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Well-posedness of a Debye type system endowed with a full wave equation.

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Abstract

We prove well-posedness for a transport-diffusion problem coupled with a wave equation for the potential. We assume that the initial data are small. A bilinear form in the spirit of Kato's proof for the Navier-Stokes equations is used, coupled with suitable estimates in Chemin-Lerner spaces. In the one dimensional case, we get well-posedness for arbitrarily large initial data by using Gagliardo-Nirenberg inequalities.

Keywords: Transport-diffusion equation, wave equation, Debye system, Chemin-Lerner spaces, Gagliardo-Nirenberg inequalities.

1 Introduction.

Transport-diffusion equations have a vast phenomenology and have been widely studied. See, among others, [2], [3], [7], [10] in the case of the semi-conductor theory, and [5] in the case of Fokker-Planck equations. The goal of this note is to prove existence and uniqueness of the solution for a modify semi-conductor equation.

In order to simplify the presentation, we restrict to the case of a single electrical charge. The novelty of our equations is that we replace the Poisson equation on the potential by a wave equation. This is a quite natural change, since the electric charge itself depends on the time. From a mathematical point of view, switching from a Poisson equation to a wave equation roughly amounts to the loss of one derivative in the estimates on the potential. Moreover, it seems that one is bound to work in L_t^p spaces with $1 \leq p \leq 2$ due to the usual Strichartz estimates.

In this paper, we prove the existence of a mild solution in Chemin-Lerner spaces $\tilde{L}^1(0, T, \dot{H}^{n/2-1}(\mathbb{R}^n))$. We first restrict to the case of small initial data ($n \geq 2$), and use a variant of the Picard fixed point theorem as in the proof of Kato's and Chemin's theorems for the Navier-Stokes (and related) equations. See [9], [6], [4] and also [8], [1]. In particular, we work in homogeneous Sobolev spaces in order to get T -independent estimates for the heat equation. Note also that our bilinear form depends on a nonlocal term, given as the solution of the wave equation on the potential.

In the case $n = 1$, well posedness is established for arbitrary large initial data (section 4). Local well posedness is obtained as in section 3. The global existence is proved by combining the usual L^1 estimate with a Gagliardo-Nirenberg inequality, in the spirit of [3].

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2 Equations and preliminary results.

We begin with some notations. In this section $n \geq 2$, $T > 0$, and $s < n/2$ are given. The homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ are often denoted by \dot{H}^s . For $p \geq 1$, we also use the Chemin-Lerner spaces $\tilde{L}^p(0, T, \dot{H}^s(\mathbb{R}^n)) = \tilde{L}^p(0, T, \dot{B}_{2,2}^s(\mathbb{R}^n))$, or simply $\tilde{L}_T^p(\dot{H}^s)$. Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the space $\tilde{L}_T^p(\dot{H}^s)$ iff $\dot{S}_j f \rightarrow 0$ in \mathcal{S}' for $j \rightarrow -\infty$, and $\|f\|_{\tilde{L}_T^p(\dot{H}^s)} := \|(2^{js} \|\dot{\Delta}_j f\|_{L_T^p(L^2)})_{j \in \mathbb{Z}}\|_{l^2(\mathbb{Z})} < \infty$. Here, $\dot{S}_j f$ and $\dot{\Delta}_j f$ are respectively the low frequency cut-off and the homogeneous dyadic block defined by the usual Paley-Littlewood decomposition. See [1] p.98 for details. Last, we write ∇ for the (spatial) gradient, div for the divergence and $\Delta = div \nabla$.

We now give the equations we are dealing with. Set $s = n/2 - 1$. Consider the Cauchy problem on the scalar valued functions u and V defined on $\mathbb{R}_+ \times \mathbb{R}_x^n$

$$\partial_t u - \Delta u = div(u \nabla V) \quad (2.1)$$

$$\partial_{tt} V - \Delta V = u \quad (2.2)$$

$$u(0) = u_0 \quad (2.3)$$

$$V(0) = V_0, V_t(0) = V_1 \quad (2.4)$$

For $u_0 \in \dot{H}^s$, $(\nabla V_0, V_1) \in \dot{H}^s \times \dot{H}^s$ and $u \in \tilde{L}_T^1(\dot{H}^s)$ given, we denote by $S(u, V_0, V_1) \in C^0(0, T, \mathcal{S}'(\mathbb{R}^n))$ the unique solution of the wave equation 2.2, 2.4. With these notations, the system 2.1 – 2.4 is interpreted as the following problem (P):

find $u \in \tilde{L}_T^1(\dot{H}^s)$ such that

$$\partial_t u - \Delta u = div(u \nabla S(u, V_0, V_1)) \quad (2.5)$$

$$u(0) = u_0 \quad (2.6)$$

For future reference, we recall a standard result on the heat equation (see [1] p.157)

Proposition 2.1. *Let $T > 0$, $\sigma \in \mathbb{R}^n$ and $1 \leq p \leq \infty$. Assume that $u_0 \in \dot{H}^\sigma$ and $f \in \tilde{L}_T^p(\dot{H}^{\sigma-2+\frac{2}{p}})$. Then the problem*

$$\partial_t u - \Delta u = f \quad (2.7)$$

$$u(0) = u_0 \quad (2.8)$$

admits a unique solution $u \in \tilde{L}_T^p(\dot{H}^{\sigma+\frac{2}{p}}) \cap \tilde{L}_T^\infty(\dot{H}^\sigma)$ and there exists $C > 0$ independent of T such that, for any $q \in [p, \infty]$

$$\|u\|_{\tilde{L}_T^q(\dot{H}^{\sigma+\frac{2}{q}})} \leq C(\|f\|_{\tilde{L}_T^p(\dot{H}^{\sigma-2+\frac{2}{p}})} + \|u_0\|_{\dot{H}^\sigma}) \quad (2.9)$$

Moreover, for $f = 0$, we have $u \in C^0([0, T], \dot{H}^\sigma) \hookrightarrow L^1([0, T], \dot{H}^\sigma)$.

The same statements hold true in nonhomogeneous Sobolev spaces with a constant $C = C_T$ depending on T .

In the sequel, we denote the solution u of proposition 2.1 by

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau$$

We will prove an existence result for problem (P) by combining proposition 2.1 with the following \dot{H}^s estimate for the solution $S(u, V_0, V_1)$ of the wave equation (see [1] pp. 360-361)

$$\|\nabla S(u, V_0, V_1)\|_{\tilde{L}_T^\infty(\dot{H}^s)} \leq C(\|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s} + \|u\|_{\tilde{L}_T^1(\dot{H}^s)}) \quad (2.10)$$

3 Existence and uniqueness in the case $n \geq 2$.

This part is devoted to the proof of existence of a mild solution to problem (P).

Theorem 3.1. *Let $n \geq 2$ and $s = n/2 - 1$. There exists $\eta > 0$ such that, for any $T > 0$, $u_0 \in \dot{H}^{s-2} \cap \dot{H}^s$, $(\nabla V_0, V_1) \in \dot{H}^s \times \dot{H}^s$ with*

$$\|u_0\|_{\dot{H}^{s-2}} + \|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s} \leq \eta \quad (3.1)$$

there exists exactly one solution to the problem

find $u \in \tilde{L}_T^1(\dot{H}^s)$ such that

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta} \operatorname{div}(u \nabla S(u, V_0, V_1))(\tau) d\tau$$

When condition 3.1 is replaced by $\|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s} \leq \eta$, we get local in time existence and uniqueness.

We will use the following classical lemma (see for instance [1] p.357). In this lemma, $\bar{B}(0, r) \subset E$ denotes the closed ball of center 0 and radius $r > 0$.

Lemma 3.1. *Let E be a Banach space. Let $\mathcal{B} : E \times E \rightarrow E$ be a continuous bilinear map and $\mathcal{L} : E \rightarrow E$ be a linear continuous map with $\|\mathcal{L}\| < 1$. Let $0 < \alpha < (1 - \|\mathcal{L}\|)^2 / (4\|\mathcal{B}\|)$. Then, for any $\gamma \in \bar{B}(0, \alpha)$, there exists exactly one $x \in \bar{B}(0, 2\alpha)$ such that $x = \gamma + \mathcal{L}(x) + \mathcal{B}(x, x)$.*

In order to use lemma 3.1, for $T > 0$ and $(\nabla V_0, V_1) \in \dot{H}^s \times \dot{H}^s$ given, set $E_T = \tilde{L}_T^1(\dot{H}^s)$ and define $\mathcal{B}_T : E_T \times E_T \rightarrow E_T$ by

$$\mathcal{B}_T(u, w) = \int_0^t e^{(t-\tau)\Delta} \operatorname{div}(u \nabla S(w, 0, 0))(\tau) d\tau \quad (3.2)$$

We also define $\mathcal{L}_T : E_T \rightarrow E_T$ by

$$\mathcal{L}_T(u) = \int_0^t e^{(t-\tau)\Delta} \operatorname{div}(u \nabla S(0, V_0, V_1))(\tau) d\tau \quad (3.3)$$

Theorem 3.1 is an immediate consequence of lemma 3.1 and the following T -independent estimates.

Lemma 3.2. *Let $n \geq 2$ and $s = n/2 - 1$, $u_0 \in \dot{H}^{s-2}$, $(\nabla V_0, V_1) \in \dot{H}^s \times \dot{H}^s$. Then, there exists $C_i > 0$ ($0 \leq i \leq 2$) such that, for any $T > 0$ and any $u \in E_T$, $w \in E_T$, we have*

$$\|\mathcal{B}_T(u, w)\|_{E_T} \leq C_0 \|u\|_{E_T} \|w\|_{E_T} \quad (3.4)$$

$$\|\mathcal{L}_T(u)\|_{E_T} \leq C_1 (\|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s}) \|u\|_{E_T} \quad (3.5)$$

$$\|e^{t\Delta}u_0\|_{E_T} \leq C_2 \|u_0\|_{\dot{H}^{s-2}} \quad (3.6)$$

Proof. Inequality 3.6 follows from proposition 2.1. Inequalities 3.4 and 3.5, amount to

$$\|\mathcal{B}_T(u, w) + \mathcal{L}_T(u)\|_{\tilde{L}_T^1(\dot{H}^s)} \leq C \|u\|_{\tilde{L}_T^1(\dot{H}^s)} (\|w\|_{L_T^1(\dot{H}^s)} + \|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s})$$

Set

$$z = \mathcal{B}_T(u, w) + \mathcal{L}_T(u) = \int_0^t e^{(t-\tau)\Delta} \operatorname{div}(u \nabla S(w, V_0, V_1))(\tau) d\tau$$

Proposition 2.1 provides

$$\|z\|_{\tilde{L}_T^1(\dot{H}^s)} \leq C \|\operatorname{div}(u \nabla S(w, V_0, V_1))\|_{\tilde{L}_T^1(\dot{H}^{s-2})}$$

hence

$$\|z\|_{\tilde{L}_T^1(\dot{H}^s)} \leq C \|u \nabla S(w, V_0, V_1)\|_{\tilde{L}_T^1(\dot{H}^{s-1})} \quad (3.7)$$

Since $-n/2 < s < n/2$, the product is continuous from $\tilde{L}_T^1(\dot{H}^s) \times \tilde{L}_T^\infty(\dot{H}^s)$ to $\tilde{L}_T^1(\dot{H}^{2s-n/2}) = \tilde{L}_T^1(\dot{H}^{s-1})$. See for instance [1] pages 90 and 98 or use Bony's decomposition. With 3.7, this implies that

$$\|z\|_{\tilde{L}_T^1(\dot{H}^s)} \leq C \|u\|_{\tilde{L}_T^1(\dot{H}^s)} \|\nabla S(w, V_0, V_1)\|_{\tilde{L}_T^\infty(\dot{H}^s)}$$

and with 2.10

$$\|z\|_{\tilde{L}_T^1(\dot{H}^s)} \leq C \|u\|_{\tilde{L}_T^1(\dot{H}^s)} (\|w\|_{\tilde{L}_T^1(\dot{H}^s)} + \|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s})$$

□

Global existence and uniqueness in theorem 3.2 is a consequence of lemmas 3.1 and 3.2 by restricting to small data, i.e $\|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s} < 1/C_1$ and $\|u_0\|_{\dot{H}^{s-2}} < [1 - C_1(\|\nabla V_0\|_{\dot{H}^s} + \|V_1\|_{\dot{H}^s})]^2 / (4C_0C_2)$. The local existence and uniqueness part is a consequence of the same lemmas once $\lim_{t \rightarrow 0} \|e^{\tau \Delta} u_0\|_{E_t} = 0$ is proved. This follows from the fact that $e^{\tau \Delta} u_0 \in L^1([0, t], \dot{H}^s(\mathbb{R}))$ (see proposition 2.1) and the inequality (see [1] p.98)

$$\|e^{\tau \Delta} u_0\|_{E_t} \leq \|e^{\tau \Delta} u_0\|_{L^1([0, t], \dot{H}^s(\mathbb{R}))} \rightarrow 0$$

when $t \rightarrow 0$.

Remark 3.1. *The proof of theorem 3.1 extends to the Debye type system (see [3], [7]): $\partial_t u_j - \Delta u_j = \text{div}(\beta_j u_j \nabla V)$, $\partial_{tt} V - \Delta V = \sum_k \alpha_k u_k$, $u_j(0) = u_{j,0}$, $V(0) = V_0$, $V_t(0) = V_1$ with $(\alpha_j, \beta_j) \in \mathbb{R}^2$ ($1 \leq j \leq m$) given.*

4 Existence and uniqueness in the case $n = 1$.

Until the end of the paper, $n = 1$. We still denote by $S(u, V_0, V_1) \in C^0(0, T, \mathcal{S}'(\mathbb{R}^n))$ the unique solution of the wave equation 2.2, 2.4, and $\mathcal{B}_T(u, w)$ and $\mathcal{L}_T(u)$ are still formally defined by formulas 3.2 and 3.3. The notation L_x^p stands for $L^p(\mathbb{R}_x)$. Last, $\nabla = \text{div} = \partial_x$.

As a building block in the proof of the existence theorem 4.1, we first establish a L^1 estimate for solutions of equations 2.5, 2.6 (lemma 4.3). We begin with two simple trace-lemmas. For $y \in \mathbb{R}$, set $D_T(y) =]0, T[\times]y, y + 1[$ and $\bar{D}_T(y) = [0, T] \times [y, y + 1]$.

Lemma 4.1. *Let $T > 0$, $y \in \mathbb{R}$. There exists $C > 0$ such that, for any $y \in \mathbb{R}$ and $f \in C^1(\bar{D}_T(y))$, we have*

$$\|f(\cdot, y)\|_{L^2(0, T)} + \|f(\cdot, y + 1)\|_{L^2(0, T)} \leq C \|f\|_{L^2(0, T, H^1(]y, y+1])} \quad (4.1)$$

Proof. We only prove inequality

$$\int_0^T |f(\tau, y)|^2 d\tau \leq C \|f\|_{L^2(0, T, H^1(]y, y+1])}^2 \quad (4.2)$$

Let $\phi \in C^1(\bar{D}_T(0))$ with $\phi(t, x) = 1$ for $(t, x) \in [0, T] \times [0, 1/4]$ and $\phi(t, x) = 0$ for $(t, x) \in [0, T] \times [3/4, 1]$. For $y \in \mathbb{R}$ fixed, define $\phi^y \in C^1(\bar{D}_T(y))$ by $\phi^y(t, x) = \phi(t, x - y)$. Let $f \in C^1(\bar{D}_T(y))$. We have

$$\begin{aligned} \int_0^T |f|^2(\tau, y) d\tau &= \int_0^T |\phi^y f|^2(\tau, y) d\tau \leq \int_0^T \int_y^{y+1} 2|\phi^y f(\phi^y f)_x|(\tau, s) ds d\tau \\ &\leq 2 \|\phi^y\|_{W^{1, \infty}(D(y))}^2 \|f\|_{L^2(0, T, H^1(]y, y+1])}^2 \end{aligned} \quad (4.3)$$

Since $\|\phi^y\|_{W^{1, \infty}(D(y))} = \|\phi\|_{W^{1, \infty}(D(0))}$, we get inequality 4.2. □

Let $y \in \mathbb{R}$. By lemma 4.1, we can define two continuous trace-operators γ_{y+1}^- and γ_y^+ : $L^2(0, T, H^1(\mathbb{R})) \rightarrow L^2(0, T)$ by $\gamma_y^+(f)(\tau) = f(\tau, y)$ and $\gamma_{y+1}^-(f)(\tau) = f(\tau, y + 1)$ for any $f \in C^1(\bar{D}_T(y))$. For future reference, notice that, for any $f \in L^2(]0, T[, H^1(]y, y + 1[))$, we have

$$\begin{aligned} & \|\gamma_{y+1}^-(f)\|_{L^1(0, T)} + \|\gamma_y^+(f)\|_{L^1(0, T)} \\ & \leq \sqrt{T}(\|\gamma_{y+1}^-(f)\|_{L^2(0, T)} + \|\gamma_y^+(f)\|_{L^2(0, T)}) \leq C\sqrt{T}\|f\|_{L^2(0, T, H^1(]y, y+1[))} \end{aligned} \quad (4.4)$$

with a constant $C > 0$ independent of $y \in \mathbb{R}$.

The second lemma is a consequence of the continuity of the trace functions γ_y^\pm and density arguments. We omit the proof. In the sequel $sign$ denotes the sign function.

Lemma 4.2. *Let $T > 0$ and $y \in \mathbb{R}$. Let also $f \in L^2(0, T, H^2(\mathbb{R}))$, $g \in L^\infty(0, T, H^2(\mathbb{R}))$ and $\phi \in L^2(0, T, H^1(\mathbb{R}))$. Then, for any $(y, z) \in \mathbb{R}^2$, $y < z$, and any $t \in [0, T]$ we have*

a)

$$\int_0^t \int_y^z \phi \partial_{xx} f dx d\tau = - \int_0^t \int_y^z \partial_x \phi \partial_x f dx d\tau + \int_0^t [\gamma_z^-(\phi \partial_x f) - \gamma_y^+(\phi \partial_x f)] d\tau \quad (4.5)$$

$$\int_0^t \int_y^z sign(f) \partial_x (f \partial_x g) dx d\tau = \int_0^t [\gamma_z^-(|f| \partial_x g) - \gamma_y^+(|f| \partial_x g)] d\tau \quad (4.6)$$

b) Let $A \in L^\infty(0, T, H^1(\mathbb{R}))$. Then $|\gamma_y^\pm(Af)|(\tau) \leq \|A\|_{L^\infty(0, T, H^1(]y, y+1[))} |\gamma_y^\pm(f)|(\tau)$ for almost every $\tau \in [0, T]$.

We are ready to prove our main L^1 lemma.

Lemma 4.3. *Let $T > 0$, $V_0 \in H^2(\mathbb{R})$, $V_1 \in H^1(\mathbb{R})$, $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $u \in L_T^2(H^2) \cap H_T^1(L^2) \cap C^0([0, T], H^1)$. Assume that $S(u, V_0, V_1) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ and assume that function $(u, S(u, V_0, V_1))$ is a solution of equations 2.5, 2.6, i.e*

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} div(u \nabla S(u, V_0, V_1))(\tau) d\tau$$

Then $u \in C^0([0, T], L^1(\mathbb{R}))$ and $\|u(t)\|_{L^1} \leq \|u_0\|_{L^1}$, for any $t \in [0, T]$. Moreover, when $\pm u_0 \geq 0$, we have $\pm u \geq 0$ and $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$.

Proof. Let $(y, z) \in \mathbb{R}^2$ with $y < z$, and $t \in [0, T]$. Since $u \in L_T^2(H^2)$ and $S(u, V_0, V_1) \in C^0([0, T], H^2)$, we can apply formula 4.6 with $f = u$, $g = S(u, V_0, V_1)$. Hence, multiplying 2.5 by $sign(u)$ and integrating on $[0, T] \times [y, z]$, we obtain

$$\begin{aligned} \int_y^z |u|(t, x) dx &= \|u_0\|_{L^1(]y, z[)} + \int_0^t [\gamma_z^-(|u| \partial_x S) - \gamma_y^+(|u| \partial_x S)] d\tau + \int_0^t \int_y^z sign(u) \Delta u dx d\tau \\ &\leq \|u_0\|_{L^1(]y, z[)} + \|\gamma_z^-(|u| \partial_x S)\|_{L^1([0, T])} + \|\gamma_y^+(|u| \partial_x S)\|_{L^1([0, T])} + \sup_{t \in [0, T]} \left(\int_0^t \int_y^z sign(u) \Delta u dx d\tau \right) \end{aligned} \quad (4.7)$$

We majorize the last three terms in inequality 4.7. Using the inequality 4.4, we get

$$\|\gamma_z^-(|u| \partial_x S)\|_{L^1([0, T])} + \|\gamma_y^+(|u| \partial_x S)\|_{L^1([0, T])} \leq C\sqrt{T} \| |u| \partial_x S \|_{L^2(0, T, H^1(]y, y+1[\cup]z-1, z[))} \quad (4.8)$$

Hence, 4.8 and $|u| \partial_x S \in L^2(0, T, H^1(\mathbb{R}))$ implies that

$$\lim_{inf(]y, |z|) \rightarrow \infty} (\|\gamma_z^-(|u| \partial_x S)\|_{L^1([0, T])} + \|\gamma_y^+(|u| \partial_x S)\|_{L^1([0, T])}) = 0 \quad (4.9)$$

We next prove that

$$\limsup_{\inf(|y|,|z|)\rightarrow\infty} \left(\sup_{t\in[0,T]} \int_0^t \int_y^z \text{sign}(u)\Delta u(\tau,x)dxd\tau \right) \leq 0 \quad (4.10)$$

Set $h_\epsilon(x) = x/\sqrt{x^2+\epsilon}$ ($x \in \mathbb{R}$, $\epsilon > 0$). Due to $\Delta u \in L^2([0,T] \times \mathbb{R}) \hookrightarrow L^1_{loc}([0,T] \times \mathbb{R})$, $\|h_\epsilon\|_{L^\infty} \leq 1$ and Lebesgue theorem, we have

$$\int_0^t \int_y^z \text{sign}(u)\Delta u(\tau,x)dxd\tau = \lim_{\epsilon\rightarrow 0} \int_0^t \int_y^z h_\epsilon(u)\Delta u(\tau,x)dxd\tau \quad (4.11)$$

Using 4.5 with $f = u \in L^2_T(H^2)$, $\phi = h_\epsilon(u) \in L^2_T(H^1)$, majorizing, and appealing to lemma 4.2 b), we obtain an (ϵ, t) -independent estimate

$$\begin{aligned} \int_0^t \int_y^z \Delta u h_\epsilon(u) dx d\tau &= - \int_0^t \int_y^z |\nabla u|^2 h'_\epsilon(u) dx d\tau + \int_0^t [\gamma_z^-(h_\epsilon(u)\nabla u) - \gamma_y^+(h_\epsilon(u)\nabla u)] d\tau \\ &\leq \int_0^T [|\gamma_z^-(\nabla u)| + |\gamma_y^+(\nabla u)|] d\tau \end{aligned} \quad (4.12)$$

Appealing to 4.4, we deduce from 4.12 and 4.11 that

$$\sup_{t\in[0,T]} \left(\int_0^t \int_y^z \text{sign}(u)\Delta u(\tau,x)dxd\tau \right) \leq C\sqrt{T}\|\nabla u\|_{L^2(0,T,H^1(|y,y+1|\cup|z-1,z|))} \quad (4.13)$$

Hence, inequality 4.10 follows from 4.13 and $u \in L^2(0,T,H^2(\mathbb{R}))$. Set $y = -z$ and let $z \rightarrow +\infty$ in 4.7. Using 4.9, 4.10 and the monotone convergence theorem, we get $\|u(t)\|_{L^1} \leq \|u_0\|_{L^1}$ for $t \in [0,T]$. Replacing the sign function by the negative or positive part functions $(\cdot)^\pm$, or the constant function 1, the same argument provides $\|(u)^\pm(t)\|_{L^1} \leq \|(u_0)^\pm\|_{L^1}$ and $\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx$. In the particular case $\pm u_0 \geq 0$, we recover $\pm u(t) \geq 0$ and $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$ for any $t \in [0,T]$.

Finally, appealing to 4.7, 4.9, 4.10 and $u_0 \in L^1$, we have $\sup_{\tau\in[0,T]} \|u(\tau)\|_{L^1(|-\infty,-x|\cup|x,+\infty|)} \rightarrow 0$ when $x \rightarrow +\infty$. Hence, noticing that $u \in C^0(0,T,L^2(\mathbb{R})) \hookrightarrow C^0(0,T,L^1_{loc}(\mathbb{R}))$, we easily obtain $u \in C^0(0,T,L^1(\mathbb{R}))$ \square

It follows that

Theorem 4.1. *Let $T > 0$, $V_0 \in H^2(\mathbb{R})$, $V_1 \in H^1(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. The problem find $u \in L^2_T(H^1)$ such that*

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta} \text{div}(u\nabla S(u, V_0, V_1))(\tau) d\tau$$

admits exactly one solution. Moreover, $u \in L^2_T(H^2) \cap H^1_T(L^2) \cap C^0([0,T], H^1 \cap L^1)$ and $\|u(t)\|_{L^1_\pm} \leq \|u_0\|_{L^1}$. Last, when $\pm u_0 \geq 0$, we have $\pm u \geq 0$ and $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$, $0 \leq t \leq T$.

Proof. Step 1 (local existence). For any $0 < T < 1$, $u \in L^2_T(H^1)$ and $w \in L^2_T(H^1)$, proposition 2.1 provides

$$\begin{aligned} \|\mathcal{B}_T(u, w) + \mathcal{L}_T(u)\|_{L^2_T(H^1)} &\leq C\|\text{div}(u\nabla S(w, V_0, V_1))\|_{L^1_T(L^2)} \\ &\leq C\|u\nabla S(w, V_0, V_1)\|_{L^1_T(H^1)} \\ &\leq C\|u\|_{L^2_T(H^1)}\|\nabla S(w, V_0, V_1)\|_{L^2_T(H^1)} \end{aligned} \quad (4.14)$$

Note that

$$S(w, V_0, V_1)(t, x) = \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} w(\tau, \xi) d\xi d\tau + \frac{1}{2}(V_0(x+t) + V_0(x-t) + \int_{x-t}^{x+t} V_1(s) ds) \quad (4.15)$$

Assume first that V_0, V_1 and w are infinitely differentiable. We have

$$\begin{aligned} \nabla S(w, V_0, V_1)(t, x) &= \int_0^t (w(\tau, x + (t - \tau)) - w(\tau, x - (t - \tau))) d\tau \\ &\quad + \frac{1}{2}(V_0'(x+t) + V_0'(x-t) + V_1(x+t) - V_1(x-t)) \end{aligned} \quad (4.16)$$

Arguing by density, the formula 4.16 holds for the distributional derivative ∇S under the assumptions of theorem 4.1, and we have

$$\|\nabla S(w, V_0, V_1)\|_{L_T^2(H^1)} \leq C\sqrt{T}(\|w\|_{L_T^2(H^1)} + \|V_0'\|_{H^1} + \|V_1\|_{H^1}) \quad (4.17)$$

From 4.14 and 4.17 we deduce that

$$\|\mathcal{B}_T(u, w) + \mathcal{L}_T(u)\|_{L_T^2(H^1)} \leq C\sqrt{T}\|u\|_{L_T^2(H^1)}(\|w\|_{L_T^2(H^1)} + \|V_0'\|_{H^1} + \|V_1\|_{H^1}) \quad (4.18)$$

Finally, notice that

$$\|e^{t\Delta}u_0\|_{L_T^2(H^1)} \leq \|u_0\|_{L^2} \quad (4.19)$$

Hence, for $T > 0$ small enough, the existence and uniqueness of a solution $u \in L_T^2(H^1)$ follows from lemma 3.1, inequalities 4.18 and 4.19.

Next, since $u \in L_T^2(H^1)$, $V_0 \in H^2$ and $V_1 \in H^1$ we get

$$S(u, V_0, V_1) \in C^0([0, T], H^2) \cap C^1([0, T], H^1) \quad (4.20)$$

Therefore $\operatorname{div}(u\nabla S(u, V_0, V_1)) \in L_T^2(L^2)$. Due to $u_0 \in H^1$, equation $\partial_t u - \Delta u = \operatorname{div}(u\nabla S(u, V_0, V_1))$, proposition 2.1 (and by interpolation), we thus obtain

$$u \in L_T^2(H^2) \cap H_T^1(L^2) \cap C^0([0, T], H^1) \quad (4.21)$$

Step 2 (global existence). Let $T^* > 0$, the maximal time of existence of a mild solution endowed with the properties 4.20, 4.21. In particular, $u \in C^0([0, T], L^1(\mathbb{R}))$ and $\|u(t)\|_{L_x^1} \leq \|u_0\|_{L^1}$ for $0 \leq t \leq T < T^*$ (see lemma 4.3). In order to prove that $T^* = \infty$, and due to 4.20, 4.21 and lemma 4.3, we essentially have to find an a priori estimate on $\|u\|_{L_T^2(H^1)}$ for $0 < T < T^*$. We multiply equation 2.5 by u and integrate with respect to x . Appealing to 4.20 and 4.21, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L_x^2}^2 + \int_{\mathbb{R}} |\partial_x u|^2 dx &\leq \left| \int_{\mathbb{R}} u \partial_x u \partial_x S dx \right| \\ &\leq \|u\|_{L_x^4} \|\partial_x u\|_{L_x^2} \|\partial_x S\|_{L_x^4} \end{aligned} \quad (4.22)$$

We now estimate $\|u\|_{L_x^4}$ and $\|\partial_x S\|_{L_x^4}$. By Gagliardo-Nirenberg inequalities and the L^1 properties of u , we have

$$\|u\|_{L_x^4}^4 \leq C \|u\|_{L_x^1}^2 \|\partial_x u\|_{L_x^2}^2 \leq C \|u_0\|_{L^1}^2 \|\partial_x u\|_{L_x^2}^2 \quad (4.23)$$

Using equation 4.16 with $w = u$, we obtain for any $0 < t < T$

$$\|\partial_x S(t)\|_{L^4} \leq C(\|u\|_{L^1(0,t,L^4)} + \|V_0'\|_{L^4} + \|V_1\|_{L^4}) \quad (4.24)$$

Invoking inequality 4.23, this implies that

$$\begin{aligned} \|\partial_x S(t)\|_{L^4} &\leq C(\|u_0\|_{L^1}^{1/2} \int_0^t \|\partial_x u(\tau)\|_{L_x^2}^{1/2} d\tau + \|V_0'\|_{L^4} + \|V_1\|_{L^4}) \\ &\leq C(\|u_0\|_{L^1}^{1/2} \|\partial_x u\|_{L^2(]0,T[\times \mathbb{R})}^{1/2} t^{3/4} + \|V_0'\|_{L^4} + \|V_1\|_{L^4}) \end{aligned} \quad (4.25)$$

Therefore, 4.22, 4.23, 4.25 and injection $H^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ provide

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L_x^2}^2 + \|\partial_x u(t)\|_{L_x^2}^2 &\leq C\|u_0\|_{L^1}^{1/2} \|\partial_x u(t)\|_{L_x^2}^{3/2} (\|u_0\|_{L^1}^{1/2} \|\partial_x u\|_{L^2(]0,T[\times \mathbb{R})}^{1/2} t^{3/4} + \|V_0'\|_{L^4} + \|V_1\|_{L^4}) \\ &\leq \eta \|u_0\|_{L^1}^{2/3} \|\partial_x u(t)\|_{L_x^2}^2 \\ &\quad + C_\eta (t^3 \|u_0\|_{L^1}^2 \|\partial_x u\|_{L^2(]0,T[\times \mathbb{R})}^2 + \|V_0\|_{H^2}^4 + \|V_1\|_{H^1}^4) \end{aligned} \quad (4.26)$$

Take η such that $0 < \eta \|u_0\|_{L^1}^{2/3} \leq 1/2$. Hiding the term $\eta \|u_0\|_{L^1}^{2/3} \|\partial_x u(t)\|_{L_x^2}^2$ in the left hand side of 4.26, setting $z(t) = \|u(t)\|_{L_x^2}^2 + \int_0^t \|\partial_x u(\tau)\|_{L_x^2}^2 d\tau$ and using the Gronwall inequality we obtain the required a priori estimate on $\|u\|_{L_T^2(H^1)}$. The end of the proof is standard and we omit further details. \square

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