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# Non-convex Mather's theory and the Conley conjecture on the cotangent bundle of the torus

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## Abstract

The aim of this paper is to use the methods and results of symplectic homogenization (see [V4]) to prove existence of periodic orbits and invariant measures with rotation number depending on the differential of the Homogenized Hamiltonian.

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# 1 Introduction

The symplectic theory of Homogenization, set up in [V4], associates to each Hamiltonian  $H(t, q, p)$  on  $T^*T^n$  a homogenized Hamiltonian,  $\overline{H}(p)$ , such that  $H_k(t, q, p) = H(kt, kq, p)$   $\gamma$ -converges to  $\overline{H}(p)$ , where the metric  $\gamma$  has been defined in previous works <sup>1</sup>. In other words if we denote by  $\varphi^t$  the flow associated to the Hamiltonian  $H$ , and by  $\overline{\varphi}^t$  the flow of  $\overline{H}(p)$  – defined in the completion  $\widehat{\mathcal{DH}}(T^*T^n)$  of the group of Hamiltonian diffeomorphisms of  $T^*T^n$  for the metric  $\gamma$  – then  $\overline{\varphi}^t$  is the  $\gamma$ -limit of  $\rho_k^{-1}\varphi^{kt}\rho_k$  where  $\rho_k(q, p) = (kq, p)$  and  $\varphi^t$  is the flow of the Hamiltonian vector field associated to  $H$ . The goal of this paper is to draw some dynamical consequences of the homogenization theorem, to prove existence of certain trajectories of the flow  $\varphi^t$  and then of invariant measures. We also apply this to the Conley conjecture on  $T^*T^n$ . Symplectic Homogenization may be summarized as the following heuristic statement

**Symplectic Homogenization Principle:** *The value of any variational problem associated to  $H_k$  will converge to the value of the same variational problem associated to  $\overline{H}$ .*

While the above sentence is vague and does not claim to be a mathematical statement, we hope it carries sufficient meaning for the reader to help him understand the substance of the method used in the present paper.

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<sup>1</sup>see [V1], and the related Hofer metric in [Ho]. See also [Hu] for the study of this metric and its completion mentioned further.

**Notations:** We denote by  $\varphi$  the time-one flow  $\varphi^1$ , by  $\Phi^t$  the lift of  $\varphi^t$  to the universal cover  $\mathbb{R}^{2n}$  of  $T^*T^n$ . The action of a trajectory  $\gamma(t) = (q(t), p(t)) = \varphi^t(q(0), p(0))$  defined on  $[0, 1]$  is

$$A(\gamma) = \int_0^1 p(t)\dot{q}(t) - H(t, q(t), p(t))dt$$

The average action for a solution defined on  $[0, T]$  is

$$A_T(\gamma) = \frac{1}{T} \int_0^T p(t)\dot{q}(t) - H(t, q(t), p(t))dt$$

Our goal is to prove the following theorems. The Clarke subdifferential  $\partial_C \overline{H}(p)$  will be defined in section 8.1 (see [Clarke 1]).

**Theorem 1.1.** *Let  $H(t, q, p)$  be a compact supported Hamiltonian in  $S^1 \times T^*T^n$ , and denote by  $\overline{H}(p)$  its homogenization defined in [V4]. Let  $\alpha \in \partial_C \overline{H}(p)$ . Then there exists, for  $k$  large enough, a solution of  $\varphi^k(q_k, p_k) = (q_k + k\alpha_k, p'_k)$  (with  $\lim_k \alpha_k = \alpha$ ) and average action*

$$A_k = \frac{1}{k} \int_0^k [\gamma_k^* \lambda - H(t, \gamma_k(t))]dt$$

where  $\gamma_k(t) = \varphi^t(q_k, p_k)$ . Moreover as  $k$  goes to infinity  $A_k$  converges to

$$\lim_k A_k = \langle p, \alpha \rangle - \overline{H}(p)$$

Therefore for each  $\alpha \in \partial_C \overline{H}(p)$  there exists an invariant measure  $\mu_\alpha$  with rotation number  $\alpha$  and average action

$$\mathcal{A}(\mu_\alpha) \stackrel{def}{=} \int_{T^*T^n} [p \frac{\partial H}{\partial p}(q, p) - H(q, p)] d\mu_\alpha = \langle p, \alpha \rangle - \overline{H}(p).$$

First of all, remember that a measure is invariant if  $(\varphi^1)_*(\mu) = \mu$ . Of course the Liouville measure  $\omega^n$  is invariant, and since  $\varphi$  is compact supported, we may truncate  $\omega^n$  to  $\chi(|p|) \cdot \omega^n$ , where  $\chi(r)$  equals 1 if the support of  $\varphi$  is contained in  $\{(q, p) \mid |p| \leq r\}$ . This gives a large family of invariant measures. However, as explained in [Ma], page 176, the rotation number of such a measure is the element of  $H_1(T^n, \mathbb{R})$  given by duality by the map

$$\begin{aligned} \rho(\mu) : H^1(T^n, \mathbb{R}) &\longrightarrow \mathbb{R} \\ \lambda &\longrightarrow \int_{T^*T^n} \langle \lambda(q), \frac{\partial H}{\partial p} \rangle d\mu = \int_{T^*T^n} \sum_{j=1}^n \lambda_j \frac{\partial H}{\partial p_j} d\mu \end{aligned}$$

But for  $\mu = \chi(|p|)\omega^n$ , using Stoke's formula, and the fact that the support of  $H$  is contained in  $\{(q, p) \mid \chi(|p|) = 1\}$ , we have

$$\int_{T^*T^n} \langle \lambda(q), \frac{\partial H}{\partial p} \rangle \omega^n = \int_{T^*T^n} \sum_{j=1}^n \frac{\partial}{\partial p_j} (\lambda_j(q) H(q, p)) \omega^n = 0$$

since  $H$  is compact supported. Therefore  $\rho(\omega^n) = 0$ .

Moreover the average action of this measure is given by

$$\int_{T^*T^n} [p \frac{\partial H}{\partial p}(q, p) - H(q, p)] \omega^n = \int_{T^*T^n} \left[ \left( \sum_{j=1}^n \frac{\partial}{\partial p_j} (p_j H(q, p)) \right) - (n+1) H(q, p) \right] \omega^n = -(n+1) \text{Cal}(\varphi)$$

where  $\text{Cal}(\varphi)$  is the Calabi invariant of  $\varphi$ . We shall see that this is at most one of the many invariant measures we find. Indeed, if  $\alpha = 0$ ,  $\mathcal{A}(\mu_0) = -\overline{H}(p_0)$ , where  $p_0$  is a critical point of  $\overline{H}$ . If  $\overline{H}(p_0) \neq -(n+1) \text{Cal}(\varphi)$ , a generic property, none of the measures given by the main theorem is of the form  $\chi(p)\omega^n$ . Otherwise, at most one of them,  $\mu_0$  is of the form  $\chi(p)\omega^n$ .

We shall also need to define  $\partial_C \overline{H}(p)$  since as we pointed out in [V4], we cannot hope that  $\overline{H}$  is better than  $C^{0,1}$ . It is thus important to figure out the set  $\partial \overline{H}(p)$  when  $\overline{H}$  is not differentiable at  $p$ . Remember also that  $\overline{H}$  coincides with Mather's  $\alpha$  function when  $H$  is strictly convex in  $p$  (see [V4], section 13.1), but in this case the set of values of  $\partial_C \overline{H}(p)$  as  $p$  describes  $\mathbb{R}^n$  is the whole of  $\mathbb{R}^n$ , so we get any rotation number, as expected from standard Aubry-Mather theory (see [Ma]). This may be generalized to

**Corollary 1.2.** *Let  $H(t, q, p)$  be a coercive Hamiltonian on  $T^*T^n$ . Then  $\overline{H}$  is coercive, so that for any  $\alpha \in \mathbb{R}^n$ , we may find an invariant measure for the flow, with rotation number  $\alpha$ .*

The main idea of the proof is to formulate the existence of intersection points in  $\Phi^k(\{q_0\} \times \mathbb{R}^n) \cap (\{q_0 + k\alpha\} \times \mathbb{R}^n)$  as a variational problem and apply our heuristic principle – i.e. that a variational problem involving  $H_k$  must converge to the variational problem involving  $\overline{H}$ .

Another consequence of our methods will be

**Theorem 1.3.** *Let us assume  $H(t, q, p)$  is a compact supported Hamiltonian on  $T^*T^n$ .*

- (a). *Assume  $\overline{H} \not\equiv 0$ . Then there exists infinitely many distinct non-contractible periodic orbits for  $\varphi^1$*

- (b). Assume  $\overline{H} \equiv 0$ . Then there exists infinitely many geometrically distinct contractible periodic orbits for  $\varphi^1$  contained in the support of  $H$ , and moreover there exists a constant  $C$  such that

$$\#\{x \in \text{supp}(H) \mid \exists k \in [1, N] \mid \varphi^k(x) = x\} \geq CN$$

Note that both cases: non-existence of contractible non-trivial periodic orbits (think of the geodesic flow for the flat metric) and non-existence of non-contractible ones (for example if  $\text{supp}(H)$  is contractible) are possible. Our result could be considered a generalization of the main result in [BPS], where the first statement is proved under the assumption that  $H$  is bounded from below on a certain Lagrangian submanifold. But this assumption implies, according to [V4] that  $\overline{H}$  is nonzero.

The Conley's conjecture proved by N. Hingston on  $T^{2n}$  and on more general manifolds by V. Ginzburg (see [Hi], [Gi]) yields existence of infinitely many contractible periodic orbits for a Hamiltonian on  $(M, \omega)$ . For Lagrangian systems in the cotangent bundle of a compact manifold, the analogous statement was proved by Y. Long and G.Lu ([L-L]) for the torus and by G.Lu ([Lu]) in the general case (see also [A-F], and [Mazz]).

*Remark 1.4.* If  $\varphi^t(x)$  is an orbit of period  $k$ , we denote by  $\nu(x, \varphi)$  the vector obtained by considering the  $q$  component of  $\frac{1}{k}(\Phi^k(q, p) - (q, p)) \in \frac{1}{k}\mathbb{Z}^{2n}$ . Then if  $\overline{H} \neq 0$ , we shall prove that the set of limit sets of  $\nu(x, \varphi)$  as  $x$  belongs to the set of  $k$ -periodic orbits is a subset  $\Omega(\varphi^1) \subset \mathbb{R}^n$  of non-empty interior.

A final comment is in order. In the convex case, Aubry-Mather theory makes two Claims:

- (a). existence of the invariant measure with given rotation number
- (b). the support of this invariant measure is a Lipschitz graph over the base of the cotangent bundle

While we believe that the present work gives the right extension of the first statement to non-convex situations (for the moment only in  $T^*T^n$  and not in a general cotangent bundle, see however [Vic1] for cotangent bundles and [Bi] for general symplectic manifolds), we say nothing close to the second statement. Of course starting from a convex Hamiltonian and applying a conjugation by a symplectic map, the support of the invariant measure will be the image by the conjugating map of a Lipschitz graph, and obviously this will not be -in general - a graph. However other statements could make sense. One plausible conjecture is to look at the action of  $\varphi^t$  over  $\widehat{\mathcal{L}}$ , the

Humilière completion of the set  $\mathcal{L}$  of Lagrangians submanifolds for the  $\gamma$ -metric. Indeed, the group of Hamiltonian diffeomorphisms acts on this set (since a Hamiltonian diffeomorphism acts as an isometry for  $\gamma$ , over  $\mathcal{L}$ , hence acts over its completion). If there is an element  $L$ , in  $\widehat{\mathcal{L}}$  fixed by  $\varphi^t$ , then  $L$  is not a Lagrangian, but  $u_L(x)$  is a continuous function, well-defined, and Lipschitz, hence differentiable a.e. The set of points  $(x, du_L(x))$  where  $u_L$  is differentiable may then be invariant by the flow  $\varphi^t$ . Note that the approach in this paper is very far from this conjecture, since we obtain the invariant measure as a limit of measures supported on trajectories, and there is no obvious way to make this into an element in  $\widehat{\mathcal{L}}$ .

## 2 Aubry-Mather theory for non-convex Hamiltonians

*Proof of theorem 1.1.* <sup>2</sup>

Remember that  $\overline{H}(p)$  is the limit of  $h_k(p)$  where  $h_k(p) = c(\mu_q \otimes 1_p, \Gamma_k)$ , where

$$\Gamma_k = \{(q, P_k(q, p), p - P_k(q, p), Q_k(q, p) - q) \mid \varphi_k(q, p) = (Q_k(q, p), P_k(q, p))\}$$

and  $\varphi_k = \rho_k^{-1} \varphi^k \rho_k$  and  $\Gamma_k$  is a Lagrangian submanifold in  $T^*(T^n \times \mathbb{R}^n)$ . Note that if  $S_k(q, P, \xi)$  is a G.F.Q.I. for  $\Gamma_k \subset T^*(T^n \times \mathbb{R}^n)$ ,  $h_k(p)$  is by definition the critical value associated to class  $\mu_q$  of  $(q, P) \mapsto S_k(q, P, \xi)$  (see [V4]). If the selector  $h_k$  is smooth at  $P$ , i.e. there is a smooth map  $P \mapsto (q(P), \xi(P))$  such that

$$\frac{\partial S_k}{\partial q}(q(P), P, \xi(P)) = 0 = \frac{\partial S_k}{\partial \xi}(q(P), P, \xi(P))$$

and  $S(q(P), P, \xi(P)) = h_k(P)$ , we have

$$\alpha_k = dh_k(P) = \frac{\partial S_k}{\partial P}(q(P), P, \xi(P))$$

so that the point of  $\Gamma_k$  corresponding to  $(q(P), P, \xi(P))$  is  $(q(P), P, 0, \alpha_k)$ , which translates into  $\varphi_k(q(P), P) = (q(P) + \alpha_k, P)$ , hence  $\varphi^k(k \cdot q(P), P) = (k \cdot q(P) + k \cdot \alpha_k, P)$ , and the trajectory  $\gamma_k = \{\varphi^{kt}(q(P), P) \mid t \in [0, 1]\}$  yields a normalized measure  $\mu_k$ , such that  $|(\Phi^1)_*(\mu_k) - \mu_k| \leq \frac{2}{k}$ .

Now in general  $h_k$  and  $\overline{H}$  are not smooth at  $P$ . However if  $\alpha_k \in \partial_C h_k(p_k)$ , we have that  $(\alpha_k, p_k)$  belongs to  $\widetilde{\text{Conv}}_x(\Gamma_k)$  (see lemma 8.3 for the definition

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<sup>2</sup>The original version of this paper, from 2010 had a more complicated proof for 1.1, which relied on Theorem 3.1.

of  $\widetilde{\text{Conv}}_x(\Gamma_k)$  and the proof of the statement), which means that there are  $\alpha_k^j$  such that  $(q_j(p_k), p_k, 0, \alpha_k^j) \in \Gamma_k$  and  $\alpha_k$  is in the convex hull of the  $\alpha_k^j$ . That is  $\varphi^k(k \cdot q_j(p_k), p_k) = (k \cdot q_j(p_k) + k \cdot \alpha_k^j, p_k)$ . Note that by Caratheodory's theorem, we may limit ourselves to  $1 \leq j \leq n + 1$ , and taking subsequences, we can assume that if  $\alpha = \lim_k \alpha_k$  we have  $\alpha^j = \lim_k \alpha_k^j$ , and  $\alpha$  is in the convex hull of the  $\alpha_j$ .

Now setting  $\gamma_k^j = \{\varphi^{kt}(q_j(P), P) \mid t \in [0, 1]\}$ , the  $\frac{1}{k}[\gamma_k^j]$  converge as measures to the probability measure  $\mu_j$ , with rotation number  $\alpha_j$  and action  $\langle p_\infty, \alpha_j \rangle - \overline{H}(p_\infty)$ , so the action of the convex hull of these measures contains a measure with rotation number  $\alpha$  and action  $\langle p_\infty, \alpha \rangle - \overline{H}(p_\infty)$ .  $\square$

### 3 Strong convergence in Symplectic homogenization

The goal of this section is to improve the convergence result of [V4]. Remember that we defined the sequence  $\varphi_k = \rho_k^{-1} \varphi^k \rho_k$  where  $\rho_k(q, p) = (k \cdot q, p)$ . Note that  $\rho_k^{-1}$  is not well-defined, but  $\varphi_k$  is well defined if  $\varphi$  is Hamiltonianly isotopic to the identity, as the unique solution of  $\rho_k \varphi_k = \varphi^k \rho_k$  obtained by continuation starting from  $\varphi = \varphi_k = \text{Id}$ .

Indeed instead of  $\gamma$ -convergence, we shall prove the following result that we call  $h$ -convergence ( $h$  stands for homological) and prove that

**Theorem 3.1.** *Let  $a < b$  be real numbers,  $L_1, L_2$  be lagrangian submanifolds Hamiltonianly isotopic to the zero section. There is a sequence  $(\varepsilon_k)_{k \geq 1}$  converging to zero, and maps*

$$i_k^{a,b} : FH^*(\varphi_k(L_1), L_2; a, b) \longrightarrow FH^*(\overline{\varphi}(L_1), L_2; a + \varepsilon_k, b + \varepsilon_k)$$

and

$$j_k^{a,b} : FH^*(\overline{\varphi}(L_1), L_2; a, b) \longrightarrow FH^*(\varphi_k(L_1), L_2; a + \varepsilon_k, b + \varepsilon_k)$$

such that in the limit of  $k$  going to infinity

$$i_k^{a+\varepsilon_k, b+\varepsilon_k} \circ j_k^{a,b} : FH^*(\overline{\varphi}(L); a, b) \longrightarrow FH^*(\overline{\varphi}(L); a + 2\varepsilon_k, b + 2\varepsilon_k)$$

converges to the identity as  $k$  goes to infinity. Moreover the maps  $i_k^{a,b}, j_k^{a,b}$  are natural, that is the following diagrams are commutative for  $a < b, c < d$  satisfying  $a < c, b < d$

$$\begin{array}{ccc}
FH^*(\varphi_k(L_1), L_2; c, d) & \xrightarrow{i_k^{c,d}} & FH^*(\bar{\varphi}(L_1), L_2; c + \varepsilon_k, d + \varepsilon_k) \\
\downarrow & & \downarrow \\
FH^*(\varphi_k(L_1), L_2; a, b) & \xrightarrow{i_k^{a,b}} & FH^*(\bar{\varphi}(L_1), L_2; a + \varepsilon_k, b + \varepsilon_k) \\
\\
FH^*(\bar{\varphi}(L), c, d) & \xrightarrow{j_k^{c,d}} & FH^*(\varphi_k(L), c + \varepsilon_k, d + \varepsilon_k) \\
\downarrow & & \downarrow \\
FH^*(\bar{\varphi}(L), a, b) & \xrightarrow{j_k^{a,b}} & FH^*(\varphi_k(L_1), L_2; a + \varepsilon_k, b + \varepsilon_k)
\end{array}$$

where the vertical maps are the natural maps.

*Remarks 3.2.* (a).  $\bar{\varphi}(L)$  is not a Lagrangian, it is an element in the completion  $\widehat{\mathcal{L}}$  for the  $\gamma$ -metric of the set  $\mathcal{L}$  of Lagrangians Hamiltonianly isotopic to the zero section. We must prove that  $FH^*(\bar{\varphi}(L); a, b)$  makes sense in this situation. This was already noticed in [V2], and we shall be more precise about that in section 4.

(b). The results in [V4] imply that  $\varphi_k \times \text{Id}$   $\gamma$ -converges to  $\bar{\varphi} \times \text{Id}$ , so if  $L$  is the graph of a Hamiltonian map,  $\psi$ , we get that the result in the proposition still holds with  $\varphi_k(L)$  and  $\bar{\varphi}(L)$  replaced by  $\varphi_k\psi$  and  $\bar{\varphi}\psi$ .

To prove Theorem 3.1, we shall use Lisa Traynor's version of Floer homology as Generating function homology (see [Tr], and [V3] for the proof of the isomorphism between Floer and Generating Homology). We will in fact compare the relative homology of generating functions corresponding to  $\bar{\varphi}(L)$  and  $\varphi_k(L)$ . For this we need to consider as in [V4] for  $S_1(x, \eta_1)$ ,  $S_2(x, \eta_2)$ , GFQI respectively for  $L_1$  and  $L_2$ , and  $F_k(x, y, \xi)$  a GFQI for  $\varphi_k$ . This means that  $\varphi_k$  is determined by

$$\varphi_k \left( x + \frac{\partial F_k}{\partial y}(x, y, \xi), y \right) = \left( x, y + \frac{\partial F_k}{\partial x}(x, y, \xi) \right) \Leftrightarrow \frac{\partial F_k}{\partial \xi}(x, y, \xi) = 0$$

and

$$G_k(x; y, u, \xi, \eta) = S_1(u; \eta) + F_k(x, y, \xi) + \langle y, x - u \rangle - S_2(x, \eta_2)$$

a generating function of  $\varphi_k(L_1) - L_2$ . Similarly if  $h_k(y) = c(\mu_x \otimes 1(y), F_k)$  and  $\bar{\varphi}_k$  the flow of the integrable Hamiltonian  $h_k$ . we have the following "generating function" of  $\bar{\varphi}_k(L_1) - L_2$

$$\bar{G}_k(u; x, y, \eta) = S_1(u; \eta_1) + h_k(y) + \langle y, x - u \rangle - S_2(x, \eta_2)$$

Remember from [V4] that the sequence  $(h_k)_{k \geq 1}$   $C^0$ -converges to  $\bar{H}$ .

First of all we have

**Definition 3.3.**

$$F_k(x, y; \xi) = \frac{1}{k} \left[ S(kx, p_1) + \sum_{j=2}^{k-1} S(kq_j, p_j) + S(kq_k, y) \right] + \widehat{B}_k(x, y, \xi) - \langle y, x \rangle =$$

$$\frac{1}{k} \left[ S(kx, p_1) + \sum_{j=2}^{k-1} S(kq_j, p_j) + S(kq_k, y) \right] + B_k(x, y, \xi)$$

where

$$\widehat{B}_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) =$$

and  $B_k(x, y, \xi) = \widehat{B}_k(x, y, \xi) - \langle y, x \rangle$ . We then set  $h_k(y) = c(\mu_x, F_{k,y}) = c(\mu_x \otimes 1(y), F_k)$  where  $F_{k,y} = F_k(x, y; \xi)$ .

**Lemma 3.4.** *Let  $\bar{\Gamma}$  be a cycle in  $H_*(\bar{G}_k^b, \bar{G}_k^a)$ . Then there is a sequence  $\varepsilon_k$  of positive numbers converging to 0 such that there exists a cycle*

$$\bar{\Gamma} \times_Y C^-$$

where  $C^- = \bigcup_y C^-(y)$  where  $C^-(y)$  is a cycle homologous to  $T_x^n \times \{y\} \times E_k^-$  such that

$$F_k(x, y, \xi) \leq h_k(y) + \alpha \chi_\delta^j(y) + \varepsilon_k$$

whenever  $(x, y, \xi) \in C^-$ .

*Proof.* The proof of the lemma follows the lines of the proof of proposition 5.13 in [V4]. Let  $F_k(u, y; \xi)$  be a GFQI for  $\Phi_k$ , and  $S_j(u; \eta_j)$  a GFQI for  $L_j$ , so that

$$G_k(x; u, y, \xi, \eta) = S_1(u; \eta_1) + F_k(x, y; \xi) + \langle y, x - u \rangle - S_2(x, \eta_2)$$

is a GFQI for  $\Phi_k(L_1) - L_2$ , and similarly for

$$\bar{G}_k(x; u, y, \eta) = S_1(u; \eta_1) + h_k(y) + \langle y, x - u \rangle - S_2(x; \eta_2)$$

a GFQI for  $\bar{\Phi}_k$  the time-one flow for  $h_k(y)$ , where  $h_k(y) = c(\mu_x \otimes 1(y), F_k)$ , and  $\lim_k h_k(y) = \bar{H}(y)$  by assumption.

Since  $h_k(y) = c(\mu_x \otimes 1(y), F_k)$ , this means there exists a cycle  $C^-(y)$  with  $[C^-(y)] = [T_x^n \times E_k^-]$  in  $H_*(F_{k,y}^\infty, F_{k,y}^{-\infty})$ , and

$$h_k(y) - \varepsilon \leq \sup_{(x,\xi) \in C^-(y)} F_k(x, y, \xi) \leq h_k(y) + \varepsilon$$

Assume first, as we did in [V4] that we can choose the map  $y \longrightarrow C^-(y)$  to be continuous. Set  $C^- = \bigcup_y \{y\} \times C^-(y)$ . Let then  $\Gamma$  be cycle representing a nonzero class in  $H^*(\bar{G}_k^b, \bar{G}_k^a)$ , and consider the cycle

$$\Gamma \times_Y \tilde{C}^- = \{(x, u, y, \xi, \eta) \mid (x, \xi) \in C^-(y), (x, u, y, \eta) \in \Gamma\}$$

Then

$$G_k(\Gamma \times_Y C^-) \leq b + \varepsilon$$

and since  $\partial(\Gamma \times_Y C^-) = \partial\Gamma \times_Y C^-$ , we have

$$G_k(\partial\Gamma \times_Y C^-) \leq a + \varepsilon$$

so that  $\Gamma \times_Y C^-$  represents a homology class in  $H_*(G_k^{b+\varepsilon}, G_k^{a+\varepsilon})$ . We must now prove that if  $\Gamma$  is a nonzero class in  $H_*(\overline{G}_k^b, \overline{G}_k^a)$  for  $k$  large enough, then  $[\Gamma \times_Y C^-]$  is nonzero in  $H_*(G_k^{b+\varepsilon}, G_k^{a+\varepsilon})$ . Indeed, denoting by  $f^{\geq \lambda}$  the set  $\{x \mid f(x) \geq \lambda\}$ , let  $\Gamma'$  be a cycle in  $H_*(\overline{G}_k^{\geq a}, \overline{G}_k^{\geq b})$  such that  $\Gamma' \cdot \Gamma = k \neq 0$ . Such a cycle exists by Alexander duality. Let  $C^+(y)$  be such that  $[C^+(y)] = [pt \times E_k^+]$  so that  $C^-(y) \cdot C^+(y) = \{pt\}$  and such that

$$\inf\{F_k(u, y, \xi) \mid (u, \xi) \in C^+(y)\} \geq h_k(y) - \varepsilon$$

We assume again that  $C^+(y)$  depends continuously on  $y$ .

Then

$$[\Gamma' \times_Y C^+] \cdot [\Gamma \times_Y C^-] = [(\Gamma' \cdot \Gamma) \times_Y (C^+ \cap C^-)] = \{pt\} \times \{pt\} \neq 0$$

And since  $\Gamma' \times_Y C^+ \subset G_k^{\geq a}$  and  $\partial(\Gamma' \times_Y C^+) = (\partial\Gamma' \times_Y C^+) \subset G_k^{\geq b}$  we get that there is a class in  $H_*(G_k^{\geq a}, G_k^{\geq b})$  such that it has nonzero intersection with the class  $[\Gamma \times_Y C^-]$  in  $H_*(G_k^{b+\varepsilon}, G_k^{a+\varepsilon})$ . This implies that  $H_*(G_k^{b+\varepsilon}, G_k^{a+\varepsilon}) \neq 0$ . This argument holds provided the cycles  $C^\pm(y)$  can be chosen to depend continuously on  $y$ , which is not usually the case. So our argument must be modified as we did in [V4]. Here is a detailed proof. As in [V4] we need the

**Lemma 3.5.** *Let  $F(u, x)$  be a smooth function on  $V \times X$  such that there exists  $f \in C^0(V, \mathbb{R})$  such that for each  $u \in V$ , there exists a cycle  $C(u) \subset \{u\} \times V$  representing a fixed class in  $H_*(X)$  with  $F(u, C(u)) \leq f(u)$ . Then for any  $\varepsilon > 0$  there is a  $\delta > 0$  so that for any subset  $U$  in  $V$ , such that each connected component of  $V \setminus U$  has diameter less than  $\delta$ , there exists a cycle  $\tilde{C}$  in  $H_*(V \times X)$  and a constant  $a$ , depending only on  $F$ , such that if we denote by  $\tilde{C}(u)$  the slice  $\tilde{C} \cap \pi^{-1}(u)$  ( $\pi : V \times X \rightarrow X$  is the first projection) we have  $[\tilde{C}(u)] = [C(u)]$  in  $H_*(X)$  and*

$$F(u, \tilde{C}(u)) \leq f(u) + a\chi_U(u) + \varepsilon$$

*Proof.* Continuity of  $F$  implies that if we take  $\tilde{C}(u)$  to be locally constant in  $V \setminus U$ , the inequality  $F(u, x) \leq f(u) + \varepsilon$  will be satisfied for  $(u, x) \in \{u\} \times C(u_0)$ , where  $u, u_0$  are close enough. Assume first that  $V$  is one dimensional, so that we take for  $V \setminus U$  a union of simplices, and for  $U$  the neighbourhood

of 0-dimensional faces (i.e. vertices). Assume  $\tilde{C}$  is defined over  $u \in T_j$ , and denote by  $\tilde{C}_j(u)$  the set  $\tilde{C} \cap \pi^{-1}(u)$ , where the  $T_j$  are edges, but do not coincide on the intersections, for example on  $T_1 \cap T_2$ . However on  $u_0 \in T_1 \cap T_2$ , we have that  $\tilde{C}_1(u_0) \neq \tilde{C}_2(u_0)$  while  $[\tilde{C}_1(u_0)] = [\tilde{C}_2(u_0)]$  in  $H^*(X)$ . We then write  $\tilde{C}_1(u_0) - \tilde{C}_2(u_0) = \partial C_{1,2}(u_0)$  where  $F(u_0, \tilde{C}_{1,2}(u_0)) \leq a_1$ . We now repeat this procedure on any adjacent pair of edges, and write

$$\tilde{C} = \bigcup_{u \in T_j} \tilde{C}_j(u) \quad \bigcup_{i \neq j, u \in T_i \cap T_j} \tilde{C}_{i,j}(u)$$

Clearly  $\tilde{C} \cap \pi^{-1}(u) = \tilde{C}(u)$  for a generic  $u$ , and  $F(u, \tilde{C}(u)) \leq f(u) + a_1$ . In the general case, we start with the top dimensional simplices, and argue by induction on the dimension of the simplices. □

We thus return to our original problem, and consider  $\tilde{C}$  but now the inequality

$$h_k(y) - \varepsilon \leq \sup_{(u, \xi) \in C^-(y)} F_k(u, y, \xi) \leq h_k(y) + \varepsilon$$

only holds outside a set  $U_{2\delta}$ , where  $U_\delta$  is a neighborhood of a fine grid in  $\mathbb{R}^n$ , while we have the general bound

$$\left| \sup_{(u, \xi) \in C^-(y)} F_k(u, y, \xi) - h_k(y) \right| \leq a\chi_\delta(y) + \varepsilon$$

where  $\chi_\delta$  is 1 in  $U_\delta$  and vanishes outside  $U_{2\delta}$ .

Now we consider  $\ell$  different such continuous families, corresponding to function  $\chi_j^\delta$ , such that their supports  $U_j^\delta$  have no more than  $n + 1$  nonempty intersections.

We can then use  $F_k$  to write a generating function for  $\Phi_{\ell k}(L_1) - L_2$  (see [V4]):

$$G_{\ell, k}(x_1; v, \bar{x}, \bar{y}, \bar{\xi}, \eta) = S_1(u, \eta_1) + \frac{1}{\ell} \sum_{j=1}^{\ell} F_k(\ell x_j, y_j, \xi_j) + Q_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle - S_2(x_1, \eta_2)$$

where

$$Q_\ell(\bar{x}, \bar{y}) = Q_\ell(x_1, y_k; y_1, x_2, \dots, x_{\ell-1}, y_{\ell-1}, x_\ell) = \sum_{j=1}^{\ell-1} \langle y_j - y_{j+1}, x_j - x_{j+1} \rangle + \langle y_\ell, x_1 \rangle$$

We then consider

$$\begin{aligned} & \overline{G}_{\ell,k}(x_1, u; \overline{x}, \overline{y}, \eta) = \\ & S_1(u, \eta_1) + \frac{1}{\ell} \sum_{j=1}^{\ell} (h_k(y_j) + a\chi_j^\delta(y_j)) + Q_\ell(\overline{x}, \overline{y}) + \langle y_\ell - v, u - x_1 \rangle - S_2(x_1, \eta_2) \end{aligned}$$

From now on we shall assume  $\varepsilon \ll b - a$ . Let  $\Gamma$  be a cycle in a nonzero homology class in  $H^*(\overline{G}_{\ell,k}^b, \overline{G}_{\ell,k}^a)$ , and consider the cycle

$$(\Gamma \times_Y C^-[\ell]) = \left\{ (u; \overline{x}, \overline{y}, \xi, \eta_1, \eta_2) \mid (u, \overline{x}, \overline{y}, \eta) \in \Gamma, (\ell x_j, \xi_j) \in \widetilde{C}_j^-(y_j) \right\} .$$

It is contained in  $G_{\ell,k}^{b+\varepsilon}$ , and its boundary is in  $G_{\ell,k}^{a+\varepsilon}$ . It thus represents a class in  $H_*(G_{\ell,k}^{b+\varepsilon}, G_{\ell,k}^{a+\varepsilon})$ .

We still have to identify the limit as  $k$  goes to infinity of  $H_*(\overline{G}_{\ell,k}^b, \overline{G}_{\ell,k}^a)$  with  $H_*(\overline{G}^b, \overline{G}^a)$ .

Let

$$K_{\ell,k}^\delta = \frac{1}{\ell} \left( \sum_{j=1}^{\ell} h_k(y) + a_k \chi_j^\delta(y) \right) .$$

be a Hamiltonian with flow  $\overline{\Psi}_{k,\ell,\delta}$ . Clearly  $\overline{G}_{\ell,k}$  is a generating function for  $\overline{\Psi}_{k,\ell,\delta}(L_1) - L_2$ .

Now at most  $(n+1)$  of the supports of  $\chi_j^\delta$  intersect, so that

$$|K_{\ell,k}^\delta(y) - h_k(y)| \leq \frac{A}{\ell}$$

and this difference goes to zero as  $\ell$  goes to infinity and since  $h_k(y)$  converges to  $\overline{H}(y)$ , thus for  $k, \ell$  large enough, we have

$$|K_{\ell,k}^\delta(y) - \overline{H}(y)| \leq \varepsilon_{k,\ell}$$

This classically implies ([V2], proposition 1.1 and Remark 1.2) that the map

$$FH_*(\overline{\Phi}(L_1), L_2; a, b) \longrightarrow FH_*(\overline{\Psi}_{\ell,k,\delta}(L_1), L_2; a + \varepsilon, b + \varepsilon) \simeq H_*(\overline{G}_{\ell,k}^{b+\varepsilon}, \overline{G}_{\ell,k}^{a+\varepsilon})$$

is an isomorphism in the limit  $\varepsilon \longrightarrow 0$ , so we finally get a map

$$FH_*(\overline{\Phi}(L_1), L_2, a, b) \longrightarrow H_*(\overline{G}_{\ell,k}^{b+\varepsilon}, \overline{G}_{\ell,k}^{a+\varepsilon}) \simeq FH_*(\Phi_{k\ell}(L_1), L_2; a + \varepsilon, b + \varepsilon)$$

This concludes the construction<sup>3</sup> of  $j_k^{a,b}$ .

Now the same argument can be carried out replacing  $\varphi$  by  $\varphi^{-1}$  and exchanging  $L_1$  and  $L_2$ . Note that by Poincaré duality

$$FH^*(\varphi_k^{-1}(L_2), L_1; a, b) \simeq FH^*(L_2, \varphi_k(L_1); a, b) \simeq FH_{-*}(\varphi_k(L_1), L_2; -b, -a)$$

One may check directly that the Floer complex associated to  $(L_2, \varphi(L_1))$  is the same as the one associated to  $(\varphi(L_1), L_2)$  but the action filtration has the opposite sign, the indices also change sign, and the differential is reversed: the coefficients of  $\langle \delta x, y \rangle$  now become those of  $\langle \delta^* y, x \rangle$ : in other words we replace the matrix of the coboundary operator by its adjoint.

From the above construction, we have a map from

$$\ell_k^{a,b} : FH^*((\overline{\varphi})^{-1}(L_2), L_1; -b, -a) \longrightarrow FH^*(\varphi_k^{-1}(L_2), L_1; -b + \varepsilon_k, -a + \varepsilon_k)$$

Note that here we use the fact that  $\overline{\varphi^{-1}} = (\overline{\varphi})^{-1}$ , or equivalently  $\overline{-H} = -\overline{H}$ , a crucial point proved in [V4], proposition 5.14. The above map is in fact a map

$$\ell_k^{a,b} : FH_{-*}(\overline{\varphi}(L_1), L_2; a, b) \longrightarrow FH_{-*}(\varphi_k(L_1), L_2; a - \varepsilon_k, b - \varepsilon_k)$$

Now we have the non degenerate Poincaré duality

$$FH^*(\Lambda_1, \Lambda_2; a, b) \otimes FH_{-*}(\Lambda_1, \Lambda_2; a, b) \longrightarrow \mathbb{Z}$$

and we have for  $u, v \in FH_{-*}(\overline{\varphi}(L_1), L_2; a, b)$  the identity  $\langle j_k^{a,b}(u), \ell_k^{a,b}(v) \rangle = \langle u, v \rangle$  and setting  $i_k^{a,b} = (\ell_k^{a,b})^*$  we have

$$j_k^{a+\varepsilon_k, b+\varepsilon_k} \circ i_k^{a,b} : FH^*(\overline{\varphi}^{-1}(L); a, b) \longrightarrow FH^*(\overline{\varphi}(L); a + 2\varepsilon_k, b + 2\varepsilon_k)$$

converging to the identity as  $k$  goes to  $+\infty$ .

□

*Remark 3.6.* In terms of barcodes (see [Z-C, PolShel, LNV]), this means that the barcodes of  $\varphi_k^1$  converge to the barcode of  $\overline{\varphi}^1$ .

---

<sup>3</sup>To be honest the construction is only made for a sequence going to infinity, but an argument similar to the argument in [V4], lemma 5.10 proves that any subsequence will have the same limit.

## 4 Floer cohomology for $C^0$ integrable Hamiltonians

Let  $H(p)$  be a smooth integrable Hamiltonian. Then the corresponding flow is  $(q, p) \mapsto (q + t\nabla H(p), p)$ . If we consider its graph  $\{(q, p, Q, P) \mid (Q, P) = \varphi^1(q, p)\}$  and its image by  $(q, p, Q, P) \mapsto (q, PP - p, q - Q)$  is  $\Gamma_H = \{(q, p, 0, \nabla H(p))\}$  and has  $S(x, y) = H(y)$  as generating function (with no fibre variable). In the following proposition, we refer to Appendix 8 for the definitions of  $\partial_C$  and  $d_s$ .

**Proposition 4.1.** *If  $\alpha \in d_s H(p)$  and  $c = H(p)$ , we have*

$$FH^*(\Gamma_{H_\alpha}, 0_{T^n \times \mathbb{R}^n}; c + \varepsilon, c - \varepsilon) \neq 0$$

*Proof.* Indeed, since  $S(x, y) = H(y)$  is a generating function for  $\Gamma_H$ , we have that  $S_\alpha(x, y) = H(y) - \langle y, \alpha \rangle$  is a generating function for  $\Gamma_{H_\alpha}$ . So we have  $FH^*(\Gamma_{H_\alpha}, 0_{T^n \times \mathbb{R}^n}; c + \varepsilon, c - \varepsilon) = H^*(H_\alpha^{c+\varepsilon}, H_\alpha^{c-\varepsilon})$ . By definition this is non-zero if  $\alpha \in d_s H(p)$ .  $\square$

Let  $u \in \mathbb{R}^n$  and set  $\Lambda_u = \{(x, y, X, Y) \mid X = 0, Y = u\}$ . Now let  $f_{u,C}(x, y) = \langle u, y \rangle \chi(\frac{y}{C})$  where  $\chi(y) = 1$  for  $|y| \leq 1$ , and vanishes for  $|y| \geq 2$ . Then  $\frac{\partial f_{u,C}}{\partial x}(x, y) = 0$  and  $\frac{\partial f_{u,C}}{\partial y}(x, y) = u$  for  $|y| \leq C$ , so  $\tilde{H}_u(y) = H(y) - f_{u,C}(x, y)$  coincides with  $H_u$  in  $\{(x, y, \xi, \eta) \mid |y| \leq C\}$

$$\tilde{\Lambda}_u = \{(x, y, \frac{\partial f_{u,C}}{\partial x}(x, y), \frac{\partial f_{u,C}}{\partial y}(x, y)) \mid (x, y) \in T^n \times \mathbb{R}^n\}$$

coincides with  $\Lambda_u$  in  $\{(x, y, \xi, \eta) \mid |y| \leq C\}$ , so  $\Lambda_u \cap \Gamma_H = \tilde{\Lambda}_u \cap \Gamma_H$ , provided the Lipschitz constant of  $H$  is less than  $C$ . The following is a consequence of the above remarks :

**Corollary 4.2.** *If  $u \in \mathbb{R}^n$  is such that  $u \in d_s H(p)$ , then*

$$FH^*(\Gamma_H, \Lambda_u, c + \varepsilon, c - \varepsilon) = FH^*(\Gamma_H, \tilde{\Lambda}_u, c + \varepsilon, c - \varepsilon) = FH^*(\Gamma_{H_u}, 0_{T^n \times \mathbb{R}^n}; c + \varepsilon, c - \varepsilon) \neq 0$$

## 5 A proof of the weak Conley conjecture on $T^*T^n$ .

*Proof.* Let us consider now the case of periodic orbits and prove Theorem 1.3. Let  $\alpha$  be a rational vector. We write  $\alpha = \frac{u}{v}$  with  $u \in \mathbb{Z}^n, v \in \mathbb{N}^*$  mutually prime. We need to find fixed points of  $\Phi^{kv} - ku$  that will yield periodic orbits

of  $\Phi$  of period  $kv$  and rotation number  $\frac{u}{v}$ . This is equivalent to finding fixed points of  $\rho_k^{-1}\Phi^{kv}\rho_k - u = \Phi_k^v - u$ .

If  $\Gamma_k^v$  is the graph of  $\Phi_k^v$  that is

$$\Gamma_k^v = (q, P_k(q, p), P_k(q, p) - p, q - Q_k(q, p)) \mid (Q_k, P_k) = \Phi_k^v(q, p)\}$$

and we look the points in  $\Gamma_k^v \cap \Lambda_u$  where  $\Lambda_u = \{(x, y, X, Y) \mid X = 0, Y = u\}$ , as before,  $f_{u,C}(x, y) = \langle u, y \rangle \chi(\frac{y}{C})$  where  $\chi = 1$  for  $|y| \leq 1$ , and vanishes for  $|y| \geq 2$ , and

$$\tilde{\Lambda}_u = \{(x, y, \frac{\partial f_{u,C}}{\partial x}(x, y), \frac{\partial f_{u,C}}{\partial y}(x, y)) \mid (x, y) \in T^n \times \mathbb{R}^n\}$$

We claim that for  $C$  large enough,  $\Gamma_k^v \cap \tilde{\Lambda}_u \subset \Gamma_k^v \cap \Lambda_u \cup 0_{T^n \times \mathbb{R}^n}$ . Indeed,  $\Lambda_u$  and  $\tilde{\Lambda}_u$  coincide in  $\{(x, y, X, Y) \mid |y| \leq C\}$ , but outside this set,  $\Gamma_k^v$  coincides with the zero section. There are actually two types of points in  $(\tilde{\Lambda}_u - \Lambda_u) \cap \Gamma_k^v$  the ones with action 0, the other with action  $A_{u,C} = f_{u,C}(y_{u,C})$  where  $f'_{u,C}(y_{u,C}) = 0$  and  $y_{u,C}$  is a non-trivial critical point of  $f_{u,C}$ . Note that setting  $F = f_{u,1}$  we have  $f_{u,C}(y) = kCF(\frac{y}{C})$ . So,  $f'_{u,C}(y) = kF'(\frac{y}{C})$  and  $y_{k,C} = Cz$  where  $z$  is a non-trivial critical point of  $F$ , and  $f_{u,C}(y_{u,C}) = kCF(z)$ . Thus if  $f_{u,C}(y_{k,C}) \neq 0$  we have that for  $C$  large enough, the critical value is outside any given interval.

Now since  $\Gamma_k^v \xrightarrow{c} \bar{\Gamma}^v$ , where  $\bar{\Gamma}^v$  is the graph of  $\bar{\Phi}^v$  in the  $\gamma$ -completion  $\widehat{\mathcal{L}}$ , and provided we have  $FH^*(\bar{\Gamma}^v, \tilde{\Lambda}_u, c - \varepsilon, c + \varepsilon) \neq 0$ , according to Theorem 3.1, this implies for  $k$  large enough  $FH^*(\Gamma_k^v, \tilde{\Lambda}_u, c - \varepsilon, c + \varepsilon) \neq 0$  and as we saw that  $FH^*(\Gamma_k, \tilde{\Lambda}_u, c - \varepsilon, c + \varepsilon) = FH^*(\Gamma_k^v, \Lambda_u, c - \varepsilon, c + \varepsilon)$  we have a fixed point with action in  $[c - \varepsilon, c + \varepsilon]$ . Now

$$\begin{aligned} FH^*(\bar{\Gamma}^v, \tilde{\Lambda}_u, a, b) &= H^*(v \cdot \bar{H}(y) - f_{u,C}(x, y); a, b) = \\ &= H^*(v \cdot \bar{H}_{u/v}; a, b) = H^*(\bar{H}_{u/v}, \frac{a}{v}, \frac{b}{v}) \end{aligned}$$

where  $H_u(y) = \bar{H}(y) - \langle u, y \rangle$ , hence  $H^*(\bar{H}_{u/v}; c/v - \varepsilon, c/v + \varepsilon) \neq 0$  is equivalent to the existence of  $p$  such that  $d_s \bar{H}(p) = u/v$  and  $\bar{H}_u(y) = c/v$  according to Appendix 8.

Let us now consider a Hamiltonian  $H$  and let  $\bar{H}$  be the homogenized Hamiltonian. We refer to [V1] for the definition of the capacities  $c_{\pm}$ . Assume first that  $\bar{H} = 0$ , this means in particular that  $\lim_k \frac{1}{k} c_{\pm}(\varphi^k) = 0$ . But since  $c_+(\varphi) = c_-(\varphi) = 0$  if and only if  $\varphi = \text{Id}$ , there is an infinite sequence of  $k$  such that either  $c_+(\varphi^k) > 0$  or  $c_-(\varphi^k) < 0$ . Replacing  $\varphi$  by  $\varphi^{-1}$  we may always assume we are in the first case. Then the fixed point  $x_k$  corresponding to  $c_+(\varphi^k)$  is such that its action,  $A(x_k, \varphi^k) = c_+(\varphi^k)$ . In case  $x_k$  is the fixed point

of  $\varphi$ , we get that  $A(x_k, \varphi^k) = k \cdot A(x_k, \varphi)$ . More generally if  $x = x_{pj} = x_{pk}$  we get  $\frac{1}{pj}A(x, \varphi^{pj}) = \frac{1}{pk}A(x, \varphi^{pk})$ , so  $\frac{1}{pj}c_+(\varphi^{pj}) = \frac{1}{pk}c_+(\varphi^{pk})$ , but since the sequence  $\frac{1}{k}c_\pm(\varphi^k)$  is positive and converges to zero, it is non constant and takes infinitely many values. Thus, there are infinitely many fixed points. In particular, we may as in [V1] (prop 4.13, page 701) show that the growth of the number of fixed points is at least linear, that is for some constant  $C$ , we have

$$\#\{x \mid \exists k \in [1, N] \mid \varphi^k(x) = x\} \geq CN$$

Assume now that  $\overline{H} \neq 0$ . Then the set  $\{d_s \overline{H}(p) \mid p \in \mathbb{R}^n\}$  has non-empty interior according to lemma 8.3 of section 8. There are thus infinitely many rational, non-colinear values of  $\alpha$  such that we have a periodic orbit of rotation number  $\alpha$ . □

## 6 On the ergodicity of the invariant measures

We consider the subsets in  $\mathbb{R}^{n+1}$  given by

$$\overline{R}(H) = \{(\alpha, \langle p, \alpha \rangle - \overline{H}(p)) \mid \alpha \in \partial_C \overline{H}(p)\}$$

and

$$R(H) = \{(\alpha, A) \mid \exists \mu, \rho(\mu) = \alpha, \varphi_H^*(\mu) = \mu, A_H(\mu) = A\}$$

Note that it follows from the computations in the previous section that these definitions are compatible, that is

$$R(\overline{H}) = \overline{R}(H)$$

We proved in the previous sections that  $R(\overline{H}) \subset R(H)$ . Moreover for each element in  $R(\overline{H})$  there is a well defined measure  $\mu_{\alpha, A}$  such that  $\rho(\mu_{\alpha, A}) = \alpha, \varphi_H^*(\mu_{\alpha, A}) = \mu_{\alpha, A}, A_H(\mu_{\alpha, A}) = A$ .

We want to figure out whether the measures thus found are ergodic, or whether we can find the minimal number of ergodic measures. Indeed, we assume there are ergodic measures  $\mu_1, \dots, \mu_q$  generating all the measures we obtained. For this, we need the measures we found to be contained in a polytope with  $q$  vertices. Since we know that the projection of  $R(H)$  on  $\mathbb{R}^n$  contains an open set, we must have  $q \geq n + 1$ . If moreover  $R(H)$  contains an open set in  $\mathbb{R}^{n+1}$ , necessarily we shall have  $q \geq n + 2$ . We could also consider a simpler question: among the invariant measures we found, which ones can be considered combination of others? Clearly, we can consider

the convex hull of  $R(\overline{H})$ : then extremal points of this hull cannot be in the convex combinations of the same. This provides a lot of such measures, for example if  $R(\overline{H})$  is strictly convex.

## 7 The case of non-compact supported Hamiltonians

A priori our results only deal with compact supported Hamiltonians. However the same truncation tricks as in [V4] allow one to extend homogenization to coercive Hamiltonians and to prove the following statements whose proofs are left to the reader

**Theorem 7.1.** *Let  $H(t, q, p)$  be a coercive Hamiltonian in  $S^1 \times T^*T^n$ , and denote by  $\overline{H}(p)$  its homogenization defined in [V4]. Let  $\alpha \in \partial_C \overline{H}(p)$ . Then there exists, for  $k$  large enough, a solution of  $\varphi^k(q_k, p_k) = (q_k + k\alpha_k, p'_k)$  (with  $\lim_k \alpha_k = \alpha$ ) and average action*

$$A_k = \frac{1}{k} \int_0^k [\gamma_k^* \lambda - H(t, \gamma_k(t))] dt$$

where  $\gamma_k(t) = \varphi^t(q_k, p_k)$ . Moreover as  $k$  goes to infinity  $A_k$  converges to

$$\lim_k A_k = p \cdot \alpha - \overline{H}(p)$$

Therefore there exists an invariant measure  $\mu_\alpha$  with rotation number  $\alpha$  and average action

$$\mathcal{A}(\mu_\alpha) \stackrel{def}{=} \int_{T^*T^n} [p \frac{\partial_C H}{\partial p}(q, p) - H(q, p)] d\mu_\alpha = p \cdot \alpha - \overline{H}(p).$$

In particular for  $H(q, p)$  strictly convex in  $p$ , we have that  $\overline{H}(p)$  is also convex in  $p$  and so for each  $\alpha$  there exists a unique  $p_\alpha$  such  $\alpha \in \partial \overline{H}(p)$ . Note that in this case Mather's theory is much more complete, and tells us that the measure obtained are minimal, and are the graph of a Lipschitz function over a subset of  $T^n$ .

## 8 Appendix: Critical point theory for non-smooth functions and subdifferentials

The aim of this section is to clarify the notions of differential that occur crucially in the previous sections. Indeed, restricting the set of rotation numbers

of invariant measures to the values corresponding to regular points is not an option, since even in the convex case, the function  $\overline{H}$  has generally dense subsets of non-differentiable values. We shall deal with two situations. The first one corresponds to Lipschitz functions: these occur as homogenization of  $C^1$  (or Lipschitz) Hamiltonians, which are the only ones we encounter in practice. This is the subject of the first subsection, and uses analytic tools, basically a notion of subdifferential and a suitable version of the Morse deformation lemma. The second one applies to any continuous function. It is best suited to our general line of work, and in principle allows us to use the main theorem in the case of Hamiltonians belonging to the Humilière completion, even though one should formalize the notion of invariant measure for such objects.

## 8.1 Analytical theory in the Lipschitz case

While the critical point theory has been studied for (smooth and non-smooth) functionals on infinite dimensional spaces, we shall here restrict ourselves to the finite dimensional case. First assume  $f$  is Lipschitz on a smooth manifold  $M$ . Then we define

**Definition 8.1.** Let  $f$  be a Lipschitz function. The vector  $w$  is in  $\partial_C f(x)$  the Clarke differential of  $f$  at  $x$ , if and only if

$$\forall v \in E \quad \limsup_{h \rightarrow 0, \lambda \rightarrow 0} \frac{1}{\lambda} [f(x + h + \lambda v) - f(x + h)] \geq \langle w, v \rangle$$

The following proposition describes the main properties of  $\partial_C f$

**Proposition 8.2** ([Clarke 2, Chang]). *We have the following properties:*

- (a).  $\partial_C f(x)$  is a non-empty convex compact set in  $T_x^* M$
- (b).  $\partial_C(f + g)(x) \subset \partial_C f(x) + \partial_C g(x)$
- (c).  $\partial_C(\alpha f)(x) = \alpha \partial_C f(x)$
- (d). The set-valued mapping  $x \rightarrow \partial_C f(x)$  is upper semi-continuous. The map  $x \rightarrow \lambda_f(x) = \min_{w \in \partial_C f(x)} |w|$  is lower semi-continuous.
- (e). Let  $\varphi \in C^1([0, 1], X)$  then  $f \circ \varphi$  is differentiable almost everywhere (according to Rademacher's theorem) hence

$$h'(t) \leq \max\{\langle w, \varphi'(t) \rangle \mid w \in \partial_C f(\varphi(t))\}$$

**Definition 8.3.** Let  $f$  be a Lipschitz function. We define the set of critical points at level  $c$  as  $K_c = \{x \in f^{-1}(c) \mid 0 \in \partial f(x)\}$ . We set  $\lambda_f(x) = \inf_{w \in \partial f(x)} \|w\|$

**Definition 8.4.** Let  $f$  be a Lipschitz function. We shall say that  $f$  satisfies the Palais-Smale condition if for all  $c$ , a sequence  $(x_n)$  such that  $f(x_n) \rightarrow c$  and  $\lim_n \lambda_f(x_n) = 0$  has a converging subsequence.

The crucial fact is the existence of a pseudo-gradient vector field in the complement of  $K_c$ . We denote by  $N_\delta(K_c)$  a  $\delta$ -neighbourhood of  $K_c$ .

**Lemma 8.5** (Lemma 3.3 in [Chang]). *There exists a Lipschitz vector field  $v(x)$  defined in a neighborhood of  $B(c, \varepsilon, \delta) = (f^{c+\varepsilon} - f^{c-\varepsilon}) \setminus N_\delta(K_c)$  such that  $\|v(x)\| \leq 1$  and  $\langle v(x), w \rangle \geq \frac{b}{2}$  for all  $w \in \partial f(x)$ , where  $0 < b = \inf\{\lambda_f(x) \mid x \in B(c, \varepsilon, \delta)\}$ .*

From this we see that following the flow of the vector field  $v$ , if  $c$  is a cycle representing a homology class in  $H_*(U \cap f^{c+\varepsilon}, U \cap f^{c-\varepsilon})$  for  $\varepsilon$  small enough, then the flow of  $v$  applied to  $c$  shows that  $c$  is homologous to a cycle in  $U \cap f^{c-\varepsilon}$ , hence  $c$  is zero. This brings us to the following subsection.

## 8.2 Topological theory (according to [Vic2])

Let  $f$  be a continuous function on  $X$ . We define a *strict critical point* of  $f$ , as follows

**Definition 8.6.** Let  $f$  be a continuous function. We define the set of *strict critical points* at level  $c$  as the set of points such that

$$\lim_{U \ni x} \lim_{\varepsilon \rightarrow 0} H_*(U \cap f^{c+\varepsilon}, U \cap f^{c-\varepsilon}) \neq 0$$

If  $f^{-1}(c)$  contains a critical point, it is called a critical level. Other points are called weakly regular points.

For example even if  $f$  is smooth, this does not coincide exactly with the usual notion of critical and regular point. For example if  $f(x) = x^3$ , the origin is critical but weakly regular, since there is not topological change for the sublevels of  $f$  at 0. .

From the above lemma the first part of the following proposition follows

**Proposition 8.7.** *Let  $f$  be Lipschitz and satisfy the Palais-Smale condition above. Then strict critical points at level  $c$  are contained in  $K_c$ . Moreover if  $f$  has a local maximum (resp. minimum) at  $x$ , then  $x$  is a strict critical point.*

*Proof.* The second statement follows obviously from the fact that for a local minimum, that is a strict minimum in  $U$ , we have  $H^0(f^{c+\varepsilon} \cap U, f^{c-\varepsilon} \cap U) = H^0(f^{c+\varepsilon} \cap U, \emptyset) \neq 0$  since  $f^{c+\varepsilon} \cap U$  is non-empty.  $\square$

**Definition 8.8.** We denote by  $d_t f(x)$  the set of  $p$  such that  $f(x) - \langle p, x \rangle$  has a strict critical point at  $x$ . This is called the topological differential at  $x$ . The set of all limits of  $d_t f(x_n)$  as  $x_n$  converges to  $x$  is denoted by  $D_t f(x)$ .

*Remark 8.9.* The set  $D_t f(x)$  coincides with  $\partial f(x)$  as defined in Definition 3.6 of [Vic2].

**Proposition 8.10.** *The set  $D_t f(x)$  is contained in  $\partial_C f(x)$  and the convex hull of  $D_t f(x)$  equals  $\partial_C f(x)$ .*

*Proof.* This is theorem 3.14 and 3.20 of [Vic2].  $\square$

The above notion is analogous to the one defined using microlocal theory of sheafs of [K-S], as is explained in [Vic2]. Indeed, the singular support of a sheaf is a classical notion in sheaf theory (see [K-S]), defined as follows:

**Definition 8.11.** Let  $\mathcal{F}$  be a sheaf on  $X$ . Then  $(x_0, p_0) \notin SS(\mathcal{F})$  if for any  $p$  close to  $p_0$ , and  $\psi$  such that  $p = d\psi(x)$  and  $\psi(x) = 0$  we have

$$R\Gamma(\{\psi \leq 0\}, \mathcal{F})_x = 0$$

This is equivalent to  $\lim_{W \ni x} H^*(W, W \cap \{\psi \leq 0\}; \mathcal{F}) = 0$ .

The connection between the two definitions is as follows. Consider the sheaf  $\mathcal{F}_f$  on  $M \times \mathbb{R}$  that is the constant sheaf on  $\{(x, t) \mid f(x) \geq t\}$  and vanishes elsewhere. Then  $SS(\mathcal{F}_f) = \{(x, t, p, \tau) \mid \tau D_t f(x) = p\}$ . It is not hard to see that as expected,  $SS(\mathcal{F}_f)$  is a conical coisotropic submanifold.

It follows from the sheaf theoretic Morse lemma from [K-S] (Corollary 5.4.19, page 239) that

**Proposition 8.12.** *Let  $f$  be a continuous function satisfying the Palais-Smale condition above. Let us assume  $c$  is a regular level. Then for  $\varepsilon$  small enough,  $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = 0$ .*

*Proof.* Let  $k_X$  be the sheaf of locally constant functions. Then according to the sheaf-theoretic Morse lemma,

$$R\Gamma(f^{c+\varepsilon}; k_X) \longrightarrow R\Gamma(f^{c-\varepsilon}; k_X)$$

is an isomorphism, but this implies by the long exact sequence in cohomology that  $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = 0$ .  $\square$

Finally we have

**Proposition 8.13.** *Let  $f$  be a continuous function satisfying the Palais-Smale condition above. Let us assume  $f^{-1}(c)$  contains an **isolated** strict critical point. Then for  $\varepsilon$  small enough,  $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ .*

*Proof.* This follows from the fact that if a sheaf is equal to a sky-scraper sheaf near  $U$  it has non-trivial sections. A more elementary approach is as follows. First notice that if we have two sets  $B \subset A$  and open sets  $\bar{U} \subset V$  and  $A \cap ((V \setminus U) \cup \bar{U}) = B \cap ((V \setminus U) \cup \bar{U})$  then

$$H^*(A, B) = H^*(A \cap U, B \cap U) \oplus H^*(A \cap (X \setminus V), B \cap (X \setminus V))$$

Now if  $x$  is an isolated critical point, of  $f$  according to lemma 8.5 (i.e. lemma 3.3 in [Chang]), we can deform  $f^{c+\varepsilon}$  to  $f^{c-\varepsilon}$  in  $V \setminus U$ , for some  $x \in U \subset \bar{U} \subset V$ . Thus

$$H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = H^*(f^{c+\varepsilon} \cap U, f^{c-\varepsilon} \cap U) \oplus H^*(f^{c+\varepsilon} \cap (X \setminus V), f^{c-\varepsilon} \cap (X \setminus V))$$

and since the first term of the right-hand side is non-zero, so is the left-hand side. □

Note that if we have an open set  $\Omega$  where  $f$  is flat, then any  $x \in \Omega$  is a strict critical point, but this does not imply  $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ . So the above proposition does not hold if the critical point is not isolated. Take as an example  $f(x) < 0$  for  $x < -1$ ,  $f(x) > 0$  for  $x > 1$  and  $f = 0$  on  $[-1, 1]$ . Then  $H^*(f^b, f^a) = 0$  for all  $a < b$ , while  $0 \in d_t f(0)$ .

This prompts the following definition

**Definition 8.14.** The real number  $c \in \mathbb{R}$  is a *strong critical value* of  $f \in C^{0,1}(X)$  if  $\lim_{\varepsilon \rightarrow 0} H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ . If  $x \in f^{-1}(c)$ ,  $x$  is a *strong critical point* if

$$\lim_{\varepsilon \rightarrow 0} H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) \longrightarrow \lim_{\varepsilon \rightarrow 0} \lim_{x \in U} H^*(f^{c+\varepsilon} \cap U, f^{c-\varepsilon} \cap U)$$

is non-zero. For  $X = \mathbb{R}^n$ , we say that the *strong differential* of  $f$  at  $x_0$  is the set of  $\alpha$  such that  $f_\alpha(x) = f(x) - \langle \alpha, x \rangle$  has a strong critical point at  $x_0$ . We denote it by  $d_s f(x_0)$ . Finally we denote by  $D_s f(x_0)$  the set of limits of strong differentials at  $x_0$ , that is the set of limits of  $d_s f(x_n)$  such that  $x_n$  converges to 0.

An obvious application of Mayer-Vietoris implies

**Proposition 8.15.** *If  $c$  is a strong critical value, then  $f^{-1}(c)$  contains a strong critical point.*

Clearly a strong critical point is a strict critical point, but the converse need not be true. Note that the notion of strong critical point is not purely local. However the converse holds if either the critical point is isolated, or the critical point is a local minimum (or a local maximum).

We now prove that for a smooth function, the various differentials coincide.

**Corollary 8.16.** *For a smooth function we have  $\{D_s f(x_0) \mid x_0 \in f^{-1}(c)\} = \{df(x_0) \mid x_0 \in f^{-1}(c)\}$ .*

*Proof.* It is clear that for a smooth function, if  $d_s f(x)$  exists, it is equal to  $df(x)$ . Now it is enough to show that if  $df(x_0) = 0$ ,  $f(x_0) = c$ , there is a sequence  $x_n$  such that  $df(x_n) = \alpha_n$ ,  $\alpha_n$  is an isolated solution of  $df(x) = \alpha_n$  and  $\lim_n f(x_n) = c$ . But by Morse-Sard's theorem, the set of values of  $df$  at which  $d^2 f(x)$  is degenerate has measure zero, so we can find a sequence  $\alpha_n \rightarrow 0$  such that  $f(x) - \langle \alpha_n, x \rangle$  is Morse, hence  $\alpha_n = d_s f(x_n) = df(x_n)$  and  $df(x_n) \rightarrow df(x_0)$ .  $\square$

**Proposition 8.17.** *Assume  $f_k \in C^0$  converges to  $f$ . Then if  $p \in d_s f(x)$  there is  $x_k$  such that for  $k$  large enough,  $p \in d_s f_k(x_k)$ . In particular if  $U \subset \bar{U} \subset V$ , where  $U, V$  are open, and  $U$  is contained in the set  $\{d_s f(x) \mid x \in N\}$ , then the same holds for  $\{d_s f_k(x) \mid x \in N\}$  for  $k$  large enough.*

*Proof.* Indeed, this follows from the fact that

$$H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = \lim_k H^*(f_k^{c+\varepsilon}, f_k^{c-\varepsilon})$$

so if  $H^*(f_p^{c+\varepsilon}, f_p^{c-\varepsilon}) \neq 0$  the same holds for  $(f_k)_p$ .  $\square$

### 8.3 A lemma on the set of subdifferentials

We have

**Lemma 8.18.** *Let  $f$  be a compact supported function on  $\mathbb{R}^n$ . If  $f$  is non-constant then the set of  $d_s f(x)$  as  $x$  describes  $\mathbb{R}^n$  must contain a neighbourhood of 0. More precisely if  $\text{supp}(f) \subset B(0, 1)$ , we have*

$$\{d_s f(x) \mid x \in B(0, 1)\} \supset B(0, \|f\|_{C^0}/4)$$

*Proof.* Assume for simplicity that  $f$  vanishes outside the unit ball,  $B$ . Consider the function  $f_p(x) = f(x) - \langle p, x \rangle$ . We claim that for  $p$  small enough this function has either a local minimum or a local maximum and therefore  $p \in \partial f(x)$ . Indeed,  $f = f_0$  has either a strictly negative minimum or strictly positive maximum. Assume we are in the first case. Then  $f(x_0) \leq -\varepsilon_0 \leq \min_{u \in \partial B} f(u) - \varepsilon_0$  for some  $x_0 \in B$  and  $\varepsilon_0 > 0$ . For  $p$  small enough (take  $|p| \leq \frac{\varepsilon_0}{4}$ ), the same holds for  $f_p$  with a smaller constant, that is

$$f_p(x_0) \leq \min_{u \in \partial B} f_p(u) - \frac{1}{2}\varepsilon_0$$

As a result  $f_p$  must have a global minimum, which is necessarily a strong critical point.  $\square$

## 8.4 Subdifferential of selectors

Let  $(L_k)_{k \geq 1}$  be a sequence of smooth Lagrangians Hamiltonianly isotopic to the zero section in  $T^*(N \times M)$  such that  $L_k$   $\gamma$ -converges to  $L \in \widehat{\mathcal{L}}$ . Let  $u_k(x) = c(\alpha \otimes 1_x, L_k)$  and  $u(x) = c(\alpha \otimes 1_x, L)$ . Set  $\text{Conv}_p(L_k)$  to be the union of the convex envelopes of the  $L_k \cap T_x^*N$ . This is a closed convex (in  $p$ ) set, and  $\widetilde{\text{Conv}}_p(L_k)$  the union of the convex envelopes of  $f_k^{-1}(c) \subset L_k$ , where  $h_k : L_k \rightarrow \mathbb{R}$  is the primitive of  $pdq$  on  $L_k$ . Note that both  $\widetilde{\text{Conv}}_p(L_k)$  and  $\text{Conv}_p(L_k)$  are closed. The following can be considered as an extension of theorem 2.1 (4) page 251 of [Clarke 1] (for the case where  $\alpha$  is the fundamental class of  $M$ ).

**Proposition 8.19.** *Let  $(L_k)_{k \geq 1}$  be a sequence of smooth Lagrangians Hamiltonianly isotopic to the zero section in  $T^*N$  such that  $L_k$   $\gamma$ -converges to  $L \in \widehat{\mathcal{L}}$ . We have  $\partial_C u_k(x) \subset \widetilde{\text{Conv}}_p(L_k)$  and  $\partial_C u(x) \subset \lim_k \widetilde{\text{Conv}}_p(L_k)$ .*

*Proof.* The second result follows from the first part and from the fact that according to [V4],  $u_k$   $C^0$ -converges to  $u$ , and according to [Clarke 1], for any sequence  $u_k$  of functions converging to  $u$ , we have  $\partial_C u(x) \subset \{\lim_k \partial_C u_k(x_k)\}$  where  $x_k$  converges to  $x$ .

Let us now prove the first part. Let  $L_k$  be such that there exists  $\Sigma_k$  of measure zero such that on  $N \setminus \Sigma_k$  we have a G.F.Q.I.  $S_k(q, x, \xi)$  of  $L_k$  and  $S_k(\bullet, x, \bullet)$  is Morse and has all distinct critical values, with critical points  $q_k^r(x), \xi_k^r(x)$  and  $1 \leq r \leq p$  ( $p$  is only constant on each connected component of  $N \setminus \Sigma_k$ ). This is generic for the  $C^\infty$  topology. Then we have on  $N \setminus \Sigma_k$  that  $c(\alpha \otimes 1_x, L_k) = S(q_k(x), x, \xi_k(x))$  where  $x \mapsto q_k^r(x), \xi_k^r(x)$  for  $r \in 1, \dots, p$  is a smooth function on  $N \setminus \Sigma_k$ . Now  $\frac{\partial S_k}{\partial q}(q_k^r(x), x, \xi_k^r(x)) = \frac{\partial S_k}{\partial \xi}(q_k^r(x), x, \xi_k^r(x)) = 0$ , so that  $\frac{d}{dx} c(\alpha \otimes 1_x, L_k) = \frac{d}{dx} S_k(q_k^r(x), x, \xi_k^r(x)) =$

$\frac{\partial S_k}{\partial x}(q_k^r(x), x, \xi_k^r(x)) \in L_k$  for some  $r$ . Since  $L_k$  is closed, and according to [Clarke 1],  $\partial_C u_k(x) = \{\lim_k du_k(x_l) \mid x_l \rightarrow x, x_l \in \Omega\}$  where  $\Omega$  is any set of full measure in the set of differentiability points of  $u_k$ , we get, using also that  $u_k(x) = S_k(q_k^r(x), x, \xi_k^r(x))$  that for all  $k$ ,  $\partial_C u_k(x) \in \widehat{\text{Conv}}_p(L_k)$ . Now, clearly by  $C^\infty$  density, we can always perturb the  $L_k$  so that they are generic in the above sense, and if  $u_{k,l} \rightarrow u_k$  converges in the  $C^\infty$  topology as  $l$  goes to infinity, we have  $\partial_C u_k = \lim_l \partial_C u_{k,l}$ , and since  $\lim_k L_{k,l} = L_k$  we get  $\partial_C u_k(x) \subset \text{Conv}_p(L_k)$ , as claimed.  $\square$

*Remark 8.20.* Let us mention here a result of Seyfaddini and the author, that is mentioned in [Vic2]. Let  $L_k$  be a sequence  $\gamma$ -converging to a smooth Lagrangian  $L$ . Then  $L \subset \lim_k L_k$ , that is for each  $z \in L$  there is a sequence  $z_k \in L_k$  such that  $\lim_k z_k = z$ . This is a trivial consequence of lemma 7 in [H-L-S]. This can be proved directly as follows. Indeed, if this was not the case, we would have  $B(z, r)$  such that  $B(z, r) \cap L_k = \emptyset$ . Then for any  $\varphi_H$  with Hamiltonian supported in  $B(z, r)$ , we have  $\gamma(L_k, \varphi(L_k)) = 0$ , hence  $\gamma(L, \varphi(L)) = 0$ . But it is easy to see by a local construction that this does not hold for all  $\varphi$  supported near  $z$ .

## 9 Questions and remarks

We may notice that our results should still hold for Hamiltonians in  $\widehat{\mathcal{H}}(T^*T^n)$  the  $\gamma$ -completion of  $\mathcal{H}(T^*T^n)$  or at least for  $H(t, q, p)$  of class  $C^0$ . However, we do not know what the proper definition of “invariant measure” should be for such an object. Note that existence of points in  $\Psi(\{q\} \times \mathbb{R}^n) \cap \{q_0 + \alpha\} \times \mathbb{R}^n$  could be defined as  $FH_*^{[a,b]}(\Psi(\{q\} \times \mathbb{R}^n), \{q_0 + \alpha\} \times \mathbb{R}^n) \neq 0$  for some  $a < b$ , but this does not seem to help for invariant measures. Note however that invariant measures are always limits of combinations of orbits, so this may be a useful tool.

Note also that it is probably the following convergence defining a stronger convergence as follows

**Definition 9.1.** We say that  $\psi_n$   $h$ -converges to  $\psi$  if  $FH_*^{[a,b]}(\psi_n \psi^{-1}) \rightarrow R(a, b)$  where  $R(a, b) = H^*(M)$  for  $a < 0 < b$  and  $R(a, b) = 0$  otherwise. that makes sense.

### 9.1 The structure of $\mu_\alpha$

It would be interesting to understand the structure of  $\mu_\alpha$ . In the convex case, the support of  $\mu_\alpha$  is a graph of the differential of a Lipschitz function, hence is Lagrangian in a generalized sense. Here, the support of  $\mu_\alpha$  cannot be a

graph, since replacing  $H$  by  $H \circ \psi$  replaces  $\mu_\alpha$  by  $\psi_*(\mu_\alpha)$ , hence  $\text{supp}(\mu_\alpha)$  is replaced by  $\psi(\text{supp}(\mu_\alpha))$ .

*Question 1.* Can one replace the support of  $\mu_\alpha$  by an invariant Lagrangian current, that is a current  $T_\alpha$  such that  $T_\alpha \wedge \omega = 0$   $\dim(\text{supp}(T_\alpha)) = n$ , and  $(\varphi^t)_*(T_\alpha) = T_\alpha$  ?

A question we did not answer until now is the location of the support of the metric with respect to the support of  $H$ .

**Proposition 9.2.** *For  $\alpha \neq 0$ , the support of  $\mu$  is contained in the interior of  $\text{supp}(H)$ , that is  $\mu_\alpha(\text{interior}(\text{supp}(H))) = 1$ .*

*Proof.* Indeed, if a trajectory meets the complement of the support of  $H$ , it is constant. Therefore the  $\gamma_k$  must all be contained in the support of  $H$  and since  $\mu_\alpha$  is the limit of the  $\frac{1}{k}[\gamma_k]$ , the proposition follows.  $\square$

It is also not difficult to say more in the case that  $H$  is time-independent. Since the orbit of a point remains in a fixed energy level, and the same will be true for the limit of the measure supported on such orbits. As a result we get the following result, proved in the Lagrangian situation in [DC]

**Proposition 9.3.** *Assume  $H$  is autonomous. Then, for any  $\alpha$  the measure  $\mu_\alpha$  is supported on a level set  $\{(x, p) \mid H(x, p) = c\}$ . Moreover  $\mathcal{A}(\mu_\alpha) = p \cdot \alpha - c$ .*

*Proof.* Indeed, each of the trajectories  $\gamma_k$  is contained in some  $H^{-1}(c_k)$ . If we select a subsequence such that  $c_k$  converges to some value  $c$ , then we have that  $\mu_\alpha$  is supported in  $H^{-1}(c)$ .  $\square$

This implies that for  $\alpha \neq 0$ , the measure is supported at a positive distance from the support.

*Question 2.* Is this still true for the time dependent case ?

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