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# A one-variable bracket polynomial for some Turk's head knots

Franck Ramaharo

Département de Mathématiques et Informatique

Université d'Antananarivo

101 Antananarivo, Madagascar

[franck.ramaharo@gmail.com](mailto:franck.ramaharo@gmail.com)

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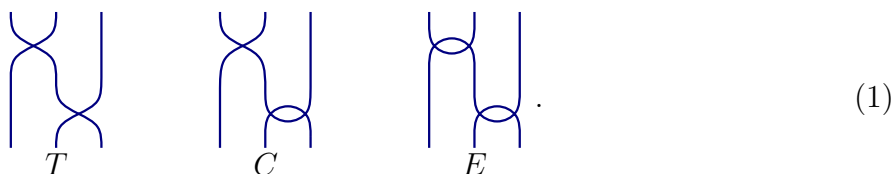
## Abstract

We compute the Kauffman bracket polynomial of *the three-lead Turk's head*, *the chain sinnet* and *the figure-eight chain* shadow diagrams. Each of these knots can in fact be constructed by repeatedly concatenating the same 3-tangle, respectively, then taking the closure. The bracket is then evaluated by expressing the state diagrams of the concerned 3-tangle by means of the Kauffman monoid diagram's elements.

Keywords: bracket polynomial, tangle shadow, Kauffman state, flat sinnet.

## 1 Introduction

The present paper is a follow-up on our previous work which aims at collecting statistics on knot shadows [5]. We would like to establish the bracket polynomial for knot diagram generated by the 3-tangle shadows below:



The knot diagrams under consideration are those obtained by repeatedly multiplying (or concatenating) the same 3-tangle, then taking the closure of the resulting 3-tangle (i.e., connecting the endpoints in a standard way, without introducing further crossings between the strands). Knots obtained from the 3-tangles pictured in (1) belong to the Ashley's *Turk's head* family [1, p. 226, Chap. 17]: the *three-lead Turk's head* [1, #1305], the *chain sinnet* [1, #1374] and the *figure-eight chain* [1, #1376], respectively (e.g. see [Figure 1](#)).

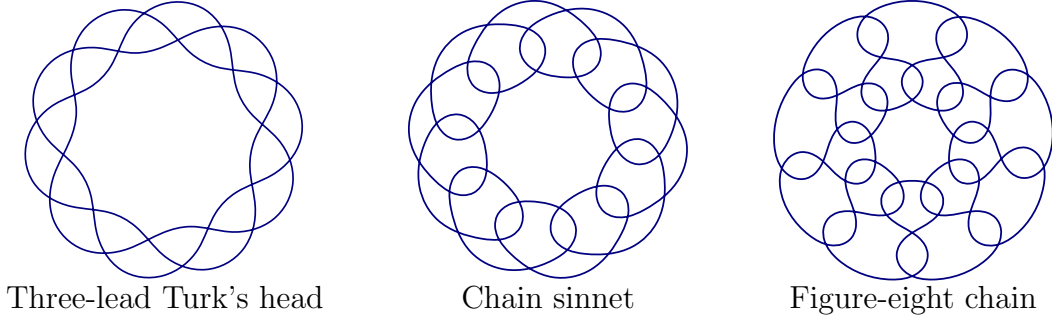


Figure 1: Some flat Turk's-head knot diagrams.

The remainder of this paper is arranged as follows. In [section 2](#), we establish the expression of the bracket polynomial for any 3-tangle shadow diagram. Then in [section 3](#), we apply those results to the flat sinnet Turk's heads mentioned earlier.

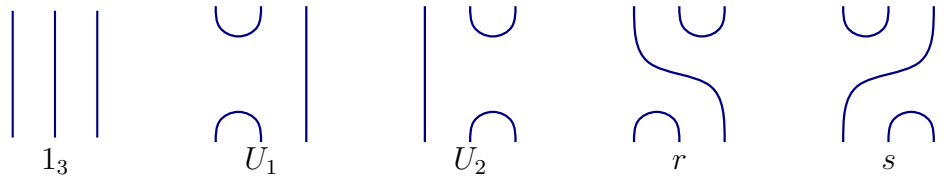
## 2 The Kauffman bracket of a 3-tangle shadow

In this paper, the Kauffman bracket maps a shadow diagram  $D$  to  $\langle D \rangle \in \mathbb{Z}[x]$  and is constructed from the following rules:

- (K1):**  $\langle \bigcirc \rangle = x$ ;
- (K2):**  $\langle \bigcirc \sqcup D \rangle = x \langle D \rangle$ ;
- (K3):**  $\langle \times \rangle = \langle \frown \rangle + \langle \smile \rangle$ .

The diagram  $\bigcirc$  in **(K1)** represents that of a single loop, and the symbol  $\sqcup$  in **(K2)** denotes the disjoint union operation. Formula in **(K3)** expresses the splitting of a crossing. Recall that the choice of such splittings for any single crossing is referred to as the so-called *Kauffman state*. Rules **(K1)**, **(K2)** and **(K3)** can be summarized by the summation which is taken over all the states for  $D$ , namely  $\langle D \rangle = \sum_S x^{|S|}$ , where  $|S|$  gives the number of loops

in the state  $S$ . Kauffman shows that the states elements of a 3-tangle diagram  $B := \boxed{\text{B}}$  are generated by the product of a loop and the following 5 elements of the 3-strand diagram monoid  $\mathcal{D}_3$  [2, 8]:



In other words, given a state  $S$ , there exist a nonnegative integer  $k$  and an element  $U$  in  $\mathcal{D}_3$  such that one writes  $S = \bigcirc^k \sqcup U$ , where  $\bigcirc^k = \bigcirc \sqcup \bigcirc \sqcup \dots \sqcup \bigcirc$  denotes the disjoint

union of  $k$  loops [3, p. 100]. The bracket of the 3-tangle  $B$  becomes  $\langle B \rangle = \sum_S \langle S \rangle$ , where  $\langle S \rangle = x^{|S|} \langle U \rangle$  for certain  $U \in \mathcal{D}_3$ .

Therefore  $\langle B \rangle$  is a linear combination of the brackets  $\langle 1_3 \rangle$ ,  $\langle U_1 \rangle$ ,  $\langle U_2 \rangle$ ,  $\langle r \rangle$  and  $\langle s \rangle$ , i.e., there exist five polynomials  $a, b, c, d, e$  in  $\mathbb{Z}[x]$  such that

$$\langle B \rangle = a \langle 1_3 \rangle + b \langle U_1 \rangle + c \langle U_2 \rangle + d \langle r \rangle + e \langle s \rangle. \quad (2)$$

**Lemma 1.** *Given two 3-tangles  $B$  and  $D$ , we have*

$$\begin{aligned} \langle BD \rangle &= a_B a_D \langle 1_3 \rangle + (b_B a_D + (a_B + b_B x + d_B) b_D + (d_B x + b_B) e_D) \langle U_1 \rangle \\ &\quad + (c_B a_D + (a_B + c_B x + e_B) c_D + (c_B + e_B x) d_D) \langle U_2 \rangle \\ &\quad + (d_B a_D + (d_B x + b_B) c_D + (a_B + b_B x + d_B) d_D) \langle r \rangle \\ &\quad + (e_B a_D + (c_B + e_B x) b_D + (a_B + c_B x + e_B) e_D) \langle s \rangle. \end{aligned}$$

.

*Proof.* We first establish the states of  $B$  leaving  $D$  intact, and then in  $D$ :

$$\begin{aligned} \langle BD \rangle &= a_B a_D \langle 1_3^2 \rangle + a_B b_D \langle 1_3 U_1 \rangle + a_B c_D \langle 1_3 U_2 \rangle + a_B d_D \langle 1_3 r \rangle + a_B e_D \langle 1_3 s \rangle \\ &\quad + b_B a_D \langle U_1 1_3 \rangle + b_B b_D \langle U_1^2 \rangle + b_B c_D \langle U_1 U_2 \rangle + b_B d_D \langle U_1 r \rangle + b_B e_D \langle U_1 s \rangle \\ &\quad + c_B a_D \langle U_2 1_3 \rangle + c_B b_D \langle U_2 U_1 \rangle + c_B c_D \langle U_2^2 \rangle + c_B d_D \langle U_2 r \rangle + c_B e_D \langle U_2 s \rangle \\ &\quad + d_B a_D \langle r 1_3 \rangle + d_B b_D \langle r U_1 \rangle + d_B c_D \langle r U_2 \rangle + d_B d_D \langle r^2 \rangle + d_B e_D \langle r s \rangle \\ &\quad + e_B a_D \langle s 1_3 \rangle + e_B b_D \langle s U_1 \rangle + e_B c_D \langle s U_2 \rangle + e_B d_D \langle s r \rangle + e_B e_D \langle s^2 \rangle. \end{aligned}$$

The brackets for the pairs in the right-hand side can be evaluated by applying the following multiplication table.

.	$1_3$	$U_1$	$U_2$	$r$	$s$
$1_3$	$1_3$	$U_1$	$U_2$	$r$	$s$
$U_1$	$U_1$	$\bigcirc \sqcup U_1$	$s$	$U_1$	$\bigcirc \sqcup s$
$U_2$	$U_2$	$r$	$\bigcirc \sqcup U_2$	$\bigcirc \sqcup r$	$U_2$
$r$	$r$	$\bigcirc \sqcup r$	$U_2$	$r$	$\bigcirc \sqcup U_2$
$s$	$s$	$U_1$	$\bigcirc \sqcup s$	$\bigcirc \sqcup U_1$	$s$

Table 1: Multiplication of elements in  $\mathcal{D}_3$ .

The proof is then completed by factoring with respect to the resulting brackets, eventually simplified according to **(K2)**.  $\square$

**Notation 2.** Let  $B_n := BB \cdots B$  denote the 3-tangle obtained by multiplying the 3-tangle  $B$   $n$  times, with  $B_0 := 1_3$ . For convenience, we shall identify the bracket formal expression in (2) by the 5-tuple  $[a, b, c, d, e]^T$ . Similarly, assume that  $\langle B_n \rangle$  is identified by  $[a_n, b_n, c_n, d_n, e_n]^T$ .

**Lemma 3.** *The bracket 5-tuple for  $B_n$  is given by*

$$\begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ b & a+bx+d & 0 & 0 & dx+b \\ c & 0 & a+cx+e & c+ex & 0 \\ d & 0 & dx+b & a+bx+d & 0 \\ e & c+ex & 0 & 0 & a+cx+e \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (3)$$

*Proof.* We write  $B_{n+1} = BB_n$ , then from Lemma 1 we have

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \\ e_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ b & a+bx+d & 0 & 0 & dx+b \\ c & 0 & a+cx+e & c+ex & 0 \\ d & 0 & dx+b & a+bx+d & 0 \\ e & c+ex & 0 & 0 & a+cx+e \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{bmatrix}. \quad (4)$$

We conclude by unfolding the recurrence and taking into consideration the initial condition  $[a_0, b_0, c_0, d_0, e_0]^T = [1, 0, 0, 0, 0]^T$ .  $\square$

We let  $M_B$  denote the  $5 \times 5$  matrix in (3), and we will later refer to it as the *states matrix* for the 3-tangle  $B$ . Using the standard method for computing (3) we obtain the characteristic polynomial for  $M_B$

$$\chi(M_B, \lambda) = -(\lambda - a) \left( \lambda - \frac{1}{2}(p - q) \right)^2 \left( \lambda - \frac{1}{2}(p + q) \right)^2,$$

then

$$a_n = a^n, \quad (5)$$

$$b_n = \frac{-1}{2q(x^2 - 1)} \left( 2a^n qx + \left( \frac{p - q}{2} \right)^n ((b - c)x^2 + (-d - e - q)x - 2b) + \left( \frac{p + q}{2} \right)^n ((-b + c)x^2 + (d + e - q)x + 2b) \right), \quad (6)$$

$$c_n = \frac{-1}{2q(x^2 - 1)} \left( 2a^n qx + \left( \frac{p + q}{2} \right)^n ((b - c)x^2 + (d + e - q)x + 2c) + \left( \frac{p - q}{2} \right)^n ((-b + c)x^2 + (-d - e - q)x - 2c) \right), \quad (7)$$

$$d_n = \frac{1}{2q(x^2 - 1)} \left( 2a^n q + \left( \frac{p - q}{2} \right)^n (-2dx^2 + (-b - c)x + d - e - q) + \left( \frac{p + q}{2} \right)^n (2dx^2 + (b + c)x - d + e - q) \right), \quad (8)$$

$$e_n = \frac{1}{2q(x^2 - 1)} \left( 2a^n q + \left( \frac{p - q}{2} \right)^n (-2ex^2 + (-b - c)x - d + e - q) + \left( \frac{p + q}{2} \right)^n (2ex^2 + (b + c)x + d - e - q) \right), \quad (9)$$

where

$$p := (b + c)x + 2a + d + e, \quad (10)$$

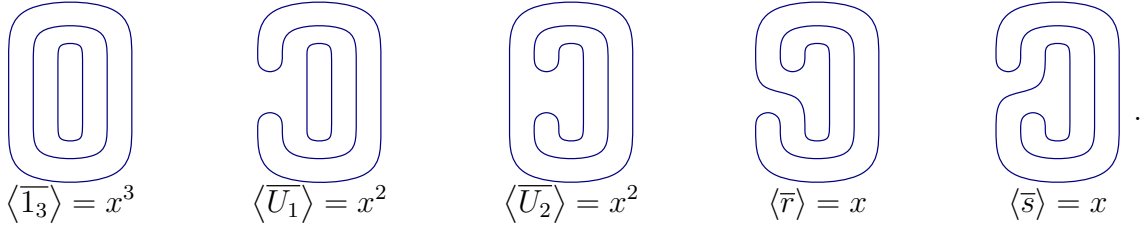
$$q := \sqrt{(b^2 - 2bc + c^2 + 4de)x^2 + (2bd + 2cd + 2be + 2ce)x + 4bc + d^2 - 2de + e^2}. \quad (11)$$

Now let  $\overline{B_n}$  denote the tangle closure of  $B_n$ . In order to evaluate  $\langle \overline{B_n} \rangle$  from formula (3) we need to apply the closure to the elements of  $\mathcal{D}_3$ .

**Lemma 4.** *The expression of the bracket polynomial for the closure  $\overline{B_n}$  is given by*

$$\langle \overline{B_n} \rangle = x^3 a_n + x^2 (b_n + c_n) + x (d_n + e_n). \quad (12)$$

The splitting at each crossing do not conflict with the closing process, hence the only point remaining concerns the evaluation of the brackets to the closure of the elements of  $\mathcal{D}_3$ , namely



Next, combining (3), (5)–(9) and (12), we obtain a better expression of the bracket:

**Lemma 5.** *The bracket polynomial for the knot  $\overline{B_n}$  is given by*

$$\langle \overline{B_n} \rangle = xa^n (x^2 - 2) + x \left( \left( \frac{p - q}{2} \right)^n + \left( \frac{p + q}{2} \right)^n \right), \quad (13)$$

where  $p$  and  $q$  are expressions defined in (10) and (11).

Finally, we let  $\overline{B}(x; y) := \sum_{n \geq 0} \langle \overline{B_n} \rangle y^n$  denote the generating function of  $(\langle \overline{B_n} \rangle)_n$ . By (13) we deduce

$$\overline{B}(x; y) = \frac{((b + c)x + 2a + d + e)y - 2}{((de - bc)x^2 + (-ac - ab)x + (-d - a)e - ad + bc - a^2)y^2 + ((b + c)x + 2a + d + e)y - 1} + \frac{x(x^2 - 2)}{1 - ay}.$$

### 3 Application

Throughout this section, let us refer to the 3-tangles in (1) as *generators*. Recall that in the expression  $\langle \overline{B_n} \rangle = \sum_{k > 0} s_B(n, k)x^k$  we have  $s_B(n, k) = \#\{S \mid S \text{ is a state of } B_n \text{ and } |S| = k\}$ , with  $B \in \{T, C, E\}$ . For each flat sinnet Turk's head below, we will give the corresponding distribution  $(s_B(n, k))_{n, k}$  for small values of  $n$  and  $k$ .

1. **Three-lead Turk's head.** Let  $\sum_{k \geq 0} s_T(n, k)x^k := \langle \overline{T}_n \rangle$ .

- Bracket for the generator  $T$ :

$$\begin{aligned} \langle \text{X} \rangle &= \langle \text{U} \rangle + \langle \text{H} \rangle \\ &= \langle \text{I} \rangle + \langle \text{U} \rangle + \langle \text{H} \rangle + \langle \text{S} \rangle \\ \langle T \rangle &= \langle 1_3 \rangle + \langle U_1 \rangle + \langle U_2 \rangle + \langle s \rangle. \end{aligned}$$

- States matrix:

$$M_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & x+1 & 0 & 0 & 1 \\ 1 & 0 & x+2 & x+1 & 0 \\ 0 & 0 & 1 & x+1 & 0 \\ 1 & x+1 & 0 & 0 & x+2 \end{bmatrix}.$$

- Bracket for  $T_n$ :

$$\langle \overline{T}_n \rangle = x(x^2 - 2) + x \left( \left( \frac{2x+3-\sqrt{4x+5}}{2} \right)^n + \left( \frac{2x+3+\sqrt{4x+5}}{2} \right)^n \right).$$

- Generating function:

$$\overline{T}(x; y) = \frac{x((-2x-3)y+2)}{(x^2+2x+1)y^2+(-2x-3)y+1} + \frac{x(x^2-2)}{1-y}.$$

- Distribution of  $(s_T(n, k))_{n,k}$ : [6, [A316659](#)]

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	1								
1	0	1	2	1								
2	0	5	8	3								
3	0	16	30	16	2							
4	0	45	104	81	24	2						
5	0	121	340	356	170	35	2					
6	0	320	1068	1411	932	315	48	2				
7	0	841	3262	5209	4396	2079	532	63	2			
8	0	2205	9760	18281	18784	11440	4144	840	80	2		
9	0	5776	28746	61786	74838	55809	26226	7602	1260	99	2	
10	0	15125	83620	202841	282980	249815	144488	54690	13080	1815	120	2

Table 2: Values of  $s_T(n, k)$  for  $0 \leq n \leq 10$  and  $0 \leq k \leq 11$ .

2. **Chain sinnet.** Let  $\sum_{k \geq 0} s_C(n, k)x^k := \langle \overline{C_n} \rangle$ .

- Bracket for the generator  $C$ :

$$\begin{aligned}
\langle \text{Diagram 1} \rangle &= \langle \text{Diagram 2} \rangle + \langle \text{Diagram 3} \rangle \\
&= \langle \text{Diagram 4} \rangle + \langle \text{Diagram 5} \rangle + \langle \text{Diagram 6} \rangle + \langle \text{Diagram 7} \rangle \\
&= \langle \text{Diagram 8} \rangle + \langle \text{Diagram 9} \rangle + \langle \text{Diagram 10} \rangle + \langle \text{Diagram 11} \rangle + \langle \text{Diagram 12} \rangle + \langle \text{Diagram 13} \rangle + \langle \text{Diagram 14} \rangle \\
\langle C \rangle &= (x+2) \langle 1_3 \rangle + (x+2) \langle U_1 \rangle + \langle U_2 \rangle + \langle s \rangle.
\end{aligned}$$

- States matrix:

$$M_C = \begin{bmatrix} x+2 & 0 & 0 & 0 & 0 \\ x+2 & x^2+3x+2 & 0 & 0 & x+2 \\ 1 & 0 & 2x+3 & x+1 & 0 \\ 0 & 0 & x+2 & x^2+3x+2 & 0 \\ 1 & x+1 & 0 & 0 & 2x+3 \end{bmatrix}.$$

- Bracket for  $C_n$ :

$$\begin{aligned}
\langle \overline{C_n} \rangle &= x(x^2-2)(x+2)^n + x \left( \left( \frac{x^2+5x+5-\sqrt{x^4+2x^3+3x^2+10x+9}}{2} \right)^n \right. \\
&\quad \left. + \left( \frac{x^2+5x+5+\sqrt{x^4+2x^3+3x^2+10x+9}}{2} \right)^n \right).
\end{aligned}$$

- Generating function

$$\overline{C}(x; y) = \frac{x((-x^2-5x-5)y+2)}{(2x^3+8x^2+10x+4)y^2+(-x^2-5x-5)y+1} + \frac{x(x^2-2)}{1-(x+2)y}.$$

- Distribution of  $(s_C(n, k))_{n, k}$ :

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	1										
1	0	1	3	3	1									
2	0	9	22	21	10	2								
3	0	49	141	164	105	42	10	1						
4	0	225	796	1186	1008	569	232	67	12	1				
5	0	961	4115	7677	8400	6205	3393	1435	461	105	15	1		
6	0	3969	20106	45481	61630	57078	39298	21239	9198	3151	822	153	18	1

Table 3: Values of  $s_C(n, k)$  for  $0 \leq n \leq 6$  and  $0 \leq k \leq 13$ .



3. **Figure-eight chain.** Let  $\sum_{k \geq 0} s_E(n, k)x^k := \langle \overline{E}_n \rangle$ .

- Bracket for the generator  $E$ :

$$\begin{aligned}
\langle \text{Figure-eight} \rangle &= \langle \text{Figure-eight with dot} \rangle + \langle \text{Figure-eight with dot on other side} \rangle \\
&= \langle \text{Figure-eight with dot} \rangle + \langle \text{Figure-eight with dot on other side} \rangle + \langle \text{Figure-eight with dot on other side} \rangle = (x+1) \langle \text{Figure-eight with dot} \rangle + \langle C \rangle \\
&= (x+1) \left( \langle \text{Figure-eight with dot} \rangle + \langle \text{Figure-eight with dot on other side} \rangle + \langle \text{Figure-eight with dot on other side} \rangle + \langle \text{Figure-eight with dot on other side} \rangle \right) + \langle C \rangle \\
\langle E \rangle &= (x^2 + 4x + 4) \langle 1_3 \rangle + (x+2) \langle U_1 \rangle + (x+2) \langle U_2 \rangle + \langle s \rangle.
\end{aligned}$$

- States matrix:

$$M_E = \begin{bmatrix} x^2 + 4x + 4 & 0 & 0 & 0 & 0 \\ x + 2 & 2x^2 + 6x + 4 & 0 & 0 & x + 2 \\ x + 2 & 0 & 2x^2 + 6x + 5 & 2x + 2 & 0 \\ 0 & 0 & x + 2 & 2x^2 + 6x + 4 & 0 \\ 1 & 2x + 2 & 0 & 0 & 2x^2 + 6x + 5 \end{bmatrix}.$$

- Bracket for  $\overline{E}_n$ :

$$\begin{aligned}
\langle \overline{E}_n \rangle &= x(x^2 - 2)(x^2 + 4x + 4)^n + x \left( \left( \frac{4x^2 + 12x + 9 - \sqrt{8x^2 + 24x + 17}}{2} \right)^n \right. \\
&\quad \left. + \left( \frac{4x^2 + 12x + 9 + \sqrt{8x^2 + 24x + 17}}{2} \right)^n \right).
\end{aligned}$$

- Generating function

$$\begin{aligned}
\overline{E}(x; y) &= \frac{x((-4x^2 - 12x - 9)y + 2)}{(4x^4 + 24x^3 + 52x^2 + 48x + 16)y^2 + (-4x^2 - 12x - 9)y + 1} \\
&\quad + \frac{x(x^2 - 2)}{1 - (x^2 + 4x + 4)y}.
\end{aligned}$$

- Distribution of  $(s_E(n, k))_{n,k}$ :

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	1										
1	0	1	4	6	4	1								
2	0	17	56	80	64	30	8	1						
3	0	169	660	1120	1096	684	280	74	12	1				
4	0	1377	6640	14112	17504	14128	7808	3008	800	142	16	1		
5	0	10201	59660	156624	244280	252460	182544	94960	35904	9800	1880	242	20	1

Table 4: Values of  $s_E(n, k)$  for  $0 \leq n \leq 5$  and  $0 \leq k \leq 13$ .

*Remark 6.* Column 1 in Table 2 is sequence [A004146](#) in the OEIS [6], the sequence of alternate Lucas numbers minus 2, which is the determinant of the Turk’s Head Knots  $THK(3, n)$  [4]. Column 2 is the  $x$ -coefficients of a generalized Jaco-Lucas polynomials for even indices [7] (see column 1 in triangle [A122076](#)) and is also a subsequence of a Fibonacci-Lucas convolution [A099920](#) for odd indices. Column 1 in Table 3 is [A060867](#) with a leading 0.

Rows 1 in Table 2, Table 3, Table 4 match the coefficients of the bracket for the 2-twist loop (see row 1 in [A300184](#), [A300192](#) and row 0 in [A300454](#)), the 3-twist loop and the 4-twist loop modulo planar isotopy and move on the 2-sphere [5], respectively (see Figure 2 (a), (b) and (d)). Row 2 in Table 2 gives those of the figure-eight knot (see Figure 2 (b) and row 1 in [A300454](#)).

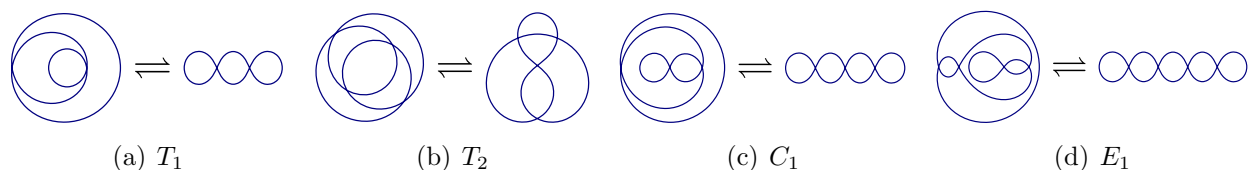


Figure 2: Equivalent knot shadow diagrams.

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