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# 1 Reachability for Two-Counter Machines with One 2 Test and One Reset

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## 12 — Abstract —

13 We prove that the reachability relation of two-counter machines with one zero-test and one reset  
14 is Presburger-definable and effectively computable. Our proof is based on the introduction of two  
15 classes of Presburger-definable relations effectively stable by transitive closure. This approach  
16 generalizes and simplifies the existing different proofs and it solves an open problem introduced  
17 by Finkel and Sutre in 2000.

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## 24 **1** Introduction

25 **Context** Vector addition systems with states (VASS) are equivalent to Petri nets and  
26 to counter machines without the ability to test counters for zero. Although VASS have  
27 been studied since the 1970's, they remain fascinating since there are still some important  
28 open problems like the complexity of reachability (known between EXPSpace and cubic-  
29 Ackermannian) or even an efficient (in practice) algorithm to solve reachability. In 1979,  
30 Hopcroft and Pansiot [13] gave an algorithm that computes the Presburger-definable reach-  
31 ability set of a 2-dim VASS, hence VASS in dimension 2 are more easy to verify and they  
32 enjoy interesting properties like reachability and equivalence of reachability sets, for instance,  
33 are both decidable. Unfortunately, these results do not extend in dimension 3 or for 2-dim  
34 VASS with zero-tests on the two counters: the reachability set (hence also the reachability  
35 relation) is not Presburger-definable for 3-dim VASS [13]; reachability, and all non-trivial  
36 problems, are undecidable for 2-dim VASS extended with zero-tests on the two counters.

37 In 2004, Leroux and Sutre proved that the reachability relation of a 2-dim VASS is  
38 also effectively Presburger-definable [17] and this is not a consequence of the Presburger-  
39 definability of the reachability set. As a matter of fact, there exist counter machines with  
40 a Presburger-definable reachability set but with a non Presburger-definable reachability



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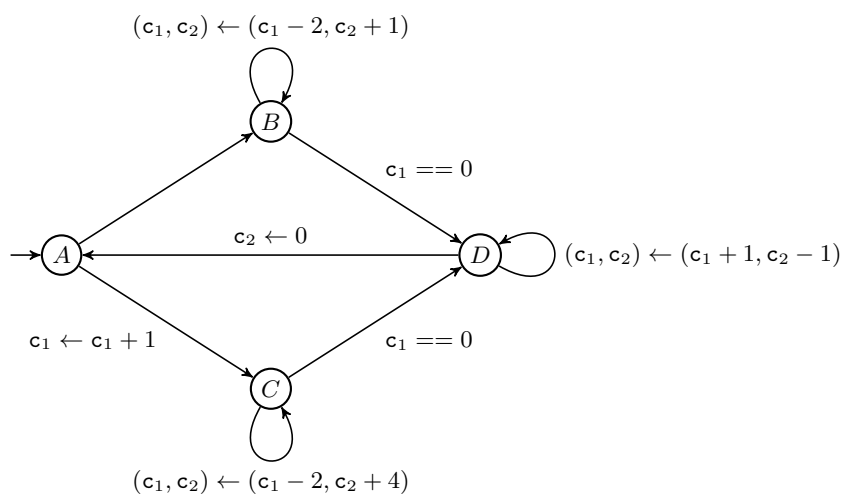
<i>Class</i>	<i>Post*</i>	<i>Pre*</i>	$\xrightarrow{*}$
$\mathbf{T}_1\mathbf{Tr}_2 \simeq \mathbf{T}_{1,2} \simeq \mathbf{T}_{1,2}\mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Not Recursive	Not Recursive	Not Recursive
$\mathbf{T}_1\mathbf{R}_2 \simeq \mathbf{T}_1\mathbf{R}_{1,2}\mathbf{Tr}_1$	<b>Eff.</b> Presburger	Eff. Presburger	<b>Eff. Presburger</b>
$\mathbf{R}_{1,2}\mathbf{Tr}_1 \simeq \mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Eff. Presburger	Eff. Presburger	<b>Eff. Presburger</b>
$\mathbf{T}_1 \simeq \mathbf{T}_1\mathbf{R}_1\mathbf{Tr}_1$	Eff. Presburger	Eff. Presburger	<b>Eff. Presburger</b>
2-dim VASS	Eff. Presburger	Eff. Presburger	Eff. Presburger

■ **Figure 1** Reachability sets ( $Post^*$  and  $Pre^*$ ) and reachability relation ( $\xrightarrow{*}$ ) for extensions of 2-dimensional VASS. We let  $\simeq$  denote the existence of mutual reductions between two classes of machines that preserve the effective Presburger-definability of the reachability sets and relation. The contributions of this paper are indicated in boldface.

41 relation. But, for all recursive 2-dim extended VASSs, the reachability sets are Presburger-  
 42 definable [11, 10]. More precisely, let us denote by  $\mathbf{T}_I\mathbf{R}_J\mathbf{Tr}_K$ , with  $I, J, K \subseteq \{1, 2\}$ , the  
 43 class of 2-dim VASS extended with zero-tests on the  $I$ -counters, resets on the  $J$ -counters  
 44 and transfers from the  $K$ -counters. For instance,  $\mathbf{T}_{\{1\}}\mathbf{R}_{\{1,2\}}\mathbf{Tr}_\emptyset$ , also written  $\mathbf{T}_1\mathbf{R}_{1,2}$  for  
 45 short, is the class of 2-dim VASS extended with zero-tests on the first counter, resets on both  
 46 counters, and no transfer. The relations between classes from [11] are recalled in Figure 1 and  
 47 the class  $\mathbf{T}_1\mathbf{R}_2$  has been shown to be the “maximal” class having Presburger-definable  $post^*$   
 48 and  $pre^*$  reachability sets [11]. However, it was unknown whether the Presburger-definable  
 49 reachability set  $post^*$  can be effectively computed or not. In fact, even the boundedness  
 50 problem (is the reachability set  $post^*$  finite?) was open for this class.

51 **Contributions** Our main contribution is a proof that the reachability relation of counter  
 52 machines in  $\mathbf{T}_1\mathbf{R}_2$  is effectively Presburger-definable. The impact of our result is threefold.  
 53 ■ We solve the main *open problem* in [11] which was the question of the existence of  
 54 an algorithm that computes the Presburger-definable reachability set for two-counter  
 55 machines in  $\mathbf{T}_1\mathbf{R}_2$ .  
 56 ■ In fact, we prove a stronger result, namely that the *reachability relation* of counter machines  
 57 in  $\mathbf{T}_1\mathbf{R}_2$  is Presburger-definable and computable. This completes the decidability picture  
 58 of 2-dim extended VASS.  
 59 ■ We provide a *simple* proof for the effective Presburger-definability of the reachability  
 60 relation in  $\mathbf{T}_1\mathbf{R}_2$ . As an immediate consequence, one may deduce all existing results [11,  
 61 10] for 2-dim extended VASS and our proof unifies all different existing proofs on 2-dim  
 62 extended VASS, including the proof in [6] that the boundedness problem is decidable for  
 63 the class  $\mathbf{R}_{1,2}$  of 2-dim VASS extended with resets on both counters.

64 **Related work** VASS have been extended with resets, transfers and zero-tests. Extended  
 65 VASS with resets and transfers are well structured transition systems [9] hence termination  
 66 and coverability are decidable; but reachability and boundedness are undecidable (except  
 67 boundedness that is decidable for extended VASS with transfers) [5, 6]. Reachability and  
 68 place-boundedness problems are decidable for extended VASS with *one* zero-test [19, 3, 8, 4].  
 69 Recently, Akshay & al. studied extended Petri nets with a hierarchy on places and with  
 70 resets, transfers and zero-tests [1]. As a counter is a particular case of a stack, it is natural  
 71 to study counter machines with one stack. Termination and boundedness are decidable for  
 72 VASS with one stack [16] but surprisingly, the decidability status of the reachability problem  
 73 is open for VASS with one stack, both in arbitrary dimension and in dimension 1. We only  
 74 know that reachability and coverability for VASS with one stack are TOWER-hard [14, 15].



■ **Figure 2** A 2-dimensional VASS extended with zero-tests on the first counter and resets on the second counter (shortly called TRVASS).

75 **Outline** We present in Section 2 an example of 2-dim extended VASS in  $\mathbf{T}_1\mathbf{R}_2$ . This  
 76 example motivates the study of two classes of binary relations on natural numbers, namely  
 77 diagonal relations in Section 3 and horizontal relations in Section 4. These two classes of  
 78 relations are combined in Section 5 into a new class of one counter automata with effectively  
 79 Presburger-definable reachability relations. These automata are used in Section 6 to compute  
 80 the reachability relations of 2-dim extended VASS in  $\mathbf{T}_1\mathbf{R}_2$ .

81 For the remainder of the paper, 2-dim extended VASS in  $\mathbf{T}_1\mathbf{R}_2$  are shortly called TRVASS.

## 82 2 Motivating Example

83 Figure 2 depicts an example of a TRVASS. There are four *states*  $A, B, C$  and  $D$ , and two  
 84 *counters*  $c_1$  and  $c_2$ . Following the standard semantics of vector addition systems, these  
 85 counters range over natural numbers. The operations labeling the three loops and the edge  
 86 from  $A$  to  $C$  are classical addition instructions of vector addition systems. In dimension 2,  
 87 these addition instructions are always of the form  $(c_1, c_2) \leftarrow (c_1 + a_1, c_2 + a_2)$  where  $a_1$  and  
 88  $a_2$  are integer constants. For instance, the instruction  $(c_1, c_2) \leftarrow (c_1 - 2, c_2 + 1)$  labeling the  
 89 loop on  $B$  means that  $c_1$  is decremented by 2 and at the same time  $c_2$  is incremented by 1.  
 90 As the counters must remain nonnegative, this instruction may be executed (i.e., the loop on  
 91  $B$  may be taken) only if  $c_1 \geq 2$ . In addition to classical addition instructions, TRVASS may  
 92 test the first counter for zero, written  $c_1 == 0$ , and reset the second counter to zero, written  
 93  $c_2 \leftarrow 0$ .

94 The operational semantics of a TRVASS is given, as for vector addition systems, by an  
 95 infinite directed graph whose nodes are called *configurations* and whose edges are called *steps*.  
 96 Formal definitions will be given in Section 6. For the TRVASS of Figure 2, configurations  
 97 are triples  $q(x_1, x_2)$  where  $q \in \{A, B, C, D\}$  is a state and  $x_1, x_2 \in \mathbb{N}$  are values of the  
 98 counters  $c_1$  and  $c_2$ , respectively. It is understood that  $\mathbb{N}$  denotes the set of natural numbers  
 99  $\{0, 1, 2, \dots\}$ . There is a step from a configuration  $p(x_1, x_2)$  to a configuration  $q(y_1, y_2)$ ,  
 100 written  $p(x_1, x_2) \rightarrow q(y_1, y_2)$ , if there is an edge from  $p$  to  $q$  labeled by an operation (1) that  
 101 can be executed from the counter values  $(x_1, x_2)$  and (2) whose execution changes the counter

## XX:4 Reachability for Test/Reset Two-Counter Machines

102 values from  $(x_1, x_2)$  to  $(y_1, y_2)$ . Here, we have the steps  $B(5, 1) \rightarrow B(3, 2)$ ,  $C(0, 2) \rightarrow D(0, 2)$   
 103 and  $D(7, 3) \rightarrow A(7, 0)$ . But there is no step from  $C(1, 2)$  and there is no step to  $A(7, 1)$ .

The *reachability relation* of a TRVASS, written  $\overset{*}{\rightarrow}$ , is the reflexive-transitive closure of the step relation  $\rightarrow$ . The reachability relation is one of the main objects of interest for verification purposes. Coming back to our example of Figure 2, we have  $A(1, 0) \overset{*}{\rightarrow} A(2, 0)$  since we have the following contiguous sequence of steps:

$$A(1, 0) \rightarrow C(2, 0) \rightarrow C(0, 4) \rightarrow D(0, 4) \rightarrow D(1, 3) \rightarrow D(2, 2) \rightarrow A(2, 0)$$

By removing the steps  $\rightarrow D(1, 3) \rightarrow D(2, 2)$ , we also get that  $A(1, 0) \overset{*}{\rightarrow} A(0, 0)$ . In fact, it can be shown that  $A(1, 0) \overset{*}{\rightarrow} A(y, 0)$  for every  $y \in \mathbb{N}$ , thanks to the following pattern, where  $k$  denotes an odd natural number and  $i \in \{1, 2\}$ :

$$A(k, 0) \rightarrow C(k + 1, 0) \overset{*}{\rightarrow} D(2k + 2, 0) \overset{*}{\rightarrow} D(k + i, k + 2 - i) \overset{*}{\rightarrow} A(k + i, 0)$$

104 One may wonder whether it also holds that  $A(x, 0) \overset{*}{\rightarrow} A(y, 0)$  for every  $x, y \in \mathbb{N}$ . A  
 105 consequence of our main result (see Theorem 14) is that we can do even better: we can  
 106 compute the set of pairs  $(x, y) \in \mathbb{N} \times \mathbb{N}$  such that  $A(x, 0) \overset{*}{\rightarrow} A(y, 0)$ , as a formula in Presburger  
 107 arithmetic<sup>1</sup>.

108 **► Remark.** It is well-known that zero-tests are more expressive than resets. Indeed, a reset  
 109  $c_1 \leftarrow 0$  can be simulated by a loop  $c_1 \leftarrow c_1 - 1$  followed by a zero-test  $c_1 == 0$ . A crucial  
 110 difference between resets and zero-tests is *monotony*. In a 2-dimensional VASS extended with  
 111 resets on both counters (shortly called RRVASS), larger counter values are always better,  
 112 in the sense that every behavior from a configuration  $q(x_1, x_2)$  can be reproduced from a  
 113 configuration  $q(x'_1, x'_2)$  with  $x'_1 \geq x_1$  and  $x'_2 \geq x_2$ . This is not true anymore in presence  
 114 of zero-tests. This difference makes the analysis of TRVASS more complex than that of  
 115 RRVASS, as illustrated in the following example. ◀

**► Example 1.** Consider the RRVASS obtained from the TRVASS of Figure 2 by replacing the two zero-tests (from  $B$  to  $D$  and from  $C$  to  $D$ ) with resets  $c_1 \leftarrow 0$ . Suppose that we want to show that  $c_1$  is unbounded in state  $A$  from  $A(1, 0)$ , i.e.,  $A(1, 0) \overset{*}{\rightarrow} A(y, 0)$  for infinitely many  $y \in \mathbb{N}$ . A natural strategy is, starting from  $A(x, 0)$  with  $x \geq 1$ , to reach  $D(0, y)$  with  $y$  as large as possible (without visiting  $A$  on the way), and then to reach  $A(y, 0)$  by taking the “transfer” loop on  $D$  as much as possible. By iterating this strategy, we get

$$A(1, 0) \overset{*}{\rightarrow} D(0, 4) \overset{*}{\rightarrow} A(4, 0) \overset{*}{\rightarrow} D(0, 8) \overset{*}{\rightarrow} A(8, 0) \overset{*}{\rightarrow} D(0, 16) \overset{*}{\rightarrow} A(16, 0) \dots$$

This witnesses that  $c_1$  is unbounded in state  $A$  from  $A(1, 0)$ . In comparison, this strategy does not work for the original TRVASS of Figure 2. Indeed, we get

$$A(1, 0) \overset{*}{\rightarrow} D(0, 4) \overset{*}{\rightarrow} A(4, 0) \overset{*}{\rightarrow} D(0, 2) \overset{*}{\rightarrow} A(2, 0) \overset{*}{\rightarrow} D(0, 1) \overset{*}{\rightarrow} A(1, 0)$$

116 by following this strategy. This is because the only way to reach  $D$  from a configuration  
 117  $A(x, 0)$  with  $x$  even is via  $B$ . ◀

118 The rest of the paper is devoted to the proof that the reachability relation of a TRVASS  
 119 is *effectively Presburger-definable*, i.e., there is an algorithm that, given a TRVASS and two  
 120 states  $p$  and  $q$ , computes a formula  $\varphi(x_1, x_2, y_1, y_2)$  in Presburger arithmetic whose models

---

<sup>1</sup> Recall that Presburger arithmetic [18] is the first-order theory of the natural numbers with addition.

121 are precisely the quadruples  $(x_1, x_2, y_1, y_2)$  of natural numbers such that  $p(x_1, x_2) \xrightarrow{*} q(y_1, y_2)$ .  
 122 It is already known that the reachability relation is effectively Presburger-definable in absence  
 123 of zero-tests and resets [17]. Obviously, the counter  $c_1$  is zero after a zero-test  $c_1 == 0$   
 124 and, similarly, the counter  $c_2$  is zero after a reset  $c_2 \leftarrow 0$ . So we focus on the reachability  
 125 subrelations between configurations where at least one of the counters is zero, for instance,  
 126  $\{(x, 0, 0, y) \mid p(x, 0) \xrightarrow{*} q(0, y)\}$ . Such a subrelation can be seen as a (binary) relation on  $\mathbb{N}$ .  
 127 This motivates our study in Sections 3 and 4 of two classes of relations on  $\mathbb{N}$  that naturally  
 128 stem from the operational semantics of TRVASS.

### 129 3 Diagonal Relations

130 We call a relation  $R \subseteq \mathbb{N} \times \mathbb{N}$  *diagonal* when  $(x, y) \in R$  implies  $(x + c, y + c) \in R$  for every  
 131  $c \in \mathbb{N}$ . For instance, the *identity relation* on  $\mathbb{N}$ , namely  $\{(x, x) \mid x \in \mathbb{N}\}$ , is a diagonal relation.  
 132 The usual order  $\leq$  on natural numbers is also a diagonal relation. It is readily seen that the  
 133 class of diagonal relations is closed under union, intersection, composition, and transitive  
 134 closure. In this section, we show that the transitive closure of a diagonal Presburger-definable  
 135 relation is effectively Presburger-definable. Our study of diagonal relations is motivated by  
 136 the following observation.

137 ► **Remark.** The reachability subrelations  $\{(x, y) \mid p(0, x) \xrightarrow{*} q(0, y)\}$ , where  $p$  and  $q$  are  
 138 states, are diagonal in a TRVASS with no reset. Analogously, the reachability subrelations  
 139  $\{(x, y) \mid p(x, 0) \xrightarrow{*} q(y, 0)\}$  are diagonal in a TRVASS with no zero-test. ◀

140 ► **Example 2.** Let us consider the diagonal relation  $R \subseteq \mathbb{N} \times \mathbb{N}$  defined by  $(x, y) \in R$  if,  
 141 and only if, the Presburger formula  $x \leq y \wedge y \leq 2x$  holds. It is routinely checked that  
 142 the transitive closure  $R^+$  of  $R$  satisfies  $(x, y) \in R^+$  if, and only if, the Presburger formula  
 143  $(x = 0 \Leftrightarrow y = 0) \wedge x \leq y$  holds. ◀

We fix, for the remainder of this section, a diagonal relation  $R \subseteq \mathbb{N} \times \mathbb{N}$ . Consider the  
 subsets  $I_R$  and  $D_R$  of  $\mathbb{N}$  defined by

$$I_R \stackrel{\text{def}}{=} \{x \mid \exists y : (x, y) \in R \wedge x < y\} \quad D_R \stackrel{\text{def}}{=} \{y \mid \exists x : (x, y) \in R \wedge x > y\}$$

144 Since  $R$  is diagonal, the sets  $I_R$  and  $D_R$  are upward-closed, meaning that  $x \in I_R$  implies  
 145  $x' \in I_R$  for every  $x' \geq x$  (and similarly for  $D_R$ ). If  $x \in I_R$  then  $(x, x + \delta) \in R$  for some  
 146 positive integer  $\delta > 0$ . Since  $R$  is diagonal,  $(x', x' + \delta) \in R$  for every  $x' \geq x$ . So the pair  
 147  $(x, x + \delta)$  can be viewed as an “increasing loop” that applies to every  $x' \geq x$ . Similarly, if  
 148  $y \in D_R$  then there is a “decreasing loop”  $(y + \delta, y) \in R$  that applies to every  $y' \geq y$ . We are  
 149 mostly interested in increasing and decreasing loops that apply to every element of  $I_R$  and  
 150  $D_R$ , respectively. This leads us to the following definitions:

$$151 \quad \alpha \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\} & \text{if } I_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$152 \quad \beta \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\} & \text{if } D_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

154 Let us explain why the natural numbers  $\alpha$  and  $\beta$  are well-defined. If  $I_R \neq \emptyset$  then there  
 155 exists  $\delta > 0$  such that  $(m, m + \delta) \in R$  where  $m = \min I_R$ . It follows from diagonality  
 156 of  $R$  that  $(x, x + \delta) \in R$  for every  $x \geq m$ , hence, for every  $x \in I_R$ . Therefore the set

## XX:6 Reachability for Test/Reset Two-Counter Machines

157  $\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\}$  is non-empty, and so it has a minimum. A similar  
 158 argument shows that  $\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\}$  is non-empty when  $D_R \neq \emptyset$ .

159 We are now almost ready to provide a characterization of the transitive closure of  $R^+$ .  
 160 To do so, we introduce the relations  $\text{Inc}_R$  and  $\text{Dec}_R$  on  $\mathbb{N}$  defined by

$$161 \quad \text{Inc}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (x \in I_R \wedge \exists h \in \mathbb{N} : y = x + h\alpha)$$

$$162 \quad \text{Dec}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (y \in D_R \wedge \exists k \in \mathbb{N} : x = y + k\beta)$$

164 We let  $\circledast$  denote *relational composition* ( $S \circledast R \stackrel{\text{def}}{=} \{(x, z) \mid \exists y : x S y R z\}$ ). The *powers* of a  
 165 relation  $R$  are inductively defined by  $R^1 \stackrel{\text{def}}{=} R$  and  $R^{n+1} \stackrel{\text{def}}{=} R \circledast R^n$ .

166 ► **Lemma 3.** *It holds that  $R^+ = \text{Inc}_R \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R$ .*

167 **Proof.** We introduce the relation  $C = \text{Inc}_R \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R$ , so as to reduce  
 168 clutter. To prove that  $C \subseteq R^+$ , we show that  $\text{Inc}_R$  and  $\text{Dec}_R$  are both contained in  $R^*$ . Let  
 169  $(x, y) \in \text{Inc}_R$ . If  $x = y$  then  $(x, y) \in R^*$ . Otherwise,  $x \in I_R$  and there exists  $h \in \mathbb{N}$  such that  
 170  $y = x + h\alpha$ . Moreover,  $h$  and  $\alpha$  are positive as  $x \neq y$ . It follows from  $x \in I_R$  and  $\alpha > 0$  that  
 171  $(x, x + \alpha) \in R$ . Since  $R$  is diagonal, we derive that  $(x, x + \alpha), \dots, (x + (h-1)\alpha, x + h\alpha)$  are all  
 172 in  $R$ . Hence,  $(x, y) \in R^+$ . We have shown that  $\text{Inc}_R \subseteq R^*$ . Now let  $(x, y) \in \text{Dec}_R$ . If  $x = y$   
 173 then  $(x, y) \in R^*$ . Otherwise,  $y \in D_R$  and there exists  $k \in \mathbb{N}$  such that  $x = y + k\beta$ . Moreover,  
 174  $k$  and  $\beta$  are positive as  $x \neq y$ . It follows from  $y \in D_R$  and  $\beta > 0$  that  $(y + \beta, y) \in R$ .  
 175 Since  $R$  is diagonal, we derive that  $(y + k\beta, y + (k-1)\beta), \dots, (y + \beta, y)$  are all in  $R$ . Hence,  
 176  $(x, y) \in R^+$ . We have shown that  $\text{Dec}_R \subseteq R^*$ . We derive from  $\text{Inc}_R \subseteq R^*$  and  $\text{Dec}_R \subseteq R^*$   
 177 that  $C \subseteq R^+$ .

178 Let us now prove the converse inclusion  $R^+ \subseteq C$ . We first observe that  $\text{Inc}_R = \text{Inc}_R^*$   
 179 and  $\text{Dec}_R = \text{Dec}_R^*$ . These equalities easily follow from the definitions of  $\text{Inc}_R$  and  $\text{Dec}_R$ . As  
 180 a consequence, we get that

$$181 \quad C = \text{Inc}_R^* \circledast (R \cup \dots \cup R^{\alpha+\beta+1}) \circledast \text{Dec}_R^* \tag{3}$$

Let us prove by induction on  $n$  that  $R^n \subseteq C$  for all  $n \geq 1$ . The base cases  $n = 1, \dots, \alpha + \beta + 1$   
 are trivial. Assume that  $R^m \subseteq C$  for all  $1 \leq m < n$ , where  $n \geq \alpha + \beta + 2$ , and let us show  
 that this inclusion also holds for  $m = n$ . Let  $(x, y) \in R^n$ . There exists  $x_0, \dots, x_n$  such that  
 $x = x_0 R x_1 R \dots R x_n = y$ . We start by showing the two following properties, as they will  
 be crucial for the rest of the proof.

$$x \notin I_R \implies x_0 \geq x_1 \geq \dots \geq x_n \quad \text{and} \quad y \notin D_R \implies x_0 \leq x_1 \leq \dots \leq x_n$$

182 We prove these properties by contraposition. If  $x_i < x_{i+1}$  for some  $0 \leq i < n$ , then we may,  
 183 w.l.o.g., choose the first such  $i$ . This entails that  $x_0 \geq \dots \geq x_i$ . Moreover,  $x_i \in I_R$  since  
 184  $x_i < x_{i+1}$  and  $x_i R x_{i+1}$ . It follows that  $x = x_0 \in I_R$  as  $I_R$  is upward-closed. Similarly, if  
 185  $x_{i-1} > x_i$  for some  $0 < i \leq n$ , then we may, w.l.o.g., choose the last such  $i$ . This entails  
 186 that  $x_i \leq \dots \leq x_n$ . Moreover,  $x_i \in D_R$  since  $x_{i-1} > x_i$  and  $x_{i-1} R x_i$ . It follows that  
 187  $y = x_n \in D_R$  as  $D_R$  is upward-closed.

188 To prove that  $(x, y) \in C$ , we consider four cases, depending on the membership of  $x$  in  
 189  $I_R$  and on the membership of  $y$  in  $D_R$ .

190 If  $x \notin I_R$  and  $y \notin D_R$  then  $x_0 = x_1 = \dots = x_n$ . This means in particular that  $x_0 R x_n$ ,  
 191 hence,  $x = x_0 C x_n = y$ .

192 If  $x \notin I_R$  and  $y \in D_R$  then  $x_0 \geq x_1 \geq \dots \geq x_n$ . Note that  $\beta > 0$  since  $D_R$  is non-empty.  
 193 Since  $n \geq \beta$ , there exists  $0 \leq i < j \leq n$  and  $k \in \mathbb{N}$  such that  $x_i = x_j + k\beta$ . Recall



194 that  $x = x_0 R^i x_i$  and  $x_j R^{n-j} x_n = y$ . As  $R$  is diagonal, we derive that  $x_i R^{n-j} y'$  where  
 195  $y' = y + k\beta$ . We obtain that  $x R^{n+i-j} y'$ . It follows from the induction hypothesis that  $x C y'$ .  
 196 Moreover, we have  $(y', y) \in \text{Dec}_R$  since  $y \in D_R$  and  $y' = y + k\beta$ . Hence,  $x (C \circ \text{Dec}_R) y$  and  
 197 we derive from Equation 3 that  $x C y$ .

198 If  $x \in I_R$  and  $y \notin D_R$  then  $x_0 \leq x_1 \leq \dots \leq x_n$ . Note that  $\alpha > 0$  since  $I_R$  is non-empty.  
 199 Since  $n \geq \alpha$ , there exists  $0 \leq i < j \leq n$  and  $h \in \mathbb{N}$  such that  $x_j = x_i + h\alpha$ . Recall that  
 200  $x = x_0 R^i x_i$  and  $x_j R^{n-j} x_n = y$ . As  $R$  is diagonal, we derive that  $x' R^i x_j$  where  $x' = x + h\alpha$ .  
 201 We obtain that  $x' R^{n+i-j} y$ . It follows from the induction hypothesis that  $x' C y$ . Moreover,  
 202 we have  $(x, x') \in \text{Inc}_R$  since  $x \in I_R$  and  $x' = x + h\alpha$ . Hence,  $x (\text{Inc}_R \circ C) y$  and we derive  
 203 from Equation 3 that  $x C y$ .

204 If  $x \in I_R$  and  $y \in D_R$  then both  $\alpha$  and  $\beta$  are positive. Since  $n \geq \alpha$ , there exists  
 205  $0 \leq i < j \leq n$  such that  $x_i \equiv x_j \pmod{\alpha}$ . If  $x_i \leq x_j$  then  $x_j = x_i + h\alpha$  for some  $h \in \mathbb{N}$  and  
 206 we may proceed as in the case  $x \in I_R \wedge y \notin D_R$  to show that  $x C y$ . Otherwise,  $x_i = x_j + k\alpha$   
 207 for some  $k \in \mathbb{N}$ . Recall that  $x = x_0 R^i x_i$  and  $x_j R^{n-j} x_n = y$ . As  $R$  is diagonal, we  
 208 derive that  $x' R^i z' R^{n-j} y'$  where  $x' = x + k\alpha(\beta - 1)$ ,  $z' = x_i + k\alpha(\beta - 1) = x_j + k\alpha\beta$  and  
 209  $y' = y + k\alpha\beta$ . We obtain that  $x' R^{n+i-j} y'$ . It follows from the induction hypothesis that  
 210  $x' C y'$ . Moreover, we have  $(x, x') \in \text{Inc}_R$  since  $x \in I_R$  and  $x' = x + k\alpha(\beta - 1)$ , and we also  
 211 have  $(y', y) \in \text{Dec}_R$  since  $y \in D_R$  and  $y' = y + k\alpha\beta$ . Hence,  $x (\text{Inc}_R \circ C \circ \text{Dec}_R) y$  and we  
 212 derive from Equation 3 that  $x C y$ . ◀

213 We derive the following theorem.

214 ▶ **Theorem 4.** *The transitive closure of a diagonal Presburger-definable relation is effectively*  
 215 *Presburger-definable.*

216 **Proof.** Assume that  $\varphi_R(x, y)$  is a Presburger formula denoting a diagonal relation  $R$ . The  
 217 sets  $I_R$  and  $D_R$  are defined by the Presburger formulas  $\exists y : \varphi_R(x, y) \wedge x < y$  and  $\exists x :$   
 218  $\varphi_R(x, y) \wedge x > y$ , respectively. The natural numbers  $\alpha$  and  $\beta$  defined in Equations 1 and 2  
 219 are obviously computable from  $\varphi_R$ . So the characterization given in Lemma 3 immediately  
 220 provides a computable Presburger formula denoting  $R^+$ . ◀

## 221 4 Horizontal Relations

222 A relation  $R \subseteq \mathbb{N} \times \mathbb{N}$  is said to be *horizontal* if  $(x, y) \in R$  implies  $(x + c, y) \in R$  for every  
 223  $c \in \mathbb{N}$ . The class of horizontal relations is clearly stable by union, intersection, composition,  
 224 and transitive closure. In this section we prove that the transitive closure of a horizontal  
 225 Presburger-definable relation is effectively Presburger-definable. Our study of horizontal  
 226 relations is motivated by the following observation.

227 ▶ **Remark.** The reachability subrelations  $\{(x, y) \mid p(0, x) \xrightarrow{*} \xrightarrow{c_2 \leftarrow 0} \xrightarrow{*} q(y, 0)\}$ , where  $p$  and  $q$   
 228 are states, are horizontal in a TRVASS. ◀

▶ **Example 5.** Let us consider the following horizontal relation  $R$ :

$$R \stackrel{\text{def}}{=} \{(x, y) \mid 2y \leq x \vee (y \in 4\mathbb{N} \wedge y \leq 2x + 2)\}$$

229 We prove that  $R^+$  is equal to  $C \stackrel{\text{def}}{=} \{(x, y) \mid x = 0 \Rightarrow y = 0\}$  as follows. Since  $R \subseteq C$  and  
 230  $C$  is transitive, we get  $R^+ \subseteq C$ . Conversely, let  $(x, y) \in C$ . If  $x = 0$  then  $y = 0$  and from  
 231  $(0, 0) \in R$  we derive  $(x, y) \in R^+$ . So, we can assume that  $x \geq 1$ . In that case  $(x, 4) \in R$  and  
 232  $(4z, 4(z+1)) \in R$  for every  $z > 0$ . It follows that  $(x, n) \in R^+$  for every  $n \in 4 + 4\mathbb{N}$ . Moreover,  
 233 there exists such an  $n$  satisfying  $2y \leq n$ . For such an  $n$ , we have  $(x, n) \in R^+$  and  $(n, y) \in R$ .  
 234 We deduce that  $(x, y) \in R^+$ . It follows that  $R^+ = C$ . ◀



## XX:8 Reachability for Test/Reset Two-Counter Machines

235 The effective Presburger-definability of the transitive closure comes from the following  
236 characterization.

237 ► **Lemma 6.** *For every horizontal relation  $R$  we have:*

$$238 \quad R^+ = \{(x, y) \mid \exists z : (z, y) \in R \wedge \forall u : x \leq u < z \Rightarrow \exists v : (u, v) \in R \wedge u < v\} \quad (4)$$

239 **Proof.** Assume first that  $(x, y) \in R^+$ . There exists a sequence  $x_0, \dots, x_k$  such that  $x =$   
240  $x_0 R x_1 \dots R x_k = y$  with  $k \geq 1$ . Let  $z = x_{k-1}$  and let us prove that for every  $u \in$   
241  $\{x, \dots, z-1\}$  there exists  $v > u$  such that  $(u, v) \in R$ . If  $z \leq x$  we are done. So we can  
242 assume that  $z > x$ . Since  $x_0 \leq u$ , there exists a maximal  $j \in \{1, \dots, k\}$  such that  $x_{j-1} \leq u$ .  
243 Let  $v = x_j$  and observe that  $(u, v) \in R$ . Since  $x_{k-1} = z > u$ , it follows that  $j < k$  and by  
244 maximality of  $j$  we deduce that  $x_j > u$ . Therefore  $v > u$ .

245 Conversely, let us consider  $(x, y) \in \mathbb{N} \times \mathbb{N}$  such that there exists  $z$  satisfying  $(z, y) \in R$   
246 and such that for every  $u \in \{x, \dots, z-1\}$  there exists  $v > u$  such that  $(u, v) \in R$ . Notice  
247 that there exists a sequence  $x_0 < \dots < x_k$  with  $k \geq 0$  such that  $x = x_0 R x_1 \dots R x_k \geq z$ . It  
248 follows that  $(x, x_k) \in R^*$ . Moreover, since  $(z, y) \in R$ ,  $z \leq x_k$ , and  $R$  is horizontal we deduce  
249 that  $(x_k, y) \in R$ . It follows that  $(x, y) \in R^+$ . ◀

250 The previous lemma shows that the transitive closure of a horizontal relation  $R$  denoted  
251 by a Presburger formula  $\varphi_R$  is denoted by the Presburger formula obtained from (4) by  
252 replacing  $(z, y) \in R$  and  $(u, v) \in R$  by  $\varphi_R(z, y)$  and  $\varphi_R(u, v)$  respectively. We have proved  
253 the following theorem.

254 ► **Theorem 7.** *The transitive closure of a horizontal Presburger-definable relation is effectively*  
255 *Presburger-definable.*

## 256 5 Presburger Automata

257 We exhibit in this section a general class of one counter automata with effectively Presburger-  
258 definable reachability relations. These automata will be used in the next section to compute  
259 the reachability relations of TRVASS.

260 A *Presburger automaton* is a pair  $\mathcal{P} = (Q, \Delta)$  where  $Q$  is a finite set of *states*, and  $\Delta$   
261 is a finite set of *transitions*  $(p, R, q)$  where  $p, q \in Q$  and  $R \subseteq \mathbb{N} \times \mathbb{N}$  is a relation denoted  
262 by a Presburger formula (which is left implicit). A *configuration* is a pair  $(q, x) \in Q \times \mathbb{N}$ ,  
263 also written as  $q(x)$  in the sequel. The *one-step* relation  $\rightarrow_{\mathcal{P}}$  is the binary relation over  
264 configurations defined by  $p(x) \rightarrow_{\mathcal{P}} q(y)$  if there exists  $(p, R, q) \in \Delta$  such that  $(x, y) \in R$ . The  
265 *reachability relation*  $\xrightarrow{*}_{\mathcal{P}}$  is defined as the reflexive-transitive closure of  $\rightarrow_{\mathcal{P}}$ .

266 ► **Remark.** The reflexive-transitive closure  $R^*$  of a Presburger-definable relation  $R \subseteq \mathbb{N} \times \mathbb{N}$   
267 need not be Presburger-definable, in general. For instance, if  $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = 2x\}$   
268 then  $R^*$  is the relation  $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \exists k \in \mathbb{N} : y = 2^k x\}$ , which is not definable in  
269 Presburger arithmetic. Worse, a simple reduction from the halting problem for Minsky  
270 machines shows that membership of a pair  $(x, y)$  in  $R^*$  is undecidable (where  $R$  is a  
271 Presburger-definable relation given as input along with  $x$  and  $y$ ). ◀

272 A consequence of the above remark is that the reachability problem for Presburger  
273 automata is undecidable, even if we restrict ourselves to Presburger automata with a single  
274 state and a single transition. This comes from the fact that transitions can use arbitrary  
275 Presburger-definable relations. We will exhibit a subclass of Presburger automata with  
276 effectively Presburger-definable reachability relations (hence, with a decidable reachability

277 problem) by limiting the expressive power of the transitions occurring on cycles. We say that  
 278 a transition  $(p, R, q)$  is *diagonal* if  $R$  is diagonal, *horizontal* if  $R$  is horizontal, and *ordinary*  
 279 if it is neither diagonal nor horizontal. Note that a relation on  $\mathbb{N}$  may be both diagonal  
 280 and horizontal, for instance  $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \leq 2x\}$ . A *cycle* is non-empty sequence of  
 281 transitions  $(p_1, R_1, q_1), \dots, (p_n, R_n, q_n)$  such that  $q_n = p_1$  and  $q_i = p_{i+1}$  for all  $1 \leq i < n$ .

282 ► **Lemma 8.** *Let  $\mathcal{P}$  be a Presburger automaton. If every cycle of  $\mathcal{P}$  contains only diagonal*  
 283 *transitions then  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable.*

284 **Proof.** We first observe that  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable when  $\mathcal{P} = (Q, \Delta)$  is  
 285 a Presburger automaton whose transitions are all diagonal. Indeed, we may view  $\mathcal{P}$  as a  
 286 finite-state automaton over the finite alphabet  $\{R \mid (p, R, q) \in \Delta\}$ . For every states  $p$  and  $q$ ,  
 287 we may compute a regular expression denoting the language accepted by  $\mathcal{P}$  with initial state  $p$   
 288 and final state  $q$ . The obvious evaluation of this regular expression (concatenation  $\cdot$  becomes  
 289 relational composition  $\circ$ , sum  $+$  becomes union  $\cup$ , and star  $\star$  becomes reflexive-transitive  
 290 closure  $*$ ) yields the relation  $\{(x, y) \mid p(x) \xrightarrow{*}_{\mathcal{P}} q(y)\}$ . This evaluation is computable because  
 291 Presburger-definable diagonal relations are effectively closed under union, composition and  
 292 reflexive-transitive closure (as an immediate consequence of Theorem 4). We have shown  
 293 that  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable when all transitions of  $\mathcal{P}$  are diagonal.

We now prove the lemma. Let  $\mathcal{P} = (Q, \Delta)$  be a Presburger automaton such that every  
 cycle of  $\mathcal{P}$  contains only diagonal transitions. Let  $\mathcal{N}$  be the Presburger automaton obtained  
 from  $\mathcal{P}$  by keeping only diagonal transitions. Consider two configurations  $p(x)$  and  $q(y)$ . It  
 is readily seen that  $p(x) \xrightarrow{*}_{\mathcal{P}} q(y)$  if, and only if, there exists  $1 \leq k \leq |Q|$ ,  $s_1, \dots, s_k \in Q$  and  
 $x_1, y_1, \dots, x_k, y_k \in \mathbb{N}$  such that  $p(x) = s_1(x_1)$ ,  $s_k(y_k) = q(y)$  and

$$s_1(x_1) \xrightarrow{*}_{\mathcal{P}} s_1(y_1) \rightarrow_{\mathcal{P}} s_2(x_2) \xrightarrow{*}_{\mathcal{P}} s_2(y_2) \cdots s_{k-1}(y_{k-1}) \rightarrow_{\mathcal{P}} s_k(x_k) \xrightarrow{*}_{\mathcal{P}} s_k(y_k)$$

294 Observe that for every state  $s \in Q$  and for every  $x, y \in \mathbb{N}$ ,  $s(x) \xrightarrow{*}_{\mathcal{P}} s(y)$  if, and only if,  
 295  $s(x) \xrightarrow{*}_{\mathcal{N}} s(y)$ . Moreover,  $\xrightarrow{*}_{\mathcal{N}}$  is effectively Presburger-definable since all transitions of  $\mathcal{N}$   
 296 are diagonal. We derive from the above characterization of  $\xrightarrow{*}_{\mathcal{P}}$  that  $\xrightarrow{*}_{\mathcal{P}}$  is also effectively  
 297 Presburger-definable. ◀

298 We say that a Presburger automaton  $\mathcal{P}$  is *shallow* if every cycle that contains an ordinary  
 299 transition also contains a horizontal transition. Shallowness of Presburger automata is  
 300 decidable. This follows from two easy observations. Firstly, diagonality and horizontality of  
 301 Presburger-definable relations on  $\mathbb{N}$  are decidable, since these properties can be expressed in  
 302 Presburger arithmetic. Secondly, a Presburger automaton is shallow if, and only if, every  
 303 *simple* cycle containing an ordinary transition also contains a horizontal transition. We now  
 304 show the main result of this section.

305 ► **Theorem 9.** *The reachability relation of a shallow Presburger automaton is effectively*  
 306 *Presburger-definable.*

**Proof.** By induction on the number of horizontal transitions. The base case follows from  
 Lemma 8. Indeed, if  $\mathcal{P}$  is a shallow Presburger automaton with no horizontal transition then  
 every cycle of  $\mathcal{P}$  contains only diagonal transitions. Assume that the theorem holds for every  
 shallow Presburger automaton with  $n$  horizontal transitions, where  $n \in \mathbb{N}$ . Let  $\mathcal{P} = (Q, \Delta)$   
 be a Presburger automaton with  $n + 1$  horizontal transitions. Pick a horizontal transition  
 $(p, R, q) \in \Delta$  and let  $\mathcal{N}$  be the Presburger automaton obtained from  $\mathcal{P}$  by removing the  
 transition  $(p, R, q)$ . Let  $S$  denote the reachability relation from  $q$  to  $p$  in  $\mathcal{N}$ , namely the

## XX:10 Reachability for Test/Reset Two-Counter Machines

relation  $S = \{(y, x) \mid q(y) \xrightarrow{*} p(x)\}$ . It is readily seen that, for every configurations  $s(x)$  and  $t(y)$  of  $\mathcal{P}$ ,  $s(x) \xrightarrow{*} t(y)$  if, and only if,  $s(x) \xrightarrow{*} t(y)$  or there exists  $x', y' \in \mathbb{N}$  such that

$$s(x) \xrightarrow{*} p(x') \quad \wedge \quad (x', y') \in ((R \circledast S)^* \circledast R) \quad \wedge \quad q(y') \xrightarrow{*} t(y)$$

307 By induction hypothesis, the relation  $\xrightarrow{*}$  is effectively Presburger-definable, and so is  $R \circledast S$ .  
 308 Moreover,  $R \circledast S$  is horizontal since  $R$  is horizontal. It follows from Theorem 7 that  $(R \circledast S)^*$   
 309 is effectively Presburger-definable. We derive from the above characterization of  $\xrightarrow{*}$  that  
 310  $\xrightarrow{*}$  is also effectively Presburger-definable. ◀

311 ▶ **Remark.** The notions of diagonal relations, horizontal relations and Presburger automata  
 312 are extended to larger dimensions in the obvious way. A relation  $R \subseteq \mathbb{N}^d \times \mathbb{N}^d$  is *diagonal*  
 313 (resp. *horizontal*) if  $(\mathbf{x}, \mathbf{y}) \in R$  implies  $(\mathbf{x} + \mathbf{c}, \mathbf{y} + \mathbf{c}) \in R$  (resp.  $(\mathbf{x} + \mathbf{c}, \mathbf{y}) \in R$ ) for every  
 314  $\mathbf{c} \in \mathbb{N}^d$ . But Theorem 9 does not extend to larger dimensions, even if we restrict ourselves  
 315 to Presburger automata with a single state and a single transition. In fact, the reflexive-  
 316 transitive closure of a Presburger-definable relation that is diagonal (resp. horizontal) need  
 317 not be Presburger-definable. Consider the relation  $R \subseteq \mathbb{N}^2 \times \mathbb{N}^2$  defined by  $(x_1, x_2) R (y_1, y_2)$   
 318 if, and only if, the Presburger formula  $y_1 \leq 2x_1 \wedge y_2 < x_2$  holds. The relation  $R$  is both  
 319 diagonal and horizontal. It is routinely checked that the reflexive-transitive closure  $R^*$  is the  
 320 set of pairs  $((x_1, x_2), (y_1, y_2)) \in \mathbb{N}^2 \times \mathbb{N}^2$  such that  $y_1 \leq 2^{x_2 - y_2} x_1$  and  $y_2 \leq x_2$ , which is not  
 321 definable in Presburger arithmetic. ◀

## 6 Reachability Relations of TRVASS

322 A TRVASS is a 2-dimensional vector addition system with states (2-dim VASS) such that  
 the first counter can be tested for zero and the second one can be reseted to zero. Formally,  
 a TRVASS is a triple  $\mathcal{V} = (Q, \Sigma, \Delta)$  where  $Q$  is a finite set of *states*,  $\Sigma \subseteq \mathbb{Z}^2 \cup \{T, R\}$  is  
 a finite set of *actions*, and  $\Delta \subseteq Q \times \Sigma \times Q$  is a finite set of *transitions*. A *configuration*  
 of  $\mathcal{V}$  is a triple  $(q, x_1, x_2) \in Q \times \mathbb{N} \times \mathbb{N}$  written as  $q(x_1, x_2)$  in the sequel. The operational  
 semantics of  $\mathcal{V}$  is given by the binary relations  $\xrightarrow{a}$  over configurations, with  $a \in \Sigma$ , defined  
 by  $p(x_1, x_2) \xrightarrow{a} q(y_1, y_2)$  if  $(p, a, q) \in \Delta$  and

$$\begin{aligned} (y_1, y_2) &= (x_1 + a_1, x_2 + a_2) & \text{if } a &= (a_1, a_2) \in \mathbb{Z}^2 \\ (y_1, y_2) &= (0, x_2) \wedge x_1 = 0 & \text{if } a &= T \\ (y_1, y_2) &= (x_1, 0) & \text{if } a &= R \end{aligned}$$

323 Given a word  $w = a_1 \dots a_k$  of actions  $a_j \in \Sigma$ , we denote by  $\xrightarrow{w}$  the binary relation over  
 324 configurations defined as the relational composition  $\xrightarrow{a_1} \circledast \dots \circledast \xrightarrow{a_k}$ . The relation  $\xrightarrow{\varepsilon}$   
 325 denotes the identity relation on configurations. Given a subset  $W \subseteq \Sigma^*$ , we let  $\xrightarrow{W}$  denote  
 326 the union  $\bigcup_{w \in W} \xrightarrow{w}$ . The relation  $\xrightarrow{\Sigma^*}$ , also written  $\xrightarrow{*}$ , is called the *reachability relation*  
 327 of  $\mathcal{V}$ . Observe that  $\xrightarrow{*}$  is the reflexive-transitive closure of the *step* relation  $\rightarrow \stackrel{\text{def}}{=} \bigcup_{a \in \Sigma} \xrightarrow{a}$ .

328 The remainder of this section is devoted to the proof that TRVASS have effectively  
 329 Presburger-definable reachability relations. Let us fix a TRVASS  $\mathcal{V} = (Q, \Sigma, \Delta)$ . We let  $A$   
 330 denote the set  $\Sigma \cap \mathbb{Z}^2$  of *addition vectors*.

The reachability relation of  $\mathcal{V}$  can be expressed in terms of the reachability relation of  
 a Presburger automaton by observing that configurations reachable just after a zero-test  
 $T$  or a reset  $R$  are restricted to  $q(0, n)$  or  $q(n, 0)$ , respectively, where  $q \in Q$  and  $n \in \mathbb{N}$ .  
 Those configurations are parametrized by introducing the set  $S = \{q^T, q^R \mid q \in Q\}$  obtained  
 as two disjoint copies of  $Q$ . Elements in  $\{q^T \mid q \in Q\}$  are called *test states*, and those in

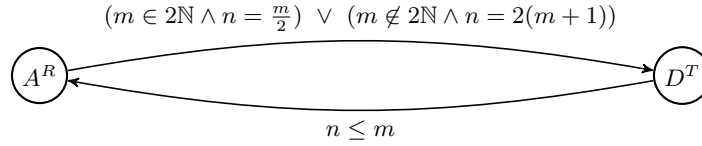
$\{q^R \mid q \in Q\}$  are called *reset states*. Given  $s \in S$  and  $n \in \mathbb{N}$ , we introduce the configuration  $\llbracket s, n \rrbracket$  in  $Q \times \mathbb{N}^2$  defined as follows:

$$\llbracket s, n \rrbracket \stackrel{\text{def}}{=} \begin{cases} q(0, n) & \text{if } s = q^T \\ q(n, 0) & \text{if } s = q^R \end{cases}$$

We also introduce, for each pair  $(s, t) \in S \times S$ , the binary relation  $R_{s,t}$  defined by

$$R_{s,t} \stackrel{\text{def}}{=} \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid \llbracket s, m \rrbracket \xrightarrow{A^*X} \llbracket t, n \rrbracket\}$$

331 where  $X = T$  if  $t$  is a test state and  $X = R$  if  $t$  is a reset state. It is known that the  
 332 reachability relation of a 2-dim VASS is effectively Presburger-definable [17, 2]. This entails  
 333 that the relation  $\xrightarrow{A^*}$  is effectively Presburger-definable, and it follows that the relations  
 334  $R_{s,t}$  are also effectively Presburger-definable. We introduce the Presburger automaton  $\mathcal{P}$   
 335 with set of states  $S$  and set of transitions  $\{(s, R_{s,t}, t) \mid (s, t) \in S \times S\}$ . Note that  $\mathcal{P}$  is  
 336 computable from  $\mathcal{V}$ .



■ **Figure 3** The Presburger automaton  $\mathcal{P}$  associated to the TRVASS of Figure 2.

337 ► **Example 10.** Let us come back to the TRVASS of Figure 2. The relations  $R_{s,t}$  are all empty  
 338 except for  $R_{A^R, D^T}$  and  $R_{D^T, A^R}$ . The corresponding automaton  $\mathcal{P}$  is depicted in Figure 3.  
 339 Each transition  $(s, R_{s,t}, t)$  is depicted by an edge from  $s$  to  $t$  labeled by a Presburger formula  
 340  $\varphi_{s,t}(m, n)$  denoting the relation  $R_{s,t}$ . The empty relations (which are both diagonal and  
 341 horizontal) and the states not in  $\{A^R, D^T\}$  are not depicted. Notice that the transition from  
 342  $A^R$  to  $D^T$  is ordinary and the one from  $D^T$  to  $A^R$  is horizontal. It follows that  $\mathcal{P}$  is shallow.  
 343 We observe that the horizontal relation  $R$  defined as the composition  $R_{D^T, A^R} \circ R_{A^R, D^T}$  is  
 344 the one introduced in Example 5. ◀

345 We first show that the Presburger automaton  $\mathcal{P}$  is shallow. By Theorem 9, this will entail  
 346 that its reachability relation  $\xrightarrow{*}$  is effectively Presburger-definable.

347 ► **Lemma 11.** *The Presburger automaton  $\mathcal{P}$  is shallow.*

348 **Proof.** It is readily seen that  $\mathcal{P}$  satisfies the following properties:

- 349 ■ Transitions from reset states to reset states are diagonal,
- 350 ■ Transitions from test states to reset states are horizontal,
- 351 ■ Transitions from test states to test states are diagonal.

352 It follows that an ordinary transition of  $\mathcal{P}$  is a transition from a reset state to a test state. If  
 353 a cycle contains such a transition then it must contain a transition from a test state to a  
 354 reset state as well. Since such a transition is horizontal, we obtain that  $\mathcal{P}$  is shallow. ◀

355 The two following lemmas show how to decompose the reachability relation of  $\mathcal{V}$  in terms  
 356 of the reachability relation of  $\mathcal{P}$ .

357 ► **Lemma 12.** *For every  $s, t \in S$  and  $m, n \in \mathbb{N}$ , if  $s(m) \xrightarrow{*} t(n)$  then  $\llbracket s, m \rrbracket \xrightarrow{*} \llbracket t, n \rrbracket$ .*

## XX:12 Reachability for Test/Reset Two-Counter Machines

358 **Proof.** It is easily seen that  $s(m) \rightarrow_{\mathcal{P}} t(n)$  implies  $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$ , for every  $s, t \in S$  and  
 359  $m, n \in \mathbb{N}$ . We derive, by an immediate induction on  $k \geq 1$ , that  $s(m) (\rightarrow_{\mathcal{P}})^k t(n)$  implies  
 360  $\llbracket s, m \rrbracket \xrightarrow{*}_{\mathcal{V}} \llbracket t, n \rrbracket$ , for every  $s, t \in S$  and  $m, n \in \mathbb{N}$ . The lemma follows. ◀

► **Lemma 13.** *Consider two configurations  $p(x_1, x_2)$  and  $q(y_1, y_2)$  of  $\mathcal{V}$ . It holds that  $p(x_1, x_2) \xrightarrow{\Sigma^* \setminus A^*}_{\mathcal{V}} q(y_1, y_2)$  if, and only if, there exist  $s, t \in S$  and  $m, n \in \mathbb{N}$  such that:*

$$p(x_1, x_2) \xrightarrow{A^* \{T, R\}}_{\mathcal{V}} \llbracket s, m \rrbracket \wedge s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \wedge \llbracket t, n \rrbracket \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

**Proof.** Lemma 12 shows the “if” direction of the equivalence. For the other direction, let  $w \in \Sigma^* \setminus A^*$  such that  $p(x_1, x_2) \xrightarrow{w}_{\mathcal{V}} q(y_1, y_2)$ . By splitting  $w$  after each occurrence of an action in  $\{T, R\}$ , we deduce that  $w = w_0 X_1 \dots w_{k-1} X_k w_k$  where  $k \geq 1$ , and  $w_0, \dots, w_k \in A^*$ . Let us introduce the configurations  $c_1, \dots, c_k$  satisfying the following relations:

$$p(x_1, x_2) \xrightarrow{w_0 X_1}_{\mathcal{V}} c_1 \cdots \xrightarrow{w_{k-1} X_k}_{\mathcal{V}} c_k \xrightarrow{w_k}_{\mathcal{V}} q(y_1, y_2)$$

361 Notice that  $c_j = \llbracket q_j^{X_j}, n_j \rrbracket$  for some  $q_j \in Q$  and some  $n_j \in \mathbb{N}$ . By definition of  $\mathcal{P}$ , we get  
 362  $q_{j-1}^{X_{j-1}}(n_{j-1}) \rightarrow_{\mathcal{P}} q_j^{X_j}(n_j)$  for every  $j \in \{1, \dots, k\}$ . We have proved the lemma. ◀

363 We deduce our main result.

364 ► **Theorem 14.** *The reachability relation of a TRVASS is effectively Presburger-definable.*

**Proof.** Lemma 13 shows that  $p(x_1, x_2) \xrightarrow{*}_{\mathcal{V}} q(y_1, y_2)$  if, and only if,  $p(x_1, x_2) \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$  or there exists  $s, t \in S$  and  $m, n \in \mathbb{N}$  such that:

$$p(x_1, x_2) \xrightarrow{A^* \{T, R\}}_{\mathcal{V}} \llbracket s, m \rrbracket \wedge s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \wedge \llbracket t, n \rrbracket \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

365 From [17, 2], the relation  $\xrightarrow{A^*}_{\mathcal{V}}$  is effectively Presburger-definable. From Lemma 11 and  
 366 Theorem 9, the relation  $\xrightarrow{*}_{\mathcal{P}}$  is effectively Presburger-definable as well. ◀

367 Coming back to the classes of 2-dim extended VASS discussed in the introduction (see  
 368 Figure 1), Theorem 14 means that the reachability relation is effectively Presburger-definable  
 369 for the “maximal” class  $\mathbf{T}_1 \mathbf{R}_2$ . This result also applies to 2-dim VASS extended with resets  
 370 and transfers on both counters (i.e., the class  $\mathbf{R}_{1,2} \mathbf{Tr}_{1,2}$ ), since they can be simulated by  
 371 machines in  $\mathbf{T}_1 \mathbf{R}_2$ .

## 372 7 Conclusion and open problems

373 We have shown that the reachability relation of 2-dim VASS extended with tests on the first  
 374 counter and resets on the second counter, is effectively Presburger-definable. This completes  
 375 the decidability picture of 2-dim extended VASS initiated in [11]. Our proof techniques may  
 376 also be used for other classes of counter machines where shallow Presburger automata would  
 377 naturally appear. Many other problems on extensions of VASS are still interesting to solve.

378 ■ The reachability problem is NP-complete [12] for 1-dim VASS, PSPACE-complete [2] for  
 379 2-dim VASS, and NL-complete [7] for unary 2-dim VASS. But we do not know what are  
 380 the complexities for the reachability problem, for the construction of the reachability set  
 381 and for the reachability relation for all 2-dim extended VASS.

382 ■ The boundedness problem is undecidable for 3-dim VASS extended with resets on all  
 383 counters [5] and it is decidable for arbitrary dimension VASS extended with resets on  
 384 two counters [6]. Is boundedness decidable for arbitrary dimension TRVASS ?

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