidable for arbitrary dimension VASS extended with resets on
 384 two counters [6]. Is boundedness decidable for arbitrary dimension TRVASS ?

385 ——— **References** ———

- 386 **1** S. Akshay, Supratik Chakraborty, Ankush Das, Vishal Jagannath, and Sai Sandeep. On
387 petri nets with hierarchical special arcs. In *CONCUR*, volume 85 of *LIPICs*, pages 40:1–
388 40:17. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.
- 389 **2** Michael Blondin, Alain Finkel, Stefan Göller, Christoph Haase, and Pierre McKenzie.
390 Reachability in two-dimensional vector addition systems with states is pspace-complete.
391 In *LICS*, pages 32–43. IEEE Computer Society, 2015.
- 392 **3** Rémi Bonnet. The reachability problem for vector addition systems with one zero-test. In
393 Filip Murlak and Piotr Sankowski, editors, *Proceedings of the 36th International Symposium*
394 *on Mathematical Foundations of Computer Science (MFCS'11)*, volume 6907 of *Lecture*
395 *Notes in Computer Science*, pages 145–157, Warsaw, Poland, August 2011. Springer.
- 396 **4** Rémi Bonnet, Alain Finkel, Jérôme Leroux, and Marc Zeitoun. Place-boundedness for
397 vector addition systems with one zero-test. In Kamal Lodaya and Meena Mahajan, editors,
398 *Proceedings of the 30th Conference on Foundations of Software Technology and Theoretical*
399 *Computer Science (FSTTCS'10)*, volume 8 of *Leibniz International Proceedings in Informa-*
400 *tics*, pages 192–203, Chennai, India, December 2010. Leibniz-Zentrum für Informatik.
- 401 **5** Catherine Dufourd, Alain Finkel, and Philippe Schnoebelen. Reset nets between decidab-
402 ility and undecidability. In *ICALP*, volume 1443 of *Lecture Notes in Computer Science*,
403 pages 103–115. Springer, 1998.
- 404 **6** Catherine Dufourd, Petr Jancar, and Philippe Schnoebelen. Boundedness of reset P/T nets.
405 In *ICALP*, volume 1644 of *Lecture Notes in Computer Science*, pages 301–310. Springer,
406 1999.
- 407 **7** Matthias Englert, Ranko Lazic, and Patrick Totzke. Reachability in two-dimensional unary
408 vector addition systems with states is nl-complete. In *LICS*, pages 477–484. ACM, 2016.
- 409 **8** Alain Finkel and Arnaud Sangnier. Mixing coverability and reachability to analyze VASS
410 with one zero-test. In David Peleg and Anca Muscholl, editors, *Proceedings of the 36th*
411 *International Conference on Current Trends in Theory and Practice of Computer Science*
412 *(SOFSEM'10)*, volume 5901 of *Lecture Notes in Computer Science*, pages 394–406, Špind-
413 lerbův Mlýn, Czech Republic, January 2010. Springer.
- 414 **9** Alain Finkel and Philippe Schnoebelen. Well-structured transition systems everywhere!
415 *Theoretical Computer Science*, 256(1-2):63–92, April 2001.
- 416 **10** Alain Finkel and Grégoire Sutre. An algorithm constructing the semilinear post^* for 2-dim
417 reset/transfer VASS. In *MFCS*, volume 1893 of *Lecture Notes in Computer Science*, pages
418 353–362. Springer, 2000.
- 419 **11** Alain Finkel and Grégoire Sutre. Decidability of reachability problems for classes of two
420 counters automata. In *STACS*, volume 1770 of *Lecture Notes in Computer Science*, pages
421 346–357. Springer, 2000.
- 422 **12** Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell. Reachability in
423 succinct and parametric one-counter automata. In *CONCUR*, volume 5710 of *Lecture Notes*
424 *in Computer Science*, pages 369–383. Springer, 2009.
- 425 **13** John Hopcroft and Jean-Jacques Pansiot. On the reachability problem for 5-dimensional
426 vector addition systems. *Theoretical Computer Science*, 8(2):135–159, 1979.
- 427 **14** Ranko Lazic. The reachability problem for vector addition systems with a stack is not
428 elementary. *CoRR*, abs/1310.1767, 2013.
- 429 **15** Ranko Lazic and Patrick Totzke. What makes petri nets harder to verify: Stack or data? In
430 *Concurrency, Security, and Puzzles*, volume 10160 of *Lecture Notes in Computer Science*,
431 pages 144–161. Springer, 2017.
- 432 **16** Jérôme Leroux, M. Praveen, and Grégoire Sutre. Hyper-ackermannian bounds for push-
433 down vector addition systems. In Thomas A. Henzinger and Dale Miller, editors, *Joint Meet-*
434 *ing of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and*

- 435 *the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS),*
436 *CSL-LICS '14, Vienna, Austria, July 14 - 18, 2014*, page 63. ACM, 2014.
- 437 **17** Jérôme Leroux and Grégoire Sutre. On flatness for 2-dimensional vector addition systems
438 with states. In *CONCUR 2004 - Concurrency Theory, 15th International Conference,*
439 *London, UK, August 31 - September 3, 2004, Proceedings*, volume 3170 of *Lecture Notes in*
440 *Computer Science*, pages 402–416. Springer, 2004.
- 441 **18** M. Presburger. Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer
442 Zahlen, in welchem die Addition als einzige Operation hervortritt. *Comptes Rendus du*
443 *premier congrès de mathématiciens des Pays Slaves, Warszawa*, pages 92–101, 1929.
- 444 **19** Klaus Reinhardt. Reachability in petri nets with inhibitor arcs. *Electr. Notes Theor.*
445 *Comput. Sci.*, 223:239–264, 2008.