# On path partitions of the divisor graph 

Paul Melotti, Eric Saias

## To cite this version:

Paul Melotti, Eric Saias. On path partitions of the divisor graph. Acta Arithmetica, 2020, 192 (4), pp.329-339. 10.4064/aa180711-26-4 . hal-01848021

## HAL Id: hal-01848021

## https://hal.science/hal-01848021

Submitted on 24 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On path partitions of the divisor graph 

Paul Melotti<br>Sorbonne Université<br>LPSM<br>4 place Jussieu<br>F-75005 PARIS<br>FRANCE<br>E-mail: paul.melotti@upmc.fr<br>Éric Saias<br>Sorbonne Université<br>LPSM<br>4 place Jussieu<br>F-75005 PARIS<br>FRANCE<br>E-mail: eric.saias@upmc.fr

[
■


#### Abstract

It is known that the longest simple path in the divisor graph that uses integers $\leq N$ is of length $\asymp N / \log N$. We study the partitions of $\{1,2, \ldots, N\}$ into a minimal number of paths of the divisor graph, and we show that in such a partition, the longest path can have length asymptotically $N^{1-o(1)}$.


## 1 Introduction

The divisor graph is the unoriented graph whose vertices are the positive integers, and edges are the $\{a, b\}$ such that $a<b$ and $a$ divides $b$. A path of length $l$ in the divisor graph is a finite sequence $n_{1}, \ldots, n_{l}$ of pairwise distinct positive integers such that $n_{i}$ is either a divisor or a multiple of $n_{i+1}$, for all $i$ such that $1 \leq i<l$. Let $F(x)$ be the minimal cardinal of a partition of $\{1,2, \ldots,\lfloor x\rfloor\}$ into paths of the divisor graph.

The asymptotic behaviour of $F(x)$ has been studied in [3, 8, 4, 1]. Thanks to the works of Mazet and Chadozeau, we know that there is a constant $c \in\left(\frac{1}{6}, \frac{1}{4}\right)$ such that

$$
\begin{equation*}
F(x)=c x\left(1+O\left(\frac{1}{\log \log x \log \log \log x}\right)\right) . \tag{1}
\end{equation*}
$$

[^0]A partition of $\{1,2, \ldots, N\}$ into paths of the divisor graph is said to be optimal if its cardinal is $F(N)$. We are interested in the length of the paths in an optimal partition.

Let us take the example $N=30$ that was considered in [5, 8. . It is known (see [8]) that $F(30)=5$, so that the following partition is optimal:
$13,26,1,11,22,2,14,28,7,21,3,27,9,18,6,12,24,8,16,4,20,10,30,15,5,25$ 17

Four of these five paths are singletons. In fact, at the end of the proof of Theorem 2 of [3], it is proven that the number of singletons in a (not necessarily optimal) partition is $\asymp N$ for $N$ large enough.

Let us look at the longest paths in an optimal partition of $\{1,2, \ldots, N\}$. Let $L(N)$ be the maximal path length, among all paths of all optimal partitions of $\{1,2, \ldots, N\}$ into paths of the divisor graph. Let also $f(N)$ denote the maximal length of a path of the divisor graph that uses integers $\leq N$.

It is known that (Theorem 2 of [7])

$$
\begin{equation*}
f(N) \asymp \frac{N}{\log N} \tag{2}
\end{equation*}
$$

Of course $L(N) \leq f(N)$. In the previous example, four of the five paths are singletons, which implies that the longest path has maximal length. In other words $L(30)=f(30)=26$. More generally, we know that for all $N \geq 1$,

$$
\begin{equation*}
F(N) \geq N-\lfloor N / 2\rfloor-\lfloor N / 3\rfloor \tag{3}
\end{equation*}
$$

(see [8]). Inspired by the case $N=30$, for any $N \in[1,33]$ it is easy to construct a partition of $\{1, \ldots, N\}$ into $N-\lfloor N / 2\rfloor-\lfloor N / 3\rfloor$ paths, all of them but one being singletons. This shows that for $1 \leq N \leq 33$, (3) is an equality and $L(N)=f(N)=\lfloor N / 2\rfloor+\lfloor N / 3\rfloor+1$.

However for larger $N$ the situation becomes more complicated. For $N$ large enough there is no optimal partition with all paths but one being singletons. This can be deduced from (2) and the fact that the constant $c$ in (11) is less than 1 . Still, it is natural to wonder if the equality $L(N)=f(N)$ holds for any $N \geq 1$.

We were unable to answer this question, but we looked for lower bounds on $L(N)$ and proved the following:

Theorem. There is a constant $A \geq 0$ such that for all $N \geq 3$,

$$
L(N) \geq \frac{N}{(\log N)^{A} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]}
$$

To prove this we introduce a new function $H(x)$. For a real number $x \geq 1$ and two distinct integers $a, b \in[1, x]$, let $L_{a, b}(x)$ be the maximal length of a path having $a$ and $b$ as endpoints and belonging to an optimal partition of $\{1,2, \ldots,\lfloor x\rfloor\}$. If there is no such path, we set $L_{a, b}(x)=0$. Then we set

$$
H(x)=\min L_{r^{\prime}, r}(x)
$$

where the min is over all couples $\left(r^{\prime}, r\right)$ of prime numbers such that

$$
\frac{x}{3}<r \leq \frac{x}{2}<r^{\prime} \leq x
$$

The theorem will be an easy consequence of the following.
Proposition. There is a constant $N_{0}$ such that for any $N \geq N_{0}$, there is a set $\mathcal{P}(N)$ of prime numbers in $(3 \sqrt{N \log N}, 4 \sqrt{N \log N}]$, of cardinal $|\mathcal{P}(N)| \geq \frac{\sqrt{N}}{19(\log N)^{3 / 2}}$, such that

$$
\begin{equation*}
H(N) \geq \sum_{p \in \mathcal{P}(N)} H\left(\frac{N}{p}\right) \tag{4}
\end{equation*}
$$

The technique used here is analogous to that of [6] in the study of the longest path. More precisely, in [6], $f^{*}(N)$ denotes the maximal length of a path that uses integers in $[\sqrt{N}, N]$. A quantity $h^{*}$ is introduced, which is to $f^{*}$ what $H$ is to $L$ in our case. The inequality (4) is analogous to Buchstab's unequality (40) from [6]. The corresponding lower bounds led to the proof that $f^{*}(N) \asymp N / \log N$ (Theorem 2 in [7]).

The analogy can be pushed further: in both the proof of (4) and of (40) in [6], we borrow a technique used by Erdős, Freud and Hegyvári who proved the following asymptotic behaviour:

$$
\min \max _{1 \leq i \leq N-1} \operatorname{lcm}\left(a_{i}, a_{i+1}\right)=\left(\frac{1}{4}+o(1)\right) \frac{N^{2}}{\log N}
$$

where the min is over all permutations $\left(a_{1}, a_{2}, \ldots, a_{N}\right)$ of $\{1,2, \ldots, N\}$; see Theorem 1 of [2]. In [2] as in [6] or in the present work, the proof goes through the construction of a sequence of integers by concatenating blocks whose largest prime factor is constant, and linking blocks together with
separating integers. In [6] as in the present work, these blocks take the form of sub-paths $p \mathcal{C}_{N / p}$, where the $\mathcal{C}_{N / p}$ is a path of integers $\leq N / p$ whose largest prime factor is $\leq p$.

It is worth mentioning that the article [2] of Erdős, Freud and Hegyvári is the origin of all works related to the divisor graph.

## 2 Notations

The letters $p, q, q^{\prime}, r, r^{\prime}$ will always denote generic prime numbers. For an integer $m \geq 2, P^{-}(m)$ denotes the smallest prime factor of $m$.

Let $N \geq 1$. A path of integers $\leq N$ of length $l$ is a $l$-uple $\mathcal{C}=\left(a_{1}, a_{2}, \ldots, a_{l}\right)$ of pairwise distinct positive integers $\leq N$, such that for all $i$ with $1 \leq i \leq$ $l-1, a_{i}$ is either a divisor or a multiple of $a_{i+1}$. For convenience, we take $\mathcal{C}$ up to global flip, i.e. we identify $\left(a_{1}, \ldots, a_{l}\right)$ with $\left(a_{l}, \ldots, a_{1}\right)$. We will denote this path by $a_{1}-a_{2}-\cdots-a_{l}$ (or $a_{l}-\cdots-a_{2}-a_{1}$ ). If $b$ and $c$ are integers such that $b=a_{i}$ and $c=a_{i \pm 1}$ for some $i$, we say that $b$ and $c$ are neighbours (in $\mathcal{C}$ ).

When a partition $\mathcal{A}(N)$ of $\{1,2, \ldots, N\}$ is fixed, for any $n \in\{1,2, \ldots, N\}$ we will simply denote by $\mathcal{C}(n)$ the path that contains $n$ in $\mathcal{A}(N)$.

A partition of $\{1,2, \ldots, N\}$ into paths is said to be optimal if it contains $F(N)$ paths (see the Introduction for the definition of $F$ ).

Let $\mathcal{C}$ be a path of integers $\leq N$ and $1 \leq n \leq N$. Then $\mathcal{C}$ is said to be $n$-factorizable if all the integers of $\mathcal{C}$ are multiple of $n$. Then $\mathcal{C}$ can be written as $\mathcal{C}=n \mathcal{D}$ where $\mathcal{D}$ is a path of integers $\leq N / n$.

For integers $1 \leq n \leq N$ and a partition $\mathcal{A}(N)$ of $\{1,2, \ldots, N\}$, we say that $n$ is factorizing for $\mathcal{A}(N)$ if every path of $\mathcal{A}(N)$ that contains a multiple of $n$ is $n$-factorizable.

## 3 Lemmas

Lemma 1. Let $N \geq 1$ and $\mathcal{A}(N)$ be an optimal partition.
(i) Let $1 \leq n \leq N$ with $n$ factorizing for $\mathcal{A}(N)$. Let $k=\lfloor N / n\rfloor$. There are exactly $F(k)$ paths in $\mathcal{A}(N)$ that contain a multiple of $n$. They are of the form $n \mathcal{D}_{1}, n \mathcal{D}_{2}, \ldots, n \mathcal{D}_{F(k)}$ where $\mathcal{D}_{1}, \mathcal{D}_{2}, \ldots, \mathcal{D}_{F(k)}$ is an optimal partition of $\{1,2, \ldots, k\}$.
(ii) Let $z>1$ be a real number. Let $M_{z}(N)$ be the set of integers $m \leq N$
that are not factorizing for $\mathcal{A}(N)$ and such that

$$
m>\frac{N}{z} \text { and } P^{-}(m)>z
$$

Then

$$
\left|M_{z}(N)\right|<\frac{2 N}{z} .
$$

Proof. (i) The set of paths that contain a multiple of $n$ is of the form $\left\{n \mathcal{D}_{1}, n \mathcal{D}_{2}, \ldots, n \mathcal{D}_{g}\right\}$ where $\mathcal{D}_{1}, \mathcal{D}_{2}, \ldots, \mathcal{D}_{g}$ is a partition of the integers $\leq k=\lfloor N / n\rfloor$. Since $\mathcal{A}(N)$ is optimal, $\mathcal{D}_{1}, \mathcal{D}_{2}, \ldots, \mathcal{D}_{g}$ is optimal, hence $g=F(k)$.
(ii) Let $m \in M_{z}(N)$. There is a path $\mathcal{C}_{m}$ in $\mathcal{A}(N)$ that contains multiples and non-multiples of $m$. Hence there is an integer $c(m)$ in $\mathcal{C}_{m}$ that is not a multiple of $m$, and is neighbour to an integer $b(m)$ which is a multiple of $m$. Then $c(m)$ has to be a divisor of $b(m)$. More precisely, if $b(m)=a m$, then $c(m)$ can be written as $c(m)=\tilde{a} \tilde{m}$ with $\tilde{a}$ a divisor of $a$ and $\tilde{m}$ a strict divisor of $m$. Since $P^{-}(m)>z, c(m)<N / z$.

Moreover, if $m, m^{\prime}$ are two distinct elements of $M_{z}(N)$, then

$$
\operatorname{lcm}\left(m, m^{\prime}\right) \geq \min \left(m P^{-}\left(m^{\prime}\right), m^{\prime} P^{-}(m)\right)>\frac{N}{z} z=N
$$

As a result the map

$$
\begin{aligned}
b: M_{z}(N) & \rightarrow\{1,2, \ldots, N\} \\
m & \mapsto b(m)
\end{aligned}
$$

is an injection.
Moreover, any integer $c<N / z$ has at most two neighbours in $\mathcal{C}(c)$. Consequently the map

$$
\begin{aligned}
c: M_{z}(N) & \rightarrow\{1 \leq n<N / z\} \\
m & \mapsto c(m)
\end{aligned}
$$

is at-most-two-to-one. Thus

$$
\left|M_{z}(N)\right|<\frac{2 N}{z} .
$$

Lemma 2. There exists a constant $N_{1}$ such that for any $N \geq N_{1}$, there is a set $\widetilde{\mathcal{P}}(N)$ of prime numbers in $(3 \sqrt{N \log N}, 4 \sqrt{N \log N}]$ of cardinal

$$
\begin{equation*}
|\widetilde{\mathcal{P}}(N)| \geq \sqrt{\frac{N}{\log N}} \tag{5}
\end{equation*}
$$

such that for any prime numbers $r, r^{\prime}$ with

$$
\frac{N}{3}<r \leq \frac{N}{2}<r^{\prime} \leq N
$$

there exists an optimal partition $\mathcal{A}(N)$ of $\{1,2, \ldots, N\}$ that contains the paths $r^{\prime}$ and $2 r-r$ and for which all the integers in $\widetilde{\mathcal{P}}(N)$ are factorizing.

Proof. Let $N_{1}$ be such that for any $N \geq N_{1}$,

$$
\begin{align*}
& \pi(4 \sqrt{N \log N})-\pi(3 \sqrt{N \log N})-\frac{2}{3} \sqrt{\frac{N}{\log N}} \geq \sqrt{\frac{N}{\log N}}  \tag{6}\\
& \pi\left(\frac{N}{2}\right)-\pi\left(\frac{N}{3}\right) \geq 8 \tag{7}
\end{align*}
$$

The existence of such a $N_{1}$ comes from the prime number theorem (more precisely the left-hand-side of (6) is equivalent to $\frac{4}{3} \sqrt{\frac{N}{\log N}}$. We also take $N_{1}$ large enough so that

$$
\begin{equation*}
(3 \sqrt{N \log N}, 4 \sqrt{N \log N}] \cap\left(\frac{N}{3}, \frac{N}{2}\right]=\emptyset \tag{8}
\end{equation*}
$$

Let $N \geq N_{1}$. We start by fixing an optimal partition $\mathcal{A}^{\prime}(N)$. We apply Lemma 1 (ii) to $\mathcal{A}^{\prime}(N)$ with $z=3 \sqrt{N \log N}$. All the prime numbers $p$ in $(3 \sqrt{N \log N}, 4 \sqrt{N \log N}]$ that are not factorizing are in $M_{z}(N)$, since they satisfy $p>3 \sqrt{N \log N} \geq \frac{N}{z}$ and $P^{-}(p)=p>z$, so there are at most $\frac{2}{3} \sqrt{\frac{N}{\log N}}$ of them. By removing these and using (6), we get a set $\widetilde{\mathcal{P}}(N)$ of prime numbers in $(3 \sqrt{N \log N}, 4 \sqrt{N \log N}]$ that are factorizing in $\mathcal{A}^{\prime}(N)$, with cardinality

$$
|\widetilde{\mathcal{P}}(N)| \geq \sqrt{\frac{N}{\log N}}
$$

We now change notations slightly and fix two prime numbers $r_{0}, r_{0}^{\prime}$ such that

$$
\frac{N}{3}<r_{0} \leq \frac{N}{2}<r_{0}^{\prime} \leq N
$$

Our goal is to go from $\mathcal{A}^{\prime}(N)$ to a new optimal partition $\mathcal{A}(N)$ that contains the paths $r_{0}^{\prime}$ and $2 r_{0}-r_{0}$ while maintaining the fact that the elements of $\widetilde{\mathcal{P}}(N)$ are factorizing.

Let us denote the set of prime numbers

$$
\mathcal{R}=\left\{\frac{N}{3}<r \leq \frac{N}{2}\right\}
$$

and $\mathcal{R}^{*}\left(\mathcal{A}^{\prime}(N)\right)$ the subset of $r \in \mathcal{R}$ such that $r$ does not have 1 as a neighbour in $\mathcal{C}(r)$ and $2 r$ does not have 1 nor 2 has a neighbour in $\mathcal{C}(2 r)$. Then for any $r \in \mathcal{R}^{*}\left(\mathcal{A}^{\prime}(N)\right)$, since the only possible neighbour of $r$ is $2 r$ and reciprocally, by optimality the path $\mathcal{C}(r)$ is equal to $r-2 r$. Moreover, since 1 and 2 have at most two neighbours,

$$
\begin{equation*}
\left|\mathcal{R} \backslash \mathcal{R}^{*}\left(\mathcal{A}^{\prime}(N)\right)\right| \leq 4 \tag{9}
\end{equation*}
$$

Now we make it so that $r_{0}^{\prime}$ is a path. If it is not the case, since the only possible neighbour of $r_{0}^{\prime}$ is $1, \mathcal{C}\left(r_{0}^{\prime}\right)$ is of the form $\mathcal{D}-r_{0}^{\prime}$ with $\mathcal{D}$ a path ending in 1. We split this path into $\mathcal{D}$ on one side and $r_{0}^{\prime}$ on the other side. By (9) and (7), there is at least one element $r^{*} \in \mathcal{R}^{*}\left(\mathcal{A}^{\prime}(N)\right)$. We stick $\mathcal{D}$ to $\mathcal{C}\left(r^{*}\right)$, thus forming the path $\mathcal{D}-\mathcal{C}\left(r^{*}\right)$. This is possible because $\mathcal{D}$ ends in 1 . Let $\mathcal{A}^{\prime \prime}(N)$ be this new partition. The total number of paths has not changed so $\mathcal{A}^{\prime}(N)$ is still optimal, furthermore it contains the path $r_{0}^{\prime}$, and the elements of $\widetilde{\mathcal{P}}(N)$ are still factorizing because the integers in the paths that changed were not multiples of any $p \in \widetilde{\mathcal{P}}(N)$.

The subset $\mathcal{R}^{*}\left(\mathcal{A}^{\prime \prime}(N)\right)$ might differ from $\mathcal{R}^{*}\left(\mathcal{A}^{\prime}(N)\right)$ by one element, but it still satisfies (9) and its elements $r$ still satisfy that $\mathcal{C}(r)$ is equal to $r-2 r$. If $r_{0} \in \mathcal{R}^{*}\left(\mathcal{A}^{\prime \prime}(N)\right)$, we can set $\mathcal{A}(N)=\mathcal{A}^{\prime \prime}(N)$ and the proof is over. We now suppose that $r_{0} \notin \mathcal{R}^{*}\left(\mathcal{A}^{\prime \prime}(N)\right)$.

By (9) and (7), there are at least four elements $r_{1}, r_{2}, r_{3}, r_{4}$ in $\mathcal{R}^{*}\left(\mathcal{A}^{\prime \prime}(N)\right)$. We cut the path $\mathcal{C}(1)$ into one, two or three paths, one of them being the singleton 1 (we will see later that we get in fact three paths). Such a move will be called an extraction of the integer 1 . We similarly extract the integer 2 . We now use these integers 1 and 2 to stick together the paths $r_{i}-2 r_{i}$ by forming

$$
r_{1}-2 r_{1}-1-2 r_{2}-r_{2} \text { and } r_{3}-2 r_{3}-2-2 r_{4}-r_{4} .
$$

We thus get a new partition $\mathcal{A}(N)$. Its number of paths is less or equal to that of $\mathcal{A}^{\prime \prime}(N)$, so it is still optimal (this shows in particular that 1 and 2 were not endpoints of their paths). It also satisfies $r_{0} \in \mathcal{R}^{*}(\mathcal{A}(N))$ since 1
and 2 are not linked to $r_{0}$ nor $2 r_{0}$, so that it contains the path $r_{0}-2 r_{0}$, as well as $r_{0}^{\prime}$, and the elements of $\widetilde{\mathcal{P}}(N)$ are still factorizing.

## 4 Proof of the Proposition

Let $N_{1}$ be the constant of Lemma 2. We fix a $N_{0}$ such that

$$
\begin{equation*}
N_{0} \geq N_{1}^{4} \tag{10}
\end{equation*}
$$

and such that for all $N \geq N_{0}$,

$$
\begin{align*}
\frac{1}{2} \sqrt{\frac{N}{\log N}} & \geq \pi\left(\frac{1}{4} \sqrt{\frac{N}{\log N}}\right)-\pi\left(\frac{1}{6} \sqrt{\frac{N}{\log N}}\right) \\
& \geq \pi\left(\frac{1}{8} \sqrt{\frac{N}{\log N}}\right)-\pi\left(\frac{1}{9} \sqrt{\frac{N}{\log N}}\right)  \tag{11}\\
& \geq\left\lfloor\frac{\sqrt{N}}{37(\log N)^{3 / 2}}\right\rfloor \geq \frac{\sqrt{N}}{38(\log N)^{3 / 2}}+\frac{1}{2} \geq 5
\end{align*}
$$

and

$$
\begin{equation*}
4 \sqrt{\log N} \leq N^{1 / 4} \tag{12}
\end{equation*}
$$

The existence of such a $N_{0}$ is again an easy consequence of the prime number theorem. Also note that since $N_{0} \geq N_{1}$, (8) still holds.

Let $N \geq N_{0}$. We chose a set $\widetilde{\mathcal{P}}(N)$ according to Lemma2. Let us denote

$$
I=\left\lfloor\frac{1}{37} \frac{\sqrt{N}}{(\log N)^{3 / 2}}\right\rfloor .
$$

By (5) and (11) we can chose $2 I$ elements in $\widetilde{\mathcal{P}}(N)$, which we denote as

$$
p_{1}, p_{2}, \ldots, p_{2 I}
$$

We set $\mathcal{P}(N)=\left\{p_{1}, \ldots, p_{2 I-1}\right\}$. By (11) again, $|\mathcal{P}(N)| \geq \frac{\sqrt{N}}{19(\log N)^{3 / 2}}$. It remains to prove that this set $\mathcal{P}(N)$ satisfies (4).

Let $r, r^{\prime}$ be two prime numbers such that

$$
\begin{equation*}
\frac{N}{3}<r \leq \frac{N}{2}<r^{\prime} \leq N \tag{13}
\end{equation*}
$$

By the property of $\widetilde{\mathcal{P}}(N)$ in Lemma 2, there exists an optimal partition $\mathcal{A}^{\prime}(N)$, that contains the paths $r^{\prime}$ and $2 r-r$, for which the elements of $\widetilde{\mathcal{P}}(N)$ (and in particular the elements of $\mathcal{P}(N)$ ) are factorizing.

We denote two sets of prime numbers

$$
\begin{aligned}
\mathcal{Q}(N) & =\left\{\frac{1}{9} \sqrt{\frac{N}{\log N}}<q \leq \frac{1}{8} \sqrt{\frac{N}{\log N}}\right\}, \\
\mathcal{Q}^{\prime}(N) & =\left\{\frac{1}{6} \sqrt{\frac{N}{\log N}}<q^{\prime} \leq \frac{1}{4} \sqrt{\frac{N}{\log N}}\right\} .
\end{aligned}
$$

For all $\left(p, q, q^{\prime}\right) \in \widetilde{\mathcal{P}}(N) \times \mathcal{Q}(N) \times \mathcal{Q}^{\prime}(N)$ we have

$$
\begin{align*}
& \frac{N}{3}<p q \leq \frac{N}{2}  \tag{14}\\
& \frac{N}{2}<p q^{\prime} \leq N \tag{15}
\end{align*}
$$

We focus on the factorizing prime number $p_{2 I}$. For any $q \in \mathcal{Q}(N)$, because of (14) the only possible neighbours of $p_{2 I} q$ are $p_{2 I}$ and $2 p_{2 I} q$. Similarly, the only possible neighbours of $2 p_{2 I} q$ are $p_{2 I}, 2 p_{2 I}$ or $p_{2 I} q$. But $p_{2 I}$ and $2 p_{2 I}$ can be linked to at most 4 elements of type $p_{2 I} q$ or $2 p_{2 I} q$. By (11) we know that $|\mathcal{Q}(N)| \geq 5$, so there exists a $q_{2 I} \in \mathcal{Q}(N)$ for which neither $p_{2 I} q_{2 I}$ nor $2 p_{2 I} q_{2 I}$ is a neighbour of $p_{2 I}$ or $2 p_{2 I}$. As a result, the only possible neighbour for $p_{2 I} q_{2 I}$ is $2 p_{2 I} q_{2 I}$, and reciprocally. By optimality, $\mathcal{A}^{\prime}(N)$ contains the path $p_{2 I} q_{2 I}-2 p_{2 I} q_{2 I}$.

Using (11) we can chose

- $I$ elements of $\mathcal{Q}^{\prime}(N)$ which we write as

$$
\begin{equation*}
q_{1}, q_{3}, \ldots, q_{2 I-1} \tag{16}
\end{equation*}
$$

- I-1 elements of $\mathcal{Q}(N) \backslash\left\{q_{2 I}\right\}$ which we write as

$$
\begin{equation*}
q_{2}, q_{4}, \ldots, q_{2 I-2} \tag{17}
\end{equation*}
$$

Let $i$ be such that $1 \leq i \leq 2 I-1$. Then the prime number $p_{i}$ is factorizing for $\mathcal{A}^{\prime}(N)$ so by Lemma 1 (i) the paths of $\mathcal{A}^{\prime}(N)$ that contain multiples of $p_{i}$ are of the form

$$
p_{i} \mathcal{C}_{i, 1}, p_{i} \mathcal{C}_{i, 2}, \ldots, p_{i} \mathcal{C}_{i, F\left(N / p_{i}\right)}
$$

where $\mathcal{C}_{i, 1}, \mathcal{C}_{i, 2}, \ldots, \mathcal{C}_{i, F\left(N / p_{i}\right)}$ is an optimal partition of $\left\{1,2, \ldots,\left\lfloor N / p_{i}\right\rfloor\right\}$. By our choice of indices (16), (17), one of the elements $q_{i}, q_{i+1}$ is in $\mathcal{Q}^{\prime}(N)$, we rename it $\widetilde{q}_{i}$, and the other is in $\mathcal{Q}(N)$, we rename it $\widetilde{q_{i+1}}$. Using (14), (15) we get

$$
\frac{N}{3 p_{i}}<\widetilde{q_{i+1}} \leq \frac{N}{2 p_{i}}<\widetilde{q_{i}} \leq \frac{N}{p_{i}}
$$

Using (12) and (10), we have $N / p_{i} \geq N^{1 / 4} \geq N_{0}^{1 / 4} \geq N_{1}$. Hence we can apply Lemma 2 with $N / p_{i}$ instead of $N$. We deduce that there exists an optimal partition of $\left\{1,2, \ldots,\left\lfloor N / p_{i}\right\rfloor\right\}$ that contains the paths $\widetilde{q_{i}}$ and $\widetilde{q_{i+1}}-$ $\widetilde{q_{i+1}}$. By extracting 1 in that partition, we can stick these two paths together into $\widetilde{q_{i}}-1-2 \widetilde{q_{i+1}}-\widetilde{q_{i+1}}$ while keeping an optimal partition. To sum up, we know now that there is an optimal partition of the integers $\leq N / p_{i}$ containing a path that has $q_{i}$ and $q_{i+1}$ as endpoints.

Let $\mathcal{D}_{i, 1}, \mathcal{D}_{i, 2}, \ldots, \mathcal{D}_{i, F\left(N / p_{i}\right)}$ be an optimal partition of the integers $\leq$ $N / p_{i}$, with $\mathcal{D}_{i, 1}$ having $q_{i}, q_{i+1}$ as endpoints and of maximal length $L_{q_{i}, q_{i+1}}\left(N / p_{i}\right)$. We can transform $\mathcal{A}^{\prime}(N)$ by replacing the paths $\left(p_{i} \mathcal{C}_{i, j}\right)_{1 \leq j \leq F\left(N / p_{i}\right)}$ by $\left(p_{i} \mathcal{D}_{i, j}\right)_{1 \leq j \leq F\left(N / p_{i}\right)}$. In this way we get a new optimal partition $\mathcal{A}^{\prime \prime}(N)$ that contains all the paths $p_{i} \mathcal{D}_{i, 1}$ for $1 \leq i \leq 2 I-1$, as well as $r^{\prime}, 2 r-r$, and $p_{2 I} q_{2 I}-2 p_{2 I} q_{2 I}$.

By extracting the integers 1,2 and the $q_{i}$ for $2 \leq i \leq 2 I$, we construct the path of Figure 1 while keeping an optimal partition of $\{1,2, \ldots, N\}$.


Figure 1: A long path with endpoints $r^{\prime}, r$.

Its length is larger than

$$
\sum_{i=1}^{2 I-1} L_{q_{i}, q_{i+1}}\left(N / p_{i}\right) \geq \sum_{p \in \mathcal{P}(N)} H\left(N / p_{i}\right)
$$

This being true for any $r, r^{\prime}$ satisfying (13), we get

$$
H(N) \geq \sum_{p \in \mathcal{P}(N)} H\left(N / p_{i}\right) .
$$

## 5 Proof of the Theorem

Let us fix a constant $N_{2}=2^{2^{k_{0}}} \geq N_{0}$, where $N_{0}$ is the constant from the Proposition. We chose a constant $B$ such that for all $N \leq 2^{2^{k_{0}+2}}$,

$$
\begin{equation*}
N \leq 4(\log N)^{B} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right] \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
B \geq 8 \tag{19}
\end{equation*}
$$

We show by induction on $k \geq k_{0}+2$ that for all $N$ such that

$$
2^{2^{k_{0}}}<N \leq 2^{2^{k}}
$$

we have

$$
\begin{equation*}
H(N) \geq \frac{N}{(\log N)^{B} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]} \tag{20}
\end{equation*}
$$

## Base case

Let $N$ be such that $2^{2^{k_{0}}}<N \leq 2^{2^{k_{0}+2}}$, then we have $N>N_{2} \geq N_{0} \geq N_{1}^{4}$ (see 10 ) with $N_{1}$ the constant of Lemma 2 Let $r, r^{\prime}$ be two prime numbers such

$$
\frac{N}{3}<r \leq \frac{N}{2}<r^{\prime} \leq N
$$

Lemma 2 implies that there is an optimal partition $\mathcal{A}(N)$ of $\{1,2, \ldots, N\}$ which contains the paths $r^{\prime}$ and $2 r-r$. By extracting 1 , we can stick them into $r^{\prime}-1-2 r-r$ while keeping an optimal partition. This implies that $H(N) \geq 4$, and (18) yields the base case.

## Induction step

Let $k \geq k_{0}+2$. We suppose that 20 holds for all $N \in\left(2^{2^{k_{0}}}, 2^{2^{k}}\right]$.
Let $N$ be such that $2^{2^{k}}<N \leq 2^{2^{k+1}}$. Since $k \geq k_{0}+2$, we also have $N^{1 / 4}>2^{2^{k_{0}}}$.

Let $p \in(3 \sqrt{N \log N}, 4 \sqrt{N \log N}]$. By 122 , we have

$$
2^{2^{k_{0}}}<N^{1 / 4} \leq \frac{N}{p} \leq \sqrt{N} \leq 2^{2^{k}}
$$

By using the induction hypothesis on $N / p$, we get

$$
\begin{aligned}
H\left(\frac{N}{p}\right) & \geq \frac{N}{p(\log (N / p))^{B} \exp \left[\frac{(\log \log (N / p))^{2}}{\log 2}\right]} \\
& \geq \frac{N}{p(\log \sqrt{N})^{B} \exp \left[\frac{(\log \log \sqrt{N})^{2}}{\log 2}\right]} \\
& =\frac{2^{B-1}(\log N)^{2} N}{p(\log N)^{B} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]}
\end{aligned}
$$

Hence by using the Proposition and (19),

$$
\begin{aligned}
H(N) & \geq \sum_{p \in \mathcal{P}(N)} H\left(\frac{N}{p}\right) \\
& \geq \frac{|\mathcal{P}(N)|}{\max \mathcal{P}(N)} \frac{2^{B-1}(\log N)^{2} N}{(\log N)^{B} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]} \\
& \geq \frac{2^{B-1}}{76} \frac{N}{(\log N)^{B} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]} \\
& \geq \frac{N}{(\log N)^{B} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]}
\end{aligned}
$$

This concludes the induction step.
Finally, since $L(N) \geq 1$ for all $N \geq 1$, we get the Theorem by chosing $A=\max \left(B, A_{0}\right)$ where $A_{0}$ is a constant such that for all $3 \leq N<N_{0}$,

$$
N \leq(\log N)^{A_{0}} \exp \left[\frac{(\log \log N)^{2}}{\log 2}\right]
$$

## References

[1] Chadozeau, A. Sur les partitions en chaînes du graphe divisoriel. Periodica Mathematica Hungarica 56, 2 (2008), 227-239.
[2] Erdôs, P., Freud, R., and Hegyvári, N. Arithmetical properties of permutations of integers. Acta Mathematica Hungarica 41, 1 (1983), 169-176.
[3] Erdôs, P., And Saias, E. Sur le graphe divisoriel. Acta Arithmetica 73, 2 (1995), 189-198.
[4] Mazet, P. Recouvrements hamiltoniens de certains graphes. European Journal of Combinatorics 27, 5 (2006), 739-749.
[5] Pomerance, C. On the longest simple path in the divisor graph. Congr. Numer. 40 (1983), 291-304.
[6] Saias, E. Sur l'utilisation de l'identité de Buchstab. In Séminaire de Théorie des Nombres de Paris 1991-1992 (1993), S. David, Ed., Birkaüser, pp. 217-245.
[7] SAIAS, E. Applications des entiers à diviseurs denses. Acta Arithmetica 83, 3 (1998), 225-240.
[8] SAiAS, E. Etude du graphe divisoriel 3. Rendiconti del Circolo Matematico di Palermo 52, 3 (2003), 481-488.


[^0]:    2010 Mathematics Subject Classification: Primary 11N37, 11B75; Secondary 05C38, 05 C 70 .

    Key words and phrases: divisor graph, path partition.

