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# On path partitions of the divisor graph

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## Abstract

It is known that the longest simple path in the divisor graph that uses integers  $\leq N$  is of length  $\asymp N/\log N$ . We study the partitions of  $\{1, 2, \dots, N\}$  into a *minimal* number of paths of the divisor graph, and we show that in such a partition, the longest path can have length asymptotically  $N^{1-o(1)}$ .

## 1 Introduction

The *divisor graph* is the unoriented graph whose vertices are the positive integers, and edges are the  $\{a, b\}$  such that  $a < b$  and  $a$  divides  $b$ . A *path* of length  $l$  in the divisor graph is a finite sequence  $n_1, \dots, n_l$  of pairwise distinct positive integers such that  $n_i$  is either a divisor or a multiple of  $n_{i+1}$ , for all  $i$  such that  $1 \leq i < l$ . Let  $F(x)$  be the minimal cardinal of a partition of  $\{1, 2, \dots, \lfloor x \rfloor\}$  into paths of the divisor graph.

The asymptotic behaviour of  $F(x)$  has been studied in [3, 8, 4, 1]. Thanks to the works of Mazet and Chadozeau, we know that there is a constant  $c \in (\frac{1}{6}, \frac{1}{4})$  such that

$$(1) \quad F(x) = cx \left( 1 + O \left( \frac{1}{\log \log x \log \log \log x} \right) \right).$$

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A partition of  $\{1, 2, \dots, N\}$  into paths of the divisor graph is said to be *optimal* if its cardinal is  $F(N)$ . We are interested in the length of the paths in an optimal partition.

Let us take the example  $N = 30$  that was considered in [5, 8]. It is known (see [8]) that  $F(30) = 5$ , so that the following partition is optimal:

13, 26, 1, 11, 22, 2, 14, 28, 7, 21, 3, 27, 9, 18, 6, 12, 24, 8, 16, 4, 20, 10, 30, 15, 5, 25  
 17  
 19  
 23  
 29

Four of these five paths are singletons. In fact, at the end of the proof of Theorem 2 of [3], it is proven that the number of singletons in a (not necessarily optimal) partition is  $\asymp N$  for  $N$  large enough.

Let us look at the longest paths in an optimal partition of  $\{1, 2, \dots, N\}$ . Let  $L(N)$  be the maximal path length, among all paths of all *optimal* partitions of  $\{1, 2, \dots, N\}$  into paths of the divisor graph. Let also  $f(N)$  denote the maximal length of a path of the divisor graph that uses integers  $\leq N$ .

It is known that (Theorem 2 of [7])

$$(2) \quad f(N) \asymp \frac{N}{\log N}.$$

Of course  $L(N) \leq f(N)$ . In the previous example, four of the five paths are singletons, which implies that the longest path has maximal length. In other words  $L(30) = f(30) = 26$ . More generally, we know that for all  $N \geq 1$ ,

$$(3) \quad F(N) \geq N - \lfloor N/2 \rfloor - \lfloor N/3 \rfloor$$

(see [8]). Inspired by the case  $N = 30$ , for any  $N \in [1, 33]$  it is easy to construct a partition of  $\{1, \dots, N\}$  into  $N - \lfloor N/2 \rfloor - \lfloor N/3 \rfloor$  paths, all of them but one being singletons. This shows that for  $1 \leq N \leq 33$ , (3) is an equality and  $L(N) = f(N) = \lfloor N/2 \rfloor + \lfloor N/3 \rfloor + 1$ .

However for larger  $N$  the situation becomes more complicated. For  $N$  large enough there is no optimal partition with all paths but one being singletons. This can be deduced from (2) and the fact that the constant  $c$  in (1) is less than 1. Still, it is natural to wonder if the equality  $L(N) = f(N)$  holds for any  $N \geq 1$ .

We were unable to answer this question, but we looked for lower bounds on  $L(N)$  and proved the following:

**Theorem.** *There is a constant  $A \geq 0$  such that for all  $N \geq 3$ ,*

$$L(N) \geq \frac{N}{(\log N)^A \exp \left[ \frac{(\log \log N)^2}{\log 2} \right]}.$$

To prove this we introduce a new function  $H(x)$ . For a real number  $x \geq 1$  and two distinct integers  $a, b \in [1, x]$ , let  $L_{a,b}(x)$  be the maximal length of a path having  $a$  and  $b$  as endpoints and belonging to an *optimal* partition of  $\{1, 2, \dots, [x]\}$ . If there is no such path, we set  $L_{a,b}(x) = 0$ . Then we set

$$H(x) = \min L_{r',r}(x)$$

where the min is over all couples  $(r', r)$  of *prime* numbers such that

$$\frac{x}{3} < r \leq \frac{x}{2} < r' \leq x.$$

The theorem will be an easy consequence of the following.

**Proposition.** *There is a constant  $N_0$  such that for any  $N \geq N_0$ , there is a set  $\mathcal{P}(N)$  of prime numbers in  $(3\sqrt{N \log N}, 4\sqrt{N \log N}]$ , of cardinal  $|\mathcal{P}(N)| \geq \frac{\sqrt{N}}{19(\log N)^{3/2}}$ , such that*

$$(4) \quad H(N) \geq \sum_{p \in \mathcal{P}(N)} H\left(\frac{N}{p}\right).$$

The technique used here is analogous to that of [6] in the study of the longest path. More precisely, in [6],  $f^*(N)$  denotes the maximal length of a path that uses integers in  $[\sqrt{N}, N]$ . A quantity  $h^*$  is introduced, which is to  $f^*$  what  $H$  is to  $L$  in our case. The inequality (4) is analogous to Buchstab's inequality (40) from [6]. The corresponding lower bounds led to the proof that  $f^*(N) \asymp N/\log N$  (Theorem 2 in [7]).

The analogy can be pushed further: in both the proof of (4) and of (40) in [6], we borrow a technique used by Erdős, Freud and Hegyvári who proved the following asymptotic behaviour:

$$\min_{1 \leq i \leq N-1} \max \text{lcm}(a_i, a_{i+1}) = \left( \frac{1}{4} + o(1) \right) \frac{N^2}{\log N},$$

where the min is over all permutations  $(a_1, a_2, \dots, a_N)$  of  $\{1, 2, \dots, N\}$ ; see Theorem 1 of [2]. In [2] as in [6] or in the present work, the proof goes through the construction of a sequence of integers by concatenating blocks whose largest prime factor is constant, and linking blocks together with

separating integers. In [6] as in the present work, these blocks take the form of sub-paths  $p\mathcal{C}_{N/p}$ , where the  $\mathcal{C}_{N/p}$  is a path of integers  $\leq N/p$  whose largest prime factor is  $\leq p$ .

It is worth mentioning that the article [2] of Erdős, Freud and Hegyvári is the origin of all works related to the divisor graph.

## 2 Notations

The letters  $p, q, q', r, r'$  will always denote generic prime numbers. For an integer  $m \geq 2$ ,  $P^-(m)$  denotes the smallest prime factor of  $m$ .

Let  $N \geq 1$ . A *path* of integers  $\leq N$  of length  $l$  is a  $l$ -uple  $\mathcal{C} = (a_1, a_2, \dots, a_l)$  of pairwise distinct positive integers  $\leq N$ , such that for all  $i$  with  $1 \leq i \leq l - 1$ ,  $a_i$  is either a divisor or a multiple of  $a_{i+1}$ . For convenience, we take  $\mathcal{C}$  up to global flip, *i.e.* we identify  $(a_1, \dots, a_l)$  with  $(a_l, \dots, a_1)$ . We will denote this path by  $a_1 - a_2 - \dots - a_l$  (or  $a_l - \dots - a_2 - a_1$ ). If  $b$  and  $c$  are integers such that  $b = a_i$  and  $c = a_{i\pm 1}$  for some  $i$ , we say that  $b$  and  $c$  are *neighbours* (in  $\mathcal{C}$ ).

When a partition  $\mathcal{A}(N)$  of  $\{1, 2, \dots, N\}$  is fixed, for any  $n \in \{1, 2, \dots, N\}$  we will simply denote by  $\mathcal{C}(n)$  the path that contains  $n$  in  $\mathcal{A}(N)$ .

A partition of  $\{1, 2, \dots, N\}$  into paths is said to be *optimal* if it contains  $F(N)$  paths (see the Introduction for the definition of  $F$ ).

Let  $\mathcal{C}$  be a path of integers  $\leq N$  and  $1 \leq n \leq N$ . Then  $\mathcal{C}$  is said to be  *$n$ -factorizable* if all the integers of  $\mathcal{C}$  are multiple of  $n$ . Then  $\mathcal{C}$  can be written as  $\mathcal{C} = n\mathcal{D}$  where  $\mathcal{D}$  is a path of integers  $\leq N/n$ .

For integers  $1 \leq n \leq N$  and a partition  $\mathcal{A}(N)$  of  $\{1, 2, \dots, N\}$ , we say that  $n$  is *factorizing* for  $\mathcal{A}(N)$  if every path of  $\mathcal{A}(N)$  that contains a multiple of  $n$  is  $n$ -factorizable.

## 3 Lemmas

**Lemma 1.** *Let  $N \geq 1$  and  $\mathcal{A}(N)$  be an optimal partition.*

(i) *Let  $1 \leq n \leq N$  with  $n$  factorizing for  $\mathcal{A}(N)$ . Let  $k = \lfloor N/n \rfloor$ . There are exactly  $F(k)$  paths in  $\mathcal{A}(N)$  that contain a multiple of  $n$ . They are of the form  $n\mathcal{D}_1, n\mathcal{D}_2, \dots, n\mathcal{D}_{F(k)}$  where  $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{F(k)}$  is an optimal partition of  $\{1, 2, \dots, k\}$ .*

(ii) *Let  $z > 1$  be a real number. Let  $M_z(N)$  be the set of integers  $m \leq N$*

that are not factorizing for  $\mathcal{A}(N)$  and such that

$$m > \frac{N}{z} \quad \text{and} \quad P^-(m) > z.$$

Then

$$|M_z(N)| < \frac{2N}{z}.$$

*Proof.* (i) The set of paths that contain a multiple of  $n$  is of the form  $\{n\mathcal{D}_1, n\mathcal{D}_2, \dots, n\mathcal{D}_g\}$  where  $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_g$  is a partition of the integers  $\leq k = \lfloor N/n \rfloor$ . Since  $\mathcal{A}(N)$  is optimal,  $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_g$  is optimal, hence  $g = F(k)$ .

(ii) Let  $m \in M_z(N)$ . There is a path  $\mathcal{C}_m$  in  $\mathcal{A}(N)$  that contains multiples and non-multiples of  $m$ . Hence there is an integer  $c(m)$  in  $\mathcal{C}_m$  that is not a multiple of  $m$ , and is neighbour to an integer  $b(m)$  which is a multiple of  $m$ . Then  $c(m)$  has to be a divisor of  $b(m)$ . More precisely, if  $b(m) = am$ , then  $c(m)$  can be written as  $c(m) = \tilde{a}\tilde{m}$  with  $\tilde{a}$  a divisor of  $a$  and  $\tilde{m}$  a strict divisor of  $m$ . Since  $P^-(m) > z$ ,  $c(m) < N/z$ .

Moreover, if  $m, m'$  are two distinct elements of  $M_z(N)$ , then

$$\text{lcm}(m, m') \geq \min(mP^-(m'), m'P^-(m)) > \frac{N}{z}z = N.$$

As a result the map

$$\begin{aligned} b : M_z(N) &\rightarrow \{1, 2, \dots, N\} \\ m &\mapsto b(m) \end{aligned}$$

is an injection.

Moreover, any integer  $c < N/z$  has at most two neighbours in  $\mathcal{C}(c)$ . Consequently the map

$$\begin{aligned} c : M_z(N) &\rightarrow \{1 \leq n < N/z\} \\ m &\mapsto c(m) \end{aligned}$$

is at-most-two-to-one. Thus

$$|M_z(N)| < \frac{2N}{z}.$$

□

**Lemma 2.** *There exists a constant  $N_1$  such that for any  $N \geq N_1$ , there is a set  $\tilde{\mathcal{P}}(N)$  of prime numbers in  $(3\sqrt{N \log N}, 4\sqrt{N \log N}]$  of cardinal*

$$(5) \quad |\tilde{\mathcal{P}}(N)| \geq \sqrt{\frac{N}{\log N}},$$

*such that for any prime numbers  $r, r'$  with*

$$\frac{N}{3} < r \leq \frac{N}{2} < r' \leq N,$$

*there exists an optimal partition  $\mathcal{A}(N)$  of  $\{1, 2, \dots, N\}$  that contains the paths  $r'$  and  $2r - r$  and for which all the integers in  $\tilde{\mathcal{P}}(N)$  are factorizing.*

*Proof.* Let  $N_1$  be such that for any  $N \geq N_1$ ,

$$(6) \quad \pi\left(4\sqrt{N \log N}\right) - \pi\left(3\sqrt{N \log N}\right) - \frac{2}{3}\sqrt{\frac{N}{\log N}} \geq \sqrt{\frac{N}{\log N}},$$

$$(7) \quad \pi\left(\frac{N}{2}\right) - \pi\left(\frac{N}{3}\right) \geq 8.$$

The existence of such a  $N_1$  comes from the prime number theorem (more precisely the left-hand-side of (6) is equivalent to  $\frac{4}{3}\sqrt{\frac{N}{\log N}}$ ). We also take  $N_1$  large enough so that

$$(8) \quad \left(3\sqrt{N \log N}, 4\sqrt{N \log N}\right] \cap \left(\frac{N}{3}, \frac{N}{2}\right] = \emptyset.$$

Let  $N \geq N_1$ . We start by fixing an optimal partition  $\mathcal{A}'(N)$ . We apply Lemma 1 (ii) to  $\mathcal{A}'(N)$  with  $z = 3\sqrt{N \log N}$ . All the prime numbers  $p$  in  $(3\sqrt{N \log N}, 4\sqrt{N \log N}]$  that are not factorizing are in  $M_z(N)$ , since they satisfy  $p > 3\sqrt{N \log N} \geq \frac{N}{z}$  and  $P^-(p) = p > z$ , so there are at most  $\frac{2}{3}\sqrt{\frac{N}{\log N}}$  of them. By removing these and using (6), we get a set  $\tilde{\mathcal{P}}(N)$  of prime numbers in  $(3\sqrt{N \log N}, 4\sqrt{N \log N}]$  that are factorizing in  $\mathcal{A}'(N)$ , with cardinality

$$|\tilde{\mathcal{P}}(N)| \geq \sqrt{\frac{N}{\log N}}.$$

We now change notations slightly and fix two prime numbers  $r_0, r'_0$  such that

$$\frac{N}{3} < r_0 \leq \frac{N}{2} < r'_0 \leq N.$$

Our goal is to go from  $\mathcal{A}'(N)$  to a new optimal partition  $\mathcal{A}(N)$  that contains the paths  $r'_0$  and  $2r_0 - r_0$  while maintaining the fact that the elements of  $\tilde{\mathcal{P}}(N)$  are factorizing.

Let us denote the set of prime numbers

$$\mathcal{R} = \left\{ \frac{N}{3} < r \leq \frac{N}{2} \right\},$$

and  $\mathcal{R}^*(\mathcal{A}'(N))$  the subset of  $r \in \mathcal{R}$  such that  $r$  does not have 1 as a neighbour in  $\mathcal{C}(r)$  and  $2r$  does not have 1 nor 2 as a neighbour in  $\mathcal{C}(2r)$ . Then for any  $r \in \mathcal{R}^*(\mathcal{A}'(N))$ , since the only possible neighbour of  $r$  is  $2r$  and reciprocally, by optimality the path  $\mathcal{C}(r)$  is equal to  $r - 2r$ . Moreover, since 1 and 2 have at most two neighbours,

$$(9) \quad |\mathcal{R} \setminus \mathcal{R}^*(\mathcal{A}'(N))| \leq 4.$$

Now we make it so that  $r'_0$  is a path. If it is not the case, since the only possible neighbour of  $r'_0$  is 1,  $\mathcal{C}(r'_0)$  is of the form  $\mathcal{D} - r'_0$  with  $\mathcal{D}$  a path ending in 1. We split this path into  $\mathcal{D}$  on one side and  $r'_0$  on the other side. By (9) and (7), there is at least one element  $r^* \in \mathcal{R}^*(\mathcal{A}'(N))$ . We stick  $\mathcal{D}$  to  $\mathcal{C}(r^*)$ , thus forming the path  $\mathcal{D} - \mathcal{C}(r^*)$ . This is possible because  $\mathcal{D}$  ends in 1. Let  $\mathcal{A}''(N)$  be this new partition. The total number of paths has not changed so  $\mathcal{A}''(N)$  is still optimal, furthermore it contains the path  $r'_0$ , and the elements of  $\tilde{\mathcal{P}}(N)$  are still factorizing because the integers in the paths that changed were not multiples of any  $p \in \tilde{\mathcal{P}}(N)$ .

The subset  $\mathcal{R}^*(\mathcal{A}''(N))$  might differ from  $\mathcal{R}^*(\mathcal{A}'(N))$  by one element, but it still satisfies (9) and its elements  $r$  still satisfy that  $\mathcal{C}(r)$  is equal to  $r - 2r$ . If  $r_0 \in \mathcal{R}^*(\mathcal{A}''(N))$ , we can set  $\mathcal{A}(N) = \mathcal{A}''(N)$  and the proof is over. We now suppose that  $r_0 \notin \mathcal{R}^*(\mathcal{A}''(N))$ .

By (9) and (7), there are at least four elements  $r_1, r_2, r_3, r_4$  in  $\mathcal{R}^*(\mathcal{A}''(N))$ . We cut the path  $\mathcal{C}(1)$  into one, two or three paths, one of them being the singleton 1 (we will see later that we get in fact three paths). Such a move will be called an *extraction* of the integer 1. We similarly *extract* the integer 2. We now use these integers 1 and 2 to stick together the paths  $r_i - 2r_i$  by forming

$$r_1 - 2r_1 - 1 - 2r_2 - r_2 \quad \text{and} \quad r_3 - 2r_3 - 2 - 2r_4 - r_4.$$

We thus get a new partition  $\mathcal{A}(N)$ . Its number of paths is less or equal to that of  $\mathcal{A}''(N)$ , so it is still optimal (this shows in particular that 1 and 2 were not endpoints of their paths). It also satisfies  $r_0 \in \mathcal{R}^*(\mathcal{A}(N))$  since 1



and 2 are not linked to  $r_0$  nor  $2r_0$ , so that it contains the path  $r_0 - 2r_0$ , as well as  $r'_0$ , and the elements of  $\tilde{\mathcal{P}}(N)$  are still factorizing.  $\square$

## 4 Proof of the Proposition

Let  $N_1$  be the constant of Lemma 2. We fix a  $N_0$  such that

$$(10) \quad N_0 \geq N_1^4$$

and such that for all  $N \geq N_0$ ,

$$(11) \quad \begin{aligned} \frac{1}{2} \sqrt{\frac{N}{\log N}} &\geq \pi \left( \frac{1}{4} \sqrt{\frac{N}{\log N}} \right) - \pi \left( \frac{1}{6} \sqrt{\frac{N}{\log N}} \right) \\ &\geq \pi \left( \frac{1}{8} \sqrt{\frac{N}{\log N}} \right) - \pi \left( \frac{1}{9} \sqrt{\frac{N}{\log N}} \right) \\ &\geq \left\lfloor \frac{\sqrt{N}}{37(\log N)^{3/2}} \right\rfloor \geq \frac{\sqrt{N}}{38(\log N)^{3/2}} + \frac{1}{2} \geq 5 \end{aligned}$$

and

$$(12) \quad 4\sqrt{\log N} \leq N^{1/4}.$$

The existence of such a  $N_0$  is again an easy consequence of the prime number theorem. Also note that since  $N_0 \geq N_1$ , (8) still holds.

Let  $N \geq N_0$ . We chose a set  $\tilde{\mathcal{P}}(N)$  according to Lemma 2. Let us denote

$$I = \left\lfloor \frac{1}{37} \frac{\sqrt{N}}{(\log N)^{3/2}} \right\rfloor.$$

By (5) and (11) we can chose  $2I$  elements in  $\tilde{\mathcal{P}}(N)$ , which we denote as

$$p_1, p_2, \dots, p_{2I}.$$

We set  $\mathcal{P}(N) = \{p_1, \dots, p_{2I-1}\}$ . By (11) again,  $|\mathcal{P}(N)| \geq \frac{\sqrt{N}}{19(\log N)^{3/2}}$ . It remains to prove that this set  $\mathcal{P}(N)$  satisfies (4).

Let  $r, r'$  be two prime numbers such that

$$(13) \quad \frac{N}{3} < r \leq \frac{N}{2} < r' \leq N.$$

By the property of  $\tilde{\mathcal{P}}(N)$  in Lemma 2, there exists an optimal partition  $\mathcal{A}'(N)$ , that contains the paths  $r'$  and  $2r - r$ , for which the elements of  $\tilde{\mathcal{P}}(N)$  (and in particular the elements of  $\mathcal{P}(N)$ ) are factorizing.

We denote two sets of prime numbers

$$\mathcal{Q}(N) = \left\{ \frac{1}{9} \sqrt{\frac{N}{\log N}} < q \leq \frac{1}{8} \sqrt{\frac{N}{\log N}} \right\},$$

$$\mathcal{Q}'(N) = \left\{ \frac{1}{6} \sqrt{\frac{N}{\log N}} < q' \leq \frac{1}{4} \sqrt{\frac{N}{\log N}} \right\}.$$

For all  $(p, q, q') \in \tilde{\mathcal{P}}(N) \times \mathcal{Q}(N) \times \mathcal{Q}'(N)$  we have

$$(14) \quad \frac{N}{3} < pq \leq \frac{N}{2},$$

$$(15) \quad \frac{N}{2} < pq' \leq N.$$

We focus on the factorizing prime number  $p_{2I}$ . For any  $q \in \mathcal{Q}(N)$ , because of (14) the only possible neighbours of  $p_{2I}q$  are  $p_{2I}$  and  $2p_{2I}q$ . Similarly, the only possible neighbours of  $2p_{2I}q$  are  $p_{2I}$ ,  $2p_{2I}$  or  $p_{2I}q$ . But  $p_{2I}$  and  $2p_{2I}$  can be linked to at most 4 elements of type  $p_{2I}q$  or  $2p_{2I}q$ . By (11) we know that  $|\mathcal{Q}(N)| \geq 5$ , so there exists a  $q_{2I} \in \mathcal{Q}(N)$  for which neither  $p_{2I}q_{2I}$  nor  $2p_{2I}q_{2I}$  is a neighbour of  $p_{2I}$  or  $2p_{2I}$ . As a result, the only possible neighbour for  $p_{2I}q_{2I}$  is  $2p_{2I}q_{2I}$ , and reciprocally. By optimality,  $\mathcal{A}'(N)$  contains the path  $p_{2I}q_{2I} - 2p_{2I}q_{2I}$ .

Using (11) we can chose

- $I$  elements of  $\mathcal{Q}'(N)$  which we write as

$$(16) \quad q_1, q_3, \dots, q_{2I-1};$$

- $I - 1$  elements of  $\mathcal{Q}(N) \setminus \{q_{2I}\}$  which we write as

$$(17) \quad q_2, q_4, \dots, q_{2I-2}.$$

Let  $i$  be such that  $1 \leq i \leq 2I - 1$ . Then the prime number  $p_i$  is factorizing for  $\mathcal{A}'(N)$  so by Lemma 1 (i) the paths of  $\mathcal{A}'(N)$  that contain multiples of  $p_i$  are of the form

$$p_i \mathcal{C}_{i,1}, p_i \mathcal{C}_{i,2}, \dots, p_i \mathcal{C}_{i,F(N/p_i)}$$

where  $\mathcal{C}_{i,1}, \mathcal{C}_{i,2}, \dots, \mathcal{C}_{i,F(N/p_i)}$  is an optimal partition of  $\{1, 2, \dots, \lfloor N/p_i \rfloor\}$ . By our choice of indices (16),(17), one of the elements  $q_i, q_{i+1}$  is in  $\mathcal{Q}'(N)$ , we rename it  $\tilde{q}_i$ , and the other is in  $\mathcal{Q}(N)$ , we rename it  $\widetilde{q_{i+1}}$ . Using (14),(15) we get

$$\frac{N}{3p_i} < \widetilde{q_{i+1}} \leq \frac{N}{2p_i} < \tilde{q}_i \leq \frac{N}{p_i}.$$

Using (12) and (10), we have  $N/p_i \geq N^{1/4} \geq N_0^{1/4} \geq N_1$ . Hence we can apply Lemma 2 with  $N/p_i$  instead of  $N$ . We deduce that there exists an optimal partition of  $\{1, 2, \dots, \lfloor N/p_i \rfloor\}$  that contains the paths  $\widetilde{q}_i$  and  $\widetilde{q}_{i+1} - 2\widetilde{q}_{i+1}$ . By extracting 1 in that partition, we can stick these two paths together into  $\widetilde{q}_i - 1 - 2\widetilde{q}_{i+1} - \widetilde{q}_{i+1}$  while keeping an optimal partition. To sum up, we know now that there is an optimal partition of the integers  $\leq N/p_i$  containing a path that has  $q_i$  and  $q_{i+1}$  as endpoints.

Let  $\mathcal{D}_{i,1}, \mathcal{D}_{i,2}, \dots, \mathcal{D}_{i,F(N/p_i)}$  be an optimal partition of the integers  $\leq N/p_i$ , with  $\mathcal{D}_{i,1}$  having  $q_i, q_{i+1}$  as endpoints and of maximal length  $L_{q_i, q_{i+1}}(N/p_i)$ . We can transform  $\mathcal{A}'(N)$  by replacing the paths  $(p_i \mathcal{C}_{i,j})_{1 \leq j \leq F(N/p_i)}$  by  $(p_i \mathcal{D}_{i,j})_{1 \leq j \leq F(N/p_i)}$ . In this way we get a new optimal partition  $\mathcal{A}''(N)$  that contains all the paths  $p_i \mathcal{D}_{i,1}$  for  $1 \leq i \leq 2I - 1$ , as well as  $r', 2r - r$ , and  $p_{2I} q_{2I} - 2p_{2I} q_{2I}$ .

By extracting the integers 1, 2 and the  $q_i$  for  $2 \leq i \leq 2I$ , we construct the path of Figure 1 while keeping an optimal partition of  $\{1, 2, \dots, N\}$ .

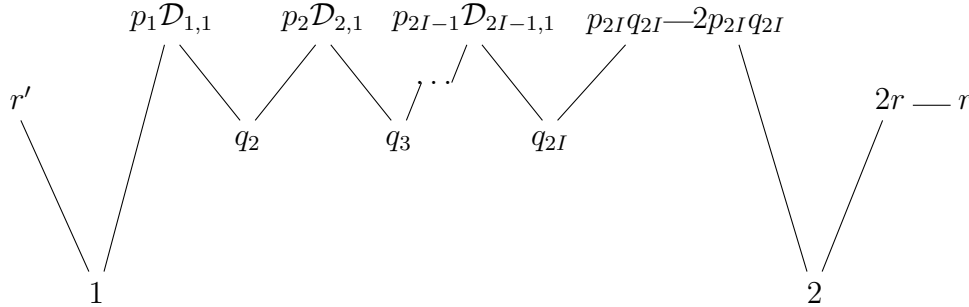


Figure 1: A long path with endpoints  $r', r$ .

Its length is larger than

$$\sum_{i=1}^{2I-1} L_{q_i, q_{i+1}}(N/p_i) \geq \sum_{p \in \mathcal{P}(N)} H(N/p_i).$$

This being true for any  $r, r'$  satisfying (13), we get

$$H(N) \geq \sum_{p \in \mathcal{P}(N)} H(N/p_i).$$

□

## 5 Proof of the Theorem

Let us fix a constant  $N_2 = 2^{2^{k_0}} \geq N_0$ , where  $N_0$  is the constant from the Proposition. We chose a constant  $B$  such that for all  $N \leq 2^{2^{k_0+2}}$ ,

$$(18) \quad N \leq 4(\log N)^B \exp \left[ \frac{(\log \log N)^2}{\log 2} \right]$$

and

$$(19) \quad B \geq 8.$$

We show by induction on  $k \geq k_0 + 2$  that for all  $N$  such that

$$2^{2^{k_0}} < N \leq 2^{2^k},$$

we have

$$(20) \quad H(N) \geq \frac{N}{(\log N)^B \exp \left[ \frac{(\log \log N)^2}{\log 2} \right]}$$

### Base case

Let  $N$  be such that  $2^{2^{k_0}} < N \leq 2^{2^{k_0+2}}$ , then we have  $N > N_2 \geq N_0 \geq N_1^4$  (see (10)) with  $N_1$  the constant of Lemma 2. Let  $r, r'$  be two prime numbers such

$$\frac{N}{3} < r \leq \frac{N}{2} < r' \leq N.$$

Lemma 2 implies that there is an optimal partition  $\mathcal{A}(N)$  of  $\{1, 2, \dots, N\}$  which contains the paths  $r'$  and  $2r - r$ . By extracting 1, we can stick them into  $r' - 1 - 2r - r$  while keeping an optimal partition. This implies that  $H(N) \geq 4$ , and (18) yields the base case.

### Induction step

Let  $k \geq k_0 + 2$ . We suppose that (20) holds for all  $N \in (2^{2^{k_0}}, 2^{2^k}]$ .

Let  $N$  be such that  $2^{2^k} < N \leq 2^{2^{k+1}}$ . Since  $k \geq k_0 + 2$ , we also have  $N^{1/4} > 2^{2^{k_0}}$ .

Let  $p \in (3\sqrt{N \log N}, 4\sqrt{N \log N}]$ . By (12), we have

$$2^{2^{k_0}} < N^{1/4} \leq \frac{N}{p} \leq \sqrt{N} \leq 2^{2^k}.$$

By using the induction hypothesis on  $N/p$ , we get

$$\begin{aligned}
H\left(\frac{N}{p}\right) &\geq \frac{N}{p(\log(N/p))^B \exp\left[\frac{(\log \log(N/p))^2}{\log 2}\right]} \\
&\geq \frac{N}{p(\log \sqrt{N})^B \exp\left[\frac{(\log \log \sqrt{N})^2}{\log 2}\right]} \\
&= \frac{2^{B-1}(\log N)^2 N}{p(\log N)^B \exp\left[\frac{(\log \log N)^2}{\log 2}\right]}.
\end{aligned}$$

Hence by using the Proposition and (19),

$$\begin{aligned}
H(N) &\geq \sum_{p \in \mathcal{P}(N)} H\left(\frac{N}{p}\right) \\
&\geq \frac{|\mathcal{P}(N)|}{\max \mathcal{P}(N)} \frac{2^{B-1}(\log N)^2 N}{(\log N)^B \exp\left[\frac{(\log \log N)^2}{\log 2}\right]} \\
&\geq \frac{2^{B-1}}{76} \frac{N}{(\log N)^B \exp\left[\frac{(\log \log N)^2}{\log 2}\right]} \\
&\geq \frac{N}{(\log N)^B \exp\left[\frac{(\log \log N)^2}{\log 2}\right]}.
\end{aligned}$$

This concludes the induction step.

Finally, since  $L(N) \geq 1$  for all  $N \geq 1$ , we get the Theorem by choosing  $A = \max(B, A_0)$  where  $A_0$  is a constant such that for all  $3 \leq N < N_0$ ,

$$N \leq (\log N)^{A_0} \exp\left[\frac{(\log \log N)^2}{\log 2}\right].$$

□

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