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Abstract

Many articles were devoted to the problem of estimating recursively the eigenvectors and eigenvalues in decreasing order of the expectation of a random matrix using an i.i.d. sample of it. The present study makes the following contributions. The convergence of a normed process is proved under more general assumptions: the random matrices are not supposed i.i.d. and a new data mini-batch or all data until the current step are taken into account at each step without storing them; three types of processes are studied; this is applied to online principal component analysis of a data stream, assuming that data are realizations of a random vector Z whose expectation is unknown and must be estimated online, as well as possibly the metrics used when it depends on unknown characteristics of Z.

Keywords: Big data, Data stream, Online estimation, Principal component analysis, Stochastic approximation.

1 Introduction

Data stream factorial analysis is defined as the factorial analysis of data that arrive continuously such as process control data, web data, telecommunication data, medical data, financial data,.... Recursive stochastic algorithms can be used for observations arriving sequentially to estimate principal components or factors, whose estimations are updated by each new arriving observation vector. When using such processes, it is not necessary to store the data and, due to the relative simplicity of the computation involved, much more data than with other methods can be taken into account during the same duration of time.

Consider the following model: suppose that p quantitative variables are observed on individuals (p may be very large); data vectors in $\mathbb{R}^p$ are thus obtained. Considering that z_n is observed at time n (or more generally that several observations, a data mini-batch, are made at time n), there is a sequence of data vectors $z_1, ..., z_n, ...$ Assume that, for $n \geq 1$, z_n is a realization of a random variable Z_n defined on a probability space (Ω, A, P) and that $(Z_1, ..., Z_n, ...)$ is an i.i.d sample of a random vector Z. Denote θ the expectation of Z and C its covariance matrix which are unknown in the case of a data stream.

Let M be a positive definite symmetric $p \times p$ matrix called metrics. Recall briefly the principal component analysis (PCA) algorithm of the random vector Z. At step l of PCA is determined a linear combination $c_l'Z$ of the components of Z, called l^{th} principal component, uncorrelated with the previous ones and of maximal variance, under the normalization constraint $c_l'M^{-1}c_l = 1$; c_l is a M^{-1}-unit eigenvector of MC corresponding to the l^{th} largest eigenvalue λ_l. For $l = 1, ..., r$, a M-unit direction vector u_l of the l^{th} principal axis is defined as $M^{-1}c_l$; the vectors u_l are M-orthonormal and are eigenvectors of the matrix CM corresponding respectively to the same eigenvalues λ_l. A particular case is normed PCA, where M is the diagonal matrix of the inverses of variances of the p components of Z. This is equivalent to use standardized data, i.e. observations of $M(Z - \theta)$, and the identity metrics. But the expectation θ and the variances of the components of Z are usually unknown and only raw data are observed. One application of this article is to recursively estimate the c_l or the u_l using stochastic approximation processes.

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Many articles were devoted to this problem when supposing M and θ known or more generally to the problem of estimating eigenvectors and eigenvalues in decreasing order of the expectation B of a random matrix, using an i.i.d. sample of it. See for example the well-known algorithms of Benzécri [1], Krasulina [2], Karhunen and Oja [3], Oja and Karhunen [4], Brandière [5],[6], Brandière and Duflo [7]. Recall the normed process studied in [3][4]:

$$X_{n+1} = \frac{(I + a_n B_n) X_n}{\|(I + a_n B_n) X_n\|},$$

with $E[B_n] = B$, $a_n > 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} a_n^2 < \infty$.

This work makes the following contributions. The convergence of this process is proved under more general assumptions: the random matrices B_n are not supposed i.i.d. and a new data mini-batch or all data until the current step are taken into account at each step without storing them; this is applied to online estimation of principal components in PCA of a random vector Z, when its expectation is unknown, as well as possibly the metrics used, and must be estimated online.

Denote Q a metrics in $\mathbb{R}^p$, $\langle .,. \rangle$ and $\|.\|$ respectively the inner product and the norm induced by Q: $\langle x, y \rangle = x'Qy$, x' denoting the transposed of the column vector x. Remind that a $p \times p$ matrix A is Q-symmetric if $(QA)' = QA$; then A has p real eigenvalues and there exists a Q-orthonormal basis of $\mathbb{R}^p$ composed of eigenvectors of A. The norm of a matrix A is the spectral norm denoted $\|A\|$.

Let B be a Q-symmetric matrix. Denote T_n the σ-field generated by the events before time n.

In the next section, the almost sure (a.s.) convergence of the normed process to eigenvectors of B is studied. Three cases are considered:

- $E[B_n|T_n]$ converges a.s. to B;
- $B_n = \omega_{1n} B_n^1 + \omega_{2n} B_n^2$ with $\omega_{1n} + \omega_{2n} = 1, B_n^2$ is T_n-measurable, $E[B_n^1|T_n]$ and B_n^2 converge a.s. to B;
- B_n converges a.s. to B.

For each case, firstly a theorem of a.s. convergence of (X_n) to a unit eigenvector of B associated to its greatest eigenvalue is proved with, in the first case, a method following that of [8] under more general assumptions, a corollary in the second case and another method of proof in the third case; secondly, using arguments of exterior algebra, the convergence of processes $\left(X_n^i\right)$, $i = 1, .., r$ of the same type, obtained by Gram-Schmidt orthonormalization, to unit eigenvectors associated to eigenvalues of B in decreasing order is proved as a corollary.

Then, in the following section, the whole results are applied to online estimation of principal components in PCA. In order to reduce computing time, particularly in the case of a data stream, and to avoid numerical explosions, it is proposed:

a) to estimate the eigenvectors a_l of the symmetric p x p matrix $B = M^{\frac{1}{2}} C M^{\frac{1}{2}}$ (symmetrization); then the orthonormalization is computed with respect to I; estimates of c_l and u_l can be obtained from that of a_l;

b) to replace Z_n by $Z_n - m$, m being an estimation of $E[Z]$ computed in a preliminary phase with a small number of observations e.g. 1000 (pseudo-centering);

c) to use a data mini-batch at step n or all observations until step n without storing them.

This yields the following definitions of B_n, $(Z_{n,1}, ..., Z_{n,m_n})$ denoting the new observations taken into account at step n, $\overline{Z}_{n-1}$ the mean of the sample $(Z_{1,1}, ..., Z_{n-1,m_{n-1}})$, M_{n-1} an estimation of M depending on this sample, for $i = 1, ..., r$, $Z_{n,i}^c = Z_{n,i} - m$ and $\overline{Z}_n^c = \overline{Z}_n - m$:

$$\begin{aligned} B_n^1 &= M_{n-1}^{\frac{1}{2}} \left(\frac{1}{m_n} \sum_{i=1}^{m_n} Z_{n,i}^c (Z_{n,i}^c)' - \overline{Z}_{n-1}^c \left(\overline{Z}_{n-1}^c \right)' \right) M_{n-1}^{\frac{1}{2}}, \\ B_n^2 &= M_{n-1}^{\frac{1}{2}} \left(\frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} Z_{i,j}^c \left(Z_{i,j}^c \right)' - \overline{Z}_n^c \left(\overline{Z}_n^c \right)' \right) M_{n-1}^{\frac{1}{2}} \end{aligned}$$

and a combination of both.

Three cases are studied: at each step are taken into account:

- a data mini-batch, supposing Z and M_{n-1} uniformly bounded,
- or all observations until this step with different weights for observations in the past, which are not stored, and observations at this step, supposing Z and M_{n-1} uniformly bounded,

- or all observations until this step with uniform weights.

The paper ends with a brief conclusion.

2 Convergence of a normed process

Let (B_n) be a sequence of random $p \times p$ matrices, B a $p \times p$ matrix, (a_n) a sequence of positive numbers, X_1 a random variable of norm 1 in $\mathbb{R}^p$ independent from the sequence of random matrices (B_n) and (X_n) a stochastic process in $\mathbb{R}^p$ recursively defined at step n by:

$$X_{n+1} = \frac{(I + a_n B_n) X_n}{||(I + a_n B_n) X_n||}.$$

2.1 First case

2.1.1 Theorem of almost sure convergence

Suppose B_n not T_n-measurable (B_n is T_{n+1}-measurable). Make the following assumptions:

(H1a) B is Q-symmetric.

(H1b) B has distinct eigenvalues: $\lambda_1 > \lambda_2 > ... > \lambda_p$. Denote V_i a unit eigenvector of B associated to $\lambda_i, i = 1, .., p$.

(H2a) There exists a positive number b such that $sup_n ||B_n|| < b \quad a.s.$

(H2b) $E\left[\sum_{n=1}^{\infty} a_n ||E[B_n|T_n] - B||\right] < \infty \quad a.s.$

(H3) $a_n > 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} a_n^2 < \infty$.

Denote $U_n = \langle X_n, BX_n \rangle, W_n = \langle X_n, B_n X_n \rangle$.

Theorem 1 *Suppose assumptions H1a,b,H2a,b,H3 hold. Then :*

1) Almost surely, U_n converges to one of the eigenvalues of B; on $E_j = \{U_n \longrightarrow \lambda_j\}$, X_n converges to V_j or $-V_j$, $\sum_{n=1}^{\infty} a_n (\lambda_j - U_n)$ and $\sum_{n=1}^{\infty} a_n (\lambda_j - W_n)$ converge.

2) If moreover on $\bigcup_{j=2}^{p} E_j$, $\liminf E\left[\langle B_n X_n, V_1 \rangle^2 \ |T_n\right] > 0$ a.s., then $P(E_1) = 1$.

State two lemmas of Duflo [8] used in the proof.

Lemma 2 *Let (M_n) be a square-integrable martingale adapted to the filtration (T_n) and $(\langle M \rangle_n)$ its increasing process defined by:*

$$\begin{aligned} \langle M \rangle_1 &= M_1^2 \\ \langle M \rangle_{n+1} - \langle M \rangle_n = E[(M_{n+1} - M_n)^2 | T_n] &= E[M_{n+1}^2 | T_n] - M_n^2. \end{aligned}$$

Let $\langle M \rangle_\infty = \lim \langle M \rangle_n$. If $E[\langle M \rangle_\infty] < \infty$, then (M_n) converges a.s. and in mean square to a finite random variable.

Lemma 3 *Let (γ_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$. Let (Z_n) and (δ_n) be two sequences of random variables adapted to a filtration (T_n), and ϵ_n a noise adapted to (T_n).*
Suppose on the set Γ:
1) For every integer n, $Z_{n+1} = Z_n(1 + \delta_n) + \gamma_n \epsilon_{n+1}$;
2) (Z_n) is bounded;
3) $\sum_{n=1}^{\infty} \delta_n^2 < \infty$, $\delta_n \geq 0$ for n sufficiently large and there exists a sequence of positive numbers (b_n) such that $\sum_{n=1}^{\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} (b_n - \delta_n)$ converges;
4) for an $a > 2$, $E[|\epsilon_{n+1}|^a | T_n] = O(1)$ and $\liminf E[\epsilon_{n+1}^2 | T_n] > 0$ a.s.
Then, $P(\Gamma) = 0$.

Proof of Theorem 1

Step 1: **expression of $\mathbf{X}_{n+1}$**

Under H2a, as $||\left(I+a_nB_n\right)X_n||^2 = 1+2a_nW_n+a_n^2\left\|B_nX_n\right\|^2$:

$$\frac{1}{||\left(I+a_nB_n\right)X_n||} = 1-a_nW_n-\frac{1}{2}a_n^2\left\|B_nX_n\right\|^2+\alpha_n, \quad \alpha_n = O(a_n^2).$$

$$\begin{aligned} X_{n+1} &= \left(I+a_nB_n\right)\left(1-a_nW_n-\frac{1}{2}a_n^2\left\|B_nX_n\right\|^2+\alpha_n\right)X_n \\ &= \left(I+a_n\left(B_n-W_nI\right)+a_n\beta_n\right)X_n, \text{ with} \\ \beta_n &= -a_nW_nB_n-\frac{1}{2}a_n\left\|B_nX_n\right\|^2 I-\frac{1}{2}a_n^2B_n\left\|B_nX_n\right\|^2+a_n^{-1}\alpha_nI+\alpha_nB_n. \end{aligned}$$

$$\begin{aligned} X_{n+1} &= \left(I+a_n\left(B-U_nI\right)+a_n\Gamma_n\right)X_n, \text{ with} \\ \Gamma_n &= \left(B_n-B\right)-\left\langle X_n,\left(B_n-B\right)X_n\right\rangle I+\beta_n, \ ||\beta_n|| = O(a_n). \end{aligned}$$

β_n and Γ_n are uniformly bounded by assumption H2a.

Step 2: convergence of $\mathbf{U}_n$

$$\begin{aligned} E[U_{n+1}|T_n] &= E\left[\left\langle\left(I+a_n\left(B-U_nI\right)+a_n\Gamma_n\right)X_n, B\left(I+a_n\left(B-U_nI\right)+a_n\Gamma_n\right)X_n\right\rangle \ |T_n\right] \\ &= U_n+2a_n\left\langle\left(B-U_nI\right)X_n, BX_n\right\rangle+2a_nE\left[\left\langle\Gamma_nX_n, BX_n\right\rangle \ |T_n\right]+a_n^2\eta_n \\ \text{with} \quad \eta_n &= \left\langle\left(B-U_nI\right)X_n, B\left(B-U_nI\right)X_n\right\rangle+2E\left[\left\langle\Gamma_nX_n, B\left(B-U_nI\right)X_n\right\rangle \ |T_n\right] \\ &\quad +E\left[\left\langle\Gamma_nX_n, B\Gamma_nX_n\right\rangle \ |T_n\right] \text{ a.s.} \end{aligned}$$

As $\left\|X_n\right\| = 1$, denoting $\mu_n = 2a_n\left\langle E\left[\Gamma_n \ |T_n\right]X_n, BX_n\right\rangle+a_n^2\eta_n$:

$$\begin{aligned} \left\langle\left(B-U_nI\right)X_n, BX_n\right\rangle &= \left\|BX_n\right\|^2-U_n^2 = \left\|BX_n-U_nX_n\right\|^2 \geqslant 0. \\ E[U_{n+1}|T_n] &\geq U_n+\mu_n \text{ a.s.} \\ E[U_{n+1}-\sum_{i=1}^{n}\mu_i \ |T_n] &\geq U_n-\sum_{i=1}^{n-1}\mu_i \text{ a.s.} \end{aligned}$$

Prove the convergence of the submartingale $U_n-\sum_{i=1}^{n-1}\mu_i$. By H2a:

$$\begin{aligned} \left\|\beta_n\right\| &\leq \frac{3}{2}a_n\left\|B_n\right\|^2+\frac{1}{2}a_n^2\left\|B_n\right\|^3+a_n^{-1}\left|\alpha_n\right|+\left|\alpha_n\right|\left\|B_n\right\| = O\left(a_n\right) \ ; \\ \left\|\Gamma_n\right\| &\leq 2\left\|B_n-B\right\|+\left\|\beta_n\right\| = O\left(1\right); \\ \left\|\eta_n\right\| &\leq 4\left\|B\right\|^3+4\left\|B\right\|^2\left\|E\left[\Gamma_n \ |T_n\right]\right\|+\left\|B\right\|E\left[\left\|\Gamma_n \ \right\|^2|T_n\right] = O\left(1\right). \end{aligned}$$

$$\left\|E\left[\Gamma_n \ |T_n\right]\right\| \leq 2\left\|E\left[B_n \ |T_n\right]-B\right\|+\left\|E\left[\beta_n \ |T_n\right]\right\| \text{ a.s.}$$

By H2b and H3:

$$\begin{aligned} E\left[\left|\sum_{i=1}^{n-1}\mu_i\right|\right] &\leq 4\left\|B\right\|E\left[\sum_{i=1}^{\infty}a_i\left\|E\left[B_i \ |T_i\right]-B\right\|\right]+2\left\|B\right\|E\left[\sum_{i=1}^{\infty}a_i\left\|E\left[\beta_i \ |T_i\right]\right\|\right]+E\left[\sum_{i=1}^{\infty}a_i^2\left\|\eta_i\right\|\right] \\ &< \infty. \end{aligned}$$

By Doob lemma the submartingale $U_n-\sum_{i=1}^{n-1}\mu_i$ converges a.s. to an integrable random variable. As $\sum_{i=1}^{n-1}\mu_i$ converges, U_n converges a.s.

Step 3: convergence of $\mathbf{X}_n^j = \left\langle\mathbf{X}_n, \mathbf{V}_j\right\rangle$

Denote $\Gamma_n^j = \left\langle\Gamma_nX_n, V_j\right\rangle$.

$$X_{n+1}^j = \left\langle\left(I+a_n(B-U_nI)+a_n\Gamma_n\right)X_n, V_j\right\rangle = X_n^j\left(1+a_n(\lambda_j-U_n)\right)+a_n\Gamma_n^j.$$

$$
\begin{aligned}
(X_{n+1}^j)^2 &= (X_n^j)^2(1+2a_n(\lambda_j-U_n)) + a_n^2(\lambda_j-U_n)^2(X_n^j)^2 \\
&\quad +2a_n(1+a_n(\lambda_j-U_n))X_n^j\Gamma_n^j + a_n^2(\Gamma_n^j)^2 \\
&= (X_n^j)^2\left(1+2a_n(\lambda_j-U_n)\right) + a_n^2\left((\lambda_j-U_n)X_n^j+\Gamma_n^j\right)^2 + 2a_nX_n^j\Gamma_n^j \\
&= (X_1^j)^2 + 2\sum_{l=1}^n a_l(\lambda_j-U_l)(X_l^j)^2 + \sum_{l=1}^n a_l^2\left((\lambda_j-U_l)X_l^j+\Gamma_l^j\right)^2 + 2\sum_{l=1}^n a_lX_l^j\Gamma_l^j \\
&= \left(X_1^j\right)^2 + (1)+(2)+(3).
\end{aligned}
$$

Study the convergence of the terms (2), (3), (1) of this decomposition.

(i) $(2) = \sum_{l=1}^n a_l^2\left((\lambda_j-U_l)X_l^j+\Gamma_l^j\right)^2$ converges a.s. by H2a and H3.

(ii) Consider now (3).

$$
\sum_{l=1}^n a_lX_l^j\Gamma_l^j = \sum_{l=1}^n a_lX_l^j(\Gamma_l^j - E[\Gamma_l^j|T_l]) + \sum_{l=1}^n a_lX_l^jE[\Gamma_l^j|T_l].
$$

$$
\begin{aligned}
\sum_{l=1}^n a_l|X_l^jE[\Gamma_l^j|T_l]| &\leq \sum_{l=1}^n a_l\left|\langle E[\Gamma_l|T_l]X_l, V_j\rangle\right| \\
&\leq \sum_{l=1}^n a_l||E[\Gamma_l|T_l]|| \leq \sum_{l=1}^n a_l\left(2||E[B_l|T_l]-B|| + ||E[\beta_l|T_l]||\right).
\end{aligned}
$$

By H2a,b and H3, as $||\beta_n|| = O(a_n)$, $\sum_{l=1}^{+\infty} a_lX_l^jE[\Gamma_l^j|T_l]$ is convergent.

Let $M_n^j = \sum_{l=1}^{n-1} a_lX_l^j(\Gamma_l^j - E[\Gamma_l^j|T_l])$; (M_n^j) is a square-integrable martingale adapted to the filtration (T_n); denote $(\langle M^j\rangle_n)$ its increasing process. By H2a:

$$
\begin{aligned}
\langle M^j\rangle_{n+1} - \langle M^j\rangle_n &= E[(M_{n+1}^j - M_n^j)^2|T_n] = a_n^2E[(X_n^j)^2(\Gamma_n^j - E[\Gamma_n^j|T_n])^2|T_n] \\
&\leq a_n^2\left(E[(\Gamma_n^j)^2|T_n] - (E[\Gamma_n^j|T_n])^2|T_n]\right) \leq a_n^2E[(\|\Gamma_n\|^2|T_n]
\end{aligned}
$$

is the general term of a convergent and uniformly bounded series; thus by lemma 2 (M_n^j) converges a.s. to a finite random variable.

Therefore (3) converges a.s.

(iii) Consider finally (1). Let ω fixed belonging to the convergence set of U_n. The writing of ω will be omitted in the following. Denote L the limit of U_n. If $L \neq \lambda_j$, the sign of $\lambda_j - U_n$ is constant from a certain rank N depending on ω. Thus there exists $A > 0$ such that:

$$
\begin{aligned}
& 2\sum_{l=N}^n a_l|\lambda_j-U_l|(X_l^j)^2 = 2\left|\sum_{l=N}^n a_l(\lambda_j-U_l)(X_l^j)^2\right| \\
= & \left|(X_{n+1}^j)^2 - (X_N^j)^2 - \sum_{l=N}^n a_l^2\left((\lambda_j-U_l)X_l^j+\Gamma_l^j\right)^2 - 2\sum_{l=N}^n a_lX_l^j\Gamma_l^j\right| \\
< & A.
\end{aligned}
$$

Then for $L \neq \lambda_j$, $2\sum_{l=N}^n a_l|\lambda_j-U_l|(X_l^j)^2$ converges.

It follows from the convergence of (1), (2) and (3) that for $L \neq \lambda_j$, $(X_n^j)^2$ converges a.s.

Step 4: convergence of $\mathbf{X}_n$

If the limit of U_n is different from λ_j, then by convergence of (1) in step 3, $\sum_{l=1}^{\infty} a_l (X_l^j)^2 < \infty$ and X_n^j converges a.s. to 0. As $||X_n|| = 1$, this can not be true for every j.

Thus the limit of U_n is one of the eigenvalues of B, λ_i.

For $j \neq i$, X_n^j converges to 0, therefore $(X_n^i)^2$ converges to 1 and since

$$X_{n+1} - X_n = a_n \left((B - U_n I) + \Gamma_n \right) X_n,$$

$X_{n+1} - X_n$ converges to 0 and the limit of X_n is V_i or $-V_i$ on $E_i = \{U_n \longrightarrow \lambda_i\}$ (first assertion of theorem 1).

Consider now the decomposition:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n(\lambda_i - W_n) &= \sum_{n=1}^{\infty} a_n(\lambda_i - U_n) + \sum_{n=1}^{\infty} a_n \langle X_n, (B_n - E[B_n|T_n])X_n \rangle \\ &\quad + \sum_{n=1}^{\infty} a_n \langle X_n, (E[B_n|T_n] - B)X_n \rangle . \end{aligned}$$

(i) Using the decomposition of $(X_n^i)^2$ in step 3, the convergence of $(X_n^i)^2$ and of (2) and (3) yields that $\sum_{n=1}^{\infty} a_n(\lambda_i - U_n)$ converges a.s. (second assertion of theorem 1).

(ii) By H2b: $\sum_{n=1}^{\infty} a_n \langle X_n, (E[B_n|T_n] - B)X_n \rangle$ converges a.s.

(iii) Let $M_n = \sum_{l=1}^{n-1} a_l \langle X_l, (B_l - E[B_l|T_l])X_l \rangle$. (M_n) is a square-integrable martingale adapted to the filtration (T_n). Its increasing process $(\langle M \rangle_n)$ converges; indeed:

$$\begin{aligned} \langle M \rangle_{n+1} - \langle M \rangle_n &= E[(M_{n+1} - M_n)^2 | T_n] \\ &= a_n^2 E[\langle X_n, (B_n - E[B_n|T_n])X_n \rangle^2 | T_n] \\ &\leq a_n^2 E[||B_n - E[B_n|T_n]||^2 | T_n] \end{aligned}$$

is the general term of a convergent and uniformly bounded series. Thus (M_n) converges a.s. to a finite random variable.

Therefore by (i), (ii) and (iii), $\sum_{n=1}^{\infty} a_n(\lambda_i - W_n)$ converges (third assertion of theorem 1).

Step 5: convergence of X_n to $\pm V_1$

Suppose $i > 1$.

$$\begin{aligned} X_{n+1}^1 &= (1 + a_n(\lambda_1 - U_n)) X_n^1 + a_n \langle \Gamma_n X_n, V_1 \rangle \\ &= (1 + a_n((\lambda_1 - \lambda_i) + (\lambda_i - U_n))) X_n^1 + a_n \langle \Gamma_n X_n, V_1 \rangle . \end{aligned}$$

In the following, apply lemma 3 to the sequence $\left(X_n^1\right)$ on $E_i = \{X_n \longrightarrow V_i\}, i > 1$, with $\gamma_n = a_n$, $\delta_n = a_n(\lambda_1 - U_n) \geqslant 0$, $b_n = a_n(\lambda_1 - \lambda_i) > 0$, $\epsilon_{n+1} = \langle \Gamma_n X_n, V_1 \rangle$.

(i) X_n^1 is bounded.

(ii) $\sum_{n=1}^{\infty} a_n^2 < \infty$, $\sum_{n=1}^{\infty} a_n^2(\lambda_1 - U_n)^2 < \infty$ by H3, $\sum_{n=1}^{\infty} a_n(\lambda_1 - \lambda_i) = \infty$, $\sum_{n=1}^{\infty} a_n(\lambda_i - U_n)$ converges a.s.

(iii) Consider the decomposition:

$$\begin{aligned} \langle \Gamma_n X_n, V_1 \rangle &= \langle (B_n - B) X_n, V_1 \rangle - \langle (B_n - B)X_n, X_n \rangle \langle X_n, V_1 \rangle + \langle \beta_n X_n, V_1 \rangle \\ &= \langle B_n X_n, V_1 \rangle - \langle X_n, V_1 \rangle (\lambda_1 + \langle (B_n - B)X_n, X_n \rangle) + \langle \beta_n X_n, V_1 \rangle . \end{aligned}$$

By H2a, there exists a positive number c such that a.s.:

$$\begin{aligned} & E[(\langle \Gamma_n X_n, V_1 \rangle - \langle B_n X_n, V_1 \rangle)^2 \ | T_n] \\ \leq \ & 2(X_n^1)^2 E[(\lambda_1 + \langle (B_n - B)X_n, X_n \rangle)^2 \ | T_n] + 2E[(\langle \beta_n X_n, V_1 \rangle)^2 \ | T_n] \\ \leq \ & c(X_n^1)^2 + 2E[||\beta_n||^2 \ | T_n] \underset{n \to +\infty}{\longrightarrow} 0 \end{aligned}$$

since $||\beta_n|| = O(a_n)$ and $X_n^1 \underset{n\to+\infty}{\longrightarrow} 0$. Likewise:

$$E[(\langle \Gamma_n X_n, V_1\rangle - \langle B_n X_n, V_1\rangle)\langle B_n X_n, V_1\rangle \ |T_n] \underset{n\to+\infty}{\longrightarrow} 0.$$

Then, if $\liminf E[\langle B_n X_n, V_1\rangle^2 \ |T_n] > 0$, $\liminf E[\langle \Gamma_n X_n, V_1\rangle^2 \ |T_n] > 0$.

By lemma 3, under (i), (ii), (iii), $P(E_i) = 0$, $i > 1$.

Then, $P(E_1) = 1$ (fourth assertion of theorem 1). ■

2.1.2 Simultaneous estimation of several eigenvectors

In this part, for $i = 1, ..., r$, X_n^i does not denote the i^{th} component of X_n, but a random variable in $\mathbb{R}^p$ recursively defined by:

$$\begin{aligned} Y_{n+1}^i &= (I + a_n B_n) X_n^i, \\ T_{n+1}^i &= Y_{n+1}^i - \sum_{j<i} \langle Y_{n+1}^i, X_{n+1}^j\rangle X_{n+1}^j, \ X_{n+1}^i = \frac{T_{n+1}^i}{||T_{n+1}^i||}. \end{aligned}$$

$(X_{n+1}^1, ..., X_{n+1}^r)$ is obtained by Gram-Schmidt orthonormalization of $(Y_{n+1}^1, ..., Y_{n+1}^r)$.

Corollary 4 *Suppose assumptions H1a,b, H2a,b and H3 hold.*

1) For $i = 1, ..., r$, almost surely X_n^i converges to one of the eigenvectors of B.

2) If moreover, for $i = 1, ..., r$, almost surely on $\underset{j=i+1}{\overset{p}{\cup}} \left\{X_n^i \longrightarrow \pm V_j\right\}$, $\liminf E[\left\langle B_n X_n^i, V_i\right\rangle^2 \ |T_n] > 0$,

then X_n^i converges a.s. to V_i or $-V_i$, $\sum_{n=1}^{\infty} a_n \left|\lambda_i - \left\langle B X_n^i, X_n^i\right\rangle\right|$ and $\sum_{n=1}^{\infty} a_n(\lambda_i - \left\langle B_n X_n^i, X_n^i\right\rangle)$ converge a.s.

Before the proof, some concepts of exterior algebra are reminded.

Let $(e_1, ..., e_p)$ be a basis of $\mathbb{R}^p$. For $r \le p$, denote ${}^r\Lambda\mathbb{R}^p$ the exterior power of order r of $\mathbb{R}^p$, generated by the C_p^r exterior products $e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_r}$, $i_1 < i_2 < ... < i_r \in \{1, ..., p\}$.

a) Let Q be a metrics in $\mathbb{R}^p$. Define the inner product in ${}^r\Lambda\mathbb{R}^p$ induced by the metrics Q, also denoted $\langle ., .\rangle$, such that:

$$\langle e_{i_1} \wedge ... \wedge e_{i_r}, e_{k_1} \wedge ... \wedge e_{k_r}\rangle = \sum_{\sigma\in G_r} (-1)^{s(\sigma)} \langle e_{i_1}, e_{\sigma(k_1)}\rangle ... \langle e_{i_r}, e_{\sigma(k_r)}\rangle,$$

G_r being the set of permutations σ of $\{k_1, ..., k_r\}$ and $s(\sigma)$ the number of inversions of σ.

Denote also $\|.\|$ the associated norm. Note that if $x_1, ..., x_r$ are Q-orthogonal, $||x_1 \wedge ... \wedge x_r|| = \prod_{i=1}^{r} ||x_i||$, and if $(e_1, ..., e_p)$ is a Q-orthonormal basis of $\mathbb{R}^p$, then the set of the C_p^r exterior products $e_{i_1} \wedge ... \wedge e_{i_r}$ is an orthonormal basis of ${}^r\Lambda\mathbb{R}^p$.

b) Let U be an endomorphism in $\mathbb{R}^p$. Define for $j = 1, ..., r$ the endomorphism ${}^{rj}U$ in ${}^r\Lambda\mathbb{R}^p$ such that:

$${}^{rj}U(x_1 \wedge ... \wedge x_r) = \sum_{1\le i_1<i_2<...<i_j\le r} x_1 \wedge ... \wedge Ux_{i_1} \wedge ... \wedge Ux_{i_j} \wedge ... \wedge x_r.$$

.

$$\text{For j=1, } {}^{r1}U(x_1 \wedge ... \wedge x_r) = \sum_{i=1}^{r} x_1 \wedge ... \wedge Ux_i \wedge ... \wedge x_r.$$

Denote ${}^r\Lambda U$ the endomorphism ${}^{rr}U$ such that:

$${}^r\Lambda U(x_1 \wedge ... \wedge x_r) = Ux_1 \wedge Ux_2 \wedge ... \wedge Ux_r.$$

c) The following properties hold:

(i) Suppose that the eigenvalues $\lambda_1 > ... > \lambda_p$ of U are distinct and denote for $j = 1, ..., r$, V_j an eigenvector associated to λ_j. Then the C_p^r vectors $V_{i_1} \wedge ... \wedge V_{i_r}$, $1 \le i_1 < ... < i_r \le p$, are eigenvectors of ${}^{r1}U$ respectively associated to the eigenvalues $\lambda_{i_1} + ... + \lambda_{i_r}$.

(ii) ${}^r\Lambda(I+U) = I + \sum_{j=1}^{r} {}^{rj}U$.

(iii) There exists $c(r) > 0$ such that, for every endomorphism U in $\mathbb{R}^p$ and for $1 \leq j \leq r$, $||{}^{rj}U|| \leq c(r)||U||^j$.

Proof

Step 1

For $i = 1, ..., r$, it follows from the orthogonality of $T_n^1, ..., T_n^i$ that:

$$||T_{n+1}^1 \wedge ... \wedge T_{n+1}^i|| = \prod_{l=1}^{i} ||T_{n+1}^l||.$$

Then, denoting ${}^iX_{n+1} = X_{n+1}^1 \wedge ... \wedge X_{n+1}^i$ and $D_n^i = {}^{i1}B_n + \sum_{j=2}^{i} a_n^{j-1}\, {}^{ij}B_n$:

$$\begin{aligned}
{}^iX_{n+1} &= \frac{T_{n+1}^1 \wedge ... \wedge T_{n+1}^i}{||T_{n+1}^1 \wedge ... \wedge T_{n+1}^i||} = \frac{Y_{n+1}^1 \wedge ... \wedge Y_{n+1}^i}{||Y_{n+1}^1 \wedge ... \wedge Y_{n+1}^i||} = \frac{{}^i\Lambda(I + a_nB_n)\ {}^iX_n}{||{}^i\Lambda(I + a_nB_n)\ {}^iX_n||} \\
&= \frac{\left(I + a_n\ {}^{i1}B_n + \sum_{j=2}^{i} a_n^j\, {}^{ij}B_n\right)\ {}^iX_n}{\left\|\left(I + a_n\ {}^{i1}B_n + \sum_{j=2}^{i} a_n^j\, {}^{ij}B_n\right)\ {}^iX_n\right\|} = \frac{\left(I + a_n\ D_n^i\right)\ {}^iX_n}{||\left(I + a_n\ D_n^i\right)\ {}^iX_n||}.
\end{aligned}$$

As $||{}^{ij}B_n|| \leq c(i)||B_n||^j$, assumptions H2a and H3 yield that there exists $b_1 > 0$ such that for all n, $||D_n^i|| \leq b_1$.

Moreover, as $U \mapsto {}^{i1}U$ is a linear application, assumptions H2a,b and H3 yield that:

$$\begin{aligned}
& E\left[\sum_{n=1}^{\infty} a_n ||E[D_n^i|T_n] - {}^{i1}B||\right] = E\left[\sum_{n=1}^{\infty} a_n \left\|E\left[{}^{i1}B_n - {}^{i1}B + \sum_{j=2}^{i} a_n^{j-1}\, {}^{ij}B_n \ |T_n\right]\right\|\right] \\
\leq\ & E\left[\sum_{n=1}^{\infty} a_n \left(||{}^{i1}E[B_n - B\ |T_n]|| + \sum_{j=2}^{i} a_n^{j-1} E[||{}^{ij}B_n||\ |T_n]\right)\right] \\
\leq\ & c(i)E\left[\sum_{n=1}^{\infty} a_n \left(||E\left[B_n|T_n\right] - B]|| + \sum_{j=2}^{i} a_n^{j-1} E[||B_n||^j\ |T_n]\right)\right] < \infty.
\end{aligned}$$

Applying first assertion of theorem 1 yields that almost surely,
${}^iX_n$ converges to a unit eigenvector $\pm V_{j_1} \wedge ... \wedge V_{j_i}$ of ${}^{i1}B$,
$\sum_{n=1}^{\infty} a_n(\lambda_{j_1} + ... + \lambda_{j_i} - \langle {}^{i1}B\ {}^iX_n, {}^iX_n\rangle)$ and
$\sum_{n=1}^{\infty} a_n(\lambda_{j_1} + ... + \lambda_{j_i} - \langle D_n^i\ {}^iX_n, {}^iX_n\rangle)$ converge.

Moreover by H2a and H3, $\sum_{n=1}^{\infty} a_n(\lambda_{j_1} + ... + \lambda_{j_i} - \langle {}^{i1}B_n\ {}^iX_n, {}^iX_n\rangle)$ converges a.s.

Step 2

Suppose that for $k = 1, ..., i-1$, $X_n^k \underset{n\to+\infty}{\longrightarrow} \pm V_k$, which is verified for $k = 1$, and prove that $X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_i$.

1) Prove that there exists $j > i-1$ such that $X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_j$. Suppose that there exists $k \in \{1, ..., i-1\}$ such that, for $l = 1, ..., i$, $V_{j_l} \neq \pm V_k$; then, for $l = 1, ..., i$, $\langle X_n^k, V_{j_l}\rangle \underset{n\to+\infty}{\longrightarrow} 0$ and $\langle X_n^1\Lambda...\Lambda X_n^i, V_{j_1}\Lambda...\Lambda V_{j_i}\rangle \underset{n\to+\infty}{\longrightarrow} 0$, a contradiction. Therefore for all $k \in \{1, ..., i-1\}$, there exists j_l such that $V_{j_l} = \pm V_k$ and there exists j such that

$${}^iX_n = X_n^1 \wedge ... \wedge X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_1\Lambda...\Lambda V_{i-1} \wedge V_j.$$

The only term which has a non-zero limit in the development of

$$\left\langle X_n^1 \wedge ... \wedge X_n^i, \pm V_1 \wedge ... \wedge V_{i-1} \wedge V_j \right\rangle,$$

whose limit is 1 as $n \longrightarrow \infty$, is $\langle X_n^1, V_1\rangle\langle X_n^2, V_2\rangle ... \langle X_n^{i-1}, V_{i-1}\rangle \left\langle X_n^i, V_j \right\rangle$ obtained for $\sigma = Id$. As for $k = 1, ..., i-1$, $\langle X_n^k, V_k\rangle \underset{n\to+\infty}{\longrightarrow} \pm 1$, then $\left\langle X_n^i, V_j \right\rangle \underset{n\to+\infty}{\longrightarrow} \pm 1$. Therefore $X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_j$.

2) Prove now that $V_j = \pm V_i$. Suppose $X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_j \neq \pm V_i$.

Denote G_i the set of permutations σ of $\{1, ..., i\}$ with $\sigma = (\sigma(1), ..., \sigma(i))$ and $s(\sigma)$ the number of inversions of σ.

$$\begin{aligned}\langle {}^{i1}B_n(X_n^1 \wedge .. \wedge X_n^i), V_1 \wedge .. \wedge V_i\rangle &= \sum_{l=1}^{i} \langle X_n^1 \wedge ... \wedge B_n X_n^l \wedge ... \wedge X_n^i, V_1 \wedge ... \wedge V_i\rangle \\ &= \sum_{l=1}^{i} \sum_{\sigma\in G_i} (-1)^{s(\sigma)} \langle X_n^1, V_{\sigma(1)}\rangle ... \langle B_n X_n^l, V_{\sigma(l)}\rangle ... \langle X_n^i, V_{\sigma(i)}\rangle.\end{aligned}$$

$$\begin{aligned}& E\left[\left\langle {}^{i1}B_n(X_n^1 \wedge ... \wedge X_n^i), V_1 \wedge ... \wedge V_i\right\rangle^2 \ |T_n\right] \\ = \ & E\left[\left(\sum_{l=1}^{i} \sum_{\sigma\in G_i} (-1)^{s(\sigma)} \langle X_n^1, V_{\sigma(1)}\rangle ... \langle B_n X_n^l, V_{\sigma(l)}\rangle ... \langle X_n^i, V_{\sigma(i)}\rangle\right)^2 \ |T_n\right].\end{aligned}$$

As for $k = 1, ..., i-1$, $X_n^k \underset{n\to+\infty}{\longrightarrow} \pm V_k$, the only term with a non-zero limit in the development of this conditional expectation is

$$\langle X_n^1, V_1\rangle^2 ... \langle X_n^{i-1}, V_{i-1}\rangle^2 E\left[\langle B_n X_n^i, V_i\rangle^2 \ |T_n\right]$$

and

$$\liminf E\left[\left\langle {}^{i1}B_n(X_n^1 \wedge ... \wedge X_n^i), V_1 \wedge ... \wedge V_i\right\rangle^2 \ |T_n\right] = \liminf E\left[\langle B_n X_n^i, V_i\rangle^2 \ |T_n\right] > 0.$$

Moreover, by H2a and $\underset{n\longrightarrow\infty}{lim}\, a_n = 0$:

$$\begin{aligned}& \liminf E\left[\left\langle \sum_{j=2}^{i} a_n^{j-1} \, {}^{ij}B_n(X_n^1 \wedge ... \wedge X_n^i), V_1 \wedge ... \wedge V_i\right\rangle^2 \ |T_n\right] \\ = \ & \liminf E\left[\left(\sum_{j=2}^{i} a_n^{j-1} \left\langle {}^{ij}B_n(X_n^1 \wedge ... \wedge X_n^i), V_1 \wedge ... \wedge V_i\right\rangle\right)^2 \ |T_n\right] = 0.\end{aligned}$$

Then $\liminf E\left[\langle D_n^i(X_n^1 \wedge ... \wedge X_n^i), V_1 \wedge ... \wedge V_i\rangle^2 | T_n\right] > 0$.

Applying second assertion of theorem 1 yields almost surely:

$$\begin{aligned}& X_n^1 \wedge ... \wedge X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_1 \wedge ... \wedge V_i, \text{ therefore } X_n^i \underset{n\to+\infty}{\longrightarrow} \pm V_i, \\ & \sum_{n=1}^{\infty} a_n \left(\sum_{l=1}^{i} \lambda_l - \langle {}^{i1}B \ {}^{i}X_n, {}^{i}X_n\rangle\right) \text{ and } \sum_{n=1}^{\infty} a_n \left(\sum_{l=1}^{i} \lambda_l - \langle D_n^i \ {}^{i}X_n, {}^{i}X_n\rangle\right) \text{ converge,}\end{aligned}$$

then by H2a and H3, $\sum\limits_{n=1}^{\infty} a_n \left(\sum\limits_{l=1}^{i} \lambda_l - \langle {}^{i1}B_n \ {}^{i}X_n, {}^{i}X_n\rangle\right)$ converges.

Step 3

$$\langle {}^{i1}B \ {}^{i}X_n, {}^{i}X_n\rangle = \sum_{k=1}^{i} \sum_{\sigma\in G_i} (-1)^{s(\sigma)} \langle X_n^1, X_n^{\sigma(1)}\rangle ... \langle BX_n^k, X_n^{\sigma(k)}\rangle ... \langle X_n^i, X_n^{\sigma(i)}\rangle.$$

As $X_n^1, ..., X_n^i$ are orthonormal, this sum is equal to

$$\sum_{k=1}^{i} \langle X_n^1, X_n^1 \rangle ... \langle BX_n^k, X_n^k \rangle ... \langle X_n^i, X_n^i \rangle = \sum_{k=1}^{i} \langle BX_n^k, X_n^k \rangle.$$

Then, as $\sum_{l=1}^{i} \lambda_l$ is the greatest eigenvalue of ${}^{i1}B$:

$$\begin{aligned} \lambda_i - \left\langle BX_n^i, X_n^i \right\rangle &= \left(\sum_{l=1}^{i} \lambda_l - \left\langle {}^{i1}B \ {}^{i}X_n, {}^{i}X_n \right\rangle \right) - \left(\sum_{l=1}^{i-1} \lambda_l - \left\langle {}^{i-1,1}B \ {}^{i-1}X_n, {}^{i-1}X_n \right\rangle \right) \\ &= \left| \sum_{l=1}^{i} \lambda_l - \left\langle {}^{i1}B \ {}^{i}X_n, {}^{i}X_n \right\rangle \right| - \left| \sum_{l=1}^{i-1} \lambda_l - \left\langle {}^{i-1,1}B \ {}^{i-1}X_n, {}^{i-1}X_n \right\rangle \right|. \end{aligned}$$

Almost surely, $\sum_{n=1}^{\infty} a_n \left| \sum_{l=1}^{i} \lambda_l - \left\langle {}^{i1}B^i X_n, {}^{i}X_n \right\rangle \right| < \infty$, then $\sum_{n=1}^{\infty} a_n \left| \lambda_i - \left\langle BX_n^i, X_n^i \right\rangle \right| < \infty$.

Likewise, as almost surely $\sum_{n=1}^{\infty} a_n \left(\sum_{l=1}^{i} \lambda_l - \left\langle {}^{i1}B_n \ {}^{i}X_n, {}^{i}X_n \right\rangle \right)$ converges, then

$\sum_{n=1}^{\infty} a_n \left(\lambda_i - \left\langle B_n X_n^i, X_n^i \right\rangle \right)$ converges. ■

2.2 Second case

Consider the same processes (X_n) and $\left(X_n^i\right)$ as in the first case.

Suppose now $B_n = \omega_{1n} B_n^1 + \omega_{2n} B_n^2$ with $\omega_{1n} > 0, \omega_{2n} \geqslant 0, \omega_{1n} + \omega_{2n} = 1$, B_n^2 T_n-measurable.

2.2.1 Theorem of almost sure convergence

Make the following assumptions:

(H2a') There exists a positive number b_1 such that $\sup_n \left\| B_n^1 \right\| < b_1$ a.s.

(H2b') $E\left[\sum_{n=1}^{\infty} a_n \left\| E\left[B_n^1 | T_n \right] - B \right\| \right] < \infty$.

(H2c') B_n^2 T_n-measurable, $B_n^2 \underset{n \to +\infty}{\longrightarrow} B$, $E\left[\sum_{n=1}^{\infty} a_n ||B_n^2 - B|| \right] < \infty$ a.s.

Theorem 5 *Suppose assumptions H1a,b, H2a',b',c' and H3 hold. Then :*

1) Almost surely, U_n converges to one of the eigenvalues of B; on $E_j = \{U_n \longrightarrow \lambda_j\}$, X_n converges to V_j or $-V_j$, $\sum_{n=1}^{\infty} a_n (\lambda_j - U_n)$ and $\sum_{n=1}^{\infty} a_n (\lambda_j - W_n)$ converge.

2) If moreover $\lim_{n \longrightarrow \infty} \omega_{1n} = \omega_1 > 0$ and on $\bigcup_{j=2}^{p} E_j$, $\liminf E\left[\left\langle B_n^1 X_n, V_1 \right\rangle^2 | T_n \right] > 0$ a.s., then $P(E_1) = 1$.

Proof

Apply theorem 1.

Under assumptions H2a',b',c', assumptions H2a,b are verified. Thus first part of theorem 1 holds.

Prove that $\liminf E[\langle B_n X_n, V_1 \rangle^2 | T_n] > 0$ a.s. when $\lim_{n \longrightarrow \infty} X_n = \pm V_j \neq \pm V_1$.

$$\begin{aligned} & E\left[\left\langle \left(\omega_{1n} B_n^1 + \omega_{2n} B_n^2 \right) X_n, V_1 \right\rangle^2 | T_n \right] \\ = \ & (\omega_{2n})^2 \left\langle B_n^2 X_n, V_1 \right\rangle^2 + 2\omega_{1n}\omega_{2n} \left\langle B_n^2 X_n, V_1 \right\rangle E\left[\left\langle B_n^1 X_n, V_1 \right\rangle | T_n \right] \\ & + (\omega_{1n})^2 E\left[\left\langle B_n^1 X_n, V_1 \right\rangle^2 | T_n \right] \text{ a.s.} \end{aligned}$$

When $\lim_{n \longrightarrow \infty} X_n = \pm V_j \neq \pm V_1$, $\lim_{n \longrightarrow \infty} \left\langle B_n^2 X_n, V_1 \right\rangle = \pm \left\langle BV_j, V_1 \right\rangle = \pm \lambda_j \langle V_j, V_1 \rangle = 0$. Then:

$\liminf E\left[\left\langle \left(\omega_{1n} B_n^1 + \omega_{2n} B_n^2 \right) X_n, V_1 \right\rangle^2 | T_n \right] = (\omega_1)^2 \liminf E\left[\left\langle B_n^1 X_n, V_1 \right\rangle^2 | T_n \right] > 0$ a.s. ■

2.2.2 Simultaneous estimation of several eigenvectors

Corollary 6 *Suppose assumptions H1a,b, H2a',b' ,c' and H3 hold. Then:*

1) For $i = 1, ..., r$, almost surely X_n^i converges to one of the eigenvectors of B.

2) If moreover $\lim_{n\longrightarrow\infty} \omega_{1n} = \omega_1 > 0$ and for $i = 1, ..., r$, a.s. on $\overset{p}{\underset{j=i+1}{\cup}} \left\{X_n^i \longrightarrow \pm V_j\right\}$, $\liminf E\left[\left\langle B_n^1 X_n^i, V_i\right\rangle^2 | T_n\right] > 0$,

then almost surely X_n^i converges to $\pm V_i$, $\sum_{n=1}^{\infty} a_n \left|\lambda_i - \left\langle BX_n^i, X_n^i\right\rangle\right|$ and $\sum_{n=1}^{\infty} a_n(\lambda_i - \left\langle B_n X_n^i, X_n^i\right\rangle)$ converge.

It is a direct application of corollary 4 whose assumptions are verified as proved above.

2.3 Third case

It is assumed in the second case that $\omega_1 > 0$. Now assume $\omega_{1n} = \omega_1 = 0$.

2.3.1 Theorem of almost sure convergence

Recursively define the process $(\widetilde{X}_n)$ such that

$$\widetilde{X}_{n+1} = (I + a_n B_n)\widetilde{X}_n$$

and the process $\left(\widetilde{U}_n\right)$ such that

$$\begin{aligned} \widetilde{U}_{n+1} &= \frac{\widetilde{X}_{n+1}}{\prod_{i=1}^{n}(1+\lambda_1 a_i)} = \frac{I + a_n B_n}{1+\lambda_1 a_n}\widetilde{U}_n \\ &= \widetilde{U}_n + \frac{a_n}{1+\lambda_1 a_n}(B_n\widetilde{U}_n - \lambda_1\widetilde{U}_n),\ \widetilde{U}_1 = \widetilde{X}_1. \end{aligned}$$

Note that $\frac{\widetilde{U}_n}{\|\widetilde{U}_n\|} = \frac{\widetilde{X}_n}{\|\widetilde{X}_n\|} = X_n$. Make the following assumptions:

(H1c) $\|B\| = \lambda_1$.

(H2c) $\sum_{n=1}^{\infty} a_n ||B_n - B|| < \infty$ a.s.

(H2d) For all n, $I + a_n B_n$ is invertible (especially verified if B_n is non-negative).

(H5) $\widetilde{X}_1$ is an absolutely continuous random variable, independent from $B_1, ..., B_n, ...$

Theorem 7 *Suppose assumptions H1a,b,c, H2c,d, H3 and H5 hold. Almost surely, $\widetilde{U}_n$ converges to a random vector colinear to V_1, therefore X_n converges to $\pm V_1$, $\sum_{n=1}^{\infty} a_n (\lambda_1 - \langle BX_n, X_n\rangle)$ and $\sum_{n=1}^{\infty} a_n |\lambda_1 - \langle B_n X_n, X_n\rangle|$ converge.*

Remark 8 *1) Note that assumption H2a is not required.*

2) Since $\omega \in \Omega$ is fixed throughout the following proof, a_n can be a positive random variable. ■

Lemma 9 *Suppose for all n, (z_n), (α_n), (β_n) and (γ_n) are four sequences of non-negative numbers such that:*

$$\text{for all } n \geqslant 1,\ z_{n+1} \leq z_n(1+\alpha_n) + \beta_n - \gamma_n, \sum_{n=1}^{\infty}\alpha_n < \infty, \sum_{n=1}^{\infty}\beta_n < \infty.$$

Then the sequence (z_n) converges and $\sum_{n=1}^{\infty}\gamma_n < \infty$.

This is a deterministic form of the Robbins-Siegmund lemma [10], whose proof is based on the convergence of the decreasing sequence

$$u_n = \frac{z_n}{\prod_{l=1}^{n-1}(1+\alpha_l)} - \sum_{k=1}^{n-1}\frac{\beta_k - \gamma_k}{\prod_{l=1}^{k}(1+\alpha_l)}.$$

Proof

Let ω be fixed, belonging to $C_1 = \left\{ \sum_{n=1}^{\infty} a_n ||B_n - B|| < \infty \right\}$. The writing of ω will be omitted in the following.

Step 1

$$\begin{aligned} ||\widetilde{U}_{n+1}||^2 &= ||\widetilde{U}_n||^2 + 2\frac{a_n}{1+\lambda_1 a_n}\left\langle \widetilde{U}_n, (B_n - \lambda_1 I)\widetilde{U}_n \right\rangle + \frac{a_n^2}{(1+\lambda_1 a_n)^2}||(B_n - \lambda_1 I)\widetilde{U}_n||^2 \\ &= ||\widetilde{U}_n||^2 + 2\frac{a_n}{1+\lambda_1 a_n}\left\langle \widetilde{U}_n, (B_n - B)\widetilde{U}_n \right\rangle + \frac{a_n^2}{(1+\lambda_1 a_n)^2}||(B_n - \lambda_1 I)\widetilde{U}_n||^2 \\ &\quad -2\frac{a_n}{1+\lambda_1 a_n}\left\langle \widetilde{U}_n, (\lambda_1 I - B)\widetilde{U}_n \right\rangle. \end{aligned}$$

$\lambda_1 I - B$ is a non-negative Q-symmetric matrix, with eigenvalues $0, \lambda_1 - \lambda_2, ..., \lambda_1 - \lambda_p$.

$$\begin{aligned} ||B_n - \lambda_1 I||^2 &\leq 2||B_n - B||^2 + 2||\lambda_1 I - B||^2. \\ ||\widetilde{U}_{n+1}||^2 &\leq ||\widetilde{U}_n||^2 \left(1 + 2a_n||B_n - B|| + 2a_n^2||B_n - B||^2 + 2a_n^2(\lambda_1 - \lambda_p)^2\right) \\ &\quad -2\frac{a_n}{1+\lambda_1 a_n}\left\langle \widetilde{U}_n, (\lambda_1 I - B)\widetilde{U}_n \right\rangle. \end{aligned}$$

By assumptions H2c and H3, applying lemma 9 yields:

$$||\widetilde{U}_n||^2 \underset{n\to+\infty}{\longrightarrow} \widetilde{U}, \quad \sum_{n=1}^{\infty} a_n \left\langle \widetilde{U}_n, (\lambda_1 I - B)\widetilde{U}_n \right\rangle = \sum_{n=1}^{\infty} a_n ||\widetilde{U}_n||^2 (\lambda_1 - \frac{\left\langle \widetilde{U}_n, B\widetilde{U}_n \right\rangle}{||\widetilde{U}_n||^2}) < \infty.$$

As $\sum_{n=1}^{\infty} a_n = \infty$, either $||\widetilde{U}_n|| \underset{n\to+\infty}{\longrightarrow} 0$ or $\sum_{n=1}^{\infty} a_n(\lambda_1 - \langle X_n, BX_n\rangle) < \infty$.

Step 2: convergence of $\widetilde{U}_n^j = \left\langle \widetilde{U}_n, V_j \right\rangle$

$$\begin{aligned} \widetilde{U}_{n+1}^j &= \left\langle V_j, \frac{I + a_n B_n}{1+\lambda_1 a_n}\widetilde{U}_n \right\rangle = \left\langle V_j, \frac{1}{1+\lambda_1 a_n}\left(I + a_n B + a_n (B_n - B)\right)\widetilde{U}_n \right\rangle \\ &= \frac{1+\lambda_j a_n}{1+\lambda_1 a_n}\widetilde{U}_n^j + \frac{a_n}{1+\lambda_1 a_n}\left\langle V_j, (B_n - B)\widetilde{U}_n \right\rangle. \end{aligned}$$

a) For $j > 1$, as $a_n \underset{n\to+\infty}{\longrightarrow} 0$, there exists $\alpha_n = O(a_n) > 0$ such that for n sufficiently large:

$$\begin{aligned} \left|\widetilde{U}_{n+1}^j\right| &\leq \frac{1+\lambda_j a_n}{1+\lambda_1 a_n}\left|\widetilde{U}_n^j\right| + a_n \|B_n - B\| \left\|\widetilde{U}_n\right\| \\ &\leq (1-\alpha_n)\left|\widetilde{U}_n^j\right| + a_n \|B_n - B\| \left\|\widetilde{U}_n\right\|. \end{aligned}$$

By H2c and as $\left\|\widetilde{U}_n\right\|$ converges, applying lemma 9 yields:

$$\left|\widetilde{U}_n^j\right| \underset{n\to+\infty}{\longrightarrow} \widetilde{U}^j, \ \sum_{n=1}^{\infty} \alpha_n \left|\widetilde{U}_n^j\right| < \infty. \text{ As } \sum_{n=1}^{\infty} a_n = \infty, \ \widetilde{U}^j = 0.$$

b) For $j = 1$, by H2c and $||\widetilde{U}_n|| \underset{n\to+\infty}{\longrightarrow} \sqrt{\widetilde{U}}$:

$$\begin{aligned} \widetilde{U}_{n+1}^1 &= \widetilde{U}_n^1 + \frac{a_n}{1+\lambda_1 a_n}\left\langle V_1, (B_n - B)\widetilde{U}_n \right\rangle = \widetilde{U}_1^1 + \sum_{i=1}^{n} \frac{a_i}{1+\lambda_1 a_i}\left\langle V_1, (B_i - B)\widetilde{U}_i \right\rangle \\ \underset{n\to+\infty}{\longrightarrow} \widetilde{U}^1 &= \widetilde{U}_1^1 + \sum_{i=1}^{\infty} \frac{a_i}{1+\lambda_1 a_i}\left\langle V_1, (B_i - B)\widetilde{U}_i \right\rangle. \end{aligned}$$

Now:

$$\begin{aligned}\widetilde{U}^1_{n+1} &= \left\langle V_1, \widetilde{U}_{n+1}\right\rangle = \left\langle V_1, \prod_{i=1}^{n}\frac{I+a_iB_i}{1+\lambda_1 a_i}\widetilde{U}_1\right\rangle \underset{n\to+\infty}{\longrightarrow} \left\langle V_1, \prod_{i=1}^{\infty}\frac{I+a_iB_i}{1+\lambda_1 a_i}\widetilde{U}_1\right\rangle \\ &= V_1'QS\widetilde{U}_1 = \widetilde{U}^1 \text{ with } S = \prod_{i=1}^{\infty}\frac{I+a_iB_i}{1+\lambda_1 a_i}.\end{aligned}$$

As $\widetilde{U}_1$ is absolutely continuous, if $V_1'QS \neq 0$, $P\left(V_1'QS\widetilde{U}_1 = 0 \mid S\right) = 0$, then $P\left(\widetilde{U}^1 = 0\right) = 0$. Prove that $V_1'QS \neq 0$.

Step 3

Denote $C_2 = \left\{\widetilde{U}_1 \neq 0\right\}$. Suppose $\omega \in C_1 \cap C_2$.

Under H2c, there exists N such that $\sum_{n=N}^{\infty} a_n||B_n - B|| < \ln 2$.

$$\begin{aligned}V_1'QS &= V_1'Q\prod_{i=N}^{\infty}\frac{I+a_iB_i}{1+\lambda_1 a_i}\prod_{i=1}^{N-1}\frac{I+a_iB_i}{1+\lambda_1 a_i} \\ &= V_1'QR\prod_{i=1}^{N-1}\frac{I+a_iB_i}{1+\lambda_1 a_i} \text{ with } R = \prod_{i=N}^{\infty}\frac{I+a_iB_i}{1+\lambda_1 a_i}.\end{aligned}$$

Under H2d, $V_1'QS \neq 0 \Leftrightarrow V_1'QR \neq 0$.

Denote $C_n = \frac{a_n\|B_n - B\|}{1+\lambda_1 a_n}$ and $(W_n, n \geqslant N)$ the process $\left(\widetilde{U}_n, n \geqslant N\right)$ with $W_N = V_1$.

As $\|B\| = \lambda_1$, $\|I + a_{i-1}B\| = 1 + \lambda_1 a_{i-1}$. By step 2, as $W_N = V_1$:

$$\begin{aligned}W^1_{n+1} &= \langle V_1, W_{n+1}\rangle = 1 + \sum_{i=N}^{n}\frac{a_i}{1+\lambda_1 a_i}\langle V_1, (B_i - B)W_i\rangle \geqslant 1 - \sum_{i=N}^{n} C_i\|W_i\|. \\ \|W_i\| &\leq \frac{\|I + a_{i-1}B_{i-1}\|}{1+\lambda_1 a_{i-1}}\|W_{i-1}\| \\ &\leq \frac{\|I + a_{i-1}B\| + a_{i-1}\|B_{i-1} - B\|}{1+\lambda_1 a_{i-1}}\|W_{i-1}\| = (1 + C_{i-1})\|W_{i-1}\| \\ &\leq \prod_{l=N}^{i-1}(1 + C_l), \ i = N+1, ..., n.\end{aligned}$$

As $\sum_{n=N}^{\infty} C_n < \ln 2$, it follows that:

$$\begin{aligned}W^1_{n+1} &\geqslant 1 - \sum_{i=N}^{n} C_i\prod_{l=N}^{i-1}(1 + C_l) = 1 - \left(\prod_{l=N}^{n}(1 + C_l) - 1\right) \\ &= 2 - \prod_{l=N}^{n}(1 + C_l) \geqslant 2 - e^{\sum_{l=N}^{n} C_l} \geqslant 2 - e^{\sum_{l=N}^{\infty} C_l} > 0.\end{aligned}$$

By step 2, W^1_n converges to $\langle V_1, RV_1\rangle = V_1'QRV_1$ which is therefore strictly positive, thus $V_1'QR \neq 0$.

Step 4: conclusion

It follows that $\left(\widetilde{U}_n\right)$ converges to $\widetilde{U}^1V_1 \neq 0$, therefore $\frac{\widetilde{U}_n}{\|\widetilde{U}_n\|} = X_n$ converges to $\pm V_1$, and by the conclusion of step 1, $\sum_{n=1}^{\infty} a_n(\lambda_1 - \langle X_n, BX_n\rangle) < \infty$.

Moreover by H2c:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \left|\lambda_1 - \langle X_n, B_n X_n\rangle\right| &= \sum_{n=1}^{\infty} a_n \left|\lambda_1 - \langle X_n, (B_n - B) X_n\rangle - \langle X_n, B X_n\rangle\right| \\ &\leq \sum_{n=1}^{\infty} a_n \left(\lambda_1 - \langle X_n, B X_n\rangle\right) + \sum_{n=1}^{\infty} a_n ||B_n - B|| < \infty. \blacksquare\end{aligned}$$

Remark 10 *Step 1 can be replaced by:*
$||\widetilde{U}_{n+1}|| \leq \frac{\|I+a_n B_n\|}{1+\lambda_1 a_n} \left\|\widetilde{U}_n\right\| \leq \left(1 + \frac{a_n \|B_n - B\|}{1+\lambda_1 a_n}\right) \left\|\widetilde{U}_n\right\|$.
Under H2c, $\left\|\widetilde{U}_n\right\|$ *converges a.s. Assumption* $\sum_{n=1}^{\infty} a_n^2 < \infty$ *is not used and can be replaced by* $a_n \underset{n\to+\infty}{\longrightarrow} 0$, *but in this case, the convergence of* $\sum_{n=1}^{\infty} a_n \left(\lambda_1 - \langle B X_n, X_n\rangle\right)$ *and* $\sum_{n=1}^{\infty} a_n \left|\lambda_1 - \langle B_n X_n, X_n\rangle\right|$ *is not proved.* $\blacksquare$

2.3.2 Simultaneous estimation of several eigenvectors

For $i = 1, ..., r$, recursively define the process $\left(\widetilde{X}_n^i\right)$ by:

$$\begin{aligned}\widetilde{Y}_{n+1}^i &= (I + a_n B_n)\widetilde{X}_n^i, \\ \widetilde{X}_{n+1}^i &= \widetilde{Y}_{n+1}^i - \sum_{j<i} \left\langle \widetilde{Y}_{n+1}^i, \frac{\widetilde{X}_{n+1}^j}{\left\|\widetilde{X}_{n+1}^j\right\|} \right\rangle \frac{\widetilde{X}_{n+1}^j}{\left\|\widetilde{X}_{n+1}^j\right\|}.\end{aligned}$$

Note that $\frac{\widetilde{X}_n^i}{\|\widetilde{X}_n^i\|} = X_n^i$.

Denote $D_n^i = {}^{i1}B_n + \sum_{j=2}^{i} a_n^{j-1}\, {}^{ij}B_n$.

Make the following assumptions:
(H1c') For $i = 1, ..., r$, $\left\|{}^{i1}B\right\| = \lambda_1 + ... + \lambda_i$.
(H2d') For $i = 1, ..., r$, $I + a_n D_n^i$ is invertible.
(H5') For $i = 1, ..., r$, X_1^i is an absolutely continuous random variable, independent from $B_1, ..., B_n, ...$

Corollary 11 *Suppose assumptions H1a,b,c', H2c,d', H3 and H5' hold. Then, for* $i = 1, ..., r$, *almost surely* X_n^i *converges to* $\pm V_i$, $\sum_{n=1}^{\infty} a_n \left|\lambda_i - \left\langle X_n^i, B X_n^i\right\rangle\right|$ *and* $\sum_{n=1}^{\infty} a_n \left|\lambda_i - \left\langle X_n^i, B_n X_n^i\right\rangle\right|$ *converge.*

Proof
ω is fixed throughout the proof, belonging to the intersection of the a.s. convergence sets. Its writing will be omitted.

Let $i \in \{1, ..., r\}$.

$$\begin{aligned}{}^{i}\widetilde{X}_{n+1} &= \widetilde{X}_{n+1}^1 \wedge ... \wedge \widetilde{X}_{n+1}^i = \widetilde{Y}_{n+1}^1 \wedge ... \wedge \widetilde{Y}_{n+1}^i = {}^{i}\Lambda (I + a_n B_n)^i \widetilde{X}_n \\ &= \left(I + a_n\, {}^{i1}B_n + a_n \sum_{j=2}^{i} a_n^{j-1}\, {}^{ij}B_n\right)\, {}^{i}\widetilde{X_n} = \left(I + a_n\, D_n^i\right)\, {}^{i}\widetilde{X}_n.\end{aligned}$$

Note that $\frac{{}^{i}\widetilde{X}_{n+1}}{\left\|{}^{i}\widetilde{X}_{n+1}\right\|} = \frac{\widetilde{X}_{n+1}^1}{\|\widetilde{X}_{n+1}^1\|} \wedge ... \wedge \frac{\widetilde{X}_{n+1}^i}{\|\widetilde{X}_{n+1}^i\|} = \frac{{}^{i}\Lambda (I+a_n B_n)^i \widetilde{X}_n}{\left\|{}^{i}\Lambda (I+a_n B_n)^i \widetilde{X}_n\right\|} = {}^{i}X_{n+1}$.
By H2c and H3:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \left\|D_n^i - {}^{i1}B\right\| &= \sum_{n=1}^{\infty} a_n \left\|{}^{i1}(B_n - B) + \sum_{j=2}^{i} a_n^{j-1}\, {}^{ij}B_n\right\| \\ &\leq c(i) \left(\sum_{n=1}^{\infty} a_n \|B_n - B\| + \sum_{j=2}^{i} \sum_{n=1}^{\infty} a_n^j \|B_n\|^j\right) < \infty.\end{aligned}$$

As B is Q-symmetric with distinct eigenvalues, ${}^{i1}B$ has the same properties ; $V_1 \wedge .. \wedge V_i$ is an eigenvector associated to its greatest eigenvalue $\lambda_1 + ... + \lambda_i$. Applying theorem 7 yields that:

$${}^{i}X_n \text{ converges to } \pm V_1 \wedge .. \wedge V_i,$$
$$\sum_{n=1}^{\infty} a_n \left(\sum_{l=1}^{i} \lambda_l - \left\langle {}^{i1}B \ {}^{i}X_n, \ {}^{i}X_n \right\rangle \right) \text{ and } \sum_{n=1}^{\infty} a_n \left| \sum_{l=1}^{i} \lambda_l - \left\langle D_n^i \ {}^{i}X_n, \ {}^{i}X_n \right\rangle \right| \text{ converge,}$$
$$\text{which implies that } \sum_{n=1}^{\infty} a_n \left| \sum_{l=1}^{i} \lambda_l - \left\langle {}^{i1}B_n \ {}^{i}X_n, \ {}^{i}X_n \right\rangle \right| \text{converges.}$$

Suppose that, for $k = 1, ..., i-1$, X_n^k converges to $\pm V_k$, which is verified for $k = 1$, and prove that it is true for $k = i$.

In the development of $\left\langle X_n^1 \wedge ... \wedge X_n^i, \pm V_1 \wedge .. \wedge V_i \right\rangle$, which converges to ± 1, the only term which has a non-zero limit is $\left\langle X_n^1, V_1 \right\rangle ... \left\langle X_n^{i-1}, V_{i-1} \right\rangle \left\langle X_n^i, V_i \right\rangle$; since for $k = 1, ..., i-1$, $\left\langle X_n^k, V_k \right\rangle$ converges to ± 1, it follows that $\left\langle X_n^i, V_i \right\rangle$ converges to ± 1.

Applying the same proof as that of corollary 4, step 3, yields:
$\sum_{n=1}^{\infty} a_n \left| \lambda_i - \left\langle X_n^i, B X_n^i \right\rangle \right| < \infty$. By H2c:

$$\sum_{n=1}^{\infty} a_n \left| \lambda_i - \left\langle X_n^i, B_n X_n^i \right\rangle \right| \leqslant \sum_{n=1}^{\infty} a_n \left| \lambda_i - \left\langle X_n^i, B X_n^i \right\rangle \right| + \sum_{n=1}^{\infty} a_n \left\| B_n - B \right\| < \infty. \ \blacksquare$$

3 Application to sequential principal component analysis of a data stream

Let $Z_{11}, ..., Z_{1m_1}, Z_{21}, ..., Z_{2m_2}, ..., Z_{n1}, ..., Z_{nm_n}, ...$ be an i.i.d sample of a random vector Z in $\mathbb{R}^p$ whose components are denoted $Z^1, ..., Z^p$. Denote M the metrics used for PCA and $B = M^{\frac{1}{2}} E\left[(Z - E[Z])(Z - E[Z])'\right] M^{\frac{1}{2}}$. Let m belonging to $\mathbb{R}^p$ (in practice m is an estimation of $E[Z]$); denoting $Z^c = Z - m$:

$$B = M^{\frac{1}{2}} \left(E\left[Z^c Z^{c\prime} \right] - E[Z^c] E[Z^c]' \right) M^{\frac{1}{2}}.$$

Denote $\overline{Z}_{n-1}$ the mean of the sample $(Z_{11}, ..., Z_{n-1,m_{n-1}})$ of Z and M_{n-1} a T_n-measurable estimation of M.

3.1 Use of a data mini-batch at each step

Note that the metrics used for orthonormalization is the identity because of the symmetrization.

Recursively define the processes $\left(X_n^i\right), i = 1, ..., r$, by

$$\begin{aligned} Y_{n+1}^i &= (I + a_n B_n) X_n^i, \\ T_{n+1}^i &= Y_{n+1}^i - \sum_{j<i} \langle Y_{n+1}^i, X_{n+1}^j \rangle X_{n+1}^j, \ X_{n+1}^i = \frac{T_{n+1}^i}{||T_{n+1}^i||}. \end{aligned}$$

.

Denote $Z_{ni}^c = Z_{ni} - m$, $\overline{Z}_{n-1}^c = \overline{Z}_{n-1} - m$. Take

$$B_n = M_{n-1}^{\frac{1}{2}} \left(\frac{1}{m_n} \sum_{i=1}^{m_n} Z_{ni}^c Z_{ni}^{c\prime} - \overline{Z}_{n-1}^c \left(\overline{Z}_{n-1}^c \right)' \right) M_{n-1}^{\frac{1}{2}}.$$

Make the following assumptions:

(H3') $a_n > 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} < \infty$, $\sum_{n=1}^{\infty} a_n^2 < \infty$.

(H4a) $\|Z\|$ is a.s. bounded.
(H4b) There is no affine or quadratic relation between the components of Z.
(H6a) There exists a positive number d such that $\sup_n \left\| M_n^{\frac{1}{2}} \right\| < d$.
(H6b) $M_n^{\frac{1}{2}} \longrightarrow M^{\frac{1}{2}}$ a.s.
(H6c) $E\left[\sum_{n=1}^{\infty} a_n \left\| M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}} \right\|\right] < \infty$.

Corollary 12 *Suppose assumptions H1b, H3', H4a,b and H6a,b,c hold. Then X_n^i converges a.s. to $\pm V_i$, $\sum_{n=1}^{\infty} a_n \left|\lambda_i - \left(X_n^i\right)' B X_n^i\right|$ and $\sum_{n=1}^{\infty} a_n \left(\lambda_i - \left(X_n^i\right)' B_n X_n^i\right)$ converge a.s. for $i = 1, ..., r$.*

Proof
Verify the assumptions of corollary 4.

(H1a) B is symmetric.
(H2a) Under H4a and H6a, $\sup_n \|B_n\|$ is a.s. uniformly bounded.
(H2b) Almost surely:

$$\begin{aligned}
& E\left[B_n \mid T_n\right] - B \\
= \; & E\left[M_{n-1}^{\frac{1}{2}} \left(\frac{1}{m_n} \sum_{i=1}^{m_n} Z_{ni}^c Z_{ni}^{c\prime} - \overline{Z}_{n-1}^c \left(\overline{Z}_{n-1}^c\right)'\right) M_{n-1}^{\frac{1}{2}} \mid T_n\right] \\
& - M^{\frac{1}{2}} \left(E\left[Z^c Z^{c'}\right] - E[Z^c] E[Z^c]'\right) M^{\frac{1}{2}} \\
= \; & M_{n-1}^{\frac{1}{2}} \left(E\left[Z^c Z^{c'}\right] - \overline{Z}_{n-1}^c \overline{Z}_{n-1}^{c\prime}\right) M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}} \left(E\left[Z^c Z^{c'}\right] - E[Z^c] E[Z^c]'\right) M^{\frac{1}{2}} \\
= \; & \left(M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}}\right) \left(E\left[Z^c Z^{c'}\right] - E[Z^c] E[Z^c]'\right) M_{n-1}^{\frac{1}{2}} \\
& + M^{\frac{1}{2}} \left(E\left[Z^c Z^{c'}\right] - E[Z^c] E[Z^c]'\right) \left(M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}}\right) \\
& - M_{n-1}^{\frac{1}{2}} \left(\overline{Z}_{n-1}^c - E[Z^c]\right) \overline{Z}_{n-1}^{c\prime} M_{n-1}^{\frac{1}{2}} - M_{n-1}^{\frac{1}{2}} E[Z^c] \left(\overline{Z}_{n-1}^c - E[Z^c]\right)' M_{n-1}^{\frac{1}{2}}.
\end{aligned}$$

If Z has 4^{th} order moments and $a_n > 0$, $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} < \infty$:

$$\sum_{n=1}^{\infty} a_n E\left[||\overline{Z}_{n-1}^c - E[Z^c]||\right] = \sum_{n=1}^{\infty} a_n E\left[||\overline{Z}_{n-1} - E[Z]||\right] < \infty. \text{ [9]}$$

Therefore, under H4a, H6a,c, $E\left[\sum_{n=1}^{\infty} a_n \left\|E\left[B_n \mid T_n\right] - B\right\|\right] < \infty$.

By corollary 4, for $k = 1, ..., r$, almost surely, X_n^k converges a.s. to one of the eigenvectors of B.

Prove now that $\lim_{n\to\infty} E[(X_n^{k\prime} B_n V_k)^2 \mid T_n] > 0$ a.s. on the set $\{X_n \longrightarrow V_j\}$ for $j \neq k$ to apply second part of corollary 4.

In the following of the proof, X_n^k is denoted X_n.

Decompose $E[(X_n' B_n V_k)^2 \mid T_n]$ into the sum of three terms (1),(2),(3):

$$
\begin{aligned}
& E\left[\left(X_n' M_{n-1}^{\frac{1}{2}}\left(\frac{1}{m_n}\sum_{i=1}^{m_n} Z_{ni}^c Z_{ni}^{c\prime} - \overline{Z}_{n-1}^c \overline{Z}_{n-1}^{c\prime}\right) M_{n-1}^{\frac{1}{2}} V_k\right)^2 \mid T_n\right] \\
= \; & E\left[\left(\frac{1}{m_n}\sum_{i=1}^{m_n}\left(X_n' M_{n-1}^{\frac{1}{2}} Z_{ni}^c\right)\left(Z_{ni}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right) - \left(X_n' M_{n-1}^{\frac{1}{2}} \overline{Z}_{n-1}^c\right)\left(\overline{Z}_{n-1}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right)\right)^2 \mid T_n\right] \\
= \; & E\left[\left(\frac{1}{m_n}\sum_{i=1}^{m_n}\left(X_n' M_{n-1}^{\frac{1}{2}} Z_{ni}^c\right)\left(Z_{ni}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right)\right)^2 \mid T_n\right] \quad (1) \\
& -2\left(X_n' M_{n-1}^{\frac{1}{2}} \overline{Z}_{n-1}^c\right)\left(\overline{Z}_{n-1}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right)\frac{1}{m_n}\sum_{i=1}^{m_n} E\left[\left(X_n' M_{n-1}^{\frac{1}{2}} Z_{ni}^c\right)\left(Z_{ni}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right) \mid T_n\right] \quad (2) \\
& +\left(X_n' M_{n-1}^{\frac{1}{2}} \overline{Z}_{n-1}^c\right)^2\left(\overline{Z}_{n-1}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right)^2. \quad (3)
\end{aligned}
$$

Note that the two random variables $R = V_j' M^{\frac{1}{2}} Z^c$ and $S = V_k' M^{\frac{1}{2}} Z^c$ are uncorrelated, then $E[RS] = E[R]E[S]$:

$$
\begin{aligned}
E[(R - E[R])(S - E[S])] &= E[V_j' M^{\frac{1}{2}}(Z - E[Z]).V_k' M^{\frac{1}{2}}(Z - E[Z])] \\
&= V_j' M^{\frac{1}{2}} E\left[(Z - E[Z])(Z - E[Z])'\right] M^{\frac{1}{2}} V_k = \lambda_k V_j' V_k = 0.
\end{aligned}
$$

Consider (1). Under H6b:

$$
\begin{aligned}
(1) \quad &= \quad \frac{1}{m_n^2}\sum_{i=1}^{m_n}\sum_{l=1}^{m_n} E\left[\left(X_n' M_{n-1}^{\frac{1}{2}} Z_{ni}^c\right)\left(Z_{ni}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right)\left(X_n' M_{n-1}^{\frac{1}{2}} Z_{nl}^c\right)\left(Z_{nl}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right) \mid T_n\right] \\
&= \quad X_n' M_{n-1}^{\frac{1}{2}} \frac{1}{m_n^2}\sum_{i=1}^{m_n}\sum_{l=1}^{m_n} E\left[\left(V_k' M_{n-1}^{\frac{1}{2}} Z_{ni}^c\right) Z_{ni}^c Z_{nl}^{c\prime}\left(Z_{nl}^{c\prime} M_{n-1}^{\frac{1}{2}} V_k\right) \mid T_n\right] M_{n-1}^{\frac{1}{2}} X_n \\
&\underset{n\to+\infty}{\longrightarrow} \quad V_j' M^{\frac{1}{2}} E\left[\left(V_k' M^{\frac{1}{2}} Z^c\right) Z^c Z^{c\prime}\left(Z^{c\prime} M^{\frac{1}{2}} V_k\right)\right] M^{\frac{1}{2}} V_j \\
&= \quad E\left[\left(V_k' M^{\frac{1}{2}} Z^c\right)^2\left(V_j' M^{\frac{1}{2}} Z^c\right)^2\right] \text{ a.s.}
\end{aligned}
$$

Consider (2):

$$
\begin{aligned}
& (2) \underset{n\to+\infty}{\longrightarrow} -2E\left[V_j' M^{\frac{1}{2}} Z^c\right] E\left[Z^{c\prime} M^{\frac{1}{2}} V_k\right] E\left[\left(V_j' M^{\frac{1}{2}} Z^c\right)\left(Z^{c\prime} M^{\frac{1}{2}} V_k\right)\right] \\
= \; & -2E\left[\left(V_j' M^{\frac{1}{2}} Z^c\right)\left(V_k' M^{\frac{1}{2}} Z^c\right)\right]^2 \text{ a.s.}
\end{aligned}
$$

Consider (3):

$$
(3) \underset{n\to+\infty}{\longrightarrow} \left(E\left[V_j' M^{\frac{1}{2}} Z^c\right] E\left[V_k' M^{\frac{1}{2}} Z^c\right]\right)^2 = E\left[\left(V_j' M^{\frac{1}{2}} Z^c\right)\left(V_k' M^{\frac{1}{2}} Z^c\right)\right]^2 \text{ a.s.}
$$

As a result:

$$
\begin{aligned}
& E[(X_n' B_n V_k)^2 \mid T_n] \underset{n\to+\infty}{\longrightarrow} E\left[\left(V_j' M^{\frac{1}{2}} Z^c\right)^2\left(V_k' M^{\frac{1}{2}} Z^c\right)^2\right] - E\left[\left(V_j' M^{\frac{1}{2}} Z^c\right)\left(V_k' M^{\frac{1}{2}} Z^c\right)\right]^2 \\
= \; & Var[V_j' M^{\frac{1}{2}} Z^c . V_k' M^{\frac{1}{2}} Z^c] > 0 \text{ a.s. by H4b. } \blacksquare
\end{aligned}
$$

3.2 Use of all observations until the current step with different weights

At each step, all observations until the current step are taken into account but with different weights for observations at the current step and observations in the past.

In the definition of processes $\left(X_n^i\right), i = 1, ..., r$, take now

$$\begin{aligned} B_n &= w_1 B_n^1 + w_2 B_n^2, w_1 + w_2 = 1, w_1 > 0, w_2 \geqslant 0, \text{ with} \\ B_n^1 &= M_{n-1}^{\frac{1}{2}} \left(\frac{1}{m_n} \sum_{j=1}^{m_n} Z_{nj}^c {Z_{nj}^c}' - \overline{Z}_{n-1}^c {\overline{Z}_{n-1}^c}' \right) M_{n-1}^{\frac{1}{2}}, \\ B_n^2 &= M_{n-1}^{\frac{1}{2}} \left(\frac{1}{\sum_{i=1}^{n-1} m_i} \sum_{i=1}^{n-1} \sum_{j=1}^{m_i} Z_{ij}^c {Z_{ij}^c}' - \overline{Z}_{n-1}^c {\overline{Z}_{n-1}^c}' \right) M_{n-1}^{\frac{1}{2}}. \end{aligned}$$

Corollary 13 *Suppose assumptions H1b, H3', H4a,b and H6a,b,c hold. Then X_n^i converges a.s. to $\pm V_i$, $\sum_{n=1}^{\infty} a_n \left| \lambda_i - \left(X_n^i\right)' B X_n^i \right|$ and $\sum_{n=1}^{\infty} a_n (\lambda_i - \left(X_n^i\right)' B_n X_n^i)$ converge a.s. for $i = 1, ..., r$.*

Proof

Verify the assumptions of corollary 6.

(i) It is established in the proof of corollary 12 that $E\left[\sum_{n=1}^{\infty} a_n \left\| E[B_n^1 | T_n] - B \right\|\right] < \infty$ a.s. under assumptions H3', H4a and H6a,c.

(ii) Prove now that $\mathrm{E}\left[\sum_{n=1}^{\infty} a_n \left\| B_n^2 - B \right\|\right] < \infty$ a.s.

$$\begin{aligned} B_n^2 &= M_{n-1}^{\frac{1}{2}} C_{n-1} M_{n-1}^{\frac{1}{2}} \text{ with} \\ C_{n-1} &= \frac{1}{\sum_{i=1}^{n-1} m_i} \sum_{i=1}^{n-1} \sum_{j=1}^{m_i} Z_{ij}^c {Z_{ij}^c}' - \overline{Z}_{n-1}^c {\overline{Z}_{n-1}^c}' = \frac{1}{\sum_{i=1}^{n-1} m_i} \sum_{i=1}^{n-1} \sum_{j=1}^{m_i} Z_{ij} {Z_{ij}}' - \overline{Z}_{n-1} {\overline{Z}_{n-1}}'. \end{aligned}$$

$$\begin{aligned} B &= M^{\frac{1}{2}} C M^{\frac{1}{2}} \text{ with } C = E[ZZ'] - E[Z]E[Z']. \\ B_n^2 - B &= M_{n-1}^{\frac{1}{2}} C_{n-1} M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}} C M^{\frac{1}{2}} \\ &= (M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}}) C_{n-1} M_{n-1}^{\frac{1}{2}} + M^{\frac{1}{2}} (C_{n-1} - C) M_{n-1}^{\frac{1}{2}} + M^{\frac{1}{2}} C (M_{n-1}^{\frac{1}{2}} - M^{\frac{1}{2}}). \\ C_{n-1} - C &= \frac{1}{\sum_{i=1}^{n-1} m_i} \sum_{i=1}^{n-1} \sum_{j=1}^{m_i} Z_{ij} Z_{ij}' - E[ZZ'] - \left(\overline{Z}_{n-1} - E[Z]\right) {\overline{Z}_{n-1}}' - E[Z] (\overline{Z}_{n-1} - E[Z])'. \end{aligned}$$

Under assumptions H3' and H4a:

$\sum_{n=1}^{\infty} a_n E\left[\left\| \overline{Z}_{n-1} - E[Z] \right\|\right] < \infty$, $\sum_{n=1}^{\infty} a_n E\left[\left\| \overline{ZZ'}_{n-1} - E[ZZ'] \right\|\right] < \infty$ [9].

Therefore, under H4a and H6a,c, $E\left[\sum_{n=1}^{\infty} a_n ||B_n^2 - B||\right] < \infty$.

(iii) Prove finally that $\lim_{n \longrightarrow \infty} E[(X_n^{k\prime} B_n V_k)^2 \ | T_n] > 0$ when $\lim_{n \longrightarrow \infty} X_n^k = \pm V_j \neq \pm V_k$ a.s. By the proof of corollary 12, as $\lim_{n \longrightarrow \infty} X_n^{k\prime} B_n^2 V_k = \pm V_j' B V_k = 0$, under H4b and H6b:

$$\begin{aligned} \lim_{n \longrightarrow \infty} E[(X_n^{k\prime} B_n V_k)^2 | T_n] &= (\omega_1)^2 \lim_{n \longrightarrow \infty} E\left[\left(X_n^{k'} B_n^1 V_k \right)^2 \ | T_n \right] \\ &= (\omega_1)^2 Var[V_j' M^{\frac{1}{2}} Z^c . V_k' M^{\frac{1}{2}} Z^c] > 0 \text{ a.s. } \blacksquare \end{aligned}$$

3.3 Use of all observations until the current step with uniform weights

Take now

$$B_n = M_n^{\frac{1}{2}} \left(\frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} Z_{ij}^c Z_{ij}^{c\prime} - \overline{Z}_n^c \overline{Z}_n^{c\prime} \right) M_n^{\frac{1}{2}}.$$

Make the following assumptions:
(H4c) Z has 4^{th} order moments.
(H6d) $\sum_{n=1}^{\infty} a_n \left\| M_n^{\frac{1}{2}} - M^{\frac{1}{2}} \right\| < \infty$ a.s.

Corollary 14 *Suppose assumptions H1b, H3', H4c, H5' and H6b,d hold. Then, for $i = 1, ..., r$, almost surely X_n^i converges to $\pm V_i$, $\sum_{n=1}^{\infty} a_n \left| \lambda_i - \left\langle X_n^i, BX_n^i \right\rangle \right|$ and $\sum_{n=1}^{\infty} a_n \left| \lambda_i - \left\langle X_n^i, B_n X_n^i \right\rangle \right|$ converge.*

Proof

It suffices to verify assumption H2c, $\sum_{n=1}^{\infty} a_n ||B_n - B|| < \infty$ a.s. to apply corollary 11. Under assumptions H4c and H6b,d, proof is similar to that of corollary 13 for B_n^2 without taking expectation. ■

In the particular case of normed principal component analysis, M is the diagonal matrix of the inverses of variances of $Z^1, ..., Z^p$. Denote for $j = 1, ..., p$, V_n^j the variance of the sample $(Z_{11}^j, ..., Z_{n,m_n}^j)$ of Z^j and M_n the diagonal matrix of order p whose element (j, j) is the inverse of $\frac{\mu_n}{\mu_n - 1} V_n^j$ with $\mu_n = \sum_{i=1}^{n} m_i$. Under H4c, H6b holds; it is established in [9] (lemma5) that H6d holds under H4c and H3'.

4 Conclusion

In this article we gave theorems of almost sure convergence of a normed stochastic approximation process to eigenvectors of a Q-symmetric matrix B associated to eigenvalues in decreasing order, assuming that $E[B_n \mid T_n]$ or B_n converges a.s. to B. This extends previous results assuming B_n i.i.d. with $E[B_n] = B$. Several observations can be used at each step or all observations until the current step.

These results are applied to online estimation of principal components in PCA when the data arrive continously. In this case, the expectation and the variance of the variables are unknown and are estimated online in parallel with the estimation of principal components. To reduce the computing time and to avoid numerical explosions, we proposed to use symmetrisation (B is I-symmetric) and pseudo-centering with respect to a preliminary estimation of the expectation.

We made a first set of experiments: several processes, with or without symmetrization, with or without pseudo-centering, with different numbers of observations used at each step or with all observations until the current step, were compared on datasets or simulations (data not shown). It appeared that processes with symmetrization, pseudo-centering and use of all observations until the current step typically yield the best results.

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