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Abstract

We bound the number of minimal hypergraph transversals that arise in tripartite 3-uniform hypergraphs, a class commonly found in applications dealing with data. Let $\mathcal{H}$ be such a hypergraph on a set of vertices V. We give a lower bound of $1.4977^{|V|}$ and an upper bound of $1.5012^{|V|}$.

1. Introduction

Hypergraphs are a generalization of graphs, where edges may have arity different than 2. They have been formalized by Berge in the seventies [1]. Formally, a *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$, where V is a set of vertices and $\mathcal{E}$ is a family of subsets of V called hyperedges. We suppose that $V = \bigcup \mathcal{E}$. From now on, we use edge in place of hyperedge. We may use the notations $V(\mathcal{H})$ to denote the set of vertices of a hypergraph $\mathcal{H}$. The order of a hypergraph is its number of vertices. When all the hyperedges of a hypergraph have the same arity k, we call it a *k-uniform hypergraph*. Graphs are exactly the 2-uniform hypergraphs. When the set of vertices of a hypergraph can be partitioned into three sets, such that every edge intersect each part at most once, the hypergraph is called *tripartite*.

A subset of the vertices of a hypergraph $\mathcal{H}$ is a *transversal* of $\mathcal{H}$ if it intersects every edge of $\mathcal{H}$. A transversal of $\mathcal{H}$ is said *minimal* if none of its strict subsets is a transversal.

Uniform hypergraphs are commonly found in applications dealing with data. For example, bipartite graphs are a known representation of data tables. Maximal bi-cliques in such bipartite graphs are of interest in data mining, and are

widely studied under the name of closed sets or concepts in the Formal Concept Analysis community [2]. Maximal bi-cliques are equivalent to minimal transversals in the bipartite complementary graph. Let n be the number of vertices of a bipartite graph. It is known that the maximum number of minimal transversals in such graphs is bounded by $2^{n/2} \approx 1.414^n$ (and this bound is reached when the bipartite graph is a perfect matching).

We are interested in the number of minimal transversals that arise in tripartite 3-uniform hypergraphs. We denote by $f_3(n)$ the maximum number of minimal transversals in such hypergraphs of order n. The construction of the matching for bipartite graphs can be taken to tripartite 3-uniform hypergraphs on n vertices, and gives $3^{n/3} \approx 1.4422^n$ minimal transversals.

Moreover, by fixing the vertices taken in two of the three sets of vertices, there is only one way to pick the remaining vertices. A naive upper bound is then found by considering that all subsets from two of the three sets of vertices can appear in minimal transversals, which gives a bound of $2^{n/3} \times 2^{n/3} = 4^{n/3} \approx 1.5874^n$ minimal transversals.

Thus, for any integer n, $1.4422^n \leq f_3(n) \leq 1.5874^n$. In this paper, we improve those bounds through two theorems.

Theorem 1. *There exists a constant c such that for any integer n,*

$$f_3(n) \geq c1.4977^n.$$

Theorem 2. *For any integer n,*

$$f_3(n) \leq 1.5012^n.$$

The proof of Theorem 2 relies on a technique introduced by Kullman [3] and used on similar objects by Lonc and Truszczyński [4]. It resembles the approach used in the analysis of exact exponential-time algorithms, *measure and conquer*, see for example by Fomin, Grandoni and Kratsch [5].

In [4], the authors consider the class of hypergraphs of rank 3 (hypergraphs in which the edges have arity at most 3). This class contains the tripartite 3-uniform hypergraphs, but the general bound they obtained (1.6702^n) is larger than the trivial one in our specific case.

2. Lower Bound

We consider tripartite 3-uniform hypergraphs. In this section, we want to improve the lower bound of 1.4422^n for $f_3(n)$. We exhibit a construction that allows to reach $c1.4977^n$. To this end, we first state an easy observation that allows us to multiply the number of minimal transversals while only summing the order of hypergraphs.

Observation 3. *Let k be an integer, and for each i between 1 and k, let $\mathcal{H}_i$ be a hypergraph of order n_i with t_i minimal transversals. Then the hypergraph $\mathcal{H}$ obtained by disjoint union of the k hypergraphs is of order $\sum_{i=1}^{k} n_i$ and has $\prod_{i=1}^{k} t_i$ minimal transversals. Moreover, if for every i, $\mathcal{H}_i$ is tripartite and 3-uniform, then $\mathcal{H}$ is tripartite and 3-uniform.*

Proof. It is sufficient to look at each connected component of n_i vertices. The minimal transversals of the whole hypergraph are exactly the sets of vertices resulting from the union of one minimal transversal from each connected component, hence the number $\prod_{i=1}^{k} t_i$. Clearly, the disjoint union of the $\mathcal{H}_i$, for all i between 1 and k is still a tripartite 3-uniform hypergraph. $\square$

A computer search in small instances allows us to make the following observation.

Claim 4. *There is a tripartite 3-uniform hypergraph on fifteen vertices with four hundred and twenty-eight minimal transversals:*

$$f_3(15) \geq 428.$$

The hypergraph mentioned in Claim 4 is described in figures 1 and 2. We denote this hypergraph by $\mathcal{H}_{15}$.

$$
\begin{array}{lllll}
\{\alpha 1a\} & \{\beta 1b\} & \{\gamma 1c\} & \{\delta 1d\} & \{\epsilon 1e\} \\
\{\alpha 2b\} & \{\beta 2c\} & \{\gamma 2d\} & \{\delta 2e\} & \{\epsilon 2a\} \\
\{\alpha 3c\} & \{\beta 3d\} & \{\gamma 3e\} & \{\delta 3a\} & \{\epsilon 3b\} \\
\{\alpha 4d\} & \{\beta 4e\} & \{\gamma 4a\} & \{\delta 4b\} & \{\epsilon 4c\} \\
\{\alpha 4e\} & \{\beta 5a\} & \{\gamma 5b\} & \{\delta 5e\} & \{\epsilon 5d\}
\end{array}
$$

Figure 1: This hypergraph has fifteen vertices that are partitioned in three sets ($\{1, 2, 3, 4, 5\}, \{a, b, c, d, e\}, \{\alpha, \beta, \gamma, \delta, \epsilon\}$). It has four hundred and twenty-eight minimal transversals.

Theorem 1. *There exists a constant c such that for any integer n,*

$$f_3(n) \geq c1.4977^n.$$

Proof. Let $c = (\frac{1.4422}{1.4977})^{12}$. We observe that $428 > 1.4977^{15}$. Now fix some integer n that is a multiple of 3. There are two integers k and r such that $n = 3(5k + r)$ with r between 0 and 4. By making k disjoint copies of $\mathcal{H}_{15}$ and then adding r disjoint edges of arity 3, we obtain a hypergraph with $428^k \times 3^r$ minimal transversals. Since $428^k \geq 1.4977^{15k}$ and $3^r \geq 1.4422^{3r}$, the number of minimal transversals is more than $1.4977^{n-3r} \times 3^{3r}$. Since r is between 0 and 4, $(\frac{1.4422}{1.4977})^{3r}$ is at least c. $\square$

We now describe the hypergraph mentioned in Claim 4. The hypergraph in Fig. 1 has fifteen vertices and four hundred and twenty-eight minimal transversals. Each set is a 3-edge. The partitions are $\{1,2,3,4,5\}$, $\{a,b,c,d,e\}$ and $\{\alpha,\beta,\gamma,\delta,\epsilon\}$.

Another way to see the hypergraph $\mathcal{H}_{15}$ is with a 3-dimensional cross table, where a cross in cell $(\alpha,1,a)$ represents the edge $\{\alpha,1,a\}$. Note that the crosses representing the 3-edges of this particular hypergraph are a solution to the 3-dimensional chess rook problem. This remark is probably devoid of any profound meaning.

	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e
1	×						×						×						×						×
2		×						×						×						×	×				
3			×						×						×	×							×		
4				×						×	×							×						×	
5					×	×							×						×						×
			α					β					γ					δ					ϵ		

Figure 2: Another way to display $\mathcal{H}_{15}$. Each cross represents a 3-edge.

3. Upper Bound

In this section, we prove Theorem 2 relying on a technical lemma (Lemma 10).

We define a rooted tree by forcing the presence or absence of some vertices in a minimal transversal. In this tree, each internal node is a hypergraph from our family, and each leaf corresponds to a minimal transversal. The next step is to count the number of leaves in this tree. We do that by using the so-called "τ-lemma" introduced by Kullmann [3, Lemma 8.2]. We associate a measure to each hypergraph of our class. Then, using this measure, we label the edges of our tree with a carefully chosen distance between the parent (hypergraph) and the child (hypergraph).

The estimation of the number of leaves is then done by computing the maximal τ, that depends on the measure for each hypergraph (for each inner node of the tree). This constitutes the basis of the exponential. The exponent is the maximal distance from the root of the tree to its leaves.

Let us dive into the more technical part. We mainly use the notion of *condition*, introduced thereafter.

Definition 5. *Given a set V of vertices, a* condition *on V is a pair of disjoint sets of vertices (A^+, A^-). A condition is* trivial *if $A^+ \cup A^- = \emptyset$, and* non-trivial *otherwise.*

All the conditions that we handle are non-trivial. A condition amounts to fixing a set of vertices to be part of the solution (the vertices in A^+) and forbidding other vertices (the vertices in A^-).

A set T of vertices *satisfies* a condition (A^+, A^-) if $A^+ \subseteq T$ and $T \cap A^- = \emptyset$.

Let $\mathcal{H}$ be a hypergraph and (A^+, A^-) a condition. The hypergraph $\mathcal{H}_{(A^+, A^-)}$ is obtained from $\mathcal{H}$ and (A^+, A^-) using the following procedure :

1. remove every edge that contains a vertex that is in A^+ (as we take the vertices of A^+, this edge is now covered);
2. remove from every remaining edge the vertices that are in A^- (we prohibit the vertices from A^-);
3. remove redundant edges (as they have the same transversals, we do not keep the duplicates).

Vertices of $\mathcal{H}$ that appear in a condition (A^+, A^-) are not in $V(\mathcal{H}_{(A^+, A^-)})$ as they are either removed from all the edges or all the edges that contained them have disappeared.

Lemma 6. *Let $\mathcal{H}$ be a hypergraph, (A^+, A^-) a condition and T a set of vertices of $\mathcal{H}$. If T is a minimal transversal of $\mathcal{H}$ and T satisfies (A^+, A^-), then $T \setminus A^+$ is a minimal transversal of $\mathcal{H}_{(A^+, A^-)}$.*

Proof. The proof is straightforward from the construction of $\mathcal{H}_{(A^+, A^-)}$. □

This echoes the parent-child relation that can be found in the exact exponential-time algorithm field [6].

A family of conditions is *complete* for the hypergraph $\mathcal{H}$ if the family is non-empty, each condition in the family is non-trivial, and every transversal of $\mathcal{H}$ satisfies at least one condition of the family.

Let $\mathcal{C}$ be the class of tripartite hypergraphs in which edges have arity at most 3 and one of the parts is a minimal transversal. We call S the part that is a minimal transversal. Tripartite 3-uniform hypergraphs belong to this class. Tripartite hypergraphs remain tripartite when vertices are removed from edges or edges are deleted. Of course, those operations do not increase the arity of edges. As such, if an hypergraph $\mathcal{H} = (V, \mathcal{E})$ belongs to the class $\mathcal{C}$ and $A = (A^+, A^-)$ is a condition on V such that the edges that contain vertices of $A^- \cap S$ also contain vertices of A^+, then $\mathcal{H}_A$ is in $\mathcal{C}$. From now on, we suppose that all the conditions we discuss respect this property.

A hypergraph is non-trivial if it is not empty. A *descendant function* for $\mathcal{C}$ is a function that assigns to each non-trivial hypergraph in $\mathcal{C}$ a complete family of conditions. Let ρ be such a function.

We define a rooted labeled tree $\mathcal{T}_{\mathcal{H}}$ for all hypergraphs $\mathcal{H}$ in $\mathcal{C}$. If $\mathcal{H} = \emptyset$, then we define $\mathcal{T}_{\mathcal{H}}$ to be a single node labeled with $\mathcal{H}$. When $\mathcal{H}$ is non-trivial, we create a node labeled $\mathcal{H}$ and make it the parent of all trees $\mathcal{T}_{\mathcal{H}_A}$ for all $A \in \rho(\mathcal{H})$. Since $\mathcal{C}$ is closed under the operation of removing edges and removing vertices from edges and since the number of vertices can only decrease when the transformation from $\mathcal{H}$ to $\mathcal{H}_A$ occurs, the inductive definition is valid.

Proposition 7. *Let ρ be a descendant function for a class closed by removing edges and removing vertices from edges. Then for all hypergraphs $\mathcal{H}$ in such a class, the number of minimal transversals is bounded above by the number of leaves of $\mathcal{T}_{\mathcal{H}}$.*

Proof. When $\mathcal{H} = \emptyset$, then it has only one transversal, the empty set. If the empty set is an edge of $\mathcal{H}$, then $\mathcal{H}$ has no transversals. In both cases, the assertion follows directly from the definition of the tree. Let us assume now that $\mathcal{H}$ is non-trivial, and that the assertion is true for all hypergraphs with fewer vertices than $\mathcal{H}$.

As $\mathcal{H}$ is non-trivial, ρ is well defined for $\mathcal{H}$ and gives a complete family of conditions. Let X be a minimal transversal of $\mathcal{H}$. Then X satisfies at least one condition A in $\rho(\mathcal{H})$. From Lemma 6, we know that there is a minimal transversal Y of $\mathcal{H}_A$ such that $Y \cup A^+ = X$. Then the number of minimal transversals of $\mathcal{H}$ is at most the sum of the number of minimal transversals in its children. $\qquad\square$

We now want to bound the number of leaves in $\mathcal{T}_{\mathcal{H}}$ for all hypergraphs in $\mathcal{C}$. To that end, we shall use Lemma 8 proved by Kullmann [3].

We denote by $L(\mathcal{T})$ the set of leaves of $\mathcal{T}$ and for a leaf ℓ, we denote by $P(\ell)$ the set of edges on the path from the root to leaf ℓ.

Lemma 8 ([3, Lemma 8.1]). *Consider rooted a tree $\mathcal{T}$ with an edge labeling w with value in the interval $[0, 1]$ such that for every internal node, the sum of the labels on the edges from that node to its children is 1 (that is a transition probability).*
Then,

$$|L(\mathcal{T})| \leq \max_{\ell \in L(\mathcal{T})} \left(\prod_{e \in P(\ell)} w(e) \right)^{-1}.$$

In order to pick an adequate probability distribution, we use a measure. A *measure* is a function that assigns to any hypergraph $\mathcal{H}$ in $\mathcal{C}$ a real number $\mu(\mathcal{H})$ such that $0 \leq \mu(\mathcal{H}) \leq |V(\mathcal{H})|$. Let $\mathcal{H}$ be such a hypergraph, A a condition on its vertices and μ a measure. We define

$$\Delta(\mathcal{H}, \mathcal{H}_A) = \mu(\mathcal{H}) - \mu(\mathcal{H}_A).$$

Let $\mathcal{H}$ be a hypergraph in $\mathcal{C}$ and μ a measure. If, for every condition A in $\rho(\mathcal{H})$, $\mu(\mathcal{H}_A) \leq \mu(\mathcal{H})$, that we say then ρ is μ-compatible. In this case, there is a unique positive real number $\tau \geq 1$ such that

$$\sum_{A \in \rho(\mathcal{H})} \tau^{-\Delta(\mathcal{H}, \mathcal{H}_A)} = 1. \tag{1}$$

When $\tau \geq 1$, $\sum_{A \in \rho(\mathcal{H})} \tau^{-\Delta(\mathcal{H}, \mathcal{H}_A)}$ is a strictly decreasing continuous function of τ. For $\tau = 1$, it is at least 1, since $\rho(\mathcal{H})$ is not empty, and it tends to 0 when τ tends to infinity.

A descendant function defined on a class $\mathcal{C}$ is μ-bounded by τ_0 if, for every non-trivial hypergraph $\mathcal{H}$ in $\mathcal{C}$, $\tau_{\mathcal{H}} \leq \tau_0$.

Now, we adapt the τ-lemma proven by Kullmann [3] to our formalism.

Theorem 9 (Kullmann [3])**.** *Let μ be a measure and ρ a descendant function, both defined on a class $\mathcal{C}$ of hypergraphs closed under the operation of removing edges and removing vertices from edges. If ρ is μ-compatible and μ-bounded by τ_0, then for every hypergraph $\mathcal{H}$ in $\mathcal{C}$,*

$$|L(\mathcal{T}_{\mathcal{H}})| \leq \tau_0^{h(\mathcal{T}_{\mathcal{H}})}$$

where $h(\mathcal{T}_{\mathcal{H}})$ is the height of $\mathcal{T}_{\mathcal{H}}$.

This theorem comes from Lemma 8. We find the worst possible branching and assume that we apply it on every node on the path from the root to the leaves.

Lemma 10. *There is a measure μ defined for every hypergraph $\mathcal{H}$ in $\mathcal{C}$ and a descendant function ρ for $\mathcal{C}$ that is μ compatible and μ-bounded by 1.8393.*

Proof. The general idea of this proof is that we give a complete family of conditions for every non-trivial hypergraph in $\mathcal{C}$.

Let $\mathcal{H}$ be a hypergraph belonging to the class $\mathcal{C}$, i.e. a tripartite hypergraph that contains a set S of vertices that is a minimal transversal such that no two vertices of S belong to a same edge.

We use Theorem 9 to bound the number of leaves in the tree $\mathcal{T}_{\mathcal{H}}$ and thus the number of minimal transversals in $\mathcal{H}$. To do so, we define a descendant function ρ that assigns a family of conditions to $\mathcal{H}$ depending on its structure. This takes the form of a case analysis.

In each case, we consider a vertex a from S and its surroundings. The conditions involve vertices from these surroundings and are sometimes strengthened to contain a if and only if its presence in A^+ or A^- is implied by our hypotheses. This causes every condition A to respect the property that A^+ intersects all the edges containing a vertex of $A^- \cap S$, which lets $\mathcal{H}_A$ remain in the class $\mathcal{C}$.

We set the measure $\mu(\mathcal{H})$ to

$$\mu(\mathcal{H}) = |V(\mathcal{H})| - \alpha m(\mathcal{H})$$

where $m(\mathcal{H})$ is the maximum number of pairwise disjoint 2-element edges in $\mathcal{H}$ and $\alpha = 0.145785$. The same measure is used in [4] (with a different α).

In each case i and for every condition A, we find a bound $k_{\mathcal{H}, A}$ such that

$$k_{\mathcal{H},A} \leq \Delta(\mathcal{H}, \mathcal{H}_A)$$

and a unique positive real number τ_i that satisfies the equation

$$\sum_{A \in \rho(\mathcal{H})} \tau_i^{-k_{\mathcal{H},A}} = 1$$

We show that $\tau_i \leq 1.8393$ for all i. Let τ_0 represent the quantity 1.8393.

As all our conditions involve at least one element from $V(\mathcal{H}) \setminus S$, the height of $\mathcal{T}_{\mathcal{H}}$ is bounded by $|V(\mathcal{H})| - |S|$. Hence, we have

$$|L(\mathcal{T}_{\mathcal{H}})| \leq \tau_0^{|V(\mathcal{H})|-|S|}$$

In the remainder of the proof, we will write conditions as sets of expressions of the form a and $\overline{b}$ where a means that a is in A^+ and $\overline{b}$ means that b is in A^-. For example, the condition $(\{a, c\}, \{b, d, e\})$ will be denoted $ac\overline{bde}$. For a vertex v, we denote by $d_2(v)$ the number of 2-edges that contain v, and by $d_3(v)$ the number of 3-edges that contain v.

Case 1: $d_2(a) \geq 2$ and $d_3(a) = 0$: the hypergraph $\mathcal{H}$ contains a vertex a from S that belongs to no 3-edge and at least two 2-edges ab and ac.

A minimal transversal of $\mathcal{H}$ either contains or does not contain b, and as such $\{b, \overline{b}\}$ is a complete family of conditions for $\mathcal{H}$. We can strengthen the conditions to obtain $\{bc, b\overline{c}, \overline{b}\}$. Minimal transversals of $\mathcal{H}$ that do not contain b or c necessarily contain a (as ab or ac would not be covered otherwise). Hence $\{bc, ab\overline{c}, a\overline{b}\}$ is a complete family of conditions for $\mathcal{H}$.

Let M be a maximum set of pairwise distinct 2-edges (matching) of $\mathcal{H}$. By removing k vertices we decrease the size of a maximum matching by at most k. Thus we have

$$\Delta(\mathcal{H}, \mathcal{H}_A) \geq \begin{cases} 2 - 2\alpha & \text{for } A \in \{\{bc\}, \{a\overline{b}\}\} \\ 3 - 3\alpha & \text{for } A \in \{\{ab\overline{c}\}\} \end{cases}. \tag{2}$$

Equation (1) becomes $2\tau_1^{2\alpha-2} + \tau_1^{3\alpha-3} = 1$. For our chosen α, we have that $\tau_1 \leq \tau_0$.

Case 2: $d_2(a) = 1$: the hypergraph $\mathcal{H}$ contains a vertex a from S that belongs to a unique 2-edge ab. We break down this case into two sub-cases depending on whether or not a belongs to some 3-edges: $d_3(a) = 0$ and $d_3(a) \geq 1$.

Since a is in a unique 2-edge, when one removes a and b, the size of a maximum matching decreases at most by 1.

- $d_3(a) = 0$: a is in a single 2-edge ab and no 3-edges. A minimal transversal of $\mathcal{H}$ either contains or does not contain b. As such, $\{b, \overline{b}\}$ is a complete family of conditions for $\mathcal{H}$. As ab is the only edge containing a, a minimal

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transversal of $\mathcal{H}$ that contains b cannot contain a. Similarly, every minimal transversal of $\mathcal{H}$ that does not contain b necessarily contains a. This makes $\{b\overline{a}, a\overline{b}\}$ a complete family of conditions for $\mathcal{H}$.

Let M be a maximum set of pairwise disjoint 2-edges of $\mathcal{H}$. As ab is the only edge containing a, b belongs to one of the edges in M. The hypergraphs $\mathcal{H}_{b\overline{a}}$ and $\mathcal{H}_{a\overline{b}}$ contain all the edges in M except for the one containing b. Thus, $m(\mathcal{H}_{b\overline{a}}) = m(\mathcal{H}_{a\overline{b}}) \geq m(\mathcal{H}) - 1$. Since $|V(\mathcal{H}_{b\overline{a}})| = |V(\mathcal{H}_{a\overline{b}})| \leq |V(\mathcal{H})| - 2$, we have

$$\Delta(\mathcal{H}, \mathcal{H}_A) \geq 2 - \alpha \text{ for } A \in \{\{b\overline{a}\}, \{a\overline{b}\}\}. \tag{3}$$

Equation (1) becomes $2\tau_{2.1}^{\alpha-2} = 1$. For our chosen α, we have that $\tau_{2.1} \leq \tau_0$.

- $d_3(a) \geq 1$: a is in a single 2-edge and in some 3-edges, one of which being acd. We start with the conditions $\{bc, bd\overline{c}, \overline{bcd}, \overline{b}\}$. Any minimal transversal of $\mathcal{H}$ that does not contain either b or both c and d necessarily contains a. This makes $\{bc, bd\overline{c}, ab\overline{c}d, a\overline{b}\}$ a complete family of conditions for $\mathcal{H}$. We obtain

$$\Delta(\mathcal{H}, \mathcal{H}_A) \geq \begin{cases} 2 - 2\alpha & \text{if } A = \{bc\} \\ 3 - 3\alpha & \text{if } A = \{bd\overline{c}\} \\ 4 - 3\alpha & \text{if } A = \{ab\overline{c}d\} \\ 2 - \alpha & \text{if } A = \{a\overline{b}\} \end{cases}. \tag{4}$$

Equation (1) becomes $\tau_{2.2}^{2\alpha-2} + \tau_{2.2}^{3\alpha-3} + \tau_{2.2}^{3\alpha-4} + \tau_{2.2}^{\alpha-2} = 1$. For our chosen α, we have that $\tau_{2.2} \leq \tau_0$.

Case 3: $d_2(a) = 0$ and $d_3(a) \geq 1$: the hypergraph $\mathcal{H}$ contains a vertex a from S that is in no 2-edge and in some 3-edges, one of which being abc. We start with the conditions $\{b, c\overline{b}, \overline{bc}\}$ and strengthen them to $\{b, c\overline{b}, a\overline{bc}\}$. Since we do not have any 2-edge anymore, we cannot decrease the size of a maximum matching. We obtain

$$\Delta(\mathcal{H}, \mathcal{H}_A) \geq \begin{cases} 1 & \text{if } A = \{b\} \\ 2 & \text{if } A = \{c\overline{b}\} \\ 3 & \text{if } A = \{a\overline{bc}\} \end{cases}. \tag{5}$$

Equation (1) becomes $\tau_3^{-1} + \tau_3^{-2} + \tau_3^{-3} = 1$. For our chosen α, we have that $\tau_3 \leq \tau_0$. $\square$

This proof ensures that there is a measure μ and a descendant function ρ for our class of hypergraphs such that ρ is μ-bounded by 1.8393. This allows us to formulate the following theorem.

Theorem 11. *The number of minimal transversals in an hypergraph belonging to the class $\mathcal{C}$ is less than $1.8393^{n-|S|}$.*

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Proof. Let μ and ρ be the measure and descendant function mentioned in Lemma 10. The height $h(\mathcal{T}_{\mathcal{H}})$ of the tree is bounded by $n - |S|$ so a straightforward application of Theorem 9 yields the result. $\qquad\square$

Theorem 2 is a straightforward corollary of Theorem 11.

Theorem 2. *For any integer n,*

$$f_3(n) \leq 1.8393^{2n/3} \leq 1.5012^n.$$

Proof. The vertices of a tripartite 3-uniform hypergraph can be partitioned into three minimal transversals so any of them can be S. The minimization of the bound is achieved by using the biggest set which, in the worst case, has size n/3. $\qquad\square$

We now state a few problems that we find of interest.

Problem 1. *Find a nice[1] extremal family (in the same feeling of perfect matchings) that is better than $3^{n/3}$.*

Problem 2. *Tighten the bound through a better choice of measure or branching.*

Problem 3. *Generalize the result to k-partite k-uniform hypergraphs.*

Acknowledgments

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