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MEAN VALUES FOR A CLASS OF ARITHMETIC FUNCTIONS IN SHORT INTERVALS

JIE WU AND QIANG WU

ABSTRACT. In this paper, we shall establish a rather general asymptotic formula in short intervals for a class of arithmetic functions and announce two applications about the distribution of divisors of square-full numbers and integers representable as sums of two squares.

1. INTRODUCTION

This is the third paper of our series on the Selberg-Delange method for *short intervals*. Roughly speaking this method applies to evaluate mean values of arithmetic functions whose associated Dirichlet series are close to *complex* powers of the Riemann ζ -function. In the first part, by using a suitable contour (located to the left of the Korobov-Vinogradov zero-free region of $\zeta(s)$) instead of Hankel's contour used in the original version of the Selberg-Delange method, Cui and Wu [2] extended this method to handle mean values of arithmetic functions over a short interval, when their corresponding Dirichlet series is close to a positive power of ζ -function. In the second one, Cui, Lü and Wu [3] treated the complex power case with the help of the well-known Hooley-Huxley-Motohashi contour. Some similar results have been appeared in Ramachandra's paper [14]. In this paper, we shall consider a more general case and give some arithmetic applications.

1.1. Assumptions.

Let us fix some notation:

- $\zeta(s)$ is the Riemann ζ -function,
- $L(s, \chi)$ is the Dirichlet L -function of χ ,
- ε is an arbitrarily small positive constant,
- $r \in \mathbb{N}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $M > 0$ (constants),
- $\mathbf{z} := (z_1, \dots, z_r) \in \mathbb{C}^r$ and $\mathbf{w} := (w_1, \dots, w_r) \in \mathbb{C}^r$,
- $\boldsymbol{\kappa} := (\kappa_1, \dots, \kappa_r) \in (\mathbb{R}^{+*})^r$ with $1 \leq \kappa_1 < \dots < \kappa_r \leq 2\kappa_1$,
- $\boldsymbol{\chi} := (\chi_1, \dots, \chi_r)$ with χ_i non principal Dirichlet characters,
- $\mathbf{B} := (B_1, \dots, B_r) \in (\mathbb{R}^{+*})^r$ and $\mathbf{C} := (C_1, \dots, C_r) \in (\mathbb{R}^{+*})^r$,
- The notation $|\mathbf{z}| \leq \mathbf{B}$ means that $|z_i| \leq B_i$ for $1 \leq i \leq r$.

Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function and its corresponding Dirichlet series is given by

$$(1.1) \quad \mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}.$$

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We say that this Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \mathbf{B}, \mathbf{C}, \alpha, \delta, A, M)$ if the following conditions are verified:

(a) For any $\varepsilon > 0$ we have

$$(1.2) \quad |f(n)| \ll_{\varepsilon} M n^{\varepsilon} \quad (n \geq 1),$$

where the implied constant depends only on ε ;

(b) We have

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \leq M(\sigma - 1/\kappa_1)^{-\alpha} \quad (\sigma > 1/\kappa_1);$$

(c) The Dirichlet series $\mathcal{F}(s)$ has the expression

$$(1.3) \quad \mathcal{F}(s) = \zeta(\boldsymbol{\kappa}s)^{\mathbf{z}} \mathbf{L}(\boldsymbol{\kappa}s; \boldsymbol{\chi})^{\mathbf{w}} \mathcal{G}(s),$$

where

$$(1.4) \quad \zeta(\boldsymbol{\kappa}s)^{\mathbf{z}} := \prod_{1 \leq i \leq r} \zeta(\kappa_i s)^{z_i},$$

$$(1.5) \quad \mathbf{L}(\boldsymbol{\kappa}s; \boldsymbol{\chi})^{\mathbf{w}} := \prod_{1 \leq i \leq r} L(\kappa_i s, \chi_i)^{w_i}$$

and the Dirichlet series $\mathcal{G}(s)$ is a holomorphic function in (some open set containing) $\sigma \geq (2\kappa_1)^{-1}$ and, in this region, $\mathcal{G}(s)$ satisfies the bound

$$(1.6) \quad |\mathcal{G}(s)| \leq M(|\tau| + 1)^{\max\{\delta(1-\kappa_1\sigma), 0\}} \log^A(|\tau| + 3)$$

uniformly for $|\mathbf{z}| \leq \mathbf{B}$ and $|\mathbf{w}| \leq \mathbf{C}$, where and in the sequel we implicitly define the real numbers σ and τ by the relation $s = \sigma + i\tau$ and choose the principal value of the complex logarithm.

As usual, we denote by $N(\sigma, T)$ and $N_{\chi}(\sigma, T)$ the number of zeros of $\zeta(s)$ and $L(s, \chi)$ in the region $\Re s \geq \sigma$ and $|\Im s| \leq T$, respectively. It is well known that there are two constants ψ and η such that

$$(1.7) \quad N(\sigma, T), N_{\chi}(\sigma, T) \ll T^{\psi(1-\sigma)} (\log T)^{\eta}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 2$. Huxley [9] showed that

$$(1.8) \quad \psi = \frac{12}{5} \quad \text{and} \quad \eta = 9$$

are admissible for $\zeta(s)$. It is not difficult to extend it for $L(s, \chi)$. The zero density hypothesis is stated as

$$(1.9) \quad \psi = 2.$$

1.2. Set-up and main results.

Our main aim of this paper is to establish, under the previous assumptions, an asymptotic formula for the summatory function

$$(1.10) \quad \sum_{x < n \leq x + x^{1-1/\kappa_1} y} f(n)$$

where $y := x^{\vartheta}$ with $\vartheta \in (0, 1]$ as small as possible. In order to state our result, it is necessary to introduce some notation. Obviously the function $Z(\boldsymbol{\kappa}s; \mathbf{z}) := \{(\boldsymbol{\kappa}s - 1)\zeta(\boldsymbol{\kappa}s)\}^{\mathbf{z}}$

is holomorphic in the disc $|s - 1/\kappa| < 1/\kappa$, and admits, in the same disc, the Taylor series expansion

$$Z(\kappa s; z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z, \kappa)}{j!} (s - 1/\kappa)^j,$$

where the $\gamma_j(z, \kappa)$'s are entire functions of z satisfying the estimate

$$(1.11) \quad \frac{\gamma_j(z, \kappa)}{j!} \ll_{B, \kappa, \varepsilon} (1 + \varepsilon)^j \quad (j \geq 0, |z| \leq B)$$

for all $B > 0$, κ and $\varepsilon > 0$. Write $\kappa_* = (\kappa_2, \dots, \kappa_r)$ and $\mathbf{z}_* = (z_2, \dots, z_r)$. Under our hypothesis, the function

$$\mathcal{G}(s) Z(\kappa_1 s; z_1) \zeta(\kappa_* s)^{z_*} \mathbf{L}(\kappa s; \boldsymbol{\chi})^{\mathbf{w}}$$

is holomorphic in the disc $|s - 1/\kappa_1| < 1/\kappa_1 - 1/\kappa_2$ and

$$(1.12) \quad |Z(\kappa_1 s; z_1) \zeta(\kappa_* s)^{z_*} \mathbf{L}(\kappa s; \boldsymbol{\chi})^{\mathbf{w}} \mathcal{G}(s)| \ll_{A, B, \mathbf{C}, \delta, \varepsilon} M$$

for $|s - 1/\kappa_1| < 1/\kappa_1 - 1/\kappa_2 - \varepsilon$, $|z| \leq B$ and $|\mathbf{w}| \leq \mathbf{C}$. Thus we can write

$$(1.13) \quad Z(\kappa_1 s; z_1) \zeta(\kappa_* s)^{z_*} \mathbf{L}(\kappa s; \boldsymbol{\chi})^{\mathbf{w}} \mathcal{G}(s) = \sum_{\ell=0}^{\infty} g_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) (s - 1/\kappa_1)^{\ell}$$

for $|s - 1/\kappa_1| \leq \frac{1}{2}(1/\kappa_1 - 1/\kappa_2)$, where

$$(1.14) \quad g_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) := \sum_{j=0}^{\ell} \frac{\gamma_j(z_1, \kappa_1)}{(\ell - j)! j!} \cdot \left. \frac{\partial^{\ell-j} (\zeta(\kappa_* s)^{z_*} \mathbf{L}(\kappa s; \boldsymbol{\chi})^{\mathbf{w}} \mathcal{G}(s))}{\partial s^{\ell-j}} \right|_{s=1/\kappa_1}.$$

The main result of this paper is as follows.

Theorem 1.1. *Let $r \in \mathbb{N}$, $\boldsymbol{\kappa} \in (\mathbb{R}^{+*})^r$, $\mathbf{z} \in \mathbb{C}^r$, $\mathbf{w} \in \mathbb{C}^r$, $\mathbf{B} \in (\mathbb{R}^{+*})^r$, $\mathbf{C} \in (\mathbb{R}^{+*})^r$, $\boldsymbol{\chi}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $M > 0$ be given as before. Suppose that the Dirichlet series $\mathcal{F}(s)$ defined as in (1.1) is of type $\mathcal{P}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \mathbf{B}, \mathbf{C}, \boldsymbol{\chi}, \alpha, \delta, A, M)$. Then for any $\varepsilon > 0$, we have*

$$(1.15) \quad \sum_{x < n \leq x + x^{1-1/\kappa_1} y} f(n) = y' (\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi})}{(\log x)^{\ell}} + O(M R_N(x)) \right\}$$

uniformly for

$$x \geq 3, \quad x^{(1-1/(\psi+\delta))/\kappa_1+\varepsilon} \leq y \leq x^{1/\kappa_1}, \quad N \geq 0, \quad |z| \leq \mathbf{B}, \quad |\mathbf{w}| \leq \mathbf{C},$$

where

$$\begin{aligned} y' &:= \kappa_1 ((x + x^{1-1/\kappa_1} y)^{1/\kappa_1} - x^{1/\kappa_1}), \\ \lambda_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) &:= \kappa_1^{-z_1} g_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) / \Gamma(z_1 - \ell), \\ R_N(x) &:= \left(\frac{c_1 N + 1}{\log x} \right)^{N+1} + \frac{(c_1 N + 1)^{N+1}}{e^{c_2 (\log x / \log_2 x)^{1/3}}} + \frac{y}{x^{1/\kappa_1} \log x} \end{aligned}$$

for some constants $c_1 > 0$ and $c_2 > 0$ depending only on $\mathbf{B}, \mathbf{C}, \delta$ and ε . The implied constant in the O -term depends only on $\boldsymbol{\kappa}, \mathbf{B}, \mathbf{C}, \boldsymbol{\chi}, A, \alpha, \delta$ and ε . In particular, $\psi = \frac{12}{5}$ is admissible.

The admissible length of short intervals in Theorem 1.1 depends only on the zero density constant ψ of $\zeta(s)$ and δ in (1.6) (for which we take $\delta = 0$ in most applications). Its independence from the power z of $\zeta(s)$ in the representation of $\mathcal{F}(s)$ seems interesting. Theorem 1.1 generalizes and improves [2, Theorem 1] to the case of complex powers and intervals of shorter length.

Taking $N = 0$ in Theorem 1.1, we obtain readily the following corollary.

Corollary 1.1. *Under the conditions of Theorem 1.1, for any $\varepsilon > 0$, we have*

$$(1.16) \quad \sum_{x < n \leq x + x^{1-1/\kappa_1} y} f(n) = y' (\log x)^{z-1} \left\{ \lambda_0(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) + O\left(\frac{M}{\log x}\right) \right\}$$

uniformly for

$$x \geq 3, \quad x^{(1-1/(\psi+\delta))/\kappa_1+\varepsilon} \leq y \leq x^{1/\kappa_1}, \quad |z| \leq \mathbf{B}, \quad |\mathbf{w}| \leq \mathbf{C},$$

where

$$\lambda_0(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) := \frac{\mathfrak{G}(1/\kappa_1)}{\kappa_1^{z_1} \Gamma(z_1)} \prod_{2 \leq i \leq r} \zeta(\kappa_i/\kappa_1)^{z_i} \prod_{1 \leq i \leq r} L(\kappa_i/\kappa_1, \chi_i)^{w_i}$$

and the implied constant in the O -term depends only on $A, \mathbf{B}, \mathbf{C}, \alpha, \delta$ and ε . Note that $\psi = \frac{12}{5}$ is admissible.

Taking $r = 2$, $\boldsymbol{\kappa} = (1, 2)$, $\mathbf{z} = (z, w)$, $\mathbf{w} = (0, 0)$ in Theorem 1.1 and Corollary 1.1, we can obtain Theorem 1.1 and Corollary 1.2 of Cui, Lü & Wu [3].

1.3. Distribution of divisors de certains integers.

In order to study the distribution of divisors of integers, Deshouillers, Dress & G. Tenenbaum [5] introduced the random variable D_n , which takes the value $(\log d)/\log n$, as d runs through the set of the $\tau(n)$ divisors of n , with the uniform probability $1/\tau(n)$, and consider its distribution function

$$F_n(t) = \text{Prob}(D_n \leq t) = \frac{1}{\tau(n)} \sum_{d|n, d \leq n^t} 1 \quad (0 \leq t \leq 1).$$

It is clear that the sequence $\{F_n\}_{n \geq 1}$ does not converge pointwisely on $[0, 1]$. However Deshouillers, Dress & Tenenbaum ([5] or [19, Theorem II.6.7]) proved that its Cesàro mean converges uniformly to the arcsin law. More precisely, they showed that the asymptotic formula

$$(1.17) \quad \frac{1}{x} \sum_{n \leq x} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right)$$

holds uniformly for $x \geq 2$ and $0 \leq t \leq 1$. Here we shall announce two applications of Corollary 1.1: the distribution of divisors of square-full numbers and of integers representable as sums of two squares. The proof will be given in another paper [22].

A. Beta law on divisors of square-full numbers in short intervals

An integer n is called *square-full* if $p \mid n \Rightarrow p^2 \mid n$. Denote by $\mathbb{1}_{\text{sf}}(n)$ the characteristic function of such integers and define

$$(1.18) \quad U(x, y) := \sum_{x < n \leq x + x^{1/2} y} \mathbb{1}_{\text{sf}}(n).$$

Our first result is as follows.

Theorem 1.2. *For any $\varepsilon \in (0, \frac{5}{48})$, we have*

$$(1.19) \quad \frac{1}{U(x, y)} \sum_{x < n \leq x + x^{1/2}y} \mathbb{1}_{\text{sf}}(n) F_n(t) = B\left(\frac{2}{3}, \frac{1}{3}\right)^{-1} \int_0^t w^{-\frac{1}{3}} (1-w)^{-\frac{2}{3}} dw + O\left(\frac{1}{\sqrt[3]{\log x}}\right)$$

uniformly for $x \geq 3$ and $x^{19/48+\varepsilon} \leq y \leq x^{1/2}$, where $B(u, v)$ is the beta function defined by

$$(1.20) \quad B(u, v) := \int_0^1 w^{u-1} (1-w)^{v-1} dw$$

for $u > 0$ and $v > 0$.

B. Beta law on divisors of integers representable as sums of two squares

Define

$$(1.21) \quad \mathbb{1}_{\square+\square}(n) := \begin{cases} 1 & \text{if } n = \square + \square \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad V(x) := \sum_{n \leq x} \mathbb{1}_{\square+\square}(n).$$

They proved that the mean of $F_n(t)$ over integers representable as sum of two squares converges to the beta law [7, Theorem 1]: For $x \rightarrow \infty$, we have

$$(1.22) \quad \frac{1}{V(x)} \sum_{n \leq x} \mathbb{1}_{\square+\square}(n) F_n(t) = B\left(\frac{1}{4}, \frac{1}{4}\right)^{-1} \int_0^t w^{-\frac{3}{4}} (1-w)^{-\frac{3}{4}} dw + O\left(\frac{1}{\sqrt[4]{\log x}}\right).$$

Here we shall generalise (1.22) to the short interval case. Put

$$(1.23) \quad V(x, y) := \sum_{x < n \leq x+y} \mathbb{1}_{\square+\square}(n).$$

Our result is as follows.

Theorem 1.3. *For any $\varepsilon \in (0, \frac{5}{12})$, we have*

$$(1.24) \quad \frac{1}{V(x, y)} \sum_{x < n \leq x+y} \mathbb{1}_{\square+\square}(n) F_n(t) = B\left(\frac{1}{4}, \frac{1}{4}\right)^{-1} \int_0^t w^{-\frac{3}{4}} (1-w)^{-\frac{3}{4}} dw + O\left(\frac{1}{\sqrt[4]{\log x}}\right)$$

uniformly for $x \geq 3$ and $x^{19/24+\varepsilon} \leq y \leq x$.

2. THE HOOLEY-HUXLEY-MOTOHASHI CONTOUR

The Hooley-Huxley-Motohashi contour appeared in Ramachandra [14] and Motohashi [13], independently. In [8], Hooley stated, without proof, his joint result with Huxley:

$$\sum_{x < n \leq x + x^\theta} \mathbb{1}_{\square+\square}(n) \sim 2^{-1/2} \prod_{p \equiv 1 \pmod{4}} (1 - p^{-2})^{-1/2} \frac{x^\theta}{\sqrt{\log x}}$$

for $x \rightarrow \infty$, provided $\theta > \frac{7}{12}$. According to Ramachandra [14, page 314], Hooley and Huxley have educated him on their method and allowed him make some comments about their method. A key point of this method is to use an ingenious contour, which is called the

Huxley-Hooley contour by Ramachandra. In [13], Motohashi constructed a essential same contour to prove

$$\sum_{x < n \leq x + x^\theta} \mu(n) = o(x^\theta) \quad (x \rightarrow \infty)$$

provided $\theta > \frac{7}{12}$. Here we shall follow Motohashi's argument and construct a Hooley-Huxley-Motohashi contour for our purpose.

2.1. Notation.

Let (κ, χ) be as before. First we write

$$(2.1) \quad \zeta(\kappa s) \mathbf{L}(\kappa s, \chi) := \prod_{1 \leq i \leq r} \zeta(\kappa_i s) L(\kappa_i s, \chi_i) =: \sum_{n \geq 1} \tau_\kappa(n; \chi) n^{-s},$$

$$(2.2) \quad (\zeta(\kappa s) \mathbf{L}(\kappa s, \chi))^{-1} := \prod_{1 \leq i \leq r} (\zeta(\kappa_i s) L(\kappa_i s, \chi_i))^{-1} =: \sum_{n \geq 1} \tau_\kappa^{(-1)}(n; \chi) n^{-s}$$

for $\sigma > 1/\kappa_1$, respectively. Here and in the sequel, we define implicitly the real numbers σ and τ by the relation $s = \sigma + i\tau$. A simple computation shows that

$$(2.3) \quad \tau_\kappa(n; \chi) = \sum_{m_1^{\kappa_1} \dots m_r^{\kappa_r} n_1^{\kappa_1} \dots n_r^{\kappa_r} = n} \prod_{1 \leq i \leq r} \chi_i(n_i),$$

$$(2.4) \quad \tau_\kappa^{(-1)}(n; \chi) = \sum_{m_1^{\kappa_1} \dots m_r^{\kappa_r} n_1^{\kappa_1} \dots n_r^{\kappa_r} = n} \prod_{1 \leq i \leq r} \mu(m_i) \mu(n_i) \chi_i(n_i).$$

From these, we deduce that

$$(2.5) \quad |\tau_\kappa(n; \chi)| \leq \tau_{\kappa, \kappa}(n), \quad |\tau_\kappa^{(-1)}(n; \chi)| \leq \tau_{\kappa, \kappa}(n), \quad \tau_\kappa * \tau_\kappa^{(-1)} = \mathbb{1}_{\{1\}}$$

for all $n \geq 1$, where

$$(2.6) \quad \tau_{\kappa, \kappa}(n) := \sum_{m_1^{\kappa_1} \dots m_r^{\kappa_r} n_1^{\kappa_1} \dots n_r^{\kappa_r} = n} 1$$

and $\mathbb{1}_{\{1\}}$ is the unit with respect to the Dirichlet convolution.

2.2. Upper bounds for $\zeta(s)^{\pm 1}$ and $L(s, \chi)^{\pm 1}$ in their zero-free regions.

It is well known that there is an absolute positive constant c such that $\zeta(s) \neq 0$ for

$$(2.7) \quad \sigma \geq 1 - c(\log |\tau|)^{-2/3} (\log_2 |\tau|)^{-1/3}, \quad |\tau| \geq 3$$

(the zero-free region, due to Korobov and Vinogradov) and in this region we have

$$(2.8) \quad \zeta(s)^{\pm 1} \ll (\log |\tau|)^{2/3} (\log_2 |\tau|)^{1/3}$$

(see [21, page 135] or [19, page 162]). For the Dirichlet L -functions, Richert [15] has established similar results.

Lemma 2.1. *Let χ be a non-principal Dirichlet character module q and let $L(s, \chi)$ be the corresponding Dirichlet L -function. Then we have*

$$(2.9) \quad |L(s, \chi)| \ll \begin{cases} |\tau|^{100(1-\sigma)^{3/2}} (\log \tau)^{4/3} & \text{if } \frac{1}{2} \leq \sigma \leq 1 \text{ and } |\tau| \geq 3, \\ \log |\tau| & \text{if } \sigma \geq 1 \text{ and } |\tau| \geq 3. \end{cases}$$

Further there is a positive constant c_χ depending on χ such that

$$(2.10) \quad |L(s, \chi)|^{-1} \ll (\log |\tau|)^{2/3} (\log_2 |\tau|)^{1/3}$$

for $\sigma \geq 1 - 500c_\chi (\log |\tau|)^{-2/3} (\log_2 |\tau|)^{-1/3}$ and $|\tau| \geq 3$. Here the implied constants depend on χ only.

Proof. Let $s = \sigma + i\tau$. Without loss of generality, we can suppose that $\tau \geq 2$. For $\sigma := \Re s > 1$ and $0 < w \leq 1$, the Hurwitz ζ -function is defined by

$$\zeta(s, w) := \sum_{n=0}^{\infty} (n+w)^{-s}.$$

This function can be extended to a meromorphic function over $\mathbb{C} \setminus \{1\}$. According to [15, Satz], there is a absolute constant $c > 0$ such that we have

$$(2.11) \quad |\zeta(s, w) - w^{-s}| \leq c |\tau|^{100(1-\sigma)^{3/2}} (\log |\tau|)^{2/3}$$

uniformly for $0 < w \leq 1$, $\frac{1}{2} \leq \sigma \leq 1$ and $|\tau| \geq 3$.

Since $\chi(n)$ is of period q , we can write, for $\sigma > 1$,

$$\begin{aligned} L(s, \chi) &= \sum_{a=1}^q \sum_{n=0}^{\infty} \chi(a+nq) (nq+a)^{-s} \\ &= q^{-s} \sum_{a=1}^q \chi(a) \{ \zeta(s, a/q) - (a/q)^{-s} \} + \sum_{a=1}^q \chi(a) a^{-s}. \end{aligned}$$

This relation also holds for all $s \in \mathbb{C} \setminus \{1\}$ by analytic continuation. Inserting (2.11), we immediately get the first inequality in (2.9). The second one is classical.

In view of (2.9), we can prove (2.10) exactly as [21, Theorem 3.11] with the choice of

$$\theta(\tau) = \left(\frac{\log_2 \tau}{100 \log \tau} \right)^{2/3} \quad \text{and} \quad \phi(\tau) = \log_2 \tau.$$

This completes the proof of Lemma 2.1. □

2.3. Definition of the Hooley-Huxley-Motohashi contour \mathcal{L}_T .

Let $(\varepsilon, \kappa, \chi)$ be as before and let $T_0 = T_0(\varepsilon, \kappa, \chi)$, $c_0 = c_0(\kappa, \chi)$ be two large constants and let $C_0 = C_0(\kappa, \chi)$ be a suitable positive constant. For $T \geq T_0$, put

$$(2.12) \quad \delta_T := C_0 (\log T)^{-2/3} (\log_2 T)^{-1/3}.$$

According to (2.8), (2.9) and (2.10) where such that

$$(2.13) \quad (\log |\tau|)^{-c_0 r} \ll |\zeta(\kappa s) \mathbf{L}(\kappa s, \chi)| \ll (\log |\tau|)^{c_0 r}$$

for $\sigma \geq (1 - 100\delta_T)/\kappa_1$ and $3 \leq |\tau| \leq 100T$, where the implied constants depend on (κ, χ) .

For $T \geq T_0$, write

$$(2.14) \quad J_T := \left[\left(\frac{1}{2} - \delta_T \right) \log T \right] \quad \text{and} \quad K_T := \lceil T (\log T)^{-1} \rceil.$$

For each pair of integers (j, k) with $0 \leq j \leq J_T$ and $0 \leq k \leq K_T$, we define

$$(2.15) \quad \sigma_j := \left(\frac{1}{2} + j (\log T)^{-1} \right) / \kappa_1, \quad \tau_k := (1 + k \log T) / \kappa_1$$

and

$$(2.16) \quad \Delta_{j,k} := \{s = \sigma + i\tau : \sigma_j \leq \sigma < \sigma_{j+1} \text{ and } \tau_k \leq \tau < \tau_{k+1}\}.$$

Define

$$(2.17) \quad M_x(s) = M_x(s; \boldsymbol{\kappa}, \boldsymbol{\chi}) := \sum_{n \leq x} \frac{\tau_{\boldsymbol{\kappa}}^{(-1)}(n; \boldsymbol{\chi})}{n^s},$$

$$(2.18) \quad \mathfrak{M}(\varsigma, T) = \mathfrak{M}(\varsigma, T; \boldsymbol{\kappa}) := \max_{\substack{\sigma \geq \varsigma \\ 1 \leq |\tau| \leq T}} |\zeta(\boldsymbol{\kappa}s)|^2.$$

Let A' be a fix large integer, and put

$$(2.19) \quad N_j := (A'(\log T)^5 \mathfrak{M}(4\sigma_j - 3/\kappa_1, 8T))^{1/2(1/\kappa_1 - \sigma_j)}.$$

We divide $\Delta_{j,k}$ into two classes (W) and (Y) as follows.

- *The case of $\sigma_j \leq (1 - \varepsilon)/\kappa_1$.*

$\Delta_{j,k} \in (W)$ if $\Delta_{j,k}$ contains at least one zero of $\zeta(\boldsymbol{\kappa}s)\mathbf{L}(\boldsymbol{\kappa}s, \boldsymbol{\chi})$, and $\Delta_{j,k} \in (Y)$ otherwise.

- *The case of $(1 - \varepsilon)/\kappa_1 < \sigma_j \leq (1 - \delta_T)/\kappa_1$.*

We write

$$(2.20) \quad \Delta_{j,k} \in (W) \Leftrightarrow \exists s \in \Delta_{j,k} \text{ such that } |\zeta(\boldsymbol{\kappa}s)\mathbf{L}(\boldsymbol{\kappa}s, \boldsymbol{\chi})M_{N_j}(s)| < \frac{1}{2}$$

and

$$(2.21) \quad \Delta_{j,k} \in (Y) \Leftrightarrow |\zeta(\boldsymbol{\kappa}s)\mathbf{L}(\boldsymbol{\kappa}s, \boldsymbol{\chi})M_{N_j}(s)| \geq \frac{1}{2} \text{ for all } s \in \Delta_{j,k}.$$

For each k , we define $j_k := \max\{j : \Delta_{j,k} \in (W)\}$ and put

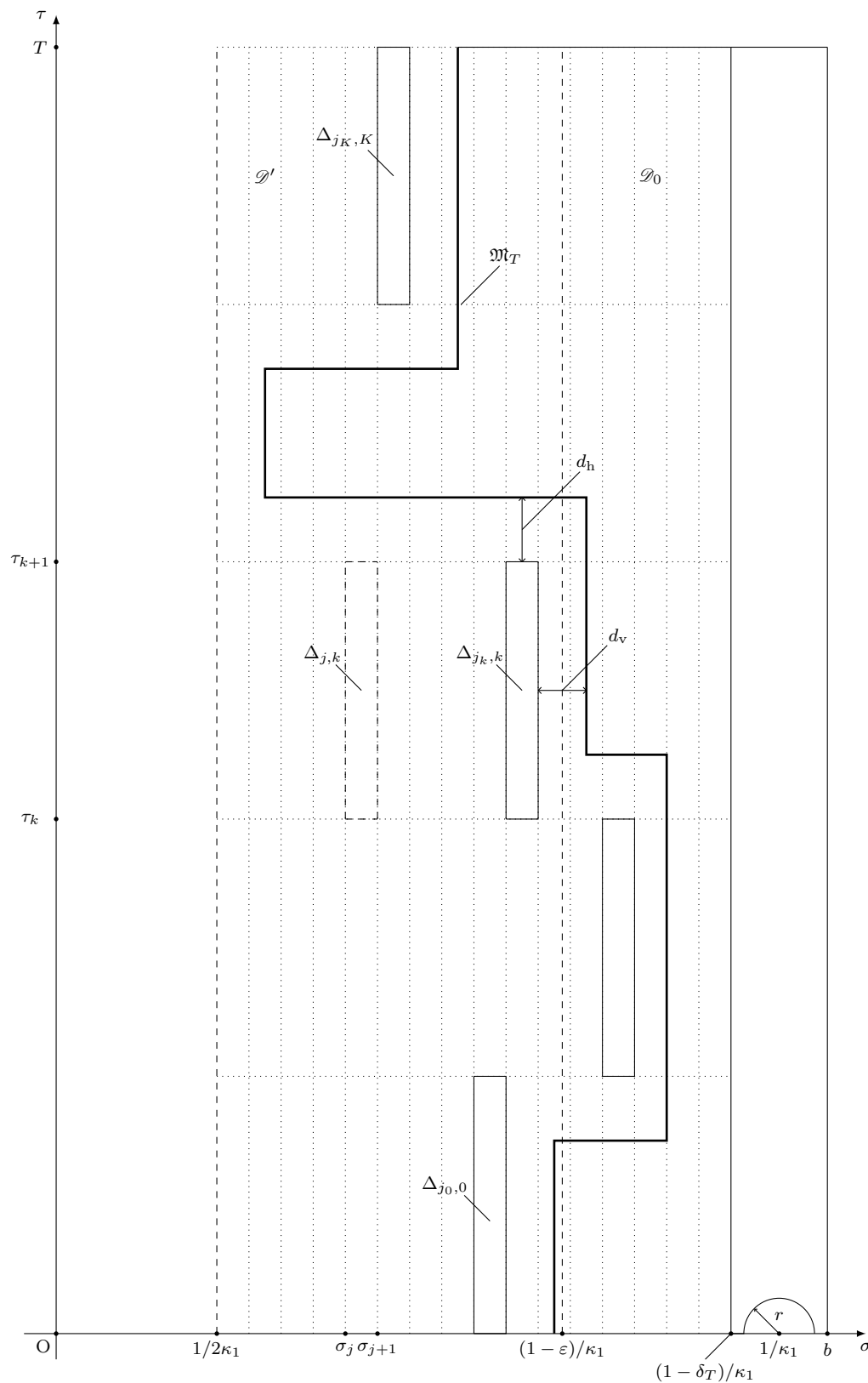
$$(2.22) \quad \mathscr{D}' := \bigcup_{0 \leq k \leq K_T} \bigcup_{0 \leq j \leq j_k} \Delta_{j,k} \quad \text{and} \quad \mathscr{D}_0 := \bigcup_{0 \leq k \leq K_T} \bigcup_{j_k < j \leq J_T} \Delta_{j,k}.$$

Clearly \mathscr{D}_0 consists of $\Delta_{j,k}$ of class (Y) only.

The Hooley-Huxley-Motohashi contour $\mathscr{L}_T = \mathscr{L}_T(\varepsilon, \boldsymbol{\kappa}, \boldsymbol{\chi})$ is symmetric about the real axe. Its sup erieur part is the path in \mathscr{D}_0 consisting of horizontal and vertical line segments whose distances away from \mathscr{D}' are respectively d_h and d_v , where d_h and d_v are defined by

$$(2.23) \quad \begin{aligned} d_h &:= (\log_2 T)/\kappa_1, \\ d_v &:= \begin{cases} \varepsilon^2/\kappa_1 & \text{if } \sigma \leq (1 - \varepsilon)/\kappa_1, \\ (\log T)^{-1}/\kappa_1 & \text{if } (1 - \varepsilon)/\kappa_1 < \sigma < (1 - \delta_T)/\kappa_1. \end{cases} \end{aligned}$$

Clearly \mathscr{L}_T is symmetric about the real axis. The following figure shows its upper part [from the point $((1 - \varepsilon)/\kappa_1, 0)$ to the point $((1 - \delta_T)/\kappa_1, T)$].


 Figure 1 – Superior part of the contour \mathcal{L}_T

3. LOWER AND UPPER BOUNDS OF $|\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)|$ ON \mathcal{L}_T

The aim of this section is to prove the following proposition.

Proposition 3.1. *Under the previous notation, we have*

$$(3.1) \quad T^{-544\sqrt{2\varepsilon}(1-\kappa_1\sigma)}(\log T)^{-4} \ll |\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| \ll T^{136\sqrt{2\varepsilon}(1-\kappa_1\sigma)}(\log T)^4.$$

for all $s \in \mathcal{L}_T$, where the implied constants depend on $(\varepsilon, \kappa, \chi)$.

First we establish two preliminary lemmas.

Lemma 3.1. *Under the previous notation, we have*

$$(3.2) \quad e^{-(\log T)^{1-\varepsilon^2}} \ll |\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| \ll e^{(\log T)^{1-\varepsilon^2}}$$

for $s \in \mathcal{L}_T$ with $\sigma \leq (1-\varepsilon)/\kappa_1$, or s with $(1-\varepsilon)/\kappa_1 < \sigma \leq \frac{1}{\kappa_1}(1-\varepsilon+\varepsilon^2)$ on the horizontal segments in \mathcal{L}_T that intersect the vertical line $\Re s = (1-\varepsilon)/\kappa_1$. Here the implied constant depends only on ε .

Proof. Let $s = \sigma + i\tau$ satisfy the conditions in this lemma. Without loss of generality, we can suppose that $\tau \geq T_0(\varepsilon, \kappa, \chi)$. Let us consider the four circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 , all centered at $s_0 := \log_2 \tau + i\tau$, with radii

$$\begin{aligned} r_1 &:= \log_2 \tau - (1 + \eta)/\kappa_1, \\ r_2 &:= \log_2 \tau - \sigma, \\ r_3 &:= \log_2 \tau - \sigma + \varepsilon^2/(2\kappa_1), \\ r_4 &:= \log_2 \tau - \sigma + \varepsilon^2/\kappa_1, \end{aligned}$$

respectively. Here $\eta > 0$ is a parameter to be chosen later. We note that these four circles pass through the points $(1 + \eta)/\kappa_1 + i\tau$, $\sigma + i\tau$, $\sigma - \varepsilon^2/(2\kappa_1) + i\tau$ and $\sigma - \varepsilon^2/\kappa_1 + i\tau$, respectively.

Clearly $\zeta(\kappa s)\mathbf{L}(\kappa s, \chi) \neq 0$ in a region containing the disc $|s - s_0| \leq r_4$. Thus we can unambiguously define $\log(\zeta(\kappa s)\mathbf{L}(\kappa s, \chi))$ in this region. We fix a branch of the logarithm throughout the remaining discussion.

Let M_i denote the maximum of $|\log(\zeta(\kappa s)\mathbf{L}(\kappa s, \chi))|$ on \mathcal{C}_i relative to this branch. By Hadamard's three-circle theorem and the fact that $s = \sigma + i\tau$ is on \mathcal{C}_2 , we have

$$(3.3) \quad |\log(\zeta(\kappa s)\mathbf{L}(\kappa s, \chi))| \leq M_2 \leq M_1^{1-\phi} M_3^\phi,$$

where

$$\begin{aligned} \phi &= \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \\ &= \frac{\log(1 + (1 + \eta - \kappa_1\sigma)/(\kappa_1 \log_2 \tau - 1 - \eta))}{\log(1 + (1 + \eta - \kappa_1\sigma + \varepsilon^2/2)/(\kappa_1 \log_2 \tau - 1 - \eta))} \\ &= \frac{1 + \eta - \kappa_1\sigma}{1 + \eta - \kappa_1\sigma + \varepsilon^2/2} + O\left(\frac{1}{\log_2 \tau}\right). \end{aligned}$$

On taking $\eta = \kappa_1\sigma - \frac{1}{2}(1 + \varepsilon^2 + \frac{\varepsilon^2}{1+\varepsilon^2})$ ($\eta \geq \frac{\varepsilon^4}{2(1+\varepsilon^2)}$, since $\sigma \geq \frac{1}{\kappa_1}(\frac{1}{2} + \varepsilon^2)$), we have

$$(3.4) \quad \phi = 1 - \varepsilon^2 - \varepsilon^4 + O((\log_2 \tau)^{-1}).$$

On the circle \mathcal{C}_1 , we have

$$(3.5) \quad M_1 \leq \max_{\Re s \geq (1+\eta)/\kappa_1} \sum_{n=2}^{\infty} \left| \frac{\Lambda(n)}{n^{\kappa_1 s} \log n} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n^{1+\eta}} \ll \frac{1}{\eta},$$

where $\Lambda(n)$ is the von Mangoldt function.

In order to bound M_3 , we shall apply the Borel-Carathéodory theorem to the function $\log(\zeta(\kappa s)\mathbf{L}(\kappa s, \chi))$ on the circles $\mathcal{C}_3, \mathcal{C}_4$. On the circle \mathcal{C}_4 , it is well known that

$$\Re(\log(\zeta(\kappa s)\mathbf{L}(\kappa s, \chi))) = \log|\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| \ll \log \tau$$

thanks to the convexity bounds of $\zeta(s)$ and of $L(s, \chi)$. Hence the Borel-Carathéodory theorem gives

$$(3.6) \quad \begin{aligned} M_3 &\leq \frac{2r_3}{r_4 - r_3} \max_{|s-s_0| \leq r_4} \log|\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| + \frac{r_4 + r_3}{r_4 - r_3} |\log(\zeta(\kappa s_0)\mathbf{L}(\kappa s_0, \chi))| \\ &\ll \frac{2(\log_2 \tau - \sigma + \varepsilon^2/(2\kappa_1))}{\varepsilon^2/(2\kappa_1)} \log \tau + \frac{2(\log_2 \tau - \sigma + \varepsilon^2/(2\kappa_1))}{\varepsilon^2/(2\kappa_1)} \\ &\ll (\log_2 \tau) \log \tau. \end{aligned}$$

From (3.3), (3.4), (3.5) and (3.6), we deduce that

$$\begin{aligned} |\log(\zeta(\kappa s)\mathbf{L}(\kappa s, \chi))| &\ll (\eta^{-1})^{1-\phi} ((\log_2 \tau) \log \tau)^\phi \\ &\ll_\varepsilon ((\log_2 \tau) \log \tau)^{1-\varepsilon^2-\varepsilon^4} \\ &\ll_\varepsilon (\log \tau)^{1-\varepsilon^2}. \end{aligned}$$

This leads to the required estimates. □

Lemma 3.2. *Under the previous notation, we have*

$$(3.7) \quad T^{-544r(1-\kappa_1\sigma_j)^{3/2}} (\log T)^{-c_0r} \ll |\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| \ll T^{136r(1-\kappa_1\sigma_j)^{3/2}} (\log T)^{c_0r}$$

for $s \in \mathcal{L}_T$ with $(1-\varepsilon)/\kappa_1 < \sigma_j \leq \sigma < \sigma_{j+1}$. Here the implied constants depend on (κ, χ) . In particular we have

$$(3.8) \quad T^{-544r\sqrt{2\varepsilon}(1-\kappa_1\sigma_j)} (\log T)^{-c_0r} \ll |\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| \ll T^{136r\sqrt{2\varepsilon}(1-\kappa_1\sigma_j)} (\log T)^{c_0r}$$

for $s \in \mathcal{L}_T$ with $(1-\varepsilon)/\kappa_1 < \sigma_j \leq \sigma < \sigma_{j+1}$. All the implied constants are absolute.

Proof. By (2.11) with $w = 1$ and Lemma 2.1, we have

$$(3.9) \quad |\zeta(\kappa s)\mathbf{L}(\kappa s, \chi)| \ll \tau^{100r(1-\kappa_1\sigma)^{3/2}} (\log \tau)^{c_0r} \quad \left(\frac{1}{2\kappa_1} \leq \sigma \leq \frac{1}{\kappa_1}, \tau \geq 2\right).$$

This immediately implies the second inequality in (3.7).

Next we consider the first inequality in (3.7). Let $s \in \mathcal{L}_T$ with $(1-\varepsilon)/\kappa_1 < \sigma_j \leq \sigma < \sigma_{j+1}$. Since $1/\kappa_1 \geq 4\sigma_j - 3/\kappa_1 \geq (1-4\varepsilon)/\kappa_1 \leq 1/2\kappa_1$, the inequality (2.11) with $w = 1$ allows us to derive that

$$(3.10) \quad \begin{aligned} N_j^{2(1/\kappa_1 - \sigma_j)} &= A'(\log T)^5 \max_{\substack{\sigma \geq 4\sigma_j - 3/\kappa_1 \\ 1 \leq |\tau| \leq 8T}} |\zeta(\kappa s)|^2 \\ &\ll T^{200\sqrt{2}r(1-\kappa_1\sigma_j)^{3/2}} (\log T)^{c_0r}. \end{aligned}$$

According to the definition of \mathcal{L}_T , there is an integer k such that $s \in \Delta_{j,k}$ and this $\Delta_{j,k}$ must be in (Y) and (2.21) holds for all s of this $\Delta_{j,k}$. On the other hand, (2.5) and (3.10) imply that for $\sigma_j \leq \sigma < \sigma_{j+1}$,

$$\begin{aligned} |M_{N_j}(s)| &\leq \sum_{n \leq N_j} \tau_{\kappa, \kappa}(n) n^{-\sigma_j} \ll (1 - \kappa_1 \sigma_j)^{-2} N_j^{1/\kappa_1 - \sigma_j} \\ &\ll T^{544(1 - \kappa_1 \sigma_j)^{3/2}} (\log T)^{c_0 r}. \end{aligned}$$

Combining this with (2.21) immediately yields

$$|\zeta(\kappa s) \mathbf{L}(\kappa s, \chi)| \geq (2|M_{N_j}(s)|)^{-1} \gg T^{-544(1 - \kappa_1 \sigma_j)^{3/2}} (\log T)^{-c_0 r}$$

for $s \in \mathcal{L}_T$ with $(1 - \varepsilon)/\kappa_1 < \sigma_j \leq \sigma < \sigma_{j+1}$.

Finally we note (3.8) is a simple consequence of (3.7) since $(1 - \varepsilon)/\kappa_1 < \sigma_j$ implies that $(1 - \kappa_1 \sigma_j)^{1/2} \leq \sqrt{\varepsilon}$. \square

Now we are ready to prove Proposition 3.1.

Proof. When $s \in \mathcal{L}_T$ with $|\tau| \leq 1$, the estimations (3.1) are trivial.

Next suppose that $s \in \mathcal{L}_T$ with $|\tau| > 1$. Then there is a j such that $\sigma_j \leq \sigma < \sigma_{j+1}$. We consider the three possibilities.

- *The case of $(1 - \varepsilon)/\kappa_1 < \sigma_j$.*

The inequality (3.1) follows immediately from (3.8) of Lemma 3.2.

- *The case of $\sigma_j \leq \sigma \leq (1 - \varepsilon)/\kappa_1$.*

In this case, the first part of Lemma 3.1 shows that (3.1) holds again since $\sqrt{\varepsilon}(1 - \kappa_j \sigma) \geq \varepsilon^{3/2} \geq (\log T)^{-\varepsilon^2}$ for $T \geq T_0(\varepsilon, \kappa, \chi)$.

- *The case of $\sigma_j \leq (1 - \varepsilon)/\kappa_1 < \sigma$.*

In this case, s must be on the horizontal segment in \mathcal{L}_T , because the vertical segment keeps the distance ε^2 from the line $\Re s = \sigma_j$ and $\sigma_j < \sigma < \sigma_{j+1}$. Thus we can apply the second part of Lemma 3.1 to get (3.1) as before. \square

4. MONTGOMERY'S METHOD

4.1. Montgomery's mean value theorem.

Let $\tau_{\kappa, \kappa}(n)$ and $\mathfrak{M}(\zeta, T)$ be defined as in (2.6) and (2.18), respectively. The following proposition is a variant of [12, Theorem 8.4] for our purpose, which will play a key role in the proof of Proposition 5.1 below.

Proposition 4.1. *Let $\sigma_0 > 0, \varrho > 0$ and $T \geq 2$ three real numbers. Let $r \in \mathbb{N}$ and κ be as before. Let $\mathcal{S} = \mathcal{S}_T(\sigma_0, \varrho)$ be a finite set of complex numbers $s = \sigma + i\tau$ such that*

$$(4.1) \quad \sigma \geq \sigma_0, \quad 1 \leq |\tau| \leq T$$

for all $s \in \mathcal{S}$ and

$$(4.2) \quad |\tau - \tau'| \geq \varrho$$

for any distinct points $s = \sigma + i\tau$ and $s' = \sigma' + i\tau'$ in \mathcal{S} . For any sequence of complex numbers $\{a_n\}$ verifying

$$(4.3) \quad a_n \neq 0 \Rightarrow \tau_{\kappa, \kappa}(n) \geq 1,$$

real numbers $\theta \in (1/\kappa_2, 1/\kappa_1)$ and $N \geq 1$, we have

$$(4.4) \quad \sum_{s \in \mathcal{S}} \left| \sum_{n \leq N} a_n n^{-s} \right|^2 \ll \left\{ (1 + \varrho^{-1}) N^{1/\kappa_1} + \left(\mathfrak{M}(\theta, 4T) + \frac{1}{(1 - \kappa_1 \theta)(\kappa_2 \theta - 1)} \right) N^\theta |\mathcal{S}| \right\} \sum_{n \leq N} \frac{|a_n|^2}{n^{2\sigma_0}},$$

where the implied constant depends on κ only.

First we prove two preliminary lemmas. The first one is due to Bombieri (see also [12, Lemma 5.1]). For the convenience of reader, we give a direct proof.

Lemma 4.1. *Let \mathcal{S} be a finite set of complex numbers s . For any $\{a_n\}_{1 \leq n \leq N} \subset \mathbb{C}$, we have*

$$(4.5) \quad \sum_{s \in \mathcal{S}} \left| \sum_{n \leq N} a_n n^{-s} \right|^2 \leq \left(\sum_{n \leq N} |a_n|^2 b_n^{-1} \right) \max_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |B(\bar{s} + s')|$$

where

$$B(s) := \sum_{n \geq 1} b_n n^{-s}$$

is absolutely convergent at the points $\bar{s} + s'$, and the b_n are non-negative real numbers for which $b_n > 0$ whenever $a_n \neq 0$. Here the implied constant is absolute.

Proof. According our hypothesis on a_n and b_n , we can write $a_n = a_n b_n^{-1/2} b_n^{1/2}$ for all $n \leq N$. Introducing the notation

$$A(s) := \sum_{n \leq N} a_n n^{-s},$$

then we can write

$$\begin{aligned} \sum_{s \in \mathcal{S}} |A(s)|^2 &= \sum_{s \in \mathcal{S}} \sum_{n \leq N} a_n b_n^{-1/2} b_n^{1/2} n^{-s} \sum_{m \leq N} \overline{a_m} m^{-\bar{s}} \\ &= \sum_{n \leq N} (a_n b_n^{-1/2}) \left(b_n^{1/2} \sum_{s \in \mathcal{S}} n^{-s} \sum_{m \leq N} \overline{a_m} m^{-\bar{s}} \right). \end{aligned}$$

By the Cauchy inequality, it follows that

$$(4.6) \quad \left(\sum_{s \in \mathcal{S}} |A(s)|^2 \right)^2 \leq \left(\sum_{n \leq N} |a_n|^2 b_n^{-1} \right) \Upsilon,$$

where

$$\Upsilon := \sum_{n \leq N} b_n \left| \sum_{s \in \mathcal{S}} n^{-s} \sum_{m \leq N} \overline{a_m} m^{-\bar{s}} \right|^2.$$

Since $B(s)$ is absolutely convergent at the points $\bar{s} + s'$, we can deduce

$$\begin{aligned} \Upsilon &\leq \sum_{n \geq 1} b_n \left| \sum_{s \in \mathcal{S}} n^{-s} \sum_{m \leq N} \overline{a_m} m^{-\bar{s}} \right|^2 \\ &= \sum_{n \geq 1} b_n \sum_{s' \in \mathcal{S}} n^{-s'} \sum_{d \leq N} \overline{a_d} d^{-\bar{s}'} \sum_{s \in \mathcal{S}} n^{-\bar{s}} \sum_{m \leq N} a_m m^{-s} \\ &= \sum_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} \sum_{d \leq N} \overline{a_d} d^{-\bar{s}'} \sum_{m \leq N} a_m m^{-s} B(\bar{s} + s'). \end{aligned}$$

On the other hand, the trivial inequality $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$ allows us to write

$$\left| \sum_{d \leq N} \overline{a_d} d^{-\bar{s}'} \sum_{m \leq N} a_m m^{-s} \right| \leq \frac{1}{2} \left(|A(s)|^2 + |A(s')|^2 \right).$$

Thus

$$\begin{aligned} \Upsilon &\leq \sum_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} \frac{1}{2} \left(|A(s)|^2 + |A(s')|^2 \right) |B(\bar{s} + s')| \\ &\leq \frac{1}{2} \sum_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |A(s)|^2 |B(\bar{s} + s')| + \frac{1}{2} \sum_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |A(s')|^2 |B(\bar{s} + s')| \\ &\leq \frac{1}{2} \sum_{s \in \mathcal{S}} |A(s)|^2 \max_{s' \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |B(\bar{s} + s')| + \frac{1}{2} \sum_{s' \in \mathcal{S}} |A(s')|^2 \max_{s \in \mathcal{S}} \sum_{s \in \mathcal{S}} |B(\bar{s} + s')|. \end{aligned}$$

Noticing that

$$|B(\bar{s} + s')| = |\overline{B(\bar{s} + s')}| = |B(\overline{\bar{s} + s'})| = |B(s + \bar{s}')|,$$

the precede inequality becomes

$$\Upsilon \leq \sum_{s \in \mathcal{S}} |A(s)|^2 \max_{s' \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |B(\bar{s} + s')|.$$

Inserting it into (4.6), we get the required result. \square

The next lemma is an analogue of [12, page 157, Theorem II.2].

Lemma 4.2. *Let r , κ , $\tau_{\kappa, \kappa}(n)$ and $\mathfrak{M}(\theta, T)$ be as before. For any $s = \sigma + i\tau$ with $\sigma \geq 0$ and $|\tau| \geq 1$, real numbers $\theta \in (1/\kappa_2, 1/\kappa_1)$ and $N \geq 1$, we have*

$$(4.7) \quad \left| \sum_{n \geq 1} \frac{\tau_{\kappa, \kappa}(n)}{n^s} (e^{-n/(2N)} - e^{-n/N}) \right| \ll N^{1/\kappa_1} e^{-|\tau|} + \{ \mathfrak{M}(\theta, 2|\tau|) + (1 - \kappa_1 \theta)^{-1} (\kappa_2 \theta - 1)^{-1} \} N^\theta,$$

where the implied constant depends on κ only.

Proof. Denote by $\mathcal{S}_N(s) = \mathcal{S}_N(s; \kappa, \chi)$ the series on the left-hand side of (4.7). By the Perron formula [21, page 151, Lemma], we can write

$$(4.8) \quad \mathcal{S}_N(s) = \frac{1}{2\pi i} \int_{2/\kappa_1 - i\infty}^{2/\kappa_1 + i\infty} \zeta(\kappa(w + s))^2 \Gamma(w) ((2N)^w - N^w) dw,$$

where $\zeta(\boldsymbol{\kappa}s)$ is defined as in (2.1) above. We take the contour to the line $\Re w = \theta - \sigma$ with $1/\kappa_2 - \sigma < \theta - \sigma < 1/\kappa_1 - \sigma$, and in doing so we pass a simple pole at $w = 1/\kappa_1 - s$. Put

$$G(w) := \prod_{2 \leq j \leq r} \zeta(\kappa_j(w+s))^2 \Gamma(w) ((2N)^w - N^w).$$

The residue of the integrand at this pole is, in view of the hypothesis $\sigma \geq 0$,

$$\begin{aligned} \operatorname{Res}_{w=1/\kappa_1-s} \zeta(\boldsymbol{\kappa}(w+s))^2 \Gamma(w) ((2N)^w - N^w) &= (\gamma G(1/\kappa_1 - s) + G'(1/\kappa_1 - s)/\kappa_1) / \kappa_1 \\ &\ll N^{1/\kappa_1} e^{-|\tau|}, \end{aligned}$$

where we have used the Stirling formula [20, page 151]:

$$(4.9) \quad |\Gamma(s)| = \sqrt{2\pi} e^{-(\pi/2)|\tau|} |\tau|^{\sigma-1/2} \left\{ 1 + O\left(\frac{|\tan(\frac{\arg s}{2})|}{|\tau|} + \frac{|a|^2 + |b|^2}{|\tau|^2} + \frac{|a|^3 + |b|^3}{|\tau|^3} \right) \right\}$$

valable uniformly for $a, b \in \mathbb{R}$ with $a < b$, $a \leq \sigma \leq b$ and $|\tau| \geq 1$, where the implied O -constant is absolute.

On the other hand, by using the Stirling formula again it is easy to see that for $\Re w = \theta - \sigma$ and $|\Im w| \leq |\tau|$, the integrand in (4.8) is

$$\ll \{ \mathfrak{M}(\theta, 2|\tau|) + (1 - \kappa_1\theta)^{-1} (\kappa_2\theta - 1)^{-1} \} N^{\theta-\sigma} e^{-|\Im w|},$$

while otherwise the Stirling formula and the convexity bounds of ζ -function imply that

$$\ll N^{\theta-\sigma} e^{-|\Im w|} \ll (\mathfrak{M}(\theta, 2|\tau|) + (1 - \kappa_1\theta)^{-1} (\kappa_2\theta - 1)^{-1}) N^{\theta-\sigma} e^{-|\Im w|}.$$

The required result follows from this. \square

Now we are ready to prove Proposition 4.1.

Proof. We shall apply Lemma 4.1 with the choice of

$$b_n := \tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}}(n) n^{2\sigma_0} (e^{-n/(2N)} - e^{-n/N}).$$

In view of the simple fact that $e^{-n/(2N)} - e^{-n/N} \asymp 1$ for $N \leq n \leq 2N$ and the hypothesis (4.3), we have

$$(4.10) \quad \sum_{s \in \mathcal{S}} \left| \sum_{N \leq n \leq 2N} a_n n^{-s} \right|^2 \ll \left(\sum_{N \leq n \leq 2N} |a_n|^2 n^{-2\sigma_0} \right) \max_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |B(\bar{s} + s')|,$$

where

$$B(s) := \sum_{n \geq 1} \frac{\tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}}(n)}{n^{s-2\sigma_0}} (e^{-n/(2N)} - e^{-n/N}).$$

In view of (4.1) and (4.2), we can apply Lemma 4.2 to deduce

$$\begin{aligned} |B(\bar{s} + s')| &= \left| \sum_{n \geq 1} \frac{\tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}}(n)}{n^{\sigma+\sigma'-2\sigma_0+i(\tau'-\tau)}} (e^{-n/(2N)} - e^{-n/N}) \right| \\ &\ll N^{1/\kappa_1} e^{-|\tau'-\tau|} + \{ \mathfrak{M}(\theta, 2|\tau' - \tau|) + (1 - \kappa_1\theta)^{-1} (\kappa_2\theta - 1)^{-1} \} N^\theta \end{aligned}$$

for any distinct points $s = \sigma + i\tau \in \mathcal{S}$ and $s' = \sigma' + i\tau' \in \mathcal{S}$. By (4.2), the contribution of the term $N^{1/\kappa_1} e^{|\tau' - \tau|}$ to the sum $\sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} |B(\bar{s} + s')|$ is

$$\ll N^{1/\kappa_1} \sum_{\substack{s' \in \mathcal{S} \\ |\tau - \tau'| \geq \varrho}} e^{-|\tau - \tau'|} \ll N^{1/\kappa_1} \sum_{n \geq 0} e^{-n} \sum_{\substack{s' \in \mathcal{S} \\ n < |\tau - \tau'| \leq n+1}} 1 \ll (1 + \varrho^{-1}) N^{1/\kappa_1}.$$

Thus

$$(4.11) \quad \sum_{s' \in \mathcal{S}, s' \neq s} |B(\bar{s} + s')| \ll (1 + \varrho^{-1}) N^{1/\kappa_1} + \{\mathfrak{M}(\theta, 2|\tau' - \tau|) + (1 - \kappa_1\theta)^{-1}(\kappa_2\theta - 1)^{-1}\} N^\theta |\mathcal{S}|.$$

When $s = s'$, we have

$$(4.12) \quad B(\bar{s} + s') = B(2\sigma) \leq \sum_{n \geq 1} \tau_{\kappa, \kappa}(n) (e^{-n/(2N)} - e^{-n/N}) \ll N^{1/\kappa_1}.$$

Now the required result follows from (4.10), (4.11) and (4.12). \square

5. DENSITY ESTIMATION OF SMALL VALUE POINTS

In [12], Montgomery developed a new method for studying zero-densities of the Riemann ζ -function and of the Dirichlet L -functions. Subsequently by modifying this method, Huxley [9] established his zero-density estimation (1.8). In [13], Motohashi noted that Montgomery's method can be adapted to estimate the density of “*small value points*” of $\zeta(s)$ (see [3, Section 2.3] for a detail description). Here we shall adapt this method to prove the following proposition.

Proposition 5.1. *Under the previous notation, for $j = 0, 1, \dots, J_T$ we have*

$$(5.1) \quad |\{k \leq K_T : \Delta_{j,k} \in (W)\}| \ll T^{\psi(1 - \kappa_1\sigma_j)} (\log T)^\eta$$

if $\kappa_1\sigma_j \leq 1 - \varepsilon$, and

$$(5.2) \quad |\{k \leq K_T : \Delta_{j,k} \in (W)\}| \ll T^{500r(1 - \kappa_1\sigma_j)^{3/2}} (\log T)^{10r}$$

if $1 - \varepsilon \leq \kappa_1\sigma_j \leq 1 - \delta_T$. Here the implied constants depend on κ and ε only.

Proof. When $\kappa_1\sigma_j \leq 1 - \varepsilon$, the number of $\Delta_{j,k}$ of type (W) does not exceed the number of non-trivial zeros of $\zeta(\kappa s) \mathbf{L}(\kappa s, \chi)$. Thus

$$|\{k \leq K_T : \Delta_{j,k} \in (W)\}| \leq \sum_{1 \leq i \leq r} (N(\kappa_i\sigma_j, 2T) + N_{\chi_i}(\kappa_i\sigma_j, 3T)).$$

Now the required bound follows from (1.7).

Next we suppose $1 - \varepsilon \leq \kappa_1\sigma_j \leq 1 - \delta_T$.

Let $\mathcal{K}_j(T)$ be a subset of the set $\{\log T \leq k \leq K_T : \Delta_{j,k} \in (W)\}$ such that the difference of two distinct integers of $\mathcal{K}_j(T)$ is at least $3A'$, where A' is the large integer specified in (2.17). Obviously

$$|\{(\log T)^2 \leq k \leq K_T : \Delta_{j,k} \in (W)\}| \leq 3A' |\mathcal{K}_j(T)|.$$

Therefore it suffices to show that

$$(5.3) \quad |\mathcal{K}_j(T)| \ll_{\varepsilon} T^{170(1-\kappa_1\sigma_j)^{3/2}} (\log T)^{13}$$

for $T \geq T_0(\varepsilon, \boldsymbol{\kappa}, \boldsymbol{\chi})$.

Let $M_x(s)$ be defined as in (2.17) and write

$$(5.4) \quad \Phi_x(s) := \zeta(\boldsymbol{\kappa}s) \mathbf{L}(\boldsymbol{\kappa}s, \boldsymbol{\chi}) M_x(s).$$

Let $\phi_x(n)$ be the n th coefficient of the Dirichlet series $\Phi_x(s)$, then

$$(5.5) \quad \phi_x(n) = \sum_{d|n, d \leq x} \tau_{\boldsymbol{\kappa}}^{(-1)}(d; \boldsymbol{\chi}) \tau_{\boldsymbol{\kappa}}(n/d; \boldsymbol{\chi}).$$

By the Perron formula [21, Lemma, page 151], we can write

$$\sum_{n \geq 1} \frac{\phi_x(n)}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2/\kappa_1 - i\infty}^{2/\kappa_1 + i\infty} \Phi_x(w+s) \Gamma(w) y^w dw$$

for $y > x \geq 3$ and $s = \sigma + i\tau \in \mathbb{C}$ with $\sigma_j \leq \sigma < \sigma_{j+1}$. We take the contour to the line $\Re w = \alpha_j - \sigma < 0$ with $\alpha_j := 4\sigma_j - 3/\kappa_1 < \sigma_j < 1/\kappa_1$, and in doing so we pass two simple poles at $w = 0$ and $w = 1/\kappa_1 - s$. Our equation becomes

$$\sum_{n \geq 1} \frac{\phi_x(n)}{n^s} e^{-n/y} = \Phi_x(s) + \Psi_{x,y}(s) + I(s; x, y),$$

where

$$\Psi_{x,y}(s) := \prod_{2 \leq j \leq r} \zeta(\kappa_j/\kappa_1) \mathbf{L}(\boldsymbol{\kappa}/\kappa_1, \boldsymbol{\chi}) M_x(1/\kappa_1) \Gamma(1/\kappa_1 - s) y^{1/\kappa_1 - s},$$

$$I(s; x, y) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_x(\alpha_j + i\tau + iu) \Gamma(\alpha_j - \sigma + iu) y^{\alpha_j - \sigma + iu} du.$$

In view of (5.5) and (2.5), we have

$$(5.6) \quad \phi_x(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } 1 < n \leq x, \end{cases}$$

and

$$(5.7) \quad |\phi_x(n)| \leq (\tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}} * \tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}})(n) \quad (n > x),$$

where $\tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}}(n)$ is defined as in (2.6) above. It is easy to see that

$$\sum_{n \leq t} (\tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}} * \tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}})(n) \ll t^{1/\kappa_1} (\log t)^3.$$

By a simple partial integration, we can deduce that

$$\begin{aligned} \left| \sum_{n > y^2} \frac{\phi_x(n)}{n^s} e^{-n/y} \right| &\leq \int_{y^2}^{\infty} t^{-\sigma} e^{-t/y} d \left(\sum_{n \leq t} (\tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}} * \tau_{\boldsymbol{\kappa}, \boldsymbol{\kappa}})(n) \right) \\ &\ll e^{-y} y^{1-2\sigma} (\log y)^3 + y^{-1} \int_{y^2}^{\infty} e^{-t/y} t^{1/\kappa_1 - \sigma} (\log t)^3 dt \\ &\ll e^{-y/2} \end{aligned}$$

for $\sigma \geq 1/(2\kappa_1)$. Inserting it into the precedent relation, we find that

$$(5.8) \quad e^{-1/y} + \sum_{x < n \leq y^2} \frac{\phi_x(n)}{n^s} e^{-n/y} + O(e^{-y/2}) = \Phi_x(s) + \Psi_{x,y}(s) + I(s; x, y)$$

for $s \in \mathbb{C}$ with $\sigma_j \leq \sigma < \sigma_{j+1}$ and $y > x \geq 3$.

If $k \in \mathcal{K}_j(T)$, then there is at least a $s_k := v_k + it_k \in \Delta_{j,k}$ such that

$$(5.9) \quad |\Phi_{N_j}(s_k)| = |\zeta(\kappa s_k) \mathbf{L}(\kappa s_k; \boldsymbol{\chi}) M_{N_j}(s_k)| \leq \frac{1}{2}.$$

By the definition of $\mathcal{K}_j(T)$, we have

$$\sigma_j \leq v_k \leq \sigma_{j+1}, \quad (\log T)^2 \leq t_k \leq T \quad \text{and} \quad |t_{k_1} - t_{k_2}| \geq 3A' \log T \quad (k_1 \neq k_2).$$

Since $|t_k| \geq (\log T)^2$, the Stirling formula (4.9) allows us to deduce

$$(5.10) \quad \begin{aligned} |\Psi_{x,y}(s_k)| &= \left| \prod_{2 \leq j \leq r} \zeta(\kappa_j/\kappa_1) \mathbf{L}(\kappa/\kappa_1, \boldsymbol{\chi}) M_x(1/\kappa_1) \Gamma(1/\kappa_1 - s_k) y^{1/\kappa_1 - s_k} \right| \\ &\ll (\log x) y^{1/2 - v_k} e^{-(\pi/2)|t_k|} |t_k|^{1/2 - v_k} \\ &\leq \frac{1}{10} \end{aligned}$$

for all $3 \leq x \leq y \leq T^{100}$.

Similarly, using the estimates

$$\begin{aligned} \zeta(\kappa(\alpha_j + it_k + iu)) &\ll (T + |u|)^r, \\ \mathbf{L}(\kappa(\alpha_j + it_k + iu), \boldsymbol{\chi}) &\ll (T + |u|)^r, \\ M_x(\alpha_j + it_k + iu) &\ll x^{1/\kappa_1 - \alpha_j} \log x \ll T^{100} \end{aligned}$$

and the Stirling formula (4.9), we derive that

$$(5.11) \quad \int_{|u| \geq A' \log T} |\Phi_x(\alpha_j + it_k + iu) \Gamma(\alpha_j - v_k + iu)| y^{\alpha_j - v_k} du \leq \frac{1}{10}$$

for all $3 \leq x \leq y \leq T^{100}$.

Taking $(s, x) = (s_k, N_j)$ in (5.8) and combining with (5.9), (5.10) and (5.11), we easily see that

$$(5.12) \quad \left| \sum_{N_j < n \leq y^2} \frac{\phi_{N_j}(n)}{n^{s_k}} e^{-n/y} \right| \geq \frac{1}{6}$$

or

$$(5.13) \quad \left| \int_{-A' \log T}^{A' \log T} \Phi_{N_j}(\alpha_j + it_k + iu) \Gamma(\alpha_j - v_k + iu) y^{\alpha_j - v_k + iu} du \right| \geq \frac{1}{6}$$

or both.

Let $\mathcal{K}'_j(T)$ and $\mathcal{K}''_j(T)$ be the subsets of $\mathcal{K}_j(T)$ for which (5.12) and (5.13) hold respectively. Then

$$(5.14) \quad |\mathcal{K}_j(T)| \leq |\mathcal{K}'_j(T)| + |\mathcal{K}''_j(T)|.$$

First we bound $|\mathcal{K}'_j(T)|$. By a dyadic argument, there is a $U \in [N_j, y^2]$ such that

$$(5.15) \quad \left| \sum_{U < n \leq 2U} \frac{\phi_{N_j}(n)}{n^{s_k}} e^{-n/y} \right| \geq \frac{1}{18 \log y}$$

holds for $\gg |\mathcal{K}'_j(T)|(\log y)^{-1}$ integers $k \in \mathcal{K}'_j(T)$. Let \mathcal{S}' be the set of corresponding points s_k . With the help of (5.6), it is easy to see that $\phi_{N_j}(n) \neq 0 \Rightarrow \tau_{\kappa, \kappa}(n) \geq 1$. Thus we can apply Proposition 4.1 with $a_n = \phi_{N_j}(n)$ and $\theta = \alpha_j := 4\sigma_j - 3/\kappa_1$. In view of the bound

$$\begin{aligned} \sum_{U < n \leq 2U} \frac{(\tau_{\kappa, \kappa} * \tau_{\kappa, \kappa})(n)^2}{n^{2\sigma_j}} e^{-2n/y} &\ll e^{-2U/y} \int_U^{2U} t^{-2\sigma_j} d\left(\sum_{U < n \leq t} (\tau_{\kappa, \kappa} * \tau_{\kappa, \kappa})(n)^2 \right) \\ &\ll U^{1/\kappa_1 - 2\sigma_j} (\log T)^3 e^{-2U/y}, \end{aligned}$$

it follows that

$$(5.16) \quad \begin{aligned} \sum_{s_k \in \mathcal{S}'} \left| \sum_{U < n \leq 2U} \frac{\phi_{N_j}(n)}{n^{s_k}} e^{-n/y} \right|^2 \\ \ll \left(U^{2(1/\kappa_1 - \sigma_j)} + |\mathcal{S}'| U^{-2(1/\kappa_1 - \sigma_j)} \mathfrak{M}(\alpha_j, 4T) \right) e^{-2U/y} (\log T)^3. \end{aligned}$$

Since $U \geq N_j$, we have

$$U^{-2(1/\kappa_1 - \sigma_j)} (\log T)^3 \mathfrak{M}(\alpha_j, 4T) \leq A'^{-1} (\log T)^{-2}.$$

On the other hand, the inequality (5.15) implies that the member on the left-hand side of (5.16) is

$$\geq |\mathcal{S}'| (18 \log y)^{-2} \geq |\mathcal{S}'| (1800 \log T)^{-2}.$$

Since A' is a fixed large integer, the last term on the right-hand side of (5.16) is smaller than this lower bound. Thus it can be simplified as

$$|\mathcal{S}'| (\log T)^{-2} \ll U^{2(1/\kappa_1 - \sigma_j)} (\log T)^3 e^{-2U/y}$$

for all $N_j \leq y \leq T^{100}$ and some $U \in [N_j, y^2]$. Noticing that

$$|\mathcal{S}'| \gg |\mathcal{K}'_j(T)| (\log T)^{-1},$$

we obtain

$$(5.17) \quad \begin{aligned} |\mathcal{K}'_j(T)| &\ll y^{2(1/\kappa_1 - \sigma_j)} (\log T)^6 \quad (\text{for all } N_j \leq y \leq T^{100}) \\ &\ll N_j^{(10/3)(1/\kappa_1 - \sigma_j)} (\log T)^{7/3} \quad (\text{for } y \text{ given by (5.19)}). \end{aligned}$$

Next we bound $|\mathcal{K}''_j(T)|$. Let $u_k \in [-A' \log T, A' \log T]$ such that

$$\Phi_{N_j}(s'_k) = \max_{|u| \leq A' \log T} |\Phi_{N_j}(\alpha_j + i(t_k + u))|$$

where $s'_k := \alpha_j + it'_k$ and $t'_k := t_k + u_k$. Thus from (5.13) we deduce that

$$\begin{aligned} \frac{1}{6} &\leq \left| \int_{-A' \log T}^{A' \log T} \Phi_{N_j}(\alpha_j + i(t_k + u)) \Gamma(\alpha_j - v_k + iu) y^{\alpha_j - v_k + iu} du \right| \\ &\leq y^{\alpha_j - v_k} |\Phi_{N_j}(s'_k)| \int_{-A' \log T}^{A' \log T} |\Gamma(\alpha_j - v_k + iu)| du. \end{aligned}$$

Since $\Gamma(s)$ has a simple pole at $s = 0$ and $|\alpha_j - v_k| \gg (\log T)^{-1}$, we can derive, via (4.9), that

$$\int_{-A' \log T}^{A' \log T} |\Gamma(\alpha_j - v_k + iu)| du \ll \log T$$

and thus

$$1 \ll y^{\alpha_j - \sigma_j} |M_{N_j}(s'_k)| \mathfrak{M}(\alpha_j, 8T) \log T,$$

or equivalently

$$|M_{N_j}(s'_k)| \gg y^{\sigma_j - \alpha} (\mathfrak{M}(\alpha_j, 8T) \log T)^{-1}.$$

Hence there is a $V \in [1, N_j]$ such that

$$\left| \sum_{V < n \leq 2V} \tau_{\kappa}^{\langle -1 \rangle}(n; \chi) n^{-s'_k} \right| \gg y^{\sigma_j - \alpha} \mathfrak{M}(\alpha_j, 8T)^{-1} (\log T)^{-2}$$

holds for $\gg |\mathcal{K}_j''(T)| (\log T)^{-1}$ integers $k \in \mathcal{K}_j''(T)$. Let \mathcal{S}'' be the corresponding set of points s'_k . We note $|t'_k| \leq 2T$ and

$$|t'_{k_1} - t'_{k_2}| \geq |t_{k_1} - t_{k_2}| - |u_{k_1} - u_{k_2}| \geq A' \log T.$$

Using Proposition 4.1 with $\theta = \alpha_j := 4\sigma_j - 3/\kappa_1$ and $a_n = \tau_{\kappa}^{\langle -1 \rangle}(n; \chi)$ and in view of the bound

$$\sum_{V < n \leq 2V} \tau_{\kappa, \kappa}(n)^2 n^{-2\alpha_j} \ll V^{1/\kappa_1 - 2\alpha_j} (\log V)^3 \ll V^{7/\kappa_1 - 8\sigma_j} (\log V)^3,$$

it follows that

$$(5.18) \quad \sum_{s'_k \in \mathcal{S}''} \left| \sum_{V < n \leq 2V} \tau_{\kappa}^{\langle -1 \rangle}(n; \chi) n^{-s'_k} \right|^2 \ll (V^{8(1/\kappa_1 - \sigma_j)} + |\mathcal{S}''| \mathfrak{M}(\alpha_j, 8T) V^{4(1/\kappa_1 - \sigma_j)}) (\log V)^3.$$

Take y such that

$$(5.19) \quad \begin{aligned} y^{2(\sigma_j - \alpha_j)} &= A'^3 N_j^{4(1/\kappa_1 - \sigma_j)} \mathfrak{M}(\alpha_j, 8T)^3 (\log T)^4 \\ &= N_j^{10(1/\kappa_1 - \sigma_j)} (\log T)^{-11}. \end{aligned}$$

The left-hand side of (5.18) is

$$\geq |\mathcal{S}''| y^{2(\sigma_j - \alpha_j)} \mathfrak{M}(\alpha_j, 8T)^{-2} (\log T)^{-4}.$$

Hence the inequality (5.18) can be simplified as

$$|\mathcal{S}''| y^{2(\sigma_j - \alpha_j)} \mathfrak{M}(\alpha_j, 8T)^{-2} (\log T)^{-4} \ll N_j^{8(1/\kappa_1 - \sigma_j)}.$$

With

$$|\mathcal{S}''| \gg |\mathcal{K}_j''(T)| (\log T)^{-1},$$

we deduce that

$$(5.20) \quad \begin{aligned} |\mathcal{K}_j''(T)| &\ll N_j^{8(1/\kappa_1 - \sigma_j)} y^{2(\alpha_j - \sigma_j)} \mathfrak{M}(\alpha_j, 8T)^2 (\log T)^5 \\ &\ll N_j^{2(1/\kappa_1 - \sigma_j)} (\log T)^7. \end{aligned}$$

On combining (5.14), (5.17) and (5.20), it follows that

$$|\mathcal{K}_j(T)| \ll N_j^{(10/3)(1/\kappa_1 - \sigma_j)} (\log T)^3.$$

Now the required inequality follows from (3.10). This completes the proof. \square

6. PROOF OF THEOREM 1.1

We shall conserve the notation in Section 2. First we prove a lemma.

Lemma 6.1. *Let $r \in \mathbb{N}$, $\boldsymbol{\kappa} := (\kappa_1, \dots, \kappa_r) \in \mathbb{N}^r$ with $1 \leq \kappa_1 < \dots < \kappa_r \leq 2\kappa_1$, $\mathbf{z} := (z_1, \dots, z_r) \in \mathbb{C}^r$, $\mathbf{B} := (B_1, \dots, B_r) \in (\mathbb{R}^{+*})^r$, and let $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $M > 0$ be some constants. Suppose that the Dirichlet series*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$$

is of type $\mathcal{P}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \mathbf{B}, \mathbf{C}, \alpha, \delta, A, M)$. Then there is an absolute positive constant D and a constant $B = B_1 + \dots + B_r + C_1 + \dots + C_r$ such that we have

$$(6.1) \quad \mathcal{F}(s) \ll MD^B T^{(\delta+B\sqrt{\varepsilon})(1-\kappa_1\sigma)} (\log T)^{A+B}$$

for all $s \in \mathcal{L}_T$, where the implied constant depends only on ε .

Proof. Since we have chosen the principal value of complex logarithm, we can write

$$\begin{aligned} |\zeta(\boldsymbol{\kappa}s)^{\mathbf{z}} \mathbf{L}(\boldsymbol{\kappa}s, \boldsymbol{\chi})^{\mathbf{w}}| &= \prod_{1 \leq i \leq r} |\zeta(\kappa_i s)|^{\Re z_i} |L(\kappa_i s, \chi_i)|^{\Re w_i} e^{-(\Im z_i) \arg \zeta(\kappa_i s) - (\Im w_i) \arg L(\kappa_i s, \chi_i)} \\ &\leq e^{\pi(B_1 + \dots + B_r + C_1 + \dots + C_r)} \prod_{1 \leq i \leq r} |\zeta(\kappa_i s)|^{\Re z_i} |L(\kappa_i s, \chi_i)|^{\Re w_i} \end{aligned}$$

for all $s \in \mathbb{C}$ verifying $\prod_{1 \leq i \leq r} \zeta(\kappa_i s) L(\kappa_i s, \chi_i) \neq 0$. Invoking Proposition 3.1, we see that there is a suitable absolute constant D and a constant $B = B(\mathbf{B}, \mathbf{C})$ depending on (\mathbf{B}, \mathbf{C}) such that

$$(6.2) \quad |\zeta(\boldsymbol{\kappa}s)^{\mathbf{z}} \mathbf{L}(\boldsymbol{\kappa}s, \boldsymbol{\chi})^{\mathbf{w}}| \ll_{\varepsilon, \boldsymbol{\chi}} D^B T^{B\sqrt{\varepsilon}(1-\kappa_1\sigma)} (\log T)^B$$

for all $s \in \mathcal{L}_T$, where the implied constant depends only on $(\varepsilon, \boldsymbol{\chi})$.

Finally the required bound (6.1) follows from (6.2) and the hypothesis (1.6). \square

Now we are ready to prove Theorem 1.1.

Since the Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \mathbf{B}, \mathbf{C}, \alpha, \delta, A, M)$, we can apply [19, Corollary II.2.2.1] with the choice of parameters $\sigma_a = 1/\kappa_1$, $\alpha = \alpha$, $\sigma = 0$ to write

$$\sum_{x < n \leq x + x^{1-1/\kappa_1} y} f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \mathcal{F}(s) \frac{(x + x^{1-1/\kappa_1} y)^s - x^s}{s} ds + O_{\varepsilon} \left(M \frac{x^{1/\kappa_1 + \varepsilon}}{T} \right),$$

where $b = 1/\kappa_1 + 1/\log x$ and $e^{\sqrt{\log x}} \leq T \leq x$ is a parameter to be chosen later.

Denote by Γ_T the path formed from the circle $|s - 1/\kappa_1| = r_0 := 1/(2\kappa_1 \log x)$ excluding the point $s = 1/\kappa_1 - r_0$, together with the segment $[(1 - \delta_T)/\kappa_1, 1/\kappa_1 - r_0]$ traced out twice with respective arguments $+\pi$ and $-\pi$. By the residue theorem, the path $[b - iT, b + iT]$ is deformed into $\Gamma_T \cup [(1 - \delta_T)/\kappa_1 - iT, (1 - \delta_T)/\kappa_1 + iT] \cup [(1 - \delta_T)/\kappa_1 \pm iT, b \pm iT]$. In view of Lemma 6.1, for any $a \in (1/(2\kappa_1), 1/\kappa_1)$, the integral over the horizontal segments

$[a \pm iT, b \pm iT]$ is

$$\begin{aligned}
& \int_{a \pm iT}^{b \pm iT} \left| \mathcal{F}(s) \frac{(x + x^{1-1/\kappa_1}y)^s - x^s}{s} \right| |ds| \\
& \ll \frac{MD^C (\log T)^{A+B}}{T} \int_a^b T^{\max\{(\delta+B\sqrt{\varepsilon})(1-\kappa_1\sigma), 0\}} x^\sigma d\sigma \\
& \ll MD^C \frac{x^{1/\kappa_1}}{T} (\log T)^{A+B} \left(\int_a^{1/\kappa_1} \left(\frac{x^{1/\kappa_1}}{T^{\delta+B\sqrt{\varepsilon}}} \right)^{\kappa_1\sigma-1} d\sigma + 1 \right) \\
& \ll MD^C \frac{x^{1/\kappa_1}}{T} (\log T)^{A+B},
\end{aligned}$$

provided

$$(6.3) \quad T^{\delta+B\sqrt{\varepsilon}} \leq x^{1/\kappa_1}.$$

Thus

$$(6.4) \quad \sum_{x < n \leq x + x^{1-1/\kappa_1}y} f(n) = I + O\left(MD^C \frac{x^{1/\kappa_1+\varepsilon}}{T} \right),$$

where

$$I := \frac{1}{2\pi i} \int_{\Gamma_T \cup [(1-\delta_T)/\kappa_1 - iT, (1-\delta_T)/\kappa_1 + iT]} \mathcal{F}(s) \frac{(x + x^{1-1/\kappa_1}y)^s - x^s}{s} ds.$$

and the implied constant depends on $(\varepsilon, \boldsymbol{\chi})$ only.

Let \mathcal{L}_T be the Motohashi contour defined as in Section 2. Consider the two symmetric simply connected regions bounded by \mathcal{L}_T , the segment $[(1-\delta_T)/\kappa_1 - iT, (1-\delta_T)/\kappa_1 + iT]$ and the two line segments $[\sigma_{j_0+1} + d_v, (1-\delta_T)/\kappa_1]$ with respective arguments $+\pi$ and $-\pi$ measured from the real axis on the right of $1-\delta_T$. It is clear that $\mathcal{F}(s)$ is analytic in these two simply connected regions. Denote by Γ_T^* the path joining (the two end-points of) Γ_T with the two line segments $[\sigma_{j_0+1} + d_v, (1-\delta_T)/\kappa_1]$ of the symmetric regions. Thanks to the residue theorem, we can write

$$(6.5) \quad I = I_1 + I_2,$$

with

$$\begin{aligned}
I_1 &:= \frac{1}{2\pi i} \int_{\Gamma_T^*} \mathcal{F}(s) \frac{(x + x^{1-1/\kappa_1}y)^s - x^s}{s} ds, \\
I_2 &:= \frac{1}{2\pi i} \int_{\mathcal{L}_T} \mathcal{F}(s) \frac{(x + x^{1-1/\kappa_1}y)^s - x^s}{s} ds.
\end{aligned}$$

A. Evaluation of I_1

According to our hypothesis, the function $s \mapsto Z(\kappa_1 s; z_1) \zeta(\boldsymbol{\kappa}_* s)^{z^*} \mathbf{L}(\boldsymbol{\kappa} s, \boldsymbol{\chi})^w \mathcal{G}(s)$ is holomorphic in the disc $|s - 1/\kappa_1| < 1/\kappa_1 - 1/\kappa_2$. In view of (1.12), the Cauchy integral formula implies that

$$(6.6) \quad g_\ell(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) \ll M c^{-\ell} \quad (\ell \geq 0, |\mathbf{z}| \leq |\mathbf{B}|, |\mathbf{w}| \leq |\mathbf{C}|),$$

where $g_\ell(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi})$ is defined as in (1.14) and $c := \frac{2}{3}(1/\kappa_1 - 1/\kappa_2)$. From this and (1.13), we deduce that for any integer $N \geq 0$ and $|s - 1/\kappa_1| \leq \frac{1}{2}(1/\kappa_1 - 1/\kappa_2)$,

$$Z(\kappa_1 s; z_1) \boldsymbol{\zeta}(\boldsymbol{\kappa}_* s)^{z_*} \mathbf{L}(\boldsymbol{\kappa} s, \boldsymbol{\chi})^w \mathcal{G}(s) = \sum_{\ell=0}^N g_\ell(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) \left(s - \frac{1}{\kappa_1}\right)^\ell + O\left(M\left(|s - \frac{1}{\kappa_1}|/c\right)^{N+1}\right).$$

Thus we have

$$(6.7) \quad I_1 = \sum_{\ell=0}^N \kappa_1^{-z_1} g_\ell(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) M_\ell(x, y) + O\left(M c^{-N} E_N(x, y)\right),$$

where

$$M_\ell(x, y) := \frac{1}{2\pi i} \int_{\Gamma_T^*} (s - 1/\kappa_1)^{\ell - z_1} \frac{(x + x^{1-1/\kappa_1} y)^s - x^s}{s} ds,$$

$$E_N(x, y) := \int_{\Gamma_T^*} \left| (s - 1/\kappa_1)^{N+1-z_1} \frac{(x + x^{1-1/\kappa_1} y)^s - x^s}{s} \right| |ds|.$$

Firstly we evaluate $M_\ell(x, y)$. Using the formula

$$(6.8) \quad \frac{(x + x^{1-1/\kappa_1} y)^s - x^s}{s} = \int_x^{x+x^{1-1/\kappa_1} y} t^{s-1} dt$$

and Corollary II.5.2.1 of [19], we write

$$\begin{aligned} M_\ell(x, y) &= \int_x^{x+x^{1-1/\kappa_1} y} \left(\frac{1}{2\pi i} \int_{\Gamma_T^*} (s - 1/\kappa_1)^{\ell - z_1} t^{s-1} ds \right) dt \\ &= \int_x^{x+x^{1-1/\kappa_1} y} t^{1/\kappa_1 - 1} (\log t)^{z_1 - 1 - \ell} \left\{ \frac{1}{\Gamma(\kappa_1 - \ell)} + O\left(\frac{(c_1 \ell + 1)^\ell}{t^{\delta_T/2}}\right) \right\} dt, \end{aligned}$$

where we have used the following inequality

$$47^{|z_1 - \ell|} \Gamma(1 + |z_1 - \ell|) \ll_{B_1} (c_1 \ell + 1)^\ell \quad (\ell \geq 0, |z_1| \leq B_1).$$

The constant c_1 and the implied constant depend at most on B_1 . Besides for $|z_1| \leq B_1$, an elementary computation shows that

$$\begin{aligned} \int_x^{x+x^{1-1/\kappa_1} y} t^{1/\kappa_1 - 1} (\log t)^{z_1 - 1 - \ell} dt &= \int_0^{x^{1-1/\kappa_1} y} (x + t)^{1/\kappa_1 - 1} (\log(x + t))^{z_1 - 1 - \ell} dt \\ &= y' (\log x)^{z_1 - 1 - \ell} \left\{ 1 + O_{B_1} \left(\frac{(\ell + 1)y}{x^{1/\kappa_1} \log x} \right) \right\}. \end{aligned}$$

Inserting this into the preceding formula, we obtain

$$(6.9) \quad M_\ell(x, y) = y' (\log x)^{z_1 - 1 - \ell} \left\{ \frac{1}{\Gamma(z_1 - \ell)} + O_{B_1} \left(\frac{(\ell + 1)y}{\Gamma(z_1 - \ell) x^{1/\kappa_1} \log x} + \frac{(c_1 \ell + 1)^\ell}{x^{\delta_T/2}} \right) \right\}$$

for $\ell \geq 0$ and $|z_1| \leq B_1$.

Next we estimate $E_N(x, y)$. In view of the trivial inequality

$$(6.10) \quad \left| \frac{(x + x^{1-1/\kappa_1} y)^s - x^s}{s} \right| \ll y x^{\sigma - 1/\kappa_1},$$

we deduce that

$$\begin{aligned}
(6.11) \quad E_N(x, y) &\ll \int_{1/2\kappa_1+\varepsilon^2}^{1/\kappa_1-1/\log x} (1/\kappa_1 - \sigma)^{N+1-\Re e z_1} x^{\sigma-1/\kappa_1} y \, d\sigma + \frac{y}{(\log x)^{N+2-\Re e z_1}} \\
&\ll \frac{y}{(\log x)^{N+2-\Re e z_1}} \left(\int_1^\infty t^{N+1-\Re e z_1} e^{-t} \, dt + 1 \right) \\
&\ll y (\log x)^{\Re e z_1-1} \left(\frac{c_1 N + 1}{\log x} \right)^{N+1}
\end{aligned}$$

uniformly for $x \geq y \geq 2$, $N \geq 0$ and $|z_1| \leq B_1$, where the constant $c_1 > 0$ and the implied constant depends only on B_1 .

Inserting (6.9) and (6.11) into (6.7) and using (6.6) and the fact that $y' \asymp y$, we find that

$$(6.12) \quad I_1 = y' (\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{\chi})}{(\log x)^\ell} + O_{B_1}(E_N^*(x, y)) \right\},$$

where

$$E_N^*(x, y) := \frac{y}{x^{1/\kappa_1}} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{\chi})|}{(\log x)^\ell} + \frac{(c_1 N + 1)^{N+1}}{x^{\delta_T/2}} + M \left(\frac{c_1 N + 1}{\log x} \right)^{N+1}.$$

B. Evaluation of I_2

Let \mathcal{L}'_T be the union of those vertical line segments of \mathcal{L}_T whose real part is equal to $\frac{1}{2\kappa_1} + \varepsilon^2$ and $\mathcal{L}''_T := \mathcal{L}_T \setminus \mathcal{L}'_T$. Denote by I'_2 and I''_2 the contribution of \mathcal{L}'_T and \mathcal{L}''_T to I_2 , respectively. Using the trivial inequality

$$\left| \frac{(x + x^{1-1/\kappa_1} y)^s - x^s}{s} \right| \ll \frac{x^{1/2\kappa_1+\varepsilon^2}}{|\tau| + 1} \quad (s \in \mathcal{L}'_T)$$

and Lemma 6.1, we can deduce

$$\begin{aligned}
(6.13) \quad I'_2 &\ll MD^B x^{1/2\kappa_1+\varepsilon^2} T^{(\delta+B\sqrt{\varepsilon})(1/2-\kappa_1\varepsilon^2)} (\log T)^{A+4B+1} \\
&\ll Mx^{(1/2+\delta/2(\psi+\delta))/\kappa_1+\sqrt{\varepsilon}} \\
&\ll Mx^{(1-1/(\psi+\delta))/\kappa_1+\sqrt{\varepsilon}}
\end{aligned}$$

with the value of T given by (6.15) below and $\psi \geq 2$.

Next we bound I''_2 . In view of (6.10), we can write that

$$\begin{aligned}
(6.14) \quad I''_2 &\ll y \int_{\mathcal{L}''_T} |\mathcal{F}(s)| x^{\sigma-1/\kappa_1} \, ds \\
&\ll y \sum_{0 \leq j \leq J_T} \sum_{\substack{0 \leq k \leq K_T \\ \Delta_{j,k} \in (W)}} \int_{\mathcal{L}_T^{[j,k]}} |\mathcal{F}(s)| x^{\sigma-1/\kappa_1} \, ds,
\end{aligned}$$

where $\mathcal{L}_T^{[j,k]}$ is the vertical line segment of \mathcal{L}_T'' around $\Delta_{j,k}$ and the horizontal line segments with $\sigma \leq \sigma_j + d_v$. Clearly the length of $\mathcal{L}_T^{[j,k]}$ is $\ll \log T$. Thus by Lemma 6.1, it follows

$$\int_{\mathcal{L}_T^{[j,k]}} |\mathcal{F}(s)| x^{\sigma-1} \, ds \ll MD^B (\log T)^{A+4B+1} T^{(\delta+B\sqrt{\varepsilon})(1-\kappa_1(\sigma_j+d_v))} x^{\sigma_j+d_v-1/\kappa_1}$$

for all $0 \leq k \leq K_T$. Inserting it into (6.14) and using Proposition 5.1, we can deduce, with the notation $J_{T,0} := [(\frac{1}{2} - \varepsilon) \log T]$, that

$$I_2'' \ll MD^B y (\log T)^{A+4B+18+\eta} (I_{2,*}'' + I_{2,\dagger}''),$$

where

$$I_{2,*}'' := \sum_{0 \leq j \leq J_{T,0}} T^{(\delta+B\sqrt{\varepsilon})(1-\kappa_1(\sigma_j+d_v))} x^{\sigma_j+d_v-1/\kappa_1} \cdot T^{\psi(1-\kappa_1\sigma_j)},$$

$$I_{2,\dagger}'' := \sum_{J_{T,0} < j \leq J_T} T^{(\delta+B\sqrt{\varepsilon})(1-\kappa_1(\sigma_j+d_v))} x^{\sigma_j+d_v-1/\kappa_1} \cdot T^{100\sqrt{\varepsilon}(1-\sigma_j)}.$$

Taking

$$(6.15) \quad T := x^{(1-\kappa_1\sqrt{\varepsilon})/\kappa_1(\psi+\delta+B\sqrt{\varepsilon})}$$

and in view of (2.23), it is easy to check that

$$I_{2,*}'' \ll x^{\varepsilon^2/\kappa_1} \sum_{0 \leq j \leq J_{T,0}} (x^{1/\kappa_1}/T^{\psi+\delta+B\sqrt{\varepsilon}})^{-(1-\kappa_1\sigma_j)} \log x \ll x^{2\varepsilon^2/\kappa_1-\varepsilon^{3/2}} \ll x^{-\varepsilon^2}$$

and

$$I_{2,\dagger}'' \ll \sum_{J_{T,0} < j \leq J_T} (x/T^{\delta+100(B+1)\sqrt{\varepsilon}})^{-(1-\kappa_1\sigma_j)} \ll e^{-2c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}.$$

Inserting it into the preceding estimate for I_2'' , we conclude that

$$(6.16) \quad I_2'' \ll_{\mathbf{B}} My e^{-c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}.$$

Now from (6.4), (6.5), (6.12), (6.13) and (6.16). we deduce that

$$\sum_{x < n \leq x+x^{1-1/\kappa_1}y} f(n) = y'(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi})}{(\log x)^{\ell}} + O(R_N^*(x, y)) \right\}$$

uniformly for $x \geq 3$, $x^{(1-1/(\psi+\delta))/\kappa_1+\varepsilon} \leq y \leq x^{1/\kappa_1}$, $N \geq 0$, $|\mathbf{z}| \leq \mathbf{B}$ and $|\mathbf{w}| \leq \mathbf{C}$, where

$$R_N^*(x, y) := \frac{y}{x^{1/\kappa_1}} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi})|}{(\log x)^{\ell}} + M \left\{ \left(\frac{c_1 N + 1}{\log x} \right)^{N+1} + \frac{(c_1 N + 1)^{N+1}}{e^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}} \right\}$$

for some constants $c_1 > 0$ and $c_2 > 0$ depending only on \mathbf{B} , \mathbf{C} , δ and ε .

It remains to prove that the first term on the right-hand side can be absorbed by the third. In view of (1.12), the Cauchy formula allows us to write $g_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) \ll_{\mathbf{A}, \mathbf{B}, \mathbf{C}, \delta} M 3^{\ell}$ for $|\mathbf{z}| \leq \mathbf{B}$, $|\mathbf{w}| \leq \mathbf{C}$ and $\ell \geq 1$. Combining this with the Stirling formula, we easily derive $\lambda_{\ell}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi}) \ll_{\mathbf{A}, \mathbf{B}, \mathbf{C}, \delta} M (9/\ell)^{\ell}$ for $|\mathbf{z}| \leq \mathbf{B}$, $|\mathbf{w}| \leq \mathbf{C}$ and $\ell \geq 1$. This implies that

$$\frac{y}{x^{1/\kappa_1}} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(\boldsymbol{\kappa}, \mathbf{z}, \mathbf{w}, \boldsymbol{\chi})|}{(\log x)^{\ell}} \ll_{\mathbf{A}, \mathbf{B}, \mathbf{C}, \delta} \frac{My}{x^{1/\kappa_1} \log x}$$

holds uniformly for $x \geq 3$, $x^{(1-1/(\psi+\delta))/\kappa_1+\varepsilon} \leq y \leq x^{1/\kappa_1}$, $N \geq 0$, $|\mathbf{z}| \leq \mathbf{B}$ and $|\mathbf{w}| \leq \mathbf{C}$. This completes the proof.

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REFERENCES

- [1] P. T. Bateman and E. Grosswald, *On a theorem of Erdős and Szekeres*, Illinois J. Math. **2** (1958), 88–89.
- [2] Z. Cui & J. Wu, *The Selberg-Delange method in short intervals with an application*, Acta Arith. **163** (2014), no. 3, 247–260.
- [3] Z. Cui, G.-S. Lü & J. Wu, *The Selberg-Delange method in short intervals with some applications*, Sci China Math, 2018, 61, <https://doi.org/10.1007/s11425-017-9172-7>.
- [4] H. Delange, *Sur les formules dues à Atle Selberg*, Bull. Sc. Math. 2^o série **83** (1959), 101–111.
- [5] J.-M. Deshouillers, F. Dress & G. Tenenbaum, *Lois de répartition des diviseurs, 1*, Acta Arith. **23** (1979), 273–283.
- [6] P. Erdős & M. Kac, *On the Gaussian law of errors in the theory of additive functions*, Proc. Nat. Acad. Sci. U.S.A. **25** (1939), 206–207.
- [7] B. Feng & J. Wu, *Beta law on divisors of integers representable as sum of two squares* (in Chinese), Preprint, 2018.
- [8] C. Hooley, *On intervals between numbers that are sums of two squares III*, J. reine angew. Math. **267** (1974), 207–218.
- [9] M. N. Huxley, *The difference between consecutive primes*, Inven. Math. **15** (1972), 164–170.
- [10] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (2 vols.), Teubner, Leipzig; 3rd edition: Chelsea, New York (1974).
- [11] H. L. Montgomery, *Zeros of L-functions*, Inven. Math. **8** (1969), 346–354.
- [12] H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics **227**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
- [13] Y. Motohashi, *On the sum of the Möbis function in a short segment*, Proc. Japan Acad. **52** (1976), 477–479.
- [14] K. Ramachandra, *Some problems of analytic number theory*, Acta Arith. **31** (1976), 313–324.
- [15] H. E. Richert, *Zur abschätzung der Riemanschen zetafunktion in der nähe der vertikalen $\sigma = 1$* , Math. Ann. **169** (1967), 97–101.
- [16] L. G. Sathe, *On a problem of Hardy and Ramanujan on the distribution of integers having a given number of prime factors*, J. Indian Math. Soc. **17** (1953), 63–141.
- [17] L. G. Sathe, *On a problem of Hardy and Ramanujan on the distribution of integers having a given number of prime factors*, J. Indian Math. Soc. **18** (1954), 27–81.
- [18] A. Selberg, *Note on the paper by L. G. Sathe*, J. Indian Math. Soc. **18** (1954), 83–87.
- [19] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Translated from the second French edition (1995) by C. B. Thomas, Cambridge Studies in Advanced Mathematics **46**, Cambridge University Press, Cambridge, 1995. xvi+448 pp.
- [20] E. C. Titchmarsh, *The theory of function*, Second edition, Oxford University Press, Oxford, 1952.
- [21] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition, Revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986. x+412 pp.
- [22] J. Wu & Q. Wu, *On the distribution of divisors de certains integers in short intervals*, Preprint 2018.

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