

Optimal control of resources for species survival

Idriss Mazari, Grégoire Nadin, Yannick Privat

▶ To cite this version:

Idriss Mazari, Grégoire Nadin, Yannick Privat. Optimal control of resources for species survival. 89th GAMM Annual Meeting, Mar 2018, munich, Germany. hal-01846136

HAL Id: hal-01846136 https://hal.science/hal-01846136

Submitted on 20 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Optimal control of resources for species survival

Idriss Mazari^{*} Grégoire Nadin[†] Yannick Privat[‡]

Abstract

Consider a species whose population density solves the steady diffusive logistic equation in a heterogeneous environment modeled with the help of a spatially non constant coefficient standing for a resources distribution in a given box. We look at maximizing the total population size with respect to resources distribution, under some biologically relevant constraints. Assuming that the diffusion rate of the species is large enough, we prove that any optimal configuration is the characteristic function of a domain standing for the resources location. Moreover, we highlight that optimal configurations look *concentrated* whenever the diffusion rate is large enough. In the one-dimensional case, this problem is deeply analyzed, and for large diffusion rates, all optimal configurations are exhibited.

1 Setting of the problem

Let $\Omega = (0,1)^n$. We investigate an optimal control problem arising in population dynamics. Consider the population density $\theta_{m,\mu}$ of a given species evolving in Ω , assumed to solve the socalled steady logistic-diffusive equation writing

$$\begin{cases} \mu \Delta \theta_{m,\mu}(x) + (m(x) - \theta_{m,\mu}(x))\theta_{m,\mu}(x) = 0 & x \in \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 & x \in \partial\Omega, \end{cases}$$
(LDE)

where m is a bounded function of Ω standing for the resources distribution and $\mu > 0$ stands for the species velocity also called *diffusion rate*. From a biological point of view, the real number m(x) is a measure of the resources available at x of the habitat Ω and can be seen as the local intrinsic growth rate of species at location $x \in \Omega$.

The pioneering works by Fisher [2], Kolmogorov-Petrovski-Piskounov [5] and Skellam [11] sparked a new interest in the study of the heterogeneity influence of the environment on the growth of a population densities. In the present work, we look at investigating the influence of heterogeneous environment on the density population. To this aim, for a given $\mu > 0$, we address the optimal control problem of maximizing the total population size, given by

$$F_{\mu}(m) = \int_{\Omega} \theta_{m,\mu},\tag{1}$$

where $\theta_{m,\mu}$ denotes the solution of equation (LDE). In the framework of population dynamics, the density $\theta_{m,\mu}$ solving Equation (LDE) can be interpreted as a steady state associated to the

^{*}Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (idriss.mazari@upmc.fr).

[†]CNRS, Université Pierre et Marie Curie (Univ. Paris 6), UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (gregoire.nadin@upmc.fr).

[‡]IRMA, Université de Strasbourg, CNRS UMR 7501, 7 rue René Descartes, 67084 Strasbourg, France (yannick.privat@unistra.fr).

following evolution equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mu \Delta u(t,x) + u(t,x)(m(x) - u(t,x)) & t > 0, \ x \in \Omega\\ \frac{\partial u}{\partial \nu}(t,x) = 0 & t > 0, \ x \in \partial \Omega\\ u(0,x) = u^{0}(x) & x \in \Omega \end{cases}$$
(LDEE)

modeling the spatiotemporal behavior of a population density u in a domain Ω with the spatially heterogeneous resource term m. More precisely, provided that $\int_{\Omega} m > 0$, the steady state $\theta_{m,\mu}$ is globally asymptotically stable meaning that, for any nonnegative and nonzero function $u^0 \in H^1(\Omega)$, $u(t, \cdot)$ converges uniformly to $\theta_{m,\mu}$ as $t \to +\infty$.

From the biological point of view, it is relevant to assume that the density m is nonnegative and bounded by a positive constant κ and that $f_{\Omega} m$, the total amount of resources, is fixed. This leads to address the issue:

For given diffusivity and total amount of resources, which weight m maximizes the total population size among all uniformly bounded elements ?

This issue rewrites more precisely

Optimal design problem. Fix $n \in \mathbb{N}^*$, $\mu > 0$, $\kappa > 0$, $m_0 \in (0, \kappa)$. We consider the optimization problem $\sup_{m \in \mathcal{M}_{\kappa,m_0}(\Omega)} F_{\mu}(m) \quad with \quad \mathcal{M}_{\kappa,m_0}(\Omega) := \left\{ m \in L^{\infty}(\Omega) , 0 \leq m \leq \kappa \text{ a.e and } f_{\Omega} m = m_0 \right\}. \quad (\mathcal{P}_{\mu})$

It is notable that similar issues for linear models have been addressed for instance in [3, 4, 6, 7, 8, 10].

2 Optimal resources sets

This section is devoted to stating two important properties of solutions of (\mathcal{P}_{μ}) , whose proof can be found in [9].

Property no. 1: pointwise constraints, bang-bang property. We investigate hereafter the *bang-bang* character of maximizers for Problem (\mathcal{P}_{μ}) , in other words, whether a solution m^* is equal to 0 or κ a.e. in Ω . Such an issue is of importance, since dealing with *bang-bang* functions enables the use of adapted numerical approaches.

Theorem 1 ([9], Theorem 1). Let $n \in \mathbb{N}^*$, $\mu > 0$, $\kappa > 0$, $m_0 \in (0, \kappa)$. There exists a positive number $\mu^* = \mu^*(\Omega, \kappa, m_0)$ such that, for every $\mu \ge \mu^*$ every maximizer of Problem (\mathcal{P}_{μ}) is bangbang. As a consequence, $m^* = \kappa \chi_E$, where χ_E is the characteristic function of a (measurable) resources set E.

The main difficulty for establishing this result rests upon the facts that $\theta_{m,\mu}$ solves a nonlinear PDE and the criterion F_{μ} does not derive from an energy, making the exploitation of adjoint approaches intricate.





Figure 1: Concentrated distribution.



Figure 2: Fragmented distribution.

Figure 3: A solution of (\mathcal{P}_{μ}) in 1D, for large diffusivities μ .

Property no. 2: concentration-fragmentation of maximizers. It is well-known (see e.g. [1, 6]) that concentrating resources, meaning that the resources distribution m is decreasing in each direction, favors the survival of species. On the contrary, we will say that a resources set is fragmented whenever it is disconnected. On Figures 1-2 below, Ω is a square, and the intuitive notion of *concentration-fragmentation* of resources distributions is illustrated.

Theorem 2 ([9], Theorem 2). Any family of maximizers $\{m_{\mu}\}_{\mu>0}$ converges in $L^{1}(\Omega)$ to the characteristic function of a set E which is concentrated (meaning that its characteristic function χ_{E} is monotone with respect to each space variable).

In the one-dimensional case, we also prove that if the diffusivity is large enough, there are only two maximizers, that are simple crenels meeting either the left or the right boundary (see Fig. 3 above). Finally, in the one-dimensional case, we obtain a surprising result: fragmentation may be better than concentration for small diffusivities.

Acknowledgement. The authors were partially supported by the Project "Analysis and simulation of optimal shapes - application to lifesciences" of the Paris City Hall.

References

- H. Berestycki, F. Hamel, and L. Roques. Analysis of the periodically fragmented environment model: I – species persistence. *Journal of Mathematical Biology*, 51(1):75–113, 2005.
- [2] R. A. Fisher. The wave of advance of advantageous genes. Annals of Eugenics, 7(4):355–369, 1937.
- [3] K. Jha and G. Porru. Minimization of the principal eigenvalue under Neumann boundary conditions. Numer. Funct. Anal. Optim., 32(11):1146–1165, 2011.
- [4] C.-Y. Kao, Y. Lou, and E. Yanagida. Principal eigenvalue for an elliptic problem with indefinite weight on cylindrical domains. *Math. Biosci. Eng.*, 5(2):315–335, 2008.
- [5] A. Kolmogorov, I. Pretrovski, and N. Piskounov. étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Moscow University Bulletin of Mathematics*, 1:1–25, 1937.
- [6] J. Lamboley, A. Laurain, G. Nadin, and Y. Privat. Properties of optimizers of the principal eigenvalue with indefinite weight and Robin conditions. *Calc. Var. Partial Differential Equations*, 55(6):55:144, 2016.
- [7] Y. Lou. Tutorials in Mathematical Biosciences IV: Evolution and Ecology, chapter Some Challenging Mathematical Problems in Evolution of Dispersal and Population Dynamics, pages 171–205. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.
- [8] Y. Lou and E. Yanagida. Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics. *Japan J. Indust. Appl. Math.*, 23(3):275–292, 2006.
- [9] I. Mazari, G. Nadin, and Y. Privat. Optimal location of resources maximizing the total population size in logistic models. Preprint HAL, 2018.
- [10] L. Roques and F. Hamel. Mathematical analysis of the optimal habitat configurations for species persistence. *Math. Biosci.*, 210(1):34–59, 2007.
- [11] J. G. Skellam. Random dispersal in theoretical populations. *Biometrika*, 38:196–218, 1951.