On Regular Schemes and Tight Frames

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Abstract

Finite frames are sequences of vectors in finite dimensional Hilbert spaces that play a key role in signal processing and coding theory. In this paper, we study the class of tight unit-norm frames for $\mathbb{C}^d$ that also form regular schemes, called tight regular schemes (TRS). Many common frames that arise in applications such as equiangular tight frames and mutually unbiased bases fall in this class. We investigate characteristic properties of TRSs and prove that for many constructions, they are intimately connected to weighted 1-designs—arising from quadrature rules for integrals over spheres in $\mathbb{C}^d$—with weights dependent on the Voronoi regions of each frame element. Aided by additional numerical evidence, we conjecture that all TRSs in fact satisfy this property.

1 Introduction

Finite frames play an important role in signal processing and coding theory as a means of providing redundant representations of elements in finite dimensional complex Hilbert spaces. For example, frames form the basis for the construction of good measurement matrices in compressed sensing [3], codebooks for vector quantization [6], and coding for erasures [12]. Connections have also been established to measurement operators in quantum information theory [10].

Despite the range of applications there are a few special classes of frames are ubiquitous, including equiangular tight frames (ETFs) [12] and mutually unbiased bases (MUBs) [16]. A key feature of these frames is that they are *tight unit-norm frames* with a low *coherence*; that is, the maximum squared absolute value of inner products, or *angles*, between distinct frame elements is small. Low coherence is a particularly important property for reconstruction of signals from a sparse approximation [3].

Construction of tight frames with a low coherence has proven to be a difficult challenge. One approach to tackle this problem is to restrict the number of distinct angles presented by the frame. In [5], constructions of real valued tight frames with at most k angles were derived. In [13], unitary representations of cyclic and dihedral groups were exploited to obtain tight frames with few angles and low coherence.

In this paper, we study finite unit-norm tight frames that are *regular schemes*, called tight regular schemes (TRSs). This class of frames includes ETFs and MUBs as special cases [8, 9], as well as a wide range of frames constructed using unitary representations of finite groups. The defining feature of frames that are regular schemes is the presence of the property that the set of angles from any frame element obtained by taking inner products with all other frame elements is the same. Due to the restricted number of distinct angles, this provides a basis to control the coherence.

We investigate special cases of TRSs for different size angle sets. A key observation is that for angle sets of size one and two, ETFs and MUBs, respectively, naturally arise. We also demonstrate that the existence of TRSs with a full angle set depends on whether or not the size of the frame is even or odd.

We also observe that certain subsets of unitary representations of finite groups yield TRSs; namely, the group covariant set. As a consequence, TRSs include a large class of unit-norm tight frames. We also show that such TRSs have Voronoi regions with equal area.

Although the coherence describes one aspect of the geometry of a frame, the Voronoi regions capture properties of the geometry beyond pairwise relationships between frame elements. As group-based TRSs have Voronoi regions with equal area, this implies that these frames form weighted 1-designs [11] with weights given by the areas of the Voronoi regions. A natural question is then whether other TRSs have the same property. We show that a finite frame obtained via Alltop's quadric polyphase construction [1] also forms a weighted 1-design with weights given by the areas of the Voronoi regions, which are not all equal. Numerical experiments also suggest that this property also holds for MUBs obtained from Alltop's quadric and cubic polyphase constructions [1] as well as ETFs obtained from difference sets [17] and Steiner systems [7]. Based on this theoretical and numerical evidence, we conjecture that in fact all TRSs form weighted 1-designs with weights determined by Voronoi region areas.

2 Preliminaries

A family of vectors $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is called a **finite frame** for $\mathbb{C}^d$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|\mathbf{x}\|^2 \leq \sum_{i=1}^{M} |\langle \mathbf{x}, \boldsymbol{\varphi}_i \rangle|^2 \leq B\|\mathbf{x}\|^2, \ \forall \mathbf{x} \in \mathbb{C}^d. \tag{1}$$

If $A = B$, then $(\boldsymbol{\varphi}_i)_{i=1}^M$ is called a **tight frame**. If $\|\boldsymbol{\varphi}_i\| = 1, \ i = 1, 2, \ldots, M$ then $(\boldsymbol{\varphi}_i)_{i=1}^M$ is called a **unit norm frame**.

An important characterization of tight frames is via the frame potential [2].

Definition 1. If $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is a unit-norm frame, the **frame potential** of $(\boldsymbol{\varphi}_i)_{i=1}^M$ is given by

$$\mathrm{FP}((\boldsymbol{\varphi}_i)_{i=1}^M) = \sum_{i,j=1}^{M} |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2. \tag{2}$$

Theorem 2. *[2] The unit-norm frame* $(\boldsymbol{\varphi}_i)_{i=1}^M$ *in* $\mathbb{C}^d$ *is tight if and only if the frame potential is minimized; that is,*

$$\mathrm{FP}((\boldsymbol{\varphi}_i)_{i=1}^M) = \frac{M^2}{d}. \tag{3}$$

The coherence of a unit-norm frame $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is defined as

$$\max_{i \neq j} |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2. \tag{4}$$

By a theorem of Welch [15], the coherence is lower bounded by

$$\max_{i \neq j} |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 \geq \frac{M-d}{d(M-1)}, \tag{5}$$

known as the Welch bound.

In this paper, we are concerned with the class of tight frames known as regular schemes [9]. These frames are defined by their **angle set**, which for the frame $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is

$$\mathcal{A} = (|\boldsymbol{\varphi}_j^\dagger \boldsymbol{\varphi}_k|^2)_{1 \leq j < k \leq M} \tag{6}$$

and define

$$\mathcal{A}_j = (|\boldsymbol{\varphi}_j^\dagger \boldsymbol{\varphi}_k|^2)_{k \neq j} \tag{7}$$

Associated to each frame element $\boldsymbol{\varphi}_j$ and angle $\alpha \in \mathcal{A}$ is the **sub-degree** of $(\boldsymbol{\varphi}_i)_{i=1}^M$, which is defined as

$$d_\alpha(j) = |\{1 \leq k \leq M : k \neq j,\ |\boldsymbol{\varphi}_j^\dagger \boldsymbol{\varphi}_k|^2 = \alpha\}|. \tag{8}$$

Definition 3. Let $(\boldsymbol{\varphi}_i)_{i=1}^M$ be a unit norm frame in $\mathbb{C}^d$ with angle set $\mathcal{A}$. If for each $\alpha \in \mathcal{A}$, the sub-degree $d_\alpha(i)$ is independent of i, then $(\boldsymbol{\varphi}_i)_{i=1}^M$ is called a **regular scheme**. If a regular scheme $(\boldsymbol{\varphi}_i)_{i=1}^M$ also forms a tight frame, then $(\boldsymbol{\varphi}_i)_{i=1}^M$ is said to be a **tight regular scheme (TRS)**.

A useful alternative characterization of TRSs is given in the following proposition, which follows immediately from the definition.

Proposition 4. *Let* $(\boldsymbol{\varphi}_i)_{i=1}^M$ *in* $\mathbb{C}^d$ *be a tight unit norm frame. Then,* $(\boldsymbol{\varphi}_i)_{i=1}^M$ *is a TRS if and only if the sets* $\mathcal{A}_j,\ j = 1, \ldots, M$ *are identical.*

3 Examples of Tight Regular Schemes

3.1 Tight Regular Schemes and Spherical t-Designs

There is a long-known connection between regular schemes and spherical t-designs (henceforth called t-designs) [8]. Following [9], let $\mathrm{Hom}(k,l)$ be the subset of the polynomial ring $\mathbb{C}[x_1, \ldots, x_d, y_1, \ldots, y_d]$ that consists of all polynomials that are homogeneous of degree k in the variables $x_1, \ldots, x_d$ and homogeneous of degree l in the variables $y_1, \ldots, y_d$. To each polynomial p in $\mathrm{Hom}(k,l)$ associate a function $p_\circ$ on the sphere S^{d-1} by defining $p_\circ(\zeta) = p(\zeta, \zeta^*)$ for $\zeta \in S^{d-1}$. Define $\mathrm{Hom}(k,k)_\circ = \{p_\circ : p \in \mathrm{Hom}(k,k)\}$.

Definition 5. Let μ be the unique normalized $\mathcal{U}(d)$-invariant Haar measure on $\mathbb{C}S^{d-1}$. A finite non-empty subset X of $\mathbb{C}S^{d-1}$ is a ***t*-design** in $\mathbb{C}S^{d-1}$ if and only if the cubature formula

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{\mu(\mathbb{C}S^{d-1})}\int_{\mathbb{C}S^{d-1}} f(x)d\mu(x) \tag{9}$$

holds for all $f \in \mathrm{Hom}(t,t)_{\circ}$.

Intuitively, a t-design provides a means of expressing the expectation of homogeneous polynomials in terms of an average over a finite set on points. The following theorem due to Hoggar [8] provides a link between regular schemes and t-designs.

Theorem 6. *[8, Theorem 2.4] Let $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ be a regular scheme with angle set $\mathcal{A}$. Then, $(\boldsymbol{\varphi}_i)_{i=1}^M$ is a t-design if and only if*

$$1 + \alpha_1^r d_1 + \cdots + \alpha_s^r d_s = M\frac{(1)_r}{(d)_r},\ r = 0, 1, \ldots, t \tag{10}$$

where $(a)_r = a(a+1)\cdots(a+r-1)$.

We now turn to classes of tight regular schemes constrained by the size of their angle sets.

3.2 $|\mathcal{A}| = 1$

Consider the case that $|\mathcal{A}| = 1$. We first observe that in this case, TRSs are intimately linked to ETFs which have a coherence achieving equality in the Welch bound.

Definition 7. Let $(\boldsymbol{\varphi}_i)_{i=1}^M$ be a unit-norm frame in $\mathbb{C}^d$. Then, $(\boldsymbol{\varphi}_i)_{i=1}^M$ is an equiangular tight frame (ETF) if and only if

$$\max_{j\neq i} |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 = \frac{M-d}{d(M-1)}. \tag{11}$$

That is, $(\boldsymbol{\varphi}_i)_{i=1}^M$ satisfies the Welch bound [12].

Proposition 8. *Suppose that $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is a TRS with $\mathcal{A} = \{\alpha\}$. Then, $(\boldsymbol{\varphi}_i)_{i=1}^M$ achieves equality in the Welch bound. That is, $(\boldsymbol{\varphi}_i)_{i=1}^M$ is an equiangular tight frame.*

Proof. It follows from the definition of $(\boldsymbol{\varphi}_i)_{i=1}^M$ and Theorem 2 that

$$1 + (M-1)\alpha = \frac{M}{d}. \tag{12}$$

Hence,

$$\alpha = \frac{M-d}{d(M-1)}, \tag{13}$$

as required. □

3.3 $|\mathcal{A}| = 2$

As in the case $|\mathcal{A}| = 1$, frames with extremal angle sets also arise for $|\mathcal{A}| = 2$. To formalize this claim, first recall the Levenshtein bound.

Theorem 9 (Levenshtein)**.** *Let $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ be a unit norm finite frame. Then,*

$$\max_{1\leq i<j\leq M} |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 \geq \frac{2M-d^2-d}{(d+1)(M-d)}. \tag{14}$$

Proposition 10. *Suppose that $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is a TRS with $\mathcal{A} = \{0, \alpha\}$. Then, $(\boldsymbol{\varphi}_i)_{i=1}^M$ forms a complex projective 2-design if and only if $(\boldsymbol{\varphi}_i)_{i=1}^M$ achieves equality in the Levenshtein bound.*

Proof. Since $(\boldsymbol{\varphi}_i)_{i=1}^M$ is a complex projective 2-design by [8, Theorem 2.4],[9, Theorem 1] if and only if

$$\frac{1}{M}(1 + d_\alpha \alpha^2) = \frac{2}{d(d+1)}. \tag{15}$$

and

$$\sum_{j=1}^M |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 = 1 + \alpha d_\alpha = \frac{M}{d},\ i = 1, \ldots, M \tag{16}$$

which implies that $d_\alpha = \frac{M-d}{d\alpha}$. It then follows that

$$\frac{1}{M}\left(1 + \frac{M-d}{d}\alpha\right) = \frac{2}{d(d+1)}. \tag{17}$$

Solving for α then yields

$$\alpha = \frac{2M-d^2-d}{(d+1)(M-d)}, \tag{18}$$

which is precisely the condition for equality in the Levenshtein bound. $\square$

Although MUBs are known to be complex projective 2-designs, the proof above is not limited to this case. Moreover, the result shows that the more fundamental properties are that MUBs are tight, have an angle set of the form $\{0, \alpha\}$ and minimize the Levenshtein bound. However, to the best of the author's knowledge, the only known examples of TRSs that satisfy the conditions in Proposition 10 are mutually unbiased bases [9].

3.4 $|\mathcal{A}| = M - 1$

We now consider TRSs with angle sets $|\mathcal{A}| = M-1$. The following example demonstrates that it is not always possible to construct such a TRS $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ for all M.

Example 11. Let $M = 3$ and $d = 2$ and suppose that $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^2$ has Gram matrix $\mathbf{G}$. Define the matrix $\mathbf{M} = \mathbf{G} \circ \mathbf{G}$, and consider the completion problem

$$\mathbf{M} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & ? \\ a_3 & ? & a_1 \end{pmatrix} \tag{19}$$

This completion problem does not have a solution such that $\mathbf{M}$ is symmetric as there is either a violation of the requirement that each column should have distinct elements or that each row should have distinct elements. As such, the TRS $(\boldsymbol{\varphi}_i)_{i=1}^M$ with $|\mathcal{A}| = 2$ does not exist.

In fact, TRSs with $|\mathcal{A}| = M - 1$ do not exist for any odd $M > 1$. This is proven in the following theorem, which relies on a connection to the existence problem of symmetric Latin squares.

Definition 12. Let $\{1, \ldots, M\}$ be the alphabet. An order M **Latin square** $\mathbf{L}$ is a $M \times M$ matrix constructed such that each row and column contains each element of the alphabet only once. The Latin square $\mathbf{L}$ is **symmetric** if $\mathbf{L} = \mathbf{L}^T$.

Theorem 13. *Let $a_1 = 1$ and $a_2, \ldots, a_M$ be the elements of the angle set $\mathcal{A}$ of the TRS $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$. If $a_1, \ldots, a_M$ are distinct. Then, M is even.*

Proof. Let $\mathbf{G}$ be the Gram matrix of $(\boldsymbol{\varphi}_i)_{i=1}^M$ and $\mathbf{M} = \mathbf{G} \circ \mathbf{G}^T$. Suppose the elements of $\mathbf{M}$ are mapped to the elements of a matrix $\mathbf{L}$ via $a_i \mapsto i$. Under the assumption that each a_i is distinct, $\mathbf{L}$ is a Latin square with alphabet $\{1, \ldots, M\}$. Suppose that only the diagonal of $\mathbf{L}$ is specified, then it is not always possible to complete the Latin square and hence guarantee that $(\boldsymbol{\varphi}_i)_{i=1}^M$ is a TRS with $|\mathcal{A}| = M - 1$. In fact by [4, Section 3], for the completion of a prescribed diagonal to a symmetric $M \times M$ Latin square to exist, it is necessary and sufficient that the diagonal contains each element exactly once for odd M and an even number of times for even M. Since a_1 appears M times on the diagonal of $\mathbf{L}$, it follows that M must be even. $\square$

It is straightforward to construct TRSs that have $M - 1$ distinct angles when M is even. We present an example below.

Example 14. An example of a TRS $(\boldsymbol{\varphi}_i)_{i=1}^4$ in $\mathbb{C}^2$ with $M - 1$ distinct elements in its angle set can be obtained as follows. Define the matrices

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{M}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{M}_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{20}$$

and the vector

$$\boldsymbol{\phi} = \begin{pmatrix} 0.9425 \\ 0.3343 \end{pmatrix}. \tag{21}$$

Then, the frame with elements

$$\boldsymbol{\varphi}_i = \mathbf{M}_i \boldsymbol{\phi} \tag{22}$$

can be readily shown to be tight and hence is a TRS with angle set $\mathcal{A} = \{0, 0.603, 0.397\}$.

In fact, the construction in Example 14 exploits the group structure of the matrices $\{M_i\}$. We further develop the connection between TRSs and groups in the following section.

4 Group Covariant Tight Regular Schemes

We now consider a class of TRSs that are obtained from unitary representations of finite groups.

Definition 15. A finite group $G \subset \mathcal{U}(\mathbb{C}^d)$ is **irreducible** if, for every $\boldsymbol{\phi} \neq 0,\ \boldsymbol{\phi} \in \mathbb{C}^d$,

$$\operatorname{span}(G\boldsymbol{\phi}) = \mathbb{C}^d. \tag{23}$$

Lemma 16. *Let $G \subset \mathcal{U}(d)$ be an irreducible finite group. Then, the G-orbit*[1] *of every unit-norm $\boldsymbol{\phi} \in \mathbb{C}^d$ is a TRS $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$.*

Proof. By [14, Theorem 6.3], $(\boldsymbol{\varphi}_i)_{i=1}^M$ is tight. All that remains is to show that each angle set is the same. To this end, observe that for all $i = 1, \ldots, M$

$$\begin{aligned}
\mathcal{A}_i &= \{|\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 : j \in \{1, 2, \ldots, M\}\} \\
&= \{|\boldsymbol{\varphi}_1^\dagger \mathbf{U}_i^\dagger \mathbf{U}_j \boldsymbol{\varphi}_1|^2 : j \in \{1, 2, \ldots, M\}\} \\
&= \{|\boldsymbol{\varphi}_1^\dagger \mathbf{U}_k \boldsymbol{\varphi}_1|^2 : k \in \{1, 2, \ldots, M\}\} \\
&= \mathcal{A}_1,
\end{aligned} \tag{24}$$

where we used the fact that $\mathbf{U}_i\mathbf{U}_j \in G$ for all $i, j \in \{1, 2, \ldots, M\}$. □

Definition 17. Let $\{\mathbf{U}_i\}_{i=1}^M$ be a set of unitary matrices in $\mathcal{U}(d)$. The set $\{\mathbf{U}_i\}_{i=1}^M$ is **group covariant** if for all $j, k = 1, \ldots, M$, $\mathbf{U}_j\mathbf{U}_k = e^{i\theta_{jk}}\mathbf{U}_l$ for some $\theta_{jk} \in [0, 2\pi)$ and $\mathbf{U}_l \in \{\mathbf{U}_i\}_{i=1}^M$.

Proposition 18. *Let G be a group, $\boldsymbol{\phi} \in \mathbb{C}^d$ be unit-norm and H be the set of group covariant matrices obtained from G. If $G\boldsymbol{\phi}$ is a tight frame, then $H\boldsymbol{\phi}$ is a TRS.*

Proof. We first establish tightness. Since $G\boldsymbol{\phi}$ is tight, it follows that

$$\sum_{\mathbf{U} \in G} \mathbf{U}\boldsymbol{\phi}\boldsymbol{\phi}^\dagger \mathbf{U}^\dagger = \frac{|G|}{d}\mathbf{I}. \tag{25}$$

Equivalently,

$$\sum_{\mathbf{U} \in H} |I(d)| \mathbf{U}\boldsymbol{\phi}\boldsymbol{\phi}^\dagger \mathbf{U}^\dagger = \frac{|G|}{d}\mathbf{I}, \tag{26}$$

[1] A G-orbit of a vector $\boldsymbol{\phi} \in \mathbb{C}^d$ is the set $\{\mathbf{g}\boldsymbol{\phi} : \mathbf{g} \in G\}$.

where $|I(d)|$ is the size of the normal subgroup of G arising from the equivalence relation $\mathbf{g} \sim e^{i\theta}\mathbf{g}$. As a consequence,

$$\sum_{\mathbf{U}\in H} \mathbf{U}\boldsymbol{\phi}\boldsymbol{\phi}^\dagger\mathbf{U}^\dagger = \frac{|H|}{d}\mathbf{I}, \tag{27}$$

since $|H||I(d)| = |G|$ by Lagrange's theorem. Hence $H\boldsymbol{\phi}$ is tight.

Now, $H\boldsymbol{\phi}$ is group covariant and hence,

$$\begin{aligned}\mathcal{A}_i &= \{|\boldsymbol{\phi}^\dagger\mathbf{U}^\dagger\mathbf{g}\boldsymbol{\phi}|^2\}_{\mathbf{U}\in H}\\ &= \{|\boldsymbol{\phi}^\dagger e^{i\theta}\mathbf{U}\boldsymbol{\phi}|^2\}_{\mathbf{U}\in H}\\ &= \mathcal{A}_1.\end{aligned} \tag{28}$$

This implies that $H\boldsymbol{\phi}$ is also a regular scheme, completing the proof. □

5 Tight Regular Schemes and Weighted 1-Designs

A feature of any TRS $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ constructed from a set of group-covariant unitary matrices is that the area of the Voronoi regions corresponding to the elements $\boldsymbol{\varphi}_i$ are equal. Let S^{d-1} be the unit sphere in $\mathbb{C}^d$. The Voronoi region corresponding to $\boldsymbol{\varphi}_i$ is then defined by

$$\{\mathbf{z}\in S^{d-1} : |\mathbf{z}^\dagger\boldsymbol{\varphi}_i|^2 \geq |\mathbf{z}^\dagger\boldsymbol{\varphi}_j|^2,\ j\neq i\}. \tag{29}$$

The area of the Voronoi region corresponding to $\boldsymbol{\varphi}_i$ is given by

$$V_i = \mu(\{\mathbf{z}\in S^{d-1} : |\mathbf{z}^\dagger\boldsymbol{\varphi}_i|^2 \geq |\mathbf{z}^\dagger\boldsymbol{\varphi}_j|^2,\ j\neq i\}), \tag{30}$$

where μ is the unique normalized $\mathcal{U}(d)$-invariant Haar measure. We then have the following result.

Theorem 19. *Let $\{\mathbf{U}_i\}_{i=1}^M$ be a group covariant set of $d\times d$ unitary matrices and $\boldsymbol{\phi}\in\mathbb{C}^d$ a unit norm vector. If $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is a frame with elements $(\mathbf{U}_i\boldsymbol{\phi})_{i=1}^M$, then $V_i = \frac{1}{M}$.*

Proof. Observe that

$$\begin{aligned}V_i &= \mu(\{\mathbf{z}\in S^{d-1} : |\mathbf{z}^\dagger\boldsymbol{\varphi}_i|^2 \geq |\mathbf{z}^\dagger\boldsymbol{\varphi}_j|^2,\ j\neq i\})\\ &= \mu(\{\mathbf{z}' = \mathbf{U}_i^\dagger\mathbf{z}\in S^{d-1} : |\mathbf{z}'^\dagger\mathbf{U}_i^\dagger\boldsymbol{\varphi}_i|^2 \geq |\mathbf{z}'^\dagger\boldsymbol{\varphi}_j|^2,\ j\neq i\})\\ &= V_1.\end{aligned} \tag{31}$$

Since μ is a normalized measure, the result follows. □

Since the frame $(\boldsymbol{\varphi}_i)_{i=1}^M$ is tight, by Theorem 2 it follows that

$$\sum_{i,j=1}^M V_iV_j|\boldsymbol{\varphi}_i^\dagger\boldsymbol{\varphi}_j|^2 = \frac{1}{d}. \tag{32}$$

As a consequence, TRSs constructed from sets of group covariant unitary matrices are closely related to a variation of t-designs, defined as follows.

Definition 20. Let $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ be a unit norm frame and $(w_i)_{i=1}^M$ be weights satisfying $\sum_{i=1}^M w_i = 1$. Then, $(\boldsymbol{\varphi}_i)_{i=1}^M$ is a **weighted 1-design** with weights $(w_i)_{i=1}^M$ if for all $1 \leq k \leq t$

$$\sum_{i,j=1}^{M} w_i w_j |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 = \frac{1}{d}. \tag{33}$$

In light of Definition 20, $(\boldsymbol{\varphi}_i)_{i=1}^M$ forms a weighted 1-design with weights $(V_i))_{i=1}^M$ in (30).

A natural question is whether other TRSs are weighted 1-designs with weights dependent on the areas of Voronoi regions. We now present an example to demonstrate that it can also hold for other classes of TRSs. Consider the frame obtained from quadric Alltop construction generated from cyclic shifts of the following four vectors [1]:

$$\begin{aligned}
\mathbf{a}_1 &= \frac{1}{\sqrt{5}}(1, \omega_5, \omega_5^4, \omega_5^4, \omega_5) \\
\mathbf{a}_2 &= \frac{1}{\sqrt{5}}(1, \omega_5^2, \omega_5^3, \omega_5^3, \omega_5^2) \\
\mathbf{a}_3 &= \frac{1}{\sqrt{5}}(1, \omega_5^3, \omega_5^2, \omega_5^2, \omega_5^3) \\
\mathbf{a}_4 &= \frac{1}{\sqrt{5}}(1, \omega_5^4, \omega_5, \omega_5, \omega_5^4),
\end{aligned} \tag{34}$$

where $\omega_5 = \exp(2\pi i/5)$. This collection of vectors is known to be equivalent to a set of MUBs [1, Theorem 1]. We have already seen that MUBs form complex projective 2-designs (see Proposition 10) and now seek to understand when they are also weighted 1-designs with non-uniform weights given by the areas of Voronoi regions of the frame elements.

To this end, observe that the elements can all be generated by applying the following unitary transformations.

$$\mathbf{U}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_5 & 0 & 0 & 0 \\ 0 & 0 & \omega_5^4 & 0 & 0 \\ 0 & 0 & 0 & \omega_5^4 & 0 \\ 0 & 0 & 0 & 0 & \omega_5 \end{pmatrix}, \qquad \mathbf{U}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_5^2 & 0 & 0 & 0 \\ 0 & 0 & \omega_5^3 & 0 & 0 \\ 0 & 0 & 0 & \omega_5^3 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^2 \end{pmatrix} \tag{35}$$

$$\mathbf{U}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_5^3 & 0 & 0 & 0 \\ 0 & 0 & \omega_5^2 & 0 & 0 \\ 0 & 0 & 0 & \omega_5^2 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^3 \end{pmatrix}, \qquad \mathbf{U}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_5^4 & 0 & 0 & 0 \\ 0 & 0 & \omega_5 & 0 & 0 \\ 0 & 0 & 0 & \omega_5 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^4 \end{pmatrix} \tag{36}$$

The frame obtained from (34) is then obtained from $\{\sigma_j \mathbf{U}_i \frac{1}{\sqrt{5}} \mathbf{1}\}_{j,i}$, where $\{\sigma_j\}$ is the unitary representation of the group of cyclic shifts on 5 letters. It then follows that

$$\mathbf{U}_1 = \sigma \mathbf{U}_4 \tag{37}$$

and

$$\mathbf{U}_2 = \sigma' \mathbf{U}_3, \tag{38}$$

where σ, σ' are elements of the unitary representation for the group of permutations on 5 letters.

Now observe that

$$\begin{aligned} V_i &= \mu(\{\mathbf{z} \in S^{d-1} | |\mathbf{z}^\dagger \boldsymbol{\varphi}_i|^2 \geq |\mathbf{z}^\dagger \boldsymbol{\varphi}_j|^2,\ j \neq i\}) \\ &= \mu(\{\mathbf{z} \in S^{d-1} | |\mathbf{z}^\dagger \sigma \boldsymbol{\varphi}_i|^2 \geq |\mathbf{z}^\dagger \boldsymbol{\varphi}_j|^2,\ j \neq i\}), \end{aligned} \tag{39}$$

which means that the permutations of $\boldsymbol{\varphi}_i$ (obtained from $\mathbf{U}_i \frac{1}{\sqrt{5}} \mathbf{1}$) all have the same Voronoi areas. Moreover, since (37) and (38) hold, it follows that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_4$ (and all their cyclic shifts) have the same Voronoi areas. Similarly $\boldsymbol{\varphi}_2$ and $\boldsymbol{\varphi}_3$ have the same Voronoi areas.

To show that equality is achieved in (32), observe that

$$\begin{aligned} \sum_i \sum_j |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 V_i V_j &= 10\alpha^2 + 10\beta^2 + 50\alpha^2/5 + 50\beta^2/5 + 200\alpha\beta/5 \\ &= 20(\alpha^2 + \beta^2) + 40\alpha\beta, \end{aligned} \tag{40}$$

where α, β are the Voronoi areas. Furthermore, $10\alpha + 10\beta = 1$, which means that

$$\sum_i \sum_j |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 V_i V_j = 20/100 = 1/5 = 1/d, \tag{41}$$

as required.

We remark that a large class of TRSs appear to be weighted 1-design with weights given by the areas of frame element Voronoi regions. To illustrate this, we have numerically estimated via Monte Carlo simulations the value

$$Q = \sum_i \sum_j |\boldsymbol{\varphi}_i^\dagger \boldsymbol{\varphi}_j|^2 V_i V_j \tag{42}$$

for several constructions of ETFs and MUBs. A selection of the results are presented in Table 1. Observe that the results suggest that for each construction, the weighted 1-design property holds.

We also remark that these numerical results complement Theorem 19, which shows that all TRSs constructed via group covariant sets are such weighted 1-designs. Moreover, the maximal ETFs corresponding to the case $M = d^2$ in [10] are obtained through the action of the Weyl-Heisenberg group and hence by Theorem 19, they are also weighted 1-designs with weights determined by the Voronoi regions.

These observations motivate the following conjecture linking TRSs and weighted 1-designs.

Conjecture 21. In $(\boldsymbol{\varphi}_i)_{i=1}^M$ in $\mathbb{C}^d$ is a TRS, then it is also a weighted 1-design with weights given by the areas of the Voronoi regions V_i in (30) for each frame element.

Table 1: Numerical Study of TRSs

Construction	Estimated Q in (42)	$\frac{1}{d}$
$(2,2,4)$-Steiner system ETF [7, Section 2]	0.1666	$\frac{1}{6}$
$\{1,2,3\}$-Difference set ETF [17]	0.3333	$\frac{1}{3}$
Alltop Quadric MUB $N=7$ [1]	0.1428	$\frac{1}{7}$
Alltop Cubic MUB $p=5$ [1]	0.2000	$\frac{1}{5}$

6 Conclusion

We have studied the class of TRSs for $\mathbb{C}^d$. Our key observation is that many TRSs form 1-designs with weights governed by the Voronoi area of each frame element. We conjecture that this is in fact a characteristic feature of all TRSs.

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