



HAL
open science

Utility maximization for Lévy switching models

Lioudmila Vostrikova, Yuchao Dong

► **To cite this version:**

Lioudmila Vostrikova, Yuchao Dong. Utility maximization for Lévy switching models. 2018. hal-01844635

HAL Id: hal-01844635

<https://hal.science/hal-01844635>

Preprint submitted on 19 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Utility maximization for Lévy switching models

Yuchao Dong

Université d'Angers,
Département de Mathématiques,
2, Bd Lavoisier,
49045 Angers Cedex 01, France,

Lioudmila Vostrikova

Université d'Angers,
Département de Mathématiques,
2, Bd Lavoisier,
49045 Angers Cedex 01, France,

July 19, 2018

Abstract

This article is devoted to the maximisation of HARA utilities of Lévy switching process on finite time interval via dual method. We give the description of all f -divergence minimal martingale measures in initially enlarged filtration, the expression of their Radon-Nikodym densities involving Hellinger and Kulback-Leibler processes, the expressions of the optimal strategies in progressively enlarged filtration for the maximisation of HARA utilities as well as the values of the corresponding maximal expected utilities. The example of Brownian switching model is presented to give the financial interpretation of the results.

KEY WORDS AND PHRASES: Lévy switching models, utility maximisation, dual approach, f -divergence minimal martingale measure, optimal strategy

MSC 2010 subject classifications: 60G07, 60G51, 91B24

1 Introduction

Regime switching processes are the processes whose parameters depend on some Markov process, taking a finite number of values. Those processes have been applied successfully since more than 20 years to model the behaviour of economic time series and financial time series. Hamilton and al. in [18] considered regime switching autoregressive conditional heteroskedastic models and proposed a way for their calibration. Cai in [1] considered regime switching ARCH models. So and al. in [28] considered the so-called Markov switching volatility models to capture the changing of the behaviour of the volatility due to the economic factors.

Lévy switching processes was widely used for pricing due to their flexibility to reflect the real data and their comodity as mathematical tool. For two state variance

gamma model, Komkov and Madan in [26] have developed a method based on fast Fourier Transform to price the vanilla options. Later, using the implementation of numerical methods of resolving PDE equations, Chordakis [8] has priced the exotic contracts, like barrier, Bermuda and American options. Then, Elliott and Osakwa in [9] used multi-state pure jump processes to price vanilla derivatives. Later, Elliott and al. [10] considered Heston Markov switching model to price variance swap and volatility swap with probabilistic and PDE approaches. Hainaut [19] gave an overview of the tools related to Lévy switching models, like moments and the conditions under which such processes are martingales, the measure transform preserving Lévy switching structure, and also the questions of option pricing and calibration.

The question of the optimal hedging in discrete time methodology that minimizes the expected value of given penalty function of the hedging error with general regime-switching framework was studied by So and al. [28]. Francois and al. [14] considered the question of the portfolio optimisation in affine models with Markov switching. Escobar and al. [11] derive the optimal investment strategies for the expected utility maximization of terminal wealth.

The utility maximization problem of the Lévy switching models can be solved by the so-called dual method. The dual method for utility maximization has been deeply studied by Goll and Ruschendorf in [16] in general semi-martingale setting. This method permits to find a semi-explicit NA conditions as well as a semi-explicit expression of the optimal (asymptotically optimal) strategies for utility maximization. This method was applied for pricing and hedging of exponential Lévy models in Miyahara [27], Fujiwara and Myahara [15], Chouli and al. [6], [7], Essche and Schweizer [12], Hubalek and Sgarra [20], Jeanblanc and al. [24], Cawston and Vostrikova [3], [4]. This method was also efficient to determine the indifference prices in Ellanskaya and Vostrikova [13], Vostrikova [29]. Then, in Cawston and Vostrikova [5] the problem of utility maximization was solved for change-point exponential Lévy models. The Lévy switching models are natural generalisations of change-point exponential Lévy models, permitting multiple changes of the behaviour of Lévy processes on the observed interval of time.

This article is devoted to the utility maximisation problem for Lévy switching models. More precisely, we consider N independent Lévy processes $X^{(j)} = (X_t^{(j)})_{0 \leq t \leq T}$, $j = 1, 2, \dots, N$, on the interval $[0, T]$, with values in \mathbb{R}^d and starting from 0, which are defined on a probability space $(\Omega_1, \mathcal{G}, P_1)$ with the natural filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ satisfying usual conditions. Each Lévy process $X^{(j)}$ has the characteristic triplet denoted by $(b^{(j)}, c^{(j)}, \nu^{(j)})$ where $b^{(j)}$ is the drift parameter, $c^{(j)}$ is the quadratic variation of its continuous martingale part and $\nu^{(j)}$ is Lévy measure which verifies the usual condition

$$\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) \nu^{(j)}(dx) < \infty$$

where $\|\cdot\|$ is euclidean norm in \mathbb{R}^d .

As known, the law of $X^{(j)}$ is entirely characterized by the characteristic function $\varphi_t^{(j)}$ of $X_t^{(j)}$ with $t > 0$, which is, for $\lambda \in \mathbb{R}^d$, given by

$$\varphi_t^{(j)}(\lambda) = \exp\{t\psi^{(j)}(\lambda)\}$$

where $\psi^{(j)}$ is the characteristic exponent of $X^{(j)}$. We recall that $\psi^{(j)}$ is determined by the Lévy-Khinchin formula, namely

$$\psi^{(j)}(\lambda) = i\langle \lambda, b^{(j)} \rangle - \frac{1}{2}\langle \lambda, c^{(j)} \lambda \rangle + \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i\langle \lambda, x \rangle I_{\{\|x\| \leq 1\}}) \nu^{(j)}(dx)$$

where $I(\cdot)$ denotes the indicator function.

Let also $\alpha = (\alpha_t)_{0 \leq t \leq T}$ be a homogeneous Markov process with the values in the set $\{1, 2, 3, \dots, N\}$, given on the probability space $(\Omega_2, \mathcal{H}, P_2)$ with the natural filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ such that $\mathcal{H}_0 = \{\emptyset, \Omega\}$, which is independent from the Lévy processes $X^{(1)}, X^{(2)}, \dots, X^{(N)}$.

Then on the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{G} \times \mathcal{H}, P_1 \times P_2)$ we can define two filtrations. One is the progressively enlarged filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ where for $t < T$

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{G}_s \otimes \mathcal{H}_s \quad \text{and} \quad \mathcal{F}_T = \mathcal{G}_T \otimes \mathcal{H}_T$$

and the second one is the initially enlarged filtration $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{0 \leq t \leq T}$ defined for $t < T$ as

$$\hat{\mathcal{F}}_t = \bigcap_{s>t} \mathcal{G}_s \otimes \mathcal{H}_T \quad \text{and} \quad \hat{\mathcal{F}}_T = \mathcal{G}_T \otimes \mathcal{H}_T.$$

For technical reasons we will work in the initially enlarged filtration, and surprisingly at the first glance, we will obtain the results in progressively enlarged filtration for minimal martingale measures and the optimal strategies for utility maximisation. Such phenomenon can be explained by the fact that firstly both filtrations coincide at the time T , and secondly, by the fact that the Lévy processes $X^{(j)}, j = 1, \dots, N$, remain independent from the Markov process α under any minimal martingale measure.

On the defined above product space, we now introduce a Lévy switching process X such that the increments of this process coincide with the increments of $X^{(j)}$ when the process α states in the state j :

$$dX_t = \sum_{j=1}^N dX_t^{(j)} I_{\{\alpha_{t-}=j\}} \tag{1}$$

or

$$X_t = \sum_{j=1}^N \int_0^t I_{\{\alpha_{s-}=j\}} dX_s^{(j)}.$$

More explicitly, if, for example, at $t = 0$, $\alpha_0 = i_0$ and τ_1 is the first time of change from the state i_0 to another state i_1 and $\tau_k = \inf\{t > \tau_{k-1} | \alpha_t \neq i_{k-1}\}$ for $k \geq 1$, then

$$X_t = \begin{cases} X_t^{(i_0)}, & \text{for } t \leq \tau_1, \\ X_{\tau_1}^{(i_0)} + X_t^{(i_1)} - X_{\tau_1}^{(i_1)}, & \text{for } \tau_1 < t \leq \tau_2, \\ \dots & \\ X_{\tau_n}^{(i_{n-1})} + X_t^{(i_n)} - X_{\tau_n}^{(i_n)}, & \text{for } \tau_n < t \leq \tau_{n+1}, \\ \dots & \end{cases}$$

The characteristic function of X can be find easily since X is a process with, conditionally to α , independent increment. Due to the mutual independence of Lévy processes and α , we have

$$\begin{aligned} E_{\mathbb{P}}[\exp(i\langle \lambda, X_t \rangle)] &= E_{\mathbb{P}}[E_{\mathbb{P}}[\exp(i\langle \lambda, X_t \rangle) | \alpha]] \\ &= E_{\mathbb{P}}[E_{\mathbb{P}}[\exp(i\langle \lambda, \sum_{j=1}^N \int_0^t I_{\{\alpha_{s-}=j\}} dX_s^{(j)} \rangle) | \alpha]] \\ &= E_{\mathbb{P}}[\prod_{j=1}^N E_{\mathbb{P}}[\exp(i\langle \lambda, \int_0^t I_{\{\alpha_{s-}=j\}} dX_s^{(j)} \rangle) | \alpha]]. \end{aligned}$$

We remark that for any real-valued deterministic function $q = (q_s)_{s \geq 0}$ such that $\int_0^t q_s dX_s^{(j)}$ exists, the characteristic function verifies

$$\begin{aligned} E_{\mathbb{P}}[\exp(i\langle \lambda, \int_0^t q_s dX_s^{(j)} \rangle)] &= \exp(i\langle \lambda, b^{(j)} \rangle) \int_0^t q_s ds - \frac{1}{2} \langle \lambda, c^{(j)} \lambda \rangle \int_0^t q_s^2 ds + \\ &\quad \int_0^t \int_{\mathbb{R}^d} (e^{i\langle \lambda q_s, x \rangle} - 1 - i\langle \lambda q_s, x \rangle I_{\{\|x\| \leq 1\}}) \nu^{(j)}(ds, dx). \end{aligned}$$

If $q_s = I_{\{\alpha_{s-}=j\}}$ for $s \geq 0$, then $q_s^2 = q_s$ and if $I_{\{\alpha_{s-}=j\}} = 1$ then

$$\int_{\mathbb{R}^d} (e^{i\langle \lambda q_s, x \rangle} - 1 - i\langle \lambda q_s, x \rangle I_{\{\|x\| \leq 1\}}) \nu^{(j)}(dx) = \int_{\mathbb{R}^d} (e^{i\langle \lambda, x \rangle} - 1 - i\langle \lambda, x \rangle I_{\{\|x\| \leq 1\}}) \nu^{(j)}(dx).$$

In addition, if $I_{\{\alpha_{s-}=j\}} = 0$ then this integral is equal to 0. Finally,

$$E_{\mathbb{P}}[\exp(i\langle \lambda, X_t \rangle)] = E_{\mathbb{P}}[\exp(\sum_{j=1}^N \psi^{(j)}(\lambda) \int_0^t I_{\{\alpha_{s-}=j\}} ds)].$$

We will consider the problem of the utility maximization for the process $S = (S_t)_{t \geq 0}$ such that the components of S denoted by $S^{(k)}$, $1 \leq k \leq d$, are Doléans-Dade exponentials of the corresponding components of X , denoted by $\bar{X}^{(k)}$, i.e. for all $t \geq 0$

$$S_t^{(k)} = S_0^{(k)} \exp\{\bar{X}_t^{(k)} - \frac{1}{2} \langle \bar{X}^{(k), c} \rangle_t\} \prod_{0 < s \leq t} e^{-\Delta \bar{X}_s^{(k)}} (1 + \Delta \bar{X}_s^{(k)})$$

where $\bar{X}^{(k),c}$ is continuous martingale part of $\bar{X}^{(k)}$ and $\langle \bar{X}^{(k),c} \rangle$ is its predictable quadratic variation.

As utility functions, we consider HARA utilities, which are logarithmic, power and exponential utilities, defined as

$$\begin{aligned} u(x) &= \ln(x) \text{ with } x > 0, \\ u(x) &= \frac{x^p}{p} \text{ with } x > 0 \text{ and } p \in (-\infty, 0) \cup (0, 1), \\ u(x) &= 1 - \exp(-x) \text{ with } x \in \mathbb{R}. \end{aligned}$$

Let us denote by \mathcal{A} a set of self-financing admissible strategies. We recall that an admissible strategy is a predictable process $\Phi = (\eta, \phi)$ taking values in \mathbb{R}^{d+1} where η represents the quantity invested in the non-risky asset B and $\phi = (\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(d)})$ are the quantities invested in the risky assets $S^{(1)}, S^{(2)}, \dots, S^{(d)}$ respectively such that η is B -integrable and there exists $a \in \mathbb{R}^+$ such that for $t \in [0, T]$

$$\sum_{k=1}^d \int_0^t \phi_s^{(k)} dS_s^{(k)} \geq -a.$$

We say that a strategy $\hat{\phi} \in \mathcal{A}$ is u -optimal on $[0, T]$ if

$$E[u(x_0 + \sum_{k=1}^d \int_0^T \hat{\phi}_s^{(k)} dS_s^{(k)})] = \sup_{\phi \in \mathcal{A}} E[u(x_0 + \sum_{k=1}^d \int_0^T \phi_s^{(k)} dS_s^{(k)})]$$

where $x > 0$ is initial capital. A sequence of admissible strategies $(\hat{\phi}^n)_{n \geq 1}$ is said to be asymptotically u -optimal on $[0, T]$ if

$$\lim_{n \rightarrow \infty} E[u(x_0 + \sum_{k=1}^d \int_0^T \hat{\phi}_s^{(k),n} dS_s^{(k)})] = \sup_{\phi \in \mathcal{A}} E[u(x_0 + \sum_{k=1}^d \int_0^T \phi_s^{(k)} dS_s^{(k)})].$$

The article is organized in the following way. In Section 2 we give a short description of the known results on dual method for the utility maximisation of Lévy processes (see Propositions 2 and 3). In Section 3 we summarize the useful information on Hellinger and Kulback-Leibler processes (see Proposition 4). In Section 4 we give a description of all f -divergence minimal martingale measures for HARA utilities in progressively enlarged filtration (see Propositions 5,6 and Theorem 1). Then, in Propositions 7, 8, 9 of Section 5 we give the expressions for the optimal strategies in progressively enlarged filtration and we calculate the value of the maximal expected utility. In section 6 we apply our results to Brownian switching model and we give the financial interpretation of our results.

2 Dual approach for utility maximisation of exponential Lévy models

The idea of this method is to replace the problem of utility maximization by the problem of minimization of the corresponding f -divergence over the set of all, equivalent to the law \mathbb{P}_T of the process X , martingale measures \mathbb{Q}_T where T is time horizon. The function f is nothing else as the dual function of u , which can be obtained by Fenchel-Legendre transform :

$$f(y) = \sup_{x \in \mathbb{R}} (u(x) - xy)$$

Simple calculations show that

$$f(x) = -\ln(x) - 1, x > 0 \text{ if } u \text{ is logarithmic,}$$

$$f(x) = -\frac{p-1}{p} x^{\frac{p}{p-1}}, x > 0 \text{ if } u \text{ is power,}$$

$$f(x) = 1 - x + x \ln(x), x > 0 \text{ if } u \text{ is exponential.}$$

We recall that for two equivalent measures \mathbb{Q}_T and \mathbb{P}_T the f -divergence of \mathbb{Q}_T w.r.t. \mathbb{P}_T is defined as

$$f(\mathbb{Q}_T|\mathbb{P}_T) = E_{\mathbb{P}} \left[f \left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \right) \right].$$

We mention now some useful definitions and results on the f -divergence problem and optimal investment problem in exponential Lévy models. For that, we take a Lévy process $L = (L_t)_{0 \leq t \leq T}$ with the values in \mathbb{R}^d , given on probability space (Ω, \mathcal{F}, P) with the filtration \mathbb{F} satisfying usual properties. We denote the characteristic triplet of the process L by (b_L, c_L, ν_L) where b_L is drift parameter, c_L is the predictable variation of the continuous martingale part of L , and ν_L is Lévy measure. We denote by \mathbb{P} the law of L and we suppose that L is integrable, i.e. $E_{\mathbb{P}}(|L_t|) < \infty$ for $t \in [0, T]$.

Definition 1. We say that \mathbb{Q}_T^* is an f -divergence minimal equivalent martingale measure for Lévy process $L = (L_t)_{0 \leq t \leq T}$ if

1. the measure \mathbb{Q}_T^* is equivalent to \mathbb{P}_T , i.e. $\mathbb{Q}_T^* \sim \mathbb{P}_T$,
2. the process L is a martingale w.r.t. $(\mathbb{F}, \mathbb{Q}_T^*)$,
3. $f(\mathbb{Q}_T^*|\mathbb{P}_T) < +\infty$ and

$$f(\mathbb{Q}_T^*|\mathbb{P}_T) = \inf_{\mathbb{Q}_T \in \mathcal{M}} f(\mathbb{Q}_T|\mathbb{P}_T)$$

where \mathcal{M} is the set of all equivalent martingale measures \mathbb{Q}_T .

Definition 2. We say that an f -divergence minimal martingale measure \mathbb{Q}_T^* is invariant under scaling if for all $x \in \mathbb{R}^+$,

$$f(x\mathbb{Q}_T^*|\mathbb{P}_T) = \inf_{\mathbb{Q}_T \in \mathcal{M}} f(x\mathbb{Q}_T|\mathbb{P}_T)$$

Definition 3. We say that an f -divergence minimal martingale measure \mathbb{Q}_T^* is time horizon invariant if for all $t \in]0, T]$

$$f(\mathbb{Q}_t^*|\mathbb{P}_t) = \inf_{\mathbb{Q}_t \in \mathcal{M}} f(\mathbb{Q}_t|\mathbb{P}_t)$$

Definition 4. We say that an f -divergence minimal martingale measure \mathbb{Q}_T^* preserves the Lévy property if L remains a Lévy process under \mathbb{Q}_T^* .

Proposition 1. (cf. [3],[4]) Let L be an integrable Lévy process, f be one of the three dual functions above and \mathbb{Q}_T^* the corresponding f -divergence minimal martingale measure. Then this measure is time horizon invariant, it is invariant under scaling and it preserves Lévy property of the process L .

From Girsanov theorem (see [23], Ch. 3, Theorem 3.24, p.159) we can also easily get the following.

Proposition 2. The Radon-Nikodym process $Z^* = (Z_t^*)_{0 \leq t \leq T}$ of the measure \mathbb{Q}_T^* w.r.t. \mathbb{P}_T verify:

$$Z_t^* = \frac{d\mathbb{Q}_t^*}{d\mathbb{P}_t} = Z_0^* \mathcal{E}(m)_t$$

where $\mathcal{E}(\cdot)$ is Doléans-Dade exponential and $m = (m_t)_{0 \leq t \leq T}$ is a martingale with

$$m_t = \int_0^t \top \beta^* dY_s^c + \int_0^t \int_{\mathbb{R}^d} (Y^*(x) - 1)(\mu_L - \nu_L)(ds, dx)$$

where μ_L and ν_L are jump measure of the process L and its compensator under (\mathbb{P}, \mathbb{F}) . The so-called Girsanov parameters (β^*, Y^*) for the change of the measure \mathbb{P}_T into \mathbb{Q}_T^* are independent of (ω, t) . Moreover, the drift $b_L^{\mathbb{Q}}$ of the process L under \mathbb{Q}_T^* is:

$$b_L^{\mathbb{Q}} = b_L + c_L \beta^* + \int_{\mathbb{R}^d} x \cdot (Y^*(x) - 1) \nu_L(dx)$$

Proposition 3. (cf.[16],[3]) Let Z_T^* be Radon-Nikodym derivative of f -divergence minimal martingale measure \mathbb{Q}_T^* with respect to \mathbb{P}_T . Let $x_0 > 0$ be the initial capital. We suppose that for $\lambda_0 > 0$ such that

$$-E_{\mathbb{P}}(Z_T^* f'(\lambda_0 Z_T^*)) = x_0 \tag{2}$$

we have

$$E_{\mathbb{P}}(|f(\lambda_0 Z_T^*)|) < \infty, \quad E_{\mathbb{P}}(Z_T^* |f'(\lambda_0 Z_T^*)|) < \infty.$$

Then there exists an optimal (respectively asymptotically optimal) strategy $\hat{\phi}$ such that

$$-f'(\lambda_0 \mathbb{Z}_T^*) = x_0 + \sum_{k=1}^d \int_0^T \hat{\phi}_s^{(k)} dS_s^{(k)}$$

where $(\int_0^\cdot \hat{\phi}_s^{(k)} dS_s^{(k)})$ are \mathbb{Q}_T^* -martingales, $1 \leq k \leq d$. If we choose $\hat{\phi}_t^0$ such that

$$B_t \hat{\phi}_t^0 = x_0 + \sum_{k=1}^d \int_0^t \hat{\phi}_s^{(k)} dS_s^{(k)} - \sum_{k=1}^d \hat{\phi}_t^{(k)} S_t^{(k)}$$

then the strategy $\Phi = (\hat{\phi}^0, \hat{\phi}^{(1)}, \dots, \hat{\phi}^{(d)})$ is self-financing and it is optimal for logarithmic and power utility and asymptotically optimal for exponential utility. Moreover, the maximal expected utility

$$U_T(x_0) = E_{\mathbb{P}}[u(-f'(\lambda_0 \mathbb{Z}_T^*))].$$

3 Hellinger integrals, Kulback-Leibler informations and the corresponding processes

We recall here some useful results on Hellinger integrals and Kulback-Leibler informations and corresponding processes. As known, these notions were introduced in semi-martingale setting and one can find the details about these processes in [23], [25]. For convenience of the readers we will restrict ourselves to the case of the processes with independent increments.

Let V be the process with independent increments which is a semi-martingale without predictable jumps, observed on the interval $[0, T]$. We denote the triplet of the semi-martingale characteristics by (B, C, ν) where B stands for the drift part, C is the predictable variation of the continuous martingale part, and ν is the compensator of the jumps of the process V . We denote P_T the law of V which is entirely defined by the triplet (B, C, ν) (see for the details [23], Ch.2, p.114).

Let $Q_T \ll P_T$ and $Z_T = \frac{dQ_T}{dP_T}$ with the Girsanov parameters (β, Y) . We recall that the Hellinger integral of order $\gamma \in]0, 1[$ of the measure Q_T w.r.t. the measure P_T

$$\mathbf{H}^{(\gamma)}(Q_T | P_T) = E_{\mathbb{P}} [Z_T^\gamma]$$

and this definition can be extended for $\gamma < 0$ when the integral exists. We define the Hellinger process $H(\gamma) = (H_t(\gamma))_{0 \leq t \leq T}$ of the order γ via the expression

$$H_t(\gamma) = \frac{1}{2} \gamma(1 - \gamma) \int_0^t \beta_s^\top dC_s \beta_s - \int_0^t \int_{\mathbb{R}^d} [Y_s^\gamma(x) - \gamma Y_s(x) - 1 + \gamma] \nu(ds, dx)$$

The Kulback-Leibler information of the measure Q_T w.r.t. the measure P_T is defined as

$$\mathbf{K}(Q_T | P_T) = E_{\mathbb{P}} [Z_T \ln(Z_T)]$$

The Kulback-Leibler process $K = (K_t)_{0 \leq t \leq T}$ is defined as

$$K_t = \frac{1}{2} \int_0^t \top \beta_s dC_s \beta + \int_0^t \int_{\mathbb{R}^d} [Y_s(x) \ln(Y_s(x)) - Y_s(x) + 1] \nu(ds, dx)$$

Proposition 4. (see [13]) For the process with independent increments V and Girsanov parameters (β, Y) of the change of the measure P into Q , which depend only on (t, x) , we have:

$$\mathbf{H}(Q_T | P_T) = \exp(-H_T(\gamma)),$$

$$\mathbf{K}(Q_T | P_T) = K_T.$$

4 F-divergence minimal martingale measures

First of all we describe the set of all equivalent martingale measures living on the space of càdlàg functions $(D^2([0, T]), \mathcal{D}^2([0, T]))$ of the trajectories of the couple of the processes (X, α) . Let \mathbb{P}_T denote the law of (X, α) and let P_T^X and P_T^α are the laws of X and α respectively, on $[0, T]$. We denote also by $P_T^X(\cdot | \alpha)$ the regular conditional law of X given α .

Proposition 5. The law $\mathbb{Q}_T \ll \mathbb{P}_T$ if and only if there exists a regular conditional law of X , denoted by $Q_T^X(\cdot | \alpha)$ such that $Q_T^X(\cdot | \alpha) \ll P_T^X(\cdot | \alpha)$ (P_T^α -a.s.) and the law of α , denoted Q_T^α , such that $Q_T^\alpha \ll P_T^\alpha$. Moreover, \mathbb{P}_T -a.s.

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = \frac{dQ_T^X(\cdot | \alpha)}{dP_T^X(\cdot | \alpha)} \frac{dQ_T^\alpha}{dP_T^\alpha}$$

Proof. For all $A, B \in \mathcal{D}([0, T])$

$$\mathbb{P}_T(A \times B) = \int_B P_T^X(A | \alpha = y) dP_T^\alpha(y) = \int_{A \times B} dP_T^X(x | \alpha = y) dP_T^\alpha(y)$$

and

$$\mathbb{Q}_T(A \times B) = \int_B Q_T^X(A | \alpha = y) dQ_T^\alpha(y) = \int_{A \times B} dQ_T^X(x | \alpha = y) dQ_T^\alpha(y)$$

At the same time, the condition $\mathbb{Q}_T \ll \mathbb{P}_T$ is equivalent to the existence of a density $f(x, y)$ such that

$$\mathbb{Q}_T(A \times B) = \int_{A \times B} f(x, y) d\mathbb{P}_T(x, y) = \int_{A \times B} f(x, y) dP_T^X(x | \alpha = y) dP_T^\alpha(y)$$

In addition, obviously $\mathbb{Q}_T \ll \mathbb{P}_T$ implies the condition $Q_T^\alpha \ll P_T^\alpha$ and

$$\mathbb{Q}_T(A \times B) = \int_{A \times B} dQ_T^X(x | \alpha = y) \xi_T^{(\alpha)}(y) dP_T^\alpha(y)$$

where $\xi_T^{(\alpha)} = \frac{dQ_T^\alpha}{dP_T^\alpha}$. Since these relations hold for all $A, B \in \mathcal{D}([0, T])$ we get that \mathbb{P}_T -a.s.

$$\frac{dQ_T}{d\mathbb{P}_T}(x, y) = f(x, y) = \frac{dQ_T^X(x | \alpha = y)}{dP_T^X(x | \alpha = y)} \xi_T^{(\alpha)}(y)$$

Conversely, if $Q_T^X(\cdot | \alpha) \ll P_T^X(\cdot | \alpha)$ (P_T^α -a.s.) and $Q_T^\alpha \ll P_T^\alpha$, then evidently $\mathbb{P}(A \times B) = 0$ implies $\mathbb{Q}(A \times B) = 0$. Since the σ -algebra $\mathcal{D}^2([0, T])$ can be generated by countable set of the events of the type $A \times B$, it gives the claim. \square

Let us suppose that the Lévy processes $X^{(j)}, j = 1, \dots, N$, are integrable, i.e. for $t \in [0, T]$, $E_{\mathbb{P}}[|X_t^{(j)}|] < \infty$. We denote by $\mathcal{M}^{(j)}$ the set of equivalent martingale measures for $X^{(j)}$. Let $Q^{(j)} \in \mathcal{M}^{(j)}$ and let $(\beta^{(j)}, Y^{(j)})$ be the Girsanov parameters for the change of the measure $P^{(j)}$ into $Q^{(j)}$.

Let \mathcal{M} denote the set of equivalent martingale measures for X such that the Girsanov parameters for the change of the measure \mathbb{P} into \mathbb{Q} verify the conditions:

$$\int_0^t |Y_s(x) - 1| \nu_X(dx) ds < \infty \text{ and } \int_0^t \|\beta_s\|^2 ds < \infty,$$

where ν_X is the compensator of the measure of jumps of the process X . We put then for all $t \geq 0$ and $x \in \mathbb{R}^d$

$$\beta_t = \sum_{j=1}^N \beta_t^{(j)} I_{\{\alpha_{t-}=j\}}, \quad (3)$$

$$Y_t(x) = \sum_{j=1}^N Y_t^{(j)}(x) I_{\{\alpha_{t-}=j\}}. \quad (4)$$

We introduce also a process $m = (m_t)_{0 \leq t \leq T}$ such that

$$m_t = \int_0^t \beta_s^\top dX_s^c + \int_0^t \int_{\mathbb{R}^d} (Y_s(x) - 1)(\mu_X - \nu_X)(ds, dx) \quad (5)$$

where μ_X and ν_X are jump measure of X and its compensator respectively.

Proposition 6. *The set $\mathcal{M} \neq \emptyset$ if and only if for all $j = 1, 2, \dots, N$, $\mathcal{M}^{(j)} \neq \emptyset$. Moreover, if $Q_T \in \mathcal{M}$, the corresponding Girsanov parameters satisfy (3) and (4) and*

$$\frac{dQ_T}{d\mathbb{P}_T} = \xi_T^{(\alpha)} \mathcal{E}(m)_T$$

with m defined by (5) and $\xi_T^{(\alpha)} = \frac{dQ_T^\alpha}{dP_T^\alpha}$.

Proof. Since the Lévy processes $X^{(1)}, X^{(2)}, \dots, X^{(N)}$ are integrable, for $j = 1, 2, \dots, N$, we have

$$X_t^{(j)} = a^{(j)}t + M_t^{(j)},$$

where $a^{(j)} = b^{(j)} + \int_{\mathbb{R}^d} x I_{\{\|x\|>1\}} \nu^{(j)}(dx)$ and

$$M_t^{(j)} = \sqrt{c^{(j)}} W_t^{(j)} + \int_0^t \int_{\mathbb{R}^d} x (\mu^{(j)} - \nu^{(j)})(ds, dx)$$

with the symmetric matrix $\sqrt{c^{(j)}}$ such that $(\sqrt{c^{(j)}})^2 = c^{(j)}$, and $\mu^{(j)}$ the measure of jumps of $X^{(j)}$. We recall that $M^{(j)}$ in this case is a martingale with respect to the filtration \mathbb{G} . But, since Lévy processes and the Markov process α are independent, it is also a martingale with respect to the filtration $\hat{\mathbb{F}}$ (cf. [2]) since the immersion property holds. Moreover, from (1) we get

$$\begin{aligned} X_t &= \sum_{j=1}^N a^{(j)} \int_0^t I_{\{\alpha_{s-}=j\}} ds + \sum_{j=1}^N \int_0^t I_{\{\alpha_{s-}=j\}} dM_s^{(j)} \\ &= A_t + M_t \end{aligned}$$

It is clear that $M = (M_t)_{0 \leq t \leq T}$ is a local martingale as a sum of integrals of bounded functions w.r.t. martingales. Since $S_t^{(k)} = \mathcal{E}(\bar{X}^{(k)})_t$, $k = 1, 2, \dots, d$, the martingale measures for $S^{(k)}$ and $\bar{X}^{(k)}$ coincide, and they are the measures which remove the drift $A = (A_t)_{0 \leq t \leq T}$ of the process X . The canonical decomposition of X in the initially enlarged filtration coincide with the canonical decomposition of X conditionally to α , as the maps on $\Omega_1 \times \Omega_2$. Then the drift of X under \mathbb{Q} is equal as a map to the drift of X under $Q_T^X(\cdot|\alpha)$.

Let $Z = (Z_t)_{0 \leq t \leq T}$ be the Radon-Nikodym process of an equivalent conditional measure $Q^X(\cdot|\alpha)$ w.r.t. $P^X(\cdot|\alpha)$. It can be always written as

$$Z_t = \mathcal{E}(m)_t,$$

where $m = (m_t)_{t \geq 0}$ is a local martingale of the form (5). According to Girsanov theorem, the drift A^Q of the process X under $Q^X(\cdot|\alpha)$ is equal to:

$$A_t^Q = A_t + \int_0^t c_s \beta_s ds + \int_0^t \int_{\mathbb{R}^d} x \cdot (Y_s(x) - 1) \nu_X(dx, ds)$$

where $c = (c_s)_{s \geq 0}$ is the density w.r.t Lebesgue measure of the quadratic variation of the continuous martingale part of X and ν_X is the compensator of the jump measure of X w.r.t. $(\mathbb{P}, \hat{\mathbb{F}})$.

Let us calculate c_t and ν_X . Since the continuous martingale part X^c of X is equal in law to $(\sum_{j=1}^N \int_0^t I_{\{\alpha_{s-}=j\}} \sqrt{c^{(j)}} dW_s^j)_{0 \leq t \leq T}$ with independent standard Brownian motions $W^{(j)}$, the quadratic variation of X^c at t is equal to

$$C_t = \sum_{j=1}^N c^{(j)} \int_0^t I_{\{\alpha_{s-}=j\}} ds$$

and its density w.r.t. Lebesgue measure is

$$c_t = \sum_{j=1}^N c^{(j)} I_{\{\alpha_{t-}=j\}} \quad (6)$$

At the same time,

$$\Delta X_t = \Delta \left(\sum_{j=1}^N \int_0^t I_{\{\alpha_{s-}=j\}} dX_s^{(j)} \right) = \sum_{j=1}^N I_{\{\alpha_{t-}=j\}} \Delta X_t^{(j)}$$

and since $I_{\{\alpha_{t-}=i\}} \cap I_{\{\alpha_{t-}=j\}} = 0$ for $i \neq j$,

$$\mu_X(dt, dx) = \sum_{j=1}^N I_{\{\alpha_{t-}=j\}} \mu^{(j)}(dt, dx) \quad (7)$$

and

$$\nu_X(dt, dx) = \sum_{j=1}^N I_{\{\alpha_{t-}=j\}} \nu^{(j)}(dt, dx) = \sum_{j=1}^N I_{\{\alpha_{t-}=j\}} \nu^{(j)}(dx) dt. \quad (8)$$

Finally,

$$A_t^Q = \sum_{j=1}^N \int_0^t \left[a^{(j)} + c^{(j)} \beta_s^{(j)} + \int_{\mathbb{R}^d} x \cdot (Y_s^{(j)}(x) - 1) \nu^{(j)}(dx) \right] I_{\{\alpha_{s-}=j\}} ds.$$

Thus, $X = (X_t)_{0 \leq t \leq T}$ is a martingale w.r.t. $Q^X(\cdot|\alpha)$ if and only if $A_t^Q = 0$ for all $t \geq 0$. We take the derivative in the previous expression to obtain that on the set $\{\alpha_{s-} = j\}$ we have that

$$a^{(j)} + c^{(j)} \beta_t^{(j)} + \int_{\mathbb{R}^d} x \cdot (Y_t^{(j)}(x) - 1) \nu^{(j)}(dx) = 0.$$

As a conclusion, on the set $\{\alpha_{t-} = j\}$ the Girsanov parameters for the change of the measure $P^X(\cdot|\alpha)$ into $Q^X(\cdot|\alpha)$, denoted (β, Y) , are equal to the Girsanov parameters $(\beta^{(j)}, Y^{(j)})$ of an equivalent martingale measure for $X^{(j)}$. This proves (3) and (4), and we conclude using the Proposition 5. \square

For all $t \in [0, T]$ due to the relations (7) and (8)

$$m_t = \sum_{j=1}^N m_t^{(j)} \quad (9)$$

with

$$m_t^{(j)} = \int_0^t I_{\{\alpha_{s-}=j\}} \top \beta^{(j)} \sqrt{(c^{(j)})} dW_s^{(j)} \quad (10)$$

$$+ \int_0^t \int_{\mathbb{R}^d} I_{\{\alpha_{s-}=j\}} (Y^{(j)}(x) - 1)(\mu^{(j)} - \nu^{(j)})(ds, dx).$$

Let us denote $Z_t^{(j)} = \mathcal{E}(m_t^{(j)})$. Now, since the processes $X^{(j)}$, $j = 1, 2, \dots, N$, are independent and cannot jump at the same time with probability 1, we conclude that

$$\begin{aligned} Z_t = \mathcal{E}(m)_t &= \exp\left\{m_t - \frac{1}{2} \langle m^c \rangle_t \prod_{0 < s \leq t} e^{-\Delta m_s} (1 + \Delta m_s)\right\} \\ &= \prod_{j=1}^N \mathcal{E}(m_t^{(j)}) = \prod_{j=1}^N Z_t^{(j)}. \end{aligned} \quad (11)$$

Theorem 1. *Suppose $\mathcal{M}^{(j)} \neq \emptyset$ for all $j = 1, 2, \dots, N$. Then for HARA utilities, the minimal martingale measure \mathbb{Q}_T^* exists and its Radon-Nikodym derivative w.r.t. \mathbb{P} is given by*

$$\mathbb{Z}_T^* = \xi_T^{(\alpha),*} \prod_{j=1}^N Z_T^{(j),*}$$

where $Z_T^{(j),*} = \mathcal{E}(m^{(j),*})_t$ with $m^{(j),*}$ defined by (10) with replacement of $(\beta^{(j)}, Y^{(j)})$ by Girsanov parameters $(\beta^{(j),*}, Y^{(j),*})$ of the minimal martingale measure and $\xi_T^{(\alpha),*}$ is the Radon-Nikodym density of an equivalent to P^α measure which minimise f -divergence. For logarithmic utility $\xi_T^{(\alpha),*} = 1$, for power utility it is given by the formula (14), and for exponential utility it is defined via (15).

Proof. 1) First of all we prove that for any density $\xi^{(\alpha)}$

$$E_{\mathbb{P}} [f(\mathbb{Z}_T)] = E_{\mathbb{P}} \left[f\left(\xi_T^{(\alpha)} \prod_{j=1}^N Z_t^{(j)}\right) \right] \geq E_{\mathbb{P}} \left[f\left(\xi_T^{(\alpha)} \prod_{j=1}^N Z_t^{(j),*}\right) \right].$$

Since the Radon-Nikodym density process $Z = (Z_t)_{0 \leq t \leq T}$ of the conditional martingale measure $Q_T^X(\cdot|\alpha)$ w.r.t. $P_T^X(\cdot|\alpha)$ is $Z_t = \mathcal{E}(m)_t$ where m is defined by (10), and $\sum_{k=0}^{\infty} I_{\{\tau_k \leq t < \tau_{k+1}\}} = 1$ for each $t > 0$ and $\tau_0 = 0$, we get

$$m_T = \sum_{k=0}^{\infty} \int_{\tau_k \wedge T}^{\tau_{k+1} \wedge T} \left[\int_{\mathbb{R}^d} \beta_t^{(j_k)} dX_t^{(j_k),c} + \int_{\mathbb{R}^d} (Y_t^{(j_k)}(x) - 1)(\mu^{(j_k)} - \nu^{(j_k)})(dt, dx) \right]$$

where $(\beta^{(j_k)}, Y^{(j_k)})$ are the Girsanov parameter of the mentioned change of the measure for the Lévy process $X^{(j_k)}$. The last formula means that on the set $\{\tau_k \leq T < \tau_{k+1}\}$ with fixed k

$$\begin{aligned} m_T &= \sum_{m=0}^{k-1} \int_{\tau_m}^{\tau_{m+1}} \left[\int_{\mathbb{R}^d} \beta_t^{(j_m)} dX_t^{(j_m),c} + \int_{\mathbb{R}^d} (Y_t^{(j_m)}(x) - 1)(\mu^{(j_m)} - \nu^{(j_m)})(dt, dx) \right] \\ &+ \int_{\tau_k}^T \left[\int_{\mathbb{R}^d} \beta_t^{(j_k)} dX_t^{(j_k),c} + \int_{\mathbb{R}^d} (Y_t^{(j_k)}(x) - 1)(\mu^{(j_k)} - \nu^{(j_k)})(dt, dx) \right] \end{aligned}$$

and the Radon-Nikodym density \mathbb{Z}_T of the measure \mathbb{Q}_T w.r.t. the measure \mathbb{P}_T verifies

$$\mathbb{Z}_T = \xi_T^{(\alpha)} \left(\prod_{m=1}^{k-1} \frac{\xi_{\tau_m}^{(j_m)}}{\xi_{\tau_{m-1}}^{(j_m)}} \right) \frac{\xi_T^{(j_k)}}{\xi_{\tau_k}^{(j_k)}}$$

where $(\xi^{(j_m)})_{0 \leq m \leq k}$, are Radon-Nikodym densities of the mentioned change of the measure for $(X^{(j_m)})_{0 \leq m \leq k}$. We denote the value of \mathbb{Z}_T on the set $\{\tau_k \leq T < \tau_{k+1}\}$ by $Z_T(k)$. Then for any f -divergence corresponding to HARA utility

$$E_{\mathbb{P}} [f(\mathbb{Z}_T)] = \sum_{k=0}^{\infty} E_{\mathbb{P}} [f(\mathbb{Z}_T) I_{\{\tau_k \leq T < \tau_{k+1}\}}] = \sum_{k=0}^{\infty} E_{\mathbb{P}} [I_{\{\tau_k \leq T < \tau_{k+1}\}} E_{\mathbb{P}} [f(Z_T(k)) | \alpha]]$$

When we take the conditional expectation w.r.t. α , the random variables $\xi_T^{(\alpha)}$ and (τ_m, j_m) , $m \geq 1$, become to be fixed, and the dependence on α will also disappear in all couple $(\beta^{(j_m)}, Y^{(j_m)})$ of the Girsanov parameters. Then we take the conditional expectation w.r.t. the processes $X^{(j_1)}, \dots, X^{(j_{k-1})}$:

$$E_{\mathbb{P}} [f(Z_T(k)) | \alpha] = E_{\mathbb{P}} [E_{\mathbb{P}} [f(Z_T(k)) | \alpha, X^{(j_1)}, \dots, X^{(j_{k-1})}]]$$

Using the scaling property of the f -divergence for Lévy processes together with the fact that conditionally to $\alpha, X^{(j_1)}, \dots, X^{(j_{k-1})}$

$$\frac{\xi_T^{(j_k)}}{\xi_{\tau_k}^{(j_k)}} \stackrel{d}{=} \xi_{T-\tau_k}^{(j_k)}$$

we can replace $\xi^{(j_k)}$ by the corresponding densities for f -divergence minimal martingale measures $\xi^{(j_k),*}$ and this procedure will decrease f -divergence. Then, we take the conditional expectation w.r.t. the processes $(X^{(j_1)}, \dots, X^{(j_{m-1})}, X^{(j_{m+1})}, \dots, X^{(j_k)})$ and use the scaling property of f -divergence and the identity in law saying that conditionally to $\alpha, X^{(j_1)}, \dots, X^{(j_{m-1})}, X^{(j_{m+1})}, \dots, X^{(j_k)}$

$$\frac{\xi_{\tau_m}^{(j_m)}}{\xi_{\tau_{m-1}}^{(j_m)}} \stackrel{d}{=} \xi_{\tau_m - \tau_{m-1}}^{(j_m)},$$

to replace $\xi^{(j_m)}$ by $\xi^{(j_m),*}$. Finally,

$$\begin{aligned} E_{\mathbb{P}} [f(\mathbb{Z}_T)] &\geq \sum_{k=0}^{\infty} E_{\mathbb{P}} \left[I_{\{\tau_k \leq t < \tau_{k+1}\}} f \left(\xi_T^{(\alpha)} \left(\prod_{m=1}^{k-1} \frac{\xi_{\tau_m}^{(j_m),*}}{\xi_{\tau_{m-1}}^{(j_m),*}} \right) \frac{\xi_T^{(j_k),*}}{\xi_{\tau_k}^{(j_k),*}} \right) \right] \\ &= E_{\mathbb{P}} \left[f \left(\xi_T^{(\alpha)} \prod_{j=1}^N Z_T^{(j),*} \right) \right] \end{aligned}$$

2) Now we will obtain the expression for $\xi^{(\alpha),*}$ minimizing each f -divergence. For logarithmic utility $f(x) = -\ln(x) - 1$ we get that

$$E_{\mathbb{P}}[f(\mathbb{Z}_T^*)] = -E_{\mathbb{P}}[\ln(\xi_T^{(\alpha)})] - \sum_{j=1}^N E_{\mathbb{P}}[\ln(Z_T^{(j),*})] - 1$$

By Jensen inequality we deduce that

$$-E_{\mathbb{P}}[\ln(\xi_T^{(\alpha)})] \geq -\ln(E_{\mathbb{P}}[\xi_T^{(\alpha)}]) = 0,$$

so the minimum is achieved for $\xi_T^{(\alpha),*} = 1$.

3) For power utility $u(x) = \frac{x^p}{p}$ we put $\gamma = \frac{p}{p-1}$. Then $f(x) = -\frac{1}{\gamma}x^\gamma$. We write

$$E_{\mathbb{P}}[f(\mathbb{Z}_T)] \geq -\frac{1}{\gamma}E_{\mathbb{P}}\left[(\xi_T^{(\alpha)})^\gamma \prod_{j=1}^N (Z_t^{(j),*})^\gamma\right] = -\frac{1}{\gamma}E_{\mathbb{P}}\left[(\xi_T^{(\alpha)})^\gamma \prod_{j=1}^N E_{\mathbb{P}}\left[\left((Z_t^{(j),*})^\gamma \mid \alpha\right)\right]\right]$$

since conditionally to α the processes $Z^{(j),*}, j = 1, \dots, N$, are independent. Now, let us denote by $Q_T^{(j),*}$ the f -divergence minimal measure of the individual Lévy process $X^{(j)}$, by $P_T^{(j)}$ its law under \mathbb{P} and by $H_t^{(j),*}(\gamma)$ the value of the corresponding Hellinger process at t . We recall that due to the homogeneity of the Lévy process $X^{(j)}$ we have

$$H_t^{(j),*}(\gamma) = t \left[\frac{\gamma(1-\gamma)}{2} \langle c^{(j)} \beta^{(j),*}, \beta^{(j),*} \rangle - \int_{\mathbb{R}^d} ((Y^{(j),*}(x))^\gamma - \gamma Y^{(j),*}(x) - 1 + \gamma) \nu^{(j)}(dx) \right].$$

We denote the derivative of the Hellinger process w.r.t. t by $h^{(j),*}(\gamma)$, i.e.

$$h^{(j),*}(\gamma) = \frac{\gamma(1-\gamma)}{2} \langle c^{(j)} \beta^{(j),*}, \beta^{(j),*} \rangle - \int_{\mathbb{R}^d} ((Y^{(j),*}(x))^\gamma - \gamma Y^{(j),*}(x) - 1 + \gamma) \nu^{(j)}(dx)$$

and we put

$$T_j^{(\alpha)} = \int_0^T I_{\{\alpha_s = j\}} ds \quad (12)$$

Using Proposition 4 we find that

$$E_{\mathbb{P}}[(Z_t^{(j),*})^\gamma \mid \alpha] = \exp\left(-\int_0^T I_{\{\alpha_s = j\}} dH_s^{(j),*}(\gamma)\right) = \exp\left(-T_j^{(\alpha)} h^{(j),*}(\gamma)\right) \quad (13)$$

and that

$$E_{\mathbb{P}}[f(\mathbb{Z}_T^*)] = -\frac{1}{\gamma}E_{\mathbb{P}}\left[(\xi_T^{(\alpha)})^\gamma \exp\left(-T_j^{(\alpha)} h^{(j),*}(\gamma)\right)\right]$$

Finally, the Lagrange method for the minimisation of the expectation under the restriction that $E_{\mathbb{P}}(\xi_T^{(\alpha)}) = 1$, performed in ω by ω way, gives that

$$\xi_T^{(\alpha),*} = \frac{\exp\left(\frac{1}{\gamma-1} \sum_{j=1}^N T_j^{(\alpha)} h^{(j),*}(\gamma)\right)}{E_{\mathbb{P}}\left[\exp\left(\frac{1}{\gamma-1} \sum_{j=1}^N T_j^{(\alpha)} h^{(j),*}(\gamma)\right)\right]} \quad (14)$$

4) For exponential utility $u(x) = 1 - \exp(-x)$ with $f(x) = 1 - x + x \ln(x)$ we write

$$E_{\mathbb{P}} \left[f(\xi_T^{(\alpha)} \prod_{j=1}^N Z_T^{(j),*}) \right] = E_{\mathbb{P}} \left[\xi_T^{(\alpha)} \ln(\xi_T^{(\alpha)}) \right] + E_{\mathbb{P}} \left[\xi_T^{(\alpha)} \sum_{j=1}^N E_{\mathbb{P}} \left[Z_T^{(j),*} \ln(Z_T^{(j),*}) | \alpha \right] \right]$$

since $E_{\mathbb{P}} \left[Z_T^{(j),*} | \alpha \right] = 1$ and since the processes $Z^{(j),*}$, conditionally to α , are independent for $j = 1, 2, \dots, n$. We apply now Proposition 4 to deduce that

$$E_{\mathbb{P}} \left[Z_T^{(j),*} \ln(Z_T^{(j),*}) | \alpha \right] = T_j^{(\alpha)} K_T^{(j),*}$$

where $K_T^{(j),*}$ is the value of the Kulback-Leibler process at T which corresponds to the Lévy process $X^{(j)}$ and defined as

$$K_T^{(j),*} = T \left[\frac{1}{2} \langle c^{(j)} \beta^{(j),*}, \beta^{(j),*} \rangle + \int_{\mathbb{R}^d} [Y^{(j),*}(x) \ln(Y^{(j),*}(x)) - Y^{(j),*}(x) + 1] \nu^{(j)}(dx) \right]$$

To simplify the notation we denote the derivative in t of this Kulback-Leibler process by $\kappa^{(j),*}$, i.e.

$$\kappa^{(j),*} = \frac{1}{2} \langle c^{(j)} \beta^{(j),*}, \beta^{(j),*} \rangle + \int_{\mathbb{R}^d} [Y^{(j),*}(x) \ln(Y^{(j),*}(x)) - Y^{(j),*}(x) + 1] \nu^{(j)}(dx)$$

So, finally we have to minimize the expression

$$E_{\mathbb{P}} \left[\xi_T^{(\alpha)} \ln(\xi_T^{(\alpha)}) \right] + E_{\mathbb{P}} \left[\xi_T^{(\alpha)} \sum_{j=1}^N T_j^{(\alpha)} \kappa^{(j),*} \right]$$

under the constraint $E_{\mathbb{P}}(\xi_T^{(\alpha)}) = 1$. Again, using Lagrange method of the minimisation of the integrals and performing the minimisation in ω by ω way, we get that

$$\xi_T^{(\alpha),*} = \frac{\exp \left(- \sum_{j=1}^n T_j^{(\alpha)} \kappa^{(j),*} \right)}{E_{\mathbb{P}} \left[\exp \left(- \sum_{j=1}^N T_j^{(\alpha)} \kappa^{(j),*} \right) \right]} \quad (15)$$

this ends the proof of this theorem. \square

5 Utility maximising strategies for Lévy switching models

In this section we will find the optimal strategies for the utility maximisation. These strategies will be automatically adapted to the progressive filtration, but for the

technical reason we will search them in the initially enlarged filtration as it was mentioned in the introduction.

To find the optimal strategy we will use Proposition 3 and also the representation of $E_{\mathbb{Q}} \left[f'(\lambda Z_T^*) \mid \hat{\mathcal{F}}_t \right]$ as a sum of the stochastic integrals using the Ito formula. For that we change the measure in the conditional expectation :

$$E_{\mathbb{Q}} \left[f'(\lambda Z_T^*) \mid \hat{\mathcal{F}}_t \right] = E_{\mathbb{P}} \left[\frac{Z_T^*}{Z_t^*} f'(\lambda Z_T^*) \mid \hat{\mathcal{F}}_t \right]$$

Since the process X , conditionally to the process α , has independent increments, $\frac{Z_T^*}{Z_t^*}$ and Z_t^* are also conditionally to the process α independent for each $t \in [0, T]$. As a conclusion,

$$E_{\mathbb{Q}} \left[f'(\lambda Z_T^*) \mid \hat{\mathcal{F}}_t \right] = E_{\mathbb{P}} \left[\frac{Z_T^*}{Z_t^*} f'(\lambda x \frac{Z_T^*}{Z_t^*}) \mid \hat{\mathcal{F}}_t \right] \Bigg|_{x=Z_t^*}$$

Let us introduce the shifted process α^t as

$$\alpha^t(s) = \alpha(s + t)$$

and let also put

$$Z_t^* = E_{\mathbb{P}} \left[Z_T^* \mid \hat{\mathcal{F}}_t \right] \quad (16)$$

and

$$Z_t(\alpha) = \frac{Z_t^*}{Z_0^*} \quad (17)$$

Then, since the processes $X^{(1)}, \dots, X^{(N)}$ and α are independent, and the processes $X^{(1)}, \dots, X^{(N)}$ are homogeneous in time,

$$\frac{Z_T^*}{Z_t^*} \stackrel{d}{=} Z_{T-t}(\alpha^t)$$

Finally,

$$E_{\mathbb{Q}} \left[f'(\lambda Z_T^*) \mid \hat{\mathcal{F}}_t \right] = E_{\mathbb{P}} \left[Z_{T-t}(\alpha^t) f'(\lambda x Z_{T-t}(\alpha^t)) \mid \hat{\mathcal{F}}_t \right] \Bigg|_{x=Z_t^*}$$

Now, we will give the expression for the optimal strategy for each HARA utility and also the corresponding maximal expected utility.

To avoid the complicated notations we will omit $*$ for the Girsanov parameters (β, Y) corresponding to the f -divergence minimal martingale measure. So, for the rest of the paper we put for all $t \geq 0$ and $x \in \mathbb{R}^d$:

$$\beta_t = \sum_{j=1}^N \beta_t^{(j),*} I_{\{\alpha_{t-}=j\}} \quad (18)$$

$$Y_t(x) = \sum_{j=1}^N Y_t^{(j),*}(x) I_{\{\alpha_{t-}=j\}} \quad (19)$$

5.1 When the utility is logarithmic

Let us suppose that the utility $u(x) = \ln(x)$. Then the corresponding f is also logarithm up to a constant, i.e. $f(x) = -\ln(x) - 1$ with $f'(x) = -\frac{1}{x}$.

Proposition 7. *Suppose that for $1 \leq j \leq N$ the matrices $c^{(j)}$ are invertible and $E_{\mathbb{P}}|\ln Z_T^*| < \infty$. Then for the initial capital $x_0 > 0$, there exists an optimal strategy and for $1 \leq k \leq d$ and $0 \leq t \leq T$, it is given by:*

$$\hat{\varphi}_t^{(k)} = -\frac{x_0 \beta_t^{(k)}}{Z_{t-}(\alpha) S_{t-}^{(k)}},$$

where β and $Z(\alpha)$ are defined by (18) and (17) respectively. This strategy is progressively adapted and the corresponding maximal expected utility is

$$U_T^{\log}(x_0) = \ln(x_0) - E_{\mathbb{P}}[\ln Z_T^*] = \ln(x_0) - E_{\mathbb{P}}[\ln(Z_T(\alpha))].$$

Proof. Simple calculus shows that in considered case

$$-E_{\mathbb{P}}[Z_{T-t}(\alpha^t) f'(\lambda_0 x Z_{T-t}(\alpha^t)) | \alpha] = \frac{1}{\lambda_0 x}$$

and from the condition (2) we get that $\lambda_0 = \frac{1}{x_0}$ and, hence,

$$-E_{\mathbb{Q}}\left(f'(\lambda_0 Z_T^*) | \hat{\mathcal{F}}_t\right) = \frac{x_0}{Z_t^*} = \frac{x_0}{Z_t(\alpha)}$$

since $Z_0^* = \xi_T^{(\alpha),*} = 1$. We have the expression

$$Z_t(\alpha) = \mathcal{E}(m)_t = \exp\left(m_t - \frac{1}{2}\langle m^c \rangle_t + \int_0^t \int_{\mathbb{R}^d} (\ln(Y_s(x)) - Y_s(x) + 1) \mu_X(ds, dx)\right),$$

where m is a $(\mathbb{P}, \hat{\mathbb{F}})$ -martingale given by (9). From the mentioned formula for m we get that

$$\begin{aligned} Z_t(\alpha) = \exp\left(\int_0^t \top \beta_s dX_s^{c, \mathbb{Q}} + \int_0^t \int_{\mathbb{R}^d} (Y_s(x) - 1)(\mu_X - \nu_X^{\mathbb{Q}})(dt, dx) + \int_0^t \frac{1}{2} \langle c_s \beta_s, \beta_s \rangle ds \right. \\ \left. + \int_0^t \int_{\mathbb{R}^d} (Y_s(x) - 1)^2 \nu_X(ds, dx) + \int_0^t \int_{\mathbb{R}^d} (\ln Y_s(x) - Y_s(x) + 1) \mu_X(ds, dx) \right), \end{aligned}$$

where $X^{c, \mathbb{Q}}$ is the continuous martingale part of X under \mathbb{Q} , c is the density of its predictable variation and $\nu_X^{\mathbb{Q}}$ is the compensator of the jump measure of X under \mathbb{Q} .

Let us define a \mathbb{Q} -martingale $M = (M_t)_{t \geq 0}$ with

$$M_t = \int_0^t \top \beta_s dX_s^{c, \mathbb{Q}} + \int_0^t \int_{\mathbb{R}^d} \ln(Y_s(x)) (\mu_X - \nu_X^{\mathbb{Q}})(dt, dx)$$

and also a predictable process $B = (B_t)_{t \geq 0}$ with

$$B_t = \int_0^t \frac{1}{2} \langle c_s \beta_s, \beta_s \rangle ds + \int_0^t \int_{\mathbb{R}^d} (Y_s(x) - 1)^2 \nu_X(ds, dx) \\ + \int_0^t \int_{\mathbb{R}^d} (\ln Y_s(x) - Y_s(x) + 1) \nu_X^{\mathbb{Q}}(ds, dx).$$

Then using the compensating formula and doing the simplifications for discontinuous martingale part we see that

$$Z_t = \exp(M_t + B_t) \quad (20)$$

Let $\tau_n = \inf\{t \geq 0 \mid Z_t(\alpha) \leq \frac{1}{n}\}$ with $n \geq 1$ and $\inf\{\emptyset\} = +\infty$. By Itô formula applied to the function $g(x) = \exp(-x)$ we get

$$\frac{1}{Z_{t \wedge \tau_n}(\alpha)} = 1 - \int_0^{t \wedge \tau_n} \frac{1}{Z_{s-}(\alpha)} (dM_s + dB_s) + \frac{1}{2} \int_0^{t \wedge \tau_n} \frac{1}{Z_{s-}(\alpha)} d\langle M^c \rangle_s \\ + \int_0^{t \wedge \tau_n} \int_{\mathbb{R}^d} \frac{1}{Z_{s-}(\alpha)} (Y_s^{-1}(x) - 1 + \ln(Y_s(x))) \mu_X(ds, dx).$$

Since $\tau_n \rightarrow +\infty$, we deduce after limit passage the similar expression for $\frac{1}{Z_t(\alpha)}$ with $t \wedge \tau_n$ replaced by t .

Using again the compensating formula and taking in account that $(Z_t(\alpha))_{t \geq 0}^{-1}$ is a \mathbb{Q} -martingale, we get that its drift part

$$- \int_0^t \frac{1}{Z_{s-}(\alpha)} dB_s + \frac{1}{2} \int_0^t \frac{1}{Z_{s-}(\alpha)} d\langle M^c \rangle_s + \\ \int_0^t \int_{\mathbb{R}^d} \frac{1}{Z_{s-}(\alpha)} (Y_s^{-1}(x) - 1 + \ln(Y_s(x))) \nu_X^{\mathbb{Q}}(ds, dx) = 0.$$

According to the Proposition 3 we obtain then that for $0 \leq t \leq T$,

$$\frac{x_0}{Z_t(\alpha)} = x_0 - \int_0^t \frac{x_0}{Z_{s-}(\alpha)} \top \beta_s dX_s^{c, \mathbb{Q}} + \int_0^t \int_{\mathbb{R}^d} \frac{x_0}{Z_{s-}(\alpha)} (Y_s^{-1}(x) - 1) (\mu_X - \nu_X^{\mathbb{Q}})(ds, dx) \\ = x_0 + \sum_{k=1}^d \int_0^t \hat{\varphi}_s^{(k)} dS_s^{(k)}.$$

But $dS_s^{(k)} = S_{s-}^{(k)} dX_s^{(k)}$ and the last equality implies that

$$\sum_{k=1}^d \int_0^t \left[\frac{x_0}{Z_{s-}(\alpha)} \beta_s^{(k)} - \hat{\varphi}_s^{(k)} S_{s-}^{(k)} \right] dX_s^{(k), c, \mathbb{Q}} \\ = \int_0^t \int_{\mathbb{R}^d} \left(\frac{x_0}{Z_{s-}(\alpha)} (Y_s^{-1}(x) - 1) + \sum_{k=1}^d \hat{\varphi}_s^{(k)} S_{s-}^{(k)} \right) (\mu_X - \nu_X^{\mathbb{Q}})(ds, dx).$$

In the left-hand side, we have a continuous martingale and in the right-hand side a purely discontinuous martingale, which is orthogonal to the first one. Hence, for $0 \leq t \leq T$, the quadratic variation of a continuous martingale

$$\left\langle \sum_{k=1}^d \int_0^t \left[\frac{x_0}{Z_{s-}(\alpha)} \beta_s^{(k)} - \hat{\varphi}_s^{(k)} S_{s-}^{(k)} \right] dX_s^{(k),c,\mathbb{Q}} \right\rangle_t = 0$$

Therefore, for $0 \leq t \leq T$,

$$\int_0^t \sum_{k=1}^d \sum_{j=1}^d \left(\frac{x_0}{Z_{s-}(\alpha)} \beta_s^{(k)} - \hat{\varphi}_s^{(k)} S_{s-}^{(k)} \right) \left(\frac{x_0}{Z_{s-}(\alpha)} \beta_s^{(l)} - \hat{\varphi}_s^{(l)} S_{s-}^{(l)} \right) c_s^{(k,l)} ds = 0,$$

where $c_s^{(k,l)}$ are the elements of the matrix c defined in (6). Then, for $0 \leq t \leq T$, the quadratic form corresponding to the strictly positive matrix c

$$\sum_{k=1}^d \sum_{j=1}^d \left(\frac{x_0}{Z_{t-}(\alpha)} \beta_t^{(k)} - \hat{\varphi}_t^{(k)} S_{t-}^{(k)} \right) \left(\frac{x_0}{Z_{t-}(\alpha)} \beta_t^{(l)} - \hat{\varphi}_t^{(l)} S_{t-}^{(l)} \right) c_t^{(k,l)} = 0,$$

which can happen only at zero. Then our result for the optimal strategy follows.

Using again Proposition 3 and doing simple calculus, we get the formula for the optimal expected utility. \square

5.2 When the utility is power

When the utility is power $u(x) = \frac{x^p}{p}$, the dual function f is also power, i.e. $f(x) = -\frac{1}{\gamma}x^\gamma$ and $f'(x) = -x^{\gamma-1}$ with $\gamma = \frac{p}{p-1}$.

Proposition 8. *Suppose that $E_{\mathbb{P}}[(Z_T^*)^\gamma] < \infty$ and that the matrices $c^{(j)}, 1 \leq j \leq N$, are invertible. Then for initial capital $x_0 > 0$, there exists an optimal strategy and for $1 \leq k \leq d$ and $0 \leq t \leq T$, it is given by*

$$\hat{\varphi}_t^{(k)} = \frac{x_0(\gamma - 1) Z_{t-}^{\gamma-1}(\alpha) \beta_t^{(k)} \exp\left(-\sum_{j=1}^N h^{(j),*}(\gamma) \int_0^t I_{\{\alpha_{s-}=j\}} ds\right)}{S_{t-}^{(k)}}$$

with β and $Z_t(\alpha)$ defined by (18) and (17). This strategy is progressively adapted and the corresponding maximal expected utility is

$$U_T^{pow}(x_0) = \frac{x_0^p}{p} \left[E_{\mathbb{P}} \exp \left(\frac{1}{\gamma-1} \sum_{j=1}^N h^{(j),*}(\gamma) T_j^{(\alpha)} \right) \right]^\gamma E_{\mathbb{P}} \exp \left(-\sum_{j=1}^N h^{(j),*}(\gamma) T_j^{(\alpha)} \right)$$

where $T_j^{(\alpha)}$ is defined by (12).

Proof. In this case,

$$-E_{\mathbb{P}} [Z_{T-t}(\alpha^t) f'(\lambda_0 x Z_{T-t}(\alpha^t)) | \alpha] = \lambda_0^{\gamma-1} x^{\gamma-1} E_{\mathbb{P}} [Z_{T-t}^{\gamma}(\alpha^t) | \alpha].$$

The last conditional expectation, according to the discussion on the Hellinger process of the section 3, is given by

$$E_{\mathbb{P}} [Z_{T-t}^{\gamma}(\alpha^t) | \alpha] = \exp \left(- \sum_{j=1}^N h^{(j),*}(\gamma) \int_t^T I_{\{\alpha_{s-}=j\}} ds \right).$$

Then,

$$-E_{\mathbb{Q}} [f'(\lambda_0 Z_T^*) | \hat{\mathcal{F}}_t] = \lambda_0^{\gamma-1} (Z_t^*)^{\gamma-1} \exp \left(- \sum_{j=1}^N h^{(j),*}(\gamma) \int_t^T I_{\{\alpha_{s-}=j\}} ds \right).$$

Let us apply Itô formula for the right hand side dropping $(\lambda_0 Z_0^*)^{\gamma-1}$. For that let $N = (N_t)_{t \geq 0}$ be a process where

$$N_t = Z_t^{\gamma-1}(\alpha) g_t$$

with $g_t = \exp \left(- \sum_{j=1}^N h^{(j),*}(\gamma) \int_t^T I_{\{\alpha_{s-}=j\}} ds \right)$. Then, by integration by part formula

$$N_t = g_0 + \int_0^t g_s dZ_s^{\gamma-1}(\alpha) + \int_0^t Z_s^{\gamma-1}(\alpha) dg_s. \quad (21)$$

since $Z_0(\alpha) = 1$. Using the representation (20) for $Z_s(\alpha)$ from the previous proposition, we then get that

$$Z_s^{\gamma-1}(\alpha) = \exp((\gamma-1)(M_s + B_s)).$$

Again by Itô formula via the localisation with the stopping times

$$\tau_n = \inf \{ t \geq 0 \mid Z_t(\alpha) \leq \frac{1}{n} \text{ or } Z_t(\alpha) \geq n \}$$

where $n \geq 1$ and $\inf \{\emptyset\} = +\infty$, and further limit passage $n \rightarrow \infty$ we find that

$$\begin{aligned} Z_t^{\gamma-1}(\alpha) &= 1 + (\gamma-1) \int_0^t Z_{s-}^{\gamma-1}(\alpha) (dM_s + dB_s) + \frac{1}{2}(\gamma-1)^2 \int_0^t Z_{s-}^{\gamma-1}(\alpha) d\langle M^c \rangle_s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} Z_{s-}^{\gamma-1} [Y_s^{\gamma-1}(x) - 1 - (\gamma-1) \ln Y_s(x)] \mu_X(ds, dx). \end{aligned}$$

We deduce from the above formula and (21) that

$$N_t = g_0 + \int_0^t Z_s^{\gamma-1}(\alpha) dg_s +$$

$$\begin{aligned}
& (\gamma-1) \int_0^t g_s Z_{s-}^{\gamma-1}(\alpha) dM_s + (\gamma-1) \int_0^t g_s Z_{s-}^{\gamma-1}(\alpha) dB_s + \frac{1}{2}(\gamma-1)^2 \int_0^t g_s Z_{s-}^{\gamma-1}(\alpha) d\langle M^c \rangle_s \\
& \quad + \int_0^t \int_{\mathbb{R}^d} g_s Z_{s-}^{\gamma-1} [Y_s^{\gamma-1}(x) - 1 - (\gamma-1) \ln Y_s(x)] \mu_X(ds, dx).
\end{aligned}$$

From the previous formula, using compensation and taking in account that N is \mathbb{Q} -martingale, we deduce that the drift part of N is equal to zero, and that

$$\begin{aligned}
N_t &= g_0 + (\gamma-1) \int_0^t g_s Z_{s-}^{\gamma-1}(\alpha) dM_s \\
& \quad + \int_0^t \int_{\mathbb{R}^d} g_s Z_{s-}^{\gamma-1}(\alpha) [Y_s^{\gamma-1}(x) - 1 - (\gamma-1) \ln Y_s(x)] (\mu_X - \nu_X^{\mathbb{Q}})(ds, dx) \\
&= g_0 + (\gamma-1) \int_0^t g_s Z_{s-}^{\gamma-1}(\alpha)^\top \beta_s dX_s^{c, \mathbb{Q}} + \\
& \quad + \int_0^t \int_{\mathbb{R}^d} g_s Z_{s-}^{\gamma-1} [Y_s^{\gamma-1}(x) - 1] (\mu_X - \nu_X^{\mathbb{Q}})(ds, dx)
\end{aligned}$$

According to Proposition 3

$$(\lambda_0 \mathbb{Z}_0^*)^{\gamma-1} N_t = x_0 + \sum_{k=1}^N \int_0^t \hat{\varphi}_s^{(k)} S_{s-}^{(k)} dX_s^{(k)}.$$

Then $(\lambda_0 \mathbb{Z}_0^*)^{\gamma-1} g_0 = x_0$ and

$$\lambda_0^{\gamma-1} = x_0 \left(E_{\mathbb{P}} \left[\exp \left(\frac{1}{\gamma-1} \sum_{i=1}^N h^{(i),*}(\gamma) \int_0^T I_{\{\alpha_{s-}=j\}} ds \right) \right] \right)^{\gamma-1}$$

Moreover, for $0 \leq t \leq T$

$$\sum_{k=1}^d \int_0^t \left[\frac{x_0(\gamma-1)g_s}{g_0} Z_{s-}^{\gamma-1}(\alpha) \beta_s^{(k)} - \varphi_s^{(k)} S_{s-}^{(k)} \right] dX_s^{(k,c,\mathbb{Q})} = V_t$$

where

$$V_t = \sum_{k=1}^N \int_0^t \hat{\varphi}_s^{(k)} S_{s-}^{(k)} dX_s^{(k),d,\mathbb{Q}} - \int_0^t \int_{\mathbb{R}^d} \frac{x_0 g_s}{g_0} Z_{s-}^{\gamma-1}(\alpha) [Y_s^{\gamma-1}(x) - 1] (\mu_X - \nu_X^{\mathbb{Q}})(ds, dx)$$

The process $V = (V_t)_{0 \leq t \leq T}$ is pure discontinuous \mathbb{Q} -martingale which is orthogonal to the continuous martingale of the left-hand side of the above equality. Then, the quadratic variation of the continuous martingale is equal to zero for $0 \leq t \leq T$:

$$\left\langle \sum_{k=1}^d \int_0^t \left[\frac{x_0(\gamma-1)g_s}{g_0} Z_{s-}^{\gamma-1}(\alpha) \beta_s^{(k)} - \hat{\varphi}_s^{(k)} S_{s-}^{(k)} \right] dX_s^{(k,c,\mathbb{Q})} \right\rangle_t = 0$$

Simple calculations show that the previous equality is equivalent to

$$\int_0^t \sum_{k=1}^d \sum_{l=1}^d \left[\frac{x_0(\gamma-1)g_s}{g_0} Z_{s-}^{\gamma-1}(\alpha) \beta_s^{(k)} - \hat{\varphi}_s^{(k)} S_{s-}^{(k)} \right] \left[\frac{x_0(\gamma-1)g_s}{g_0} Z_{s-}^{\gamma-1}(\alpha) \beta_s^{(l)} - \hat{\varphi}_s^{(l)} S_{s-}^{(l)} \right] c_s^{(k,l)} = 0,$$

where the matrix c_s is defined by (6). Then, the quadratic form associated with the strictly positive matrix c_t

$$\sum_{k=1}^d \sum_{l=1}^d \left[\frac{x_0(\gamma-1)g_t}{g_0} Z_{t-}^{\gamma-1}(\alpha) \beta_t^{(k)} - \hat{\varphi}_t^{(k)} S_{t-}^{(k)} \right] \left[\frac{x_0(\gamma-1)g_t}{g_0} Z_{t-}^{\gamma-1}(\alpha) \beta_t^{(l)} - \hat{\varphi}_t^{(l)} S_{t-}^{(l)} \right] c_t^{(k,l)} = 0$$

and it proves that

$$\hat{\varphi}_t^{(k)} = \frac{x_0(\gamma-1)g_t Z_{t-}^{\gamma-1}(\alpha) \beta_t^{(k)}}{g_0 S_{t-}^{(k)}}$$

and this gives the formula of the proposition.

Using again Proposition 3 and doing simple calculus, we get the formula for the optimal expected utility. \square

5.3 When the utility is exponential

When u is exponential, $f(x) = x \ln x - x + 1$ and $f'(x) = \ln(x)$.

Proposition 9. *Suppose that $E_{\mathbb{P}}[|Z_T^* \ln Z_T^*|] < \infty$ and that the matrices $c^{(j)}$ are invertible for $1 \leq j \leq N$. Then for the initial capital $x_0 > 0$, there exists an asymptotically optimal strategy such that for $1 \leq k \leq d$ and $0 \leq t \leq T$*

$$\hat{\varphi}_t^{(k)} = \frac{-\beta_t^{(k)}}{S_{t-}^{(k)}}$$

where β defined by formula (18). This strategy is progressively adapted and the corresponding maximal expected utility is

$$U_T^{exp}(x_0) = 1 - \exp(-x_0) \left(E_{\mathbb{P}} \left[\exp \left(- \sum_{j=1}^N \kappa^{(j),*} \int_0^T I_{\{\alpha_{s-}=j\}} ds \right) \right] \right).$$

Proof. In this case,

$$\begin{aligned} E_{\mathbb{P}} [Z_{T-t}(\alpha^t) f'(\lambda_0 x Z_{T-t}(\alpha^t)) | \alpha] &= \ln(\lambda_0 x) E_{\mathbb{P}} [Z_{T-t}(\alpha^t) | \alpha] + E_{\mathbb{P}} [Z_{T-t}(\alpha^t) \ln Z_{T-t}(\alpha^t) | \alpha] \\ &= \ln(\lambda_0 x) + \sum_{j=1}^N \kappa^{(j),*} \int_t^T I_{\{\alpha_{s-}=j\}} ds \end{aligned}$$

since $E_P(Z_{T-t}(\alpha^t)|\alpha) = 1$ due to the fact that it is a martingale starting from 1. Hence,

$$-E_{\mathbb{Q}}(f'(\lambda_0 Z_T^*)|\hat{\mathcal{F}}_t) = -\ln(\lambda_0) - \ln(Z_t^*) - \sum_{j=1}^N \kappa^{(j),*} \int_t^T I_{\{\alpha_{s-}=j\}} ds.$$

As it was mentioned $Z_t(\alpha) = \exp(M_t + B_t)$ w.r.t. the measure \mathbb{Q} . Hence,

$$\ln Z_t(\alpha) = M_t + B_t. \quad (22)$$

and the right-hand side of (22) should be a martingale. Since the processes $B = (B_t)_{t \geq 0}$ and $(\sum_{j=1}^N \int_0^t I_{\{\alpha_{s-}=j\}} dK_s^{(j)})_{t \geq 0}$ are the predictable processes, we obtain via Proposition 3 that

$$-\int_0^t \beta_s dX_s^{c, \mathbb{Q}} - \int_0^t \int_{\mathbb{R}^d} \ln(Y_s(x)) (\mu_X - \nu_X^{\mathbb{Q}})(ds, dx) = \sum_{k=1}^d \int_0^t \hat{\varphi}_s^{(k)} S_{s-}^{(k)} dX_s^{(k)}$$

and that

$$\ln(\lambda_0) + E_{\mathbb{P}}(\ln(Z_0^*)) + E_{\mathbb{P}}\left(\sum_{j=1}^N \kappa^{(j),*} \int_0^T I_{\{\alpha_{s-}=j\}} ds\right) = -x_0$$

Performing the separation of continuous and discontinuous part in the right-hand side of the previous expression, and using the orthogonality of continuous and pure discontinuous martingales, we deduce that the quadratic variation of the continuous martingale part is zero, and, hence,

$$\sum_{k=0}^d \sum_{l=0}^d \left(\beta_t^{(k)} + \hat{\varphi}_t^{(k)} S_{t-}^{(k)} \right) \left(\beta_t^{(l)} + \hat{\varphi}_t^{(l)} S_{t-}^{(l)} \right) c_t^{(k,l)} ds = 0$$

where $c_t^{(k,l)}$ are the elements of the strictly positive matrix c_t given in (6). It proves the formula for the optimal strategy.

Using again Proposition 3 and doing simple calculus, we get the formula for the optimal expected utility. \square

6 Example

In this section, we shall consider one important example that each Lévy process is a Brownian motion with drift. More precisely,

$$X_t^{(j)} = b^{(j)}t + \sigma^{(j)}W_t^{(j)},$$

where $(W^{(j)})_{1 \leq j \leq N}$ states for independent d -dimensional Brownian motions, $(b^{(j)})_{1 \leq j \leq N}$ are d -dimensional vectors and $(\sigma^{(j)})_{1 \leq j \leq N}$ are $d \times d$ real-valued matrix. We assume

that $\sigma^{(j)}$ is invertible for each j . In this case, the set $\mathcal{M}^{(j)}$ of the martingale measures consists of only one element with Girsanov parameter

$$\beta^{(j),*} = -(\top\sigma^{(j)}\sigma^{(j)})^{-1}b^{(j)}.$$

Then we see that the function related to Hellinger integrals and Kulback-Leibler informations are

$$h^{(j),*}(\gamma) = \frac{\gamma(1-\gamma)}{2} \langle (\top\sigma^{(j)}\sigma^{(j)})^{-1}b^{(j)}, b^{(j)} \rangle$$

and

$$\kappa^{(j),*} = \frac{1}{2} \langle (\top\sigma^{(j)}\sigma^{(j)})^{-1}b^{(j)}, b^{(j)} \rangle.$$

As before, β_s is defined by the formula (18).

For the power utility, from Proposition 8, we know that the optimal investment strategy $\hat{\varphi}$ is given by

$$\hat{\varphi}_t^{(k)} = \frac{x_0(\gamma-1)Z_{t-}^{\gamma-1}(\alpha)\beta_t^{(k)} \exp\left(-\sum_{j=1}^N h^{(j),*}(\gamma) \int_0^t I_{\{\alpha_{s-}=j\}} ds\right)}{S_{t-}^{(k)}}$$

where $1 \leq k \leq N$ and it satisfies

$$(\lambda_0 Z_T^*)^{\gamma-1} = x_0 + \sum_{k=1}^d \int_0^T \hat{\varphi}_s^{(k)} dS_s^{(k)}. \quad (23)$$

Denote by $V^{\hat{\varphi}} = (V_t^{\hat{\varphi}})_{0 \leq t \leq T}$ the corresponding value process of the optimal portfolio. Then

$$V_T^{\hat{\varphi}} = (\lambda_0 Z_T^*)^{\gamma-1}.$$

Since the stochastic integral in (23) is a \mathbb{Q} -martingale, we have

$$V_t^{\hat{\varphi}} = E_{\mathbb{Q}}[(\lambda_0 Z_T^*)^{\gamma-1} | \hat{\mathcal{F}}_t] = \lambda_0^{\gamma-1} Z_t^{*,\gamma-1} E_{\mathbb{P}}[Z_{T-t}^{\gamma}(\alpha^t) | \alpha].$$

In the proof of Proposition 8, we have already calculated that

$$V_t^{\hat{\varphi}} = x_0 Z_t^{\gamma-1}(\alpha) \exp\left(-\sum_{j=1}^N h^{(j),*}(\gamma) \int_0^t I_{\{\alpha_{s-}=j\}} ds\right).$$

Thus, the optimal strategy can also be written as

$$\hat{\varphi}_t^{(k)} = \frac{(\gamma-1)\beta_t^{(k)} V_{t-}^{\hat{\varphi}}}{S_{t-}^{(k)}}.$$

With similar calculation, we can also rewrite the optimal value process for the logarithm utility in a similar form as

$$\hat{\varphi}_t^{(k)} = \frac{\beta_t^{(k)} V_{t-}^{\hat{\varphi}}}{S_{t-}^{(k)}}.$$

In fact, from the discussion above, we see that the same is true for the general case where the Lévy process admits jumps but with a non-degenerated diffusion. This suggests a very simple structure of the investment strategy. In both cases, the investor should keep a constant proportion of money in each risky asset when the market state is fixed. The proportion of money depends on the excess return and volatility. In some case with negative excess return, the investor should choose to short-sell such assets. Once the market state switched to another state, the investor should rebalance his portfolio and keep the proportion constant until next switching. In each state, we see that the total proportion that invested in the risky assets is a multiple of $\sum_{k=1}^d \beta_t^{(k)}$. By the definition of β , we see that, comparing to bull market (high return, low volatility), the investor will put more money in the riskless account in the bear market (high volatility, low return). For the exponential utility, the strategy is similar except that the investor should keep a constant amount of money in each asset.

7 Acknowledgements

This research was partially supported by Defimath project of the Research Federation of "Mathématiques des Pays de la Loire" and by PANORisk project "Pays de la Loire" region.

References

- [1] J. CAI (1994) *A markov model of switching-regime arch*. Journal of Business and Economic Statistics, 12(3), 309-316.
- [2] G. CALLEGARO, M. JEANBLANC, B. ZARGARI (2011) *Carthaginian enlargement of filtrations*. ESAIM: Probability and Statistics, Volume 17, 2013, 550 - 566.
- [3] S. CAWSTON, L. VOSTRIKOVA (2013) *An f -divergence approach for optimal portfolios in exponential Lévy models*. In Kabanov Yu., Zariphopoulou T., Rutkowski M. "Inspired by Finance", Springer-Verlag.
- [4] S. CAWSTON, L. VOSTRIKOVA (2012) *Lévy preservation and associated properties for the f -divergence minimal equivalent martingale measures*. In Shiryaev A., Presman E., Yor M. "Prokhorov and Contemporary Probability Theory", Springer-Verlag.
- [5] S. CAWSTON, L. VOSTRIKOVA (2012) *F -divergence minimal martingale measures and optimal portfolios for exponential Lévy models with a change-point*. In R. Dalang, M. Dozzi, F. Russo (Editors). Stochastic analysis, random fields and applications VI. Progress in Probability 67, 2013, 285-305.

- [6] T. CHOULLI, C. STRICKER (2005) *Minimal entropy-Hellinger martingale measure in incomplete markets*. Math. Finance. 15, 465-490.
- [7] T. CHOULLI, C. STRICKER, J. LI (2007) *Minimal Hellinger martingale measures of order q* . Finance and Stoch., 11.3, 399-427.
- [8] K. CHOURDAKIS (2005) *Switching Lévy models in continuous time : Finite distributions and option pricing*. WORKING PAPER SSRN : 838924.
- [9] R. ELIOTT, C.J. OSAKWE (2006) *Option pricing for pure jump processes with Markov Switching compensators*. Finance and Stochastics, 10(2), 250-275.
- [10] R. ELIOTT, K.T. SIU, L. CHAN (2007) *Pricing volatility swaps under Heston's stochastic volatility model with regime switching*. Applied Mathematical Finance, 14(1), 41-62.
- [11] M. ESCOBAR, D. NEYKOVA, R. ZAGST (2014) *Portfolio optimization in affine models with Markov switching*. International Journal of Theoretical & Applied Finance, 18(05), 413-461.
- [12] F. ESSCHE, M. SCHWEIZER (2005) *Minimal entropy preserves the Lévy property : how and why*. Stoch. Proc. Appl., 115.2, 299-327.
- [13] A. ELLANSKAYA , L. VOSTRIKOVA (2014) *Utility maximisation and utility indifference price for exponential semimartingale models and HARA utilities*. Proceedings of Steklov Institute, 287, 75-102.
- [14] P. FRANCOIS, G. GAUTIER, F. GODIN (2014) *Optimal hedging when the underlying asset follows a regime-switching Markov process*. European journal of Operational Research, 237, 312-322.
- [15] T.FUJIWARA, Y.MIYAHARA (2003) *The Minimal Entropy Martingale Measures for Geometric Lévy Processes*. Finance Stoch. 7, 509-531
- [16] T. GOLL, L. RUSCHENDORF (2001) *Minimax and minimal distance martingale measures and their relationship to portfolio optimisation*. Finance and Stochastics, Vol. V.4, 557-581.
- [17] D. GASBARRA, E. VALKEILA, L. VOSTRIKOVA(2006) *Enlargement of filtration and additional information in pricing models: Bayesian approach*. In Kabanov Yu., Liptser R., Stoyanov D. "From Stochastic Calculus to Mathematical Finance" , 257-285, Springer-Verlag.
- [18] J.D. HAMILTON, R. SUSMEL (1994) *A stochastic volatility model with markov switching*. Journal of Econometrics, 64(1-2), 307-333.
- [19] D. HAINAUT (2010) *Switching Lévy processes : a toolbox for financial applications*. Working paper.

- [20] F. HUBALEK, C. SGARRA (2009) *Esscher transforms and the minimal entropy martingale measure for exponential Lévy models*. Quantitative finance, 6.2 125-145.
- [21] K.R. JAKSON, S. JAIMUNGAL, V. SURKOV (2007) *Option pricing with regime switching Lévy processes using Fourier space time stepping*. Proceedings of the Fourth IASTED International Conference on Financial Engineering and Applications. Berkeley, California.
- [22] J. JACOD (1980) *Grossissement initiale, hypothèse (H') et théorème de Girsanov*. In: Jeulin, T. and Yor, M. (eds) Grossissements de filtrations: exemples et applications, Lecture Notes in Mathematics, 118, Springer, Berlin.
- [23] J. JACOD, A.N. SHIRYAEV (2003) *Limit theorems for stochastic processes*. Springer, Berlin.
- [24] M. JEANBLANC, S. KLÖPPEL, Y. MIYAHARA (2007) *Minimal f^q -martingale measures for exponential Lévy processes*. Ann. Appl. Probab. Vol. 17, 5/6 1615-1638.
- [25] E.I. KOLOMIETZ (1984) *Relations between triplets of local characteristics of semimartingales*. Russian Math. Surveys 39, no.4, 123 -124.
- [26] M. KONIKOV, D. MADAN (2002) *Option pricing using variance Gamma Markov Chain*. Review of derivatives research, 5, 81-115.
- [27] Y. MIYAHARA (1999) *Minimal Entropy Martingale Measures of Jump Type Price Processes in Incomplete Assets Markets*. Asian-Pacific Financial Markets, 6, 2, 97-113.
- [28] M.K.P. SO, K. LAM, W.K. LI (1994) *A stochastic volatility model with Markov switching*. Journal of Business and Economic Statistics, 16(2), 244-253.
- [29] L. VOSTRIKOVA (2016) *Expected utility maximisation for exponential Lévy models with option and information processes*. Theory of probability and their applications, 61, no. 1, 26-52.