

PIECEWISE DETERMINISTIC MARKOV PROCESSES AND THEIR INVARIANT MEASURE

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ABSTRACT. Piecewise Deterministic Markov Processes (PDMPs) are studied in a general framework. First, different constructions are proven to be equivalent. Second, we introduce a coupling between two PDMPs following the same differential flow which implies quantitative bounds on the total variation between the marginal distributions of the two processes. Finally two results are established regarding the invariant measures of PDMPs. A practical condition to show that a probability measure is invariant for the associated PDMP semi-group is presented. In a second time, a bound on the invariant probability measures in V -norm of two PDMPs following the same differential flow is established. This last result is then applied to study the asymptotic bias of some non-exact PDMP MCMC methods.

1. INTRODUCTION

Piecewise Deterministic Markov Processes (PDMP), similarly to diffusion processes, form an important class of Markov processes, which are used to model random dynamical systems in numerous fields (see e.g. [18, 1]). Recently, interest has grown for their use in MCMC algorithms [4, 22, 5]. To this end, natural questions arise as to the stationarity of the target measure, the ergodicity of the corresponding process and possible bias introduced by the method. In mathematical physics [6] and biology [7], the long time behaviour of these processes has been the subject of several works. In this context, these studies are done through the Kolmogorov Fokker Planck operator \mathcal{A}^* of the PDMP of interest given for all smooth density ρ on \mathbb{R}^{2d} by

$$\mathcal{A}^* \rho = - \langle \Xi, \nabla \rho \rangle + K(\lambda \rho) - \lambda \rho ,$$

where Ξ is a smooth vector field of \mathbb{R}^{2d} , $\lambda : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+$ and K is a non-local collision operator.

The relevance of the present work emerged while writing the companion paper [12], concerned with the geometric ergodicity of the Bouncy Particle Sampler (BPS) [5], an MCMC algorithm which, given a target distribution π on \mathbb{R}^d , introduces a PDMP for which π is invariant. In order to make rigorous several arguments in [12], technical lemmas had to be established, in particular to cope with the fact that Markov semi-groups associated to PDMP lack the regularity properties of (hypo-)elliptic diffusions, and thus implies additional difficulties and technicalities. These results, of interest in a more general framework, are gathered here with the hope that it will set a framework where for example verification of the invariance of a measure becomes a mere calculus via the generator (as for diffusion). The BPS is used as a recurrent example.

Let us present these different results, together with the organization of the paper. Section 2 contains the basic definitions of our framework, and in particular presents

the construction of a PDMP. Alternative constructions are shown in Sections 3 and 4 to give the same process (i.e. to give a random variable with the same law on the Skorokhod space). Conditions which ensure that PDMPs are non explosive are presented in Section 5. The synchronous coupling of two PDMPs is defined in Section 6, which aims to construct simultaneously two different PDMPs, starting at the same initial state, in such a way that they have some probability to stay equal for some time. It yields estimates on the difference of the corresponding semi-groups in total variation norm. In Section 8, conditions are established under which the semi-group associated to a PDMP leaves invariant the space of compactly-supported smooth functions. Using this result, a practical criterion to ensure that a given probability measure μ is invariant for a PDMP is obtained. Indeed, it is classical that, denoting by $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$ the strong generator of the Markov semi-group associated to the PDMP, then μ is invariant if and only if $\int \bar{\mathcal{A}}f d\mu = 0$ for all f in a core of $\bar{\mathcal{A}}$. Nevertheless, due to the lack of regularization properties of the semi-group, it is generally impossible to determine such a core. We will prove that, under some simple assumptions, it is enough to consider compactly-supported smooth functions f . Finally, in Section 10, we are interested in bounding the V -norm between two invariant probability measures μ_1 and μ_2 of two PDMPs sharing the same differential flow but with different jump rates and Markov kernels, sometimes called perturbation theory in the litterature (see for example the recent [23]). This question is here mainly motivated by the thinning method used to sample trajectories of PDMPs [17, 16]. Indeed, a PDMP can be exactly sampled (in the sense that no time discretization is needed) provided that the associated differential flow can be computed and a simple upper bound on the jump rate is known. When this is not the case, a PDMP with a truncated jump rate can be sampled, and our result gives a control on the ensuing error.

Notations and conventions. For all $a, b \in \mathbb{R}$, we denote $a_+ = \max(0, a)$, $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$. Id stands for the identity matrix on \mathbb{R}^d .

For all $x, y \in \mathbb{R}^d$, the scalar product between x and y is denoted by $\langle x, y \rangle$ and the Euclidean norm of x by $\|x\|$. For all $x \in \mathbb{R}^d$, $r > 0$, we denote by $B(x, r) = \{w \in \mathbb{R}^d : \|w - x\| \leq r\}$ the ball centered at x with radius r . The closed ball centered in x with radius r is denoted by $\bar{B}(x, r)$. For any d -dimensional matrix M , define by $\|M\| = \sup_{w \in B(0,1)} \|Mw\|$ the operator norm associated with M .

Let (M, \mathfrak{g}) be a smooth closed Riemannian sub-manifold of \mathbb{R}^N and $\mathcal{B}(M)$ the associated Borel σ -field. The distance induced by \mathfrak{g} is denoted by dist . With a slight abuse of notations, the ball (respectively closed ball) centered at $x \in M$ with radius $r > 0$ is denoted by $B(x, r)$ (respectively $\bar{B}(x, r)$).

For all function $F : M \rightarrow \mathbb{R}^m$ and compact set $K \subset M$, denote $\|F\|_\infty = \sup_{x \in M} \|F(x)\|$, $\|F\|_{\infty, K} = \sup_{x \in K} \|F(x)\|$. Denote by $B(M)$ the set of all measurable and bounded functions from M to \mathbb{R} . The space $B(M)$ is endowed with the topology associated with the uniform norm $\|\cdot\|_\infty$. Let $C(M)$ stand for the set of continuous function from M to \mathbb{R} and, for all $k \in \mathbb{N}^*$, let $C^k(M)$ be the set of k -times continuously differentiable function from M to \mathbb{R} . Denote for all $k \in \mathbb{N}$, $C_c^k(M)$ and $C_b^k(M)$ the set of functions in $C^k(M)$ with compact support and the set of bounded functions in $C^k(M)$ respectively.

For $f \in C^k(\mathbb{M})$, we denote by $D^k f$, the k^{th} differential of f . For all function $f : \mathbb{M} \rightarrow \mathbb{R}$, we denote by ∇f and $\nabla^2 f$ the gradient and the Hessian of f respectively, if they exist.

We denote by $\mathcal{P}(\mathbb{M})$ the set of probability measures on \mathbb{M} . For $\mu, \nu \in \mathcal{P}(\mathbb{M})$, $\xi \in \mathcal{P}(\mathbb{M}^2)$ is called a transference plan between μ and ν if for all $A \in \mathcal{B}(\mathbb{M})$, $\xi(A \times \mathbb{M}) = \mu(A)$ and $\xi(\mathbb{M} \times A) = \nu(A)$. The set of transference plan between μ and ν is denoted $\Gamma(\mu, \nu)$. The random variables X and Y on \mathbb{M} are a coupling between μ and ν if the distribution of (X, Y) belongs to $\Gamma(\mu, \nu)$. The total variation norm between μ and ν is defined by

$$\|\mu - \nu\|_{\text{TV}} = 2 \inf_{\xi \in \Gamma(\mu, \nu)} \int_{\mathbb{M}^2} \mathbb{1}_{\Delta_{\mathbb{M}}}(x, y) d\xi(x, y),$$

where $\Delta_{\mathbb{M}} = \{(x, y) \in \mathbb{M}^2 : x = y\}$. For all $\mu \in \mathcal{P}(\mathbb{M})$, define the support of μ by

$$\text{supp } \mu = \overline{\{x \in \mathbb{M} : \text{for all open set } U \ni x, \mu(U) > 0\}}.$$

In the sequel, we take the convention that $\inf \emptyset = +\infty$. All the random variables considered in this paper are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2. A FIRST DEFINITION OF PIECEWISE DETERMINISTIC MARKOV PROCESSES

Definitions and further notations.

Let $(\mathbb{M}, \mathfrak{g})$ be a smooth closed Riemannian sub-manifold of \mathbb{R}^N . A PDMP on \mathbb{M} is defined using a triple $(\varphi, (\lambda, Q)_{i \in \llbracket 1, \ell \rrbracket})$, $\ell \in \mathbb{N}^*$, referred to as the local characteristics of a PDMP, where

- φ is a differential flow on \mathbb{M} : $\varphi : (t, h, x) \mapsto \varphi_{t, t+h}(x)$ is a measurable function from $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{M}$ to \mathbb{M} , such that for all $t, h_1, h_2 \geq 0$ $\varphi_{t+h_1, t+h_1+h_2} \circ \varphi_{t, t+h_1} = \varphi_{t, t+h_1+h_2}$, $\varphi_{t, t} = \text{Id}$. Moreover, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{M}$, $\tilde{h} \mapsto \varphi_{t, t+\tilde{h}}(x)$ is continuous differentiable from \mathbb{R}_+ to \mathbb{M} and for all $t, h \in \mathbb{R}_+$, $\tilde{x} \mapsto \varphi_{t, t+h}(\tilde{x})$ is a C^1 -diffeomorphism of \mathbb{M} . The flow φ is (time)-homogeneous if for all $t, h \in \mathbb{R}_+$, $\varphi_{t, t+h} = \varphi_{0, h}$, in which case we denote $\varphi_h = \varphi_{0, h}$.
- For all $i \in \llbracket 1, \ell \rrbracket$, $\lambda_i : \mathbb{R}_+ \times \mathbb{M} \rightarrow \mathbb{R}_+$ is a measurable function referred to as a jump rate on \mathbb{M} which is locally bounded, in the sense that $\|\lambda_i\|_{\infty, K} < \infty$ for all compact $K \subset \mathbb{R}_+ \times \mathbb{M}$. The jump rate λ_i is (time)-homogeneous if it does not depend on t .
- For all $i \in \llbracket 1, \ell \rrbracket$, $Q_i : \mathbb{R}_+ \times \mathbb{M} \times \mathcal{B}(\mathbb{M}) \rightarrow [0, 1]$ is an inhomogeneous Markov kernel on \mathbb{M} : for all $A \in \mathcal{B}(\mathbb{M})$, $(t, x) \mapsto Q_i(t, x, A)$ is measurable, and for all $(t, x) \in \mathbb{R}_+ \times \mathbb{M}$, $Q_i(t, x, \cdot) \in \mathcal{P}(\mathbb{M})$. The Markov kernel Q_i is (time)-homogeneous if it does not depend on t .

If φ is a homogeneous differential flow and, for all $i \in \llbracket 1, \ell \rrbracket$, λ_i, Q_i are homogeneous as well, the local characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ are said to be homogeneous. A (homogeneous) jump mechanism on \mathbb{M} is a pair (λ, Q) constituted of a (homogeneous) jump rate and a (homogeneous) Markov kernel on \mathbb{M} .

A first construction of a PDMP. For all $i \in \llbracket 1, \ell \rrbracket$, consider a representation \mathbf{G}_i of the Markov kernel Q_i , *i.e.* a measurable function $\mathbf{G}_i : t, x, u \mapsto \mathbf{G}(t, x, u)$ from $\mathbb{R}_+ \times \mathbb{M} \times [0, 1]$ to \mathbb{M} such that for all $(t, x, A) \in \mathbb{R}_+ \times \mathbb{M} \times \mathcal{B}(\mathbb{M})$, $Q_i(t, x, A) = \mathbb{P}(\mathbf{G}_i(t, x, U) \in A)$, where U is a random variable uniformly distributed on $[0, 1]$. By [3, Corollary 7.16.1], such a representation always exists.

Then, a PDMP $(X_t)_{t \geq 0}$ based on the local characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and the initial distribution μ_0 can be defined recursively through a Markov chain $(X'_k, S_k)_{k \in \mathbb{N}}$ on $(M \cup \{\infty\}) \times (\mathbb{R}_+ \cup \{+\infty\})$. For all $k \in \mathbb{N}$, X'_k will be the state of the process $(X_t)_{t \geq 0}$ at times S_k . Between two times S_k and S_{k+1} , $(X_t)_{t \geq 0}$ will be a deterministic function of X'_k and S_k . More precisely, consider the following construction.

Construction 1. Let W_0 be a random variable with distribution $\mu_0 \in \mathcal{P}(M)$ and $((E_{j,k})_{j \in \llbracket 1, \ell \rrbracket}, U_k)_{k \in \mathbb{N}^*}$ be an i.i.d. sequence, independent of W_0 , such that for all $k \in \mathbb{N}^*$ and $j \in \llbracket 1, \ell \rrbracket$, U_k is uniformly distributed on $[0, 1]$ and $E_{j,k}$ is an exponential random variable with parameter 1, independent of U_k and from $E_{i,k}$ for $i \neq j$. Let $\infty \notin M$ be a cemetery point.

Set $S_0 = 0$, $X'_0 = W_0$, and suppose that (X'_k, S_k) and $(X_t)_{t \leq S_k}$ have been defined for some $k \in \mathbb{N}$, with $X'_k \in M$ and $S_k \in \mathbb{R}_+$. For all $j \in \llbracket 1, \ell \rrbracket$, set

$$S_{j,k+1} = \inf \left\{ t \geq S_k : E_{j,k+1} < \int_{S_k}^t \lambda_j(s, \varphi_{S_k, s}(X'_k)) ds \right\}, \quad S_{k+1} = \min_{j \in \llbracket 1, \ell \rrbracket} S_{j,k+1}.$$

- If $S_{k+1} = +\infty$, set $S_m = +\infty$, $X'_m = \infty$, $I_m = 1$ for all $m > k$, and $X_t = \varphi_{S_k, t}(X'_k)$ for all $t \geq S_k$.
- If $S_{k+1} < +\infty$, set

$$I_{k+1} = \min\{j \in \llbracket 1, \ell \rrbracket, S_{j,k+1} = S_{k+1}\}, \quad X'_{k+1} = \mathbf{G}_{I_{k+1}}(S_{k+1}, \varphi_{S_k, S_{k+1}}(X'_k), U_{k+1}).$$

For $t \in (S_k, S_{k+1}]$, set $X_t = \varphi_{S_k, t}(X'_k)$ and $X'_{k+1} = X_{S_{k+1}}$.

For $t \geq \sup_{k \in \mathbb{N}} S_k$, set $X_t = \infty$.

Note that, when $S_{k+1} < +\infty$, $k \in \mathbb{N}$, the probability of $\{S_{j,k+1} = S_{i,k+1} = S_{k+1}\}$ for two indexes $i \neq j$ in $\llbracket 1, \ell \rrbracket$ is zero, but the definition of I_{k+1} ensures that the process (X'_k, S_k) is defined not only almost everywhere on Ω , but in fact on all Ω .

Let $(\mathcal{F}'_k)_{k \in \mathbb{N}}$ be the filtration associated with $(X'_k, S_k, I_k)_{k \in \mathbb{N}}$. Then taking a random variable I_0 on $\llbracket 1, \ell \rrbracket$, $(X'_k, S_k, I_k)_{k \in \mathbb{N}}$ is an inhomogeneous Markov chains since for all $k \in \mathbb{N}$, $A \in \mathcal{B}(M)$, $t \geq S_k$, $j \in \llbracket 1, \ell \rrbracket$,

$$\begin{aligned} \mathbb{P}(X'_{k+1} \in A, S_{k+1} \leq t, I_{k+1} = j \mid \mathcal{F}'_k) &= \mathbb{1}_M(X'_k) \int_{S_k}^t Q_j(s, \varphi_{S_k, s}(X'_k), A) \lambda_j(s, \varphi_{S_k, s}(X'_k)) \\ (1) \quad &\times \exp \left\{ - \sum_{i=1}^{\ell} \int_{S_k}^s \lambda_i(u, \varphi_{S_k, u}(X'_k)) du \right\} ds. \end{aligned}$$

Note that the sequence $(X'_k, S_k)_{k \in \mathbb{N}}$ is an inhomogeneous Markov chain as well, whose kernel can be straightforwardly deduced from (1).

Then, $(X_t)_{t \geq 0}$ is a stochastic process on $M \cup \{\infty\}$, *i.e.* it is a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to the space $D(\mathbb{R}_+, M \cup \{\infty\})$ of càdlàg functions from \mathbb{R}_+ to $M \cup \{\infty\}$, endowed with the Skorokhod topology, see [15, Chapter 6]. Moreover, $(X_t)_{t \geq 0}$ is a Markov process [14, Theorem 7.3.1], from the class of piecewise deterministic Markov processes (PDMPs). We say that a stochastic process $(\tilde{X}_t)_{t \geq 0}$ is a PDMP with local characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, n \rrbracket})$ and initial distribution μ_0 if it has the same distribution on

$D(\mathbb{R}_+, \mathbf{M} \cup \{\infty\})$ as $(X_t)_{t \geq 0}$. We will denote by PDMP $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket}, \mu_0)$ this distribution. In the sequel, we will see that a given PDMP can admit several local characteristics. Note that, as φ is a C^1 -diffeomorphism, $(X_t)_{t \geq 0}$ is completely determined by the Markov chain $(X'_k, S_k)_{k \in \mathbb{N}}$, referred to as the embedded chain associated to the process. The sequence $(S_k)_{k \in \mathbb{N}}$ is said to be the jump times of the process $(X_t)_{t \geq 0}$.

A PDMP is said to be homogeneous if its local characteristics are (time) homogeneous.

For $(\tilde{X}_t)_{t \geq 0} \in D(\mathbb{R}_+, \mathbf{M} \cup \{\infty\})$, we call $\tau_\infty(\tilde{X}) = \inf\{t \geq 0 : \tilde{X}_t = \infty\}$ the explosion time of the process $(\tilde{X}_t)_{t \geq 0}$. A process $(\tilde{X}_t)_{t \geq 0}$ is said to be non-explosive if $\tau_\infty(\tilde{X}) = +\infty$ almost surely. PDMP characteristics are said to be non-explosive if for all initial distribution the associated PDMP is non-explosive.

Construction 1 associated with the characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ defines a Markov semi-group $(P_{s,t})_{t \geq s \geq 0}$ for all $x \in \mathbf{M}$, $A \in \mathcal{B}(\mathbf{M})$ and $t \geq s \geq 0$ by

$$P_{s,t}(x, A) = \mathbb{P}(\bar{X}_{t-s}^{s,x} \in A),$$

where $(\bar{X}_u^{s,x})_{u \geq 0}$ is a PDMP started from x with characteristics $((\varphi_{s+u, s+t})_{t \geq u \geq 0}, (\lambda_i(s + \cdot, \cdot), Q_i(s + \cdot, \cdot, \cdot))_{i \in \llbracket 1, \ell \rrbracket})$. Its left-action on $C(\mathbf{M})$ and right-action on $\mathcal{P}(\mathbf{M})$ are then given by

$$P_{s,t}f(x) = \mathbb{E}[f(\bar{X}_t^{s,x})], \quad \nu P_{s,t}(A) = \int_{\mathbf{M}} \mathbb{P}(\bar{X}_t^{x,s} \in A) d\nu(x),$$

for all $f \in C(\mathbf{M})$, $x \in \mathbf{M}$, $\nu \in \mathcal{P}(\mathbf{M})$, $A \in \mathcal{B}(\mathbf{M})$ and $t \geq s \geq 0$. The Markov property of $(X_t)_{t \geq 0}$ is equivalent to the semi-group property $P_{u,s}P_{s,t} = P_{u,t}$ for all $t \geq s \geq u \geq 0$. If $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ is non explosive, then $P_{s,t}$ is a Markov kernel for all $t \geq s \geq 0$ and we say that $(P_{s,t})_{t \geq s \geq 0}$ is non explosive. Else, it is only a sub-Markovian kernel. For a homogeneous process, we simply write $P_t = P_{0,t}$ for all $t \geq 0$.

For a PDMP $(X_t)_{t \geq 0}$ with jump times $(S_k)_{k \in \mathbb{N}}$, we say that at time S_{k+1} , $k \geq 0$, a true jump occurred if $X_{S_{k+1}} \neq \varphi_{S_k, S_{k+1}}(X_{S_k})$. Else, we say that at time S_{k+1} a phantom jump occurred. Note that, in the definition of homogeneous PDMPs with characteristics (φ, λ, Q) given in [9, standard conditions p. 62], phantom jumps are impossible, since it is assumed that for all $x \in \mathbf{M}$, $Q(x, \{x\}) = 0$. This is not the case with the definition we gave in Section 2, where the notion of jump times depends on the jump mechanisms used to define the process. We will see that in Section 4 that under our settings, based on characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ which define a PDMP $(X_t)_{t \geq 0}$, we can always define some characteristics (φ, λ, Q) which define a PDMP $(Z_t)_{t \geq 0}$ with the same distribution as $(X_t)_{t \geq 0}$ but no phantom jump.

The condition imposed by [9] implying that a PDMP has no phantom jump can be very useful since it allows a one-to-one correspondence between the path of the continuous-time process $(X_t)_{t \geq 0}$ and of its embedded chain $(X'_k, S_k)_{k \in \mathbb{N}}$. With our construction, the continuous process is completely determined by its embedded chain but not the opposite.

On the other hand, adding phantom jumps sometimes turns out to be convenient. Here is an example: let (φ, λ, Q) be the characteristics of a PDMP $(X_t)_{t \geq 0}$, and suppose that there exists $\lambda_* > 0$ such that $\lambda(t, x) \leq \lambda_*$ for all $t \geq 0$ and $x \in \mathbf{M}$. From Proposition 5 below, $(X_t)_{t \geq 0}$ has the same distribution as the PDMP $(Z_t)_{t \geq 0}$ obtained through Construction 1 from the characteristics $(\varphi, \lambda_*, \tilde{Q})$ with for all $(t, x, A) \in \mathbb{R}_+ \times \mathbf{M} \times \mathcal{B}(\mathbf{M})$,

$$\tilde{Q}(t, x, A) = \frac{\lambda(t, x)}{\lambda_*} Q(t, x, A) + \left\{ 1 - \frac{\lambda(t, x)}{\lambda_*} \right\} \delta_x(A).$$

The jump times of $(Z_t)_{t \geq 0}$ are given by a Poisson process with intensity λ_* . The method of adding phantom jumps so that the distribution of the jump times get simpler (for sampling purpose, for instance) is called thinning (see [16] and references therein for more details).

Another use of phantom jump is presented in [8]. The stability or ergodicity of a PDMP $(X_t)_{t \geq 0}$ and of its embedded chain $(X'_k, T_k)_{k \in \mathbb{N}}$ may differ, but this is no more the case if phantom jumps are added at constant rate, *i.e.* if we consider the PDMP with characteristics $(\varphi, \lambda + 1, \tilde{Q})$, where \tilde{Q} is given for all $(t, x, A) \in \mathbb{R}_+ \times M \times \mathcal{B}(M)$ by

$$\tilde{Q}(t, x, A) = \frac{\lambda(t, x)}{1 + \lambda(t, x)} Q(t, x, A) + \frac{1}{1 + \lambda(t, x)} \delta_x(A)$$

and its embedded chain. See [8] for more details.

There are a other differences between the assumptions we made on the characteristics of a PDMP and those made in [9, standard conditions p. 62]. For simplicity, we consider that the flow cannot exit M contrary to [9]. In addition, to prevent the artificial problem of an infinity of phantom jumps in a finite time, we assume that λ is locally bounded, instead of the following weaker condition that would be sufficient to define $(X_t)_{t \geq 0}$: for all $(t, x) \in \mathbb{R}_+ \times M$, there exists $h > 0$ such that $\int_t^{t+h} \lambda(s, \varphi_{t,s}(x)) ds < +\infty$. On the other hand, we don't assume a priori that PDMPs are non-explosive.

Examples.

Several examples of PDMP can be found in [18] and references therein. In the present paper, special attention will be paid to the family of velocity jump PDMP, described as follows. Let $V \subset \mathbb{R}^d$ be a smooth complete Riemannian submanifold, and set $M = \mathbb{R}^d \times V$. Then, M is a smooth complete Riemannian submanifold of \mathbb{R}^{2d} endowed with the canonical Euclidean distance and tensor metric. We say that a PDMP $(X_t, Y_t)_{t \geq 0}$ on M (where $X_t \in \mathbb{R}^d$ and $Y_t \in V$ for all $t \geq 0$) with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ is a velocity jump PDMP if φ is homogeneous and given for any $t \in \mathbb{R}_+$ and $(x, v) \in \mathbb{R}_+ \times V$ by

$$\varphi_t(x, y) = (x + ty, y)$$

and if for all $i \in \llbracket 1, \ell \rrbracket$, all $A \in \mathcal{B}(\mathbb{R}^d)$ and all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times V$,

$$Q_i(t, (x, y), A \times V) = \delta_x(A).$$

Consider the PDMP $(X_t, Y_t)_{t \geq 0}$ associated with this choice of characteristics and $(X'_k, Y'_k, S_k)_{k \geq 0}$ the corresponding embedded chain. Note that by construction for all $t \in [S_k, S_{k+1})$, $k \in \mathbb{N}$, $X_t = X'_k + (t - S_k)Y'_k$ and $Y_t = Y'_k$. Therefore for all $t < \sup_{k \in \mathbb{N}} S_k$, $X_t = \int_0^t Y_s ds$ and only $(Y_t)_{t \geq 0}$ can be discontinuous in time.

The class of velocity jump processes gathers the Zig-Zag process [4], the Bouncy Particle Sampler (BPS) [22] and many of their variants. The choice for the jump rates and Markov kernels of these different (but similar) processes are mainly of one of the following type (here we only consider homogeneous mechanisms):

- refreshment mechanism: the rate $\lambda(x, y)$ only depends on $x \in \mathbb{R}^d$, and the kernel Q is constant, *i.e.* there exists $\nu \in \mathcal{P}(V)$ such that for all $(x, y) \in \mathbb{R}^d \times V$ and all $(A, A') \in \mathcal{B}(M) \times \mathcal{B}(V)$

$$Q((x, y), A \times A') = \delta_x(A) \nu(A').$$

- deterministic bounce mechanism: there exists a measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for all $(x, y) \in \mathbf{M}$, $\lambda(x, y) = \langle g(x), y \rangle_+$ and $Q((x, y), \{x\} \times \{\Psi(x, y)\}) = 1$, for a measurable function $\Psi : \mathbf{M} \rightarrow \mathbf{Y}$. A particular example in the case $\mathbf{Y} = \mathbb{S}^d$ or $\mathbf{Y} = \mathbb{R}^d$, is $\Psi = \mathbf{R}$ where \mathbf{R} is given for all $(x, y) \in \mathbf{M}$ by

$$(2) \quad \mathbf{R}(x, y) = \begin{cases} y - 2 \|g(x)\|^{-2} \langle g(x), y \rangle g(x) & \text{if } g(x) \neq 0, \\ y & \text{otherwise.} \end{cases}$$

Note that $\mathbf{R}(x, y)$ is simply the orthogonal reflection of y with respect to $g(x)$ if $g(x) \neq 0$.

- randomized bounce mechanism: there exists a measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for all $(x, y) \in \mathbf{M}$ and $\mathbf{A} \in \mathcal{B}(\mathbb{R}^d)$, $\mathbf{A}' \in \mathcal{B}(\mathbf{Y})$, $\lambda(x, y) = \langle g(x), y \rangle_+$ and $Q((x, y), \mathbf{A} \times \mathbf{A}') = \delta_x(\mathbf{A}) \tilde{Q}((x, y), \mathbf{A}')$, where \tilde{Q} is a Markov kernel on $\mathbf{M} \times \mathcal{B}(\mathbf{Y})$.

For instance, [6] studies the velocity jump process associated with the linear Boltzmann equation, which gives an example of refreshment mechanism. The Zig-Zag (ZZ) process [4] and the Bouncy Particle Sampler (BPS) [22, 21, 10] are recently proposed PDMP used to sample from a target density $\pi \propto \exp(-U)$, where $U : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuously differentiable function. The ZZ process is a velocity jump process with $\mathbf{Y} = \{-1, 1\}^d$ and d deterministic bounce mechanisms $(\lambda_i, Q_i)_{i \in \llbracket 1, d \rrbracket}$ given for all $i \in \llbracket 1, d \rrbracket$, $x \in \mathbb{R}^d$, $y \in \{-1, 1\}^d$ and $\mathbf{A} \in \mathcal{B}(\mathbb{R}^d)$ by

$$\lambda_i(x, y) = (y_i \partial U(x) / \partial x_i)_+, \quad Q_i((x, y), \{x\} \times \{-y\}) = 1.$$

Note that in this case, for all $x \in \mathbb{R}^d$, $g_i(x) = (\partial U(x) / \partial x_i) \mathbf{e}_i$ where \mathbf{e}_i is the i^{th} vector of the standard basis of \mathbb{R}^d . Additional refreshment mechanisms can be added to the process. In the rest of this paper, we will repeatedly use the BPS process as an illustration to our different results.

Example-Bouncy Particle Sampler 1. Let \mathbf{V} be a smooth closed sub-manifold of \mathbb{R}^d rotation invariant, *i.e.* for any rotation O of \mathbb{R}^d , $O\mathbf{V} = \mathbf{V}$. Let $\lambda_c > 0$ and $\mu_v \in \mathcal{P}(\mathbf{V})$. The BPS process associated with the potential U , refreshment rate λ_c and refreshment distribution μ_v is the PDMP on $\mathbf{M} = \mathbb{R}^d \times \mathbf{V}$ with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, 2 \rrbracket})$ where φ is given by (2) and for all $(x, y) \in \mathbb{R}^d \times \mathbf{V}$, $\mathbf{A} \in \mathcal{B}(\mathbb{R}^d)$, $\lambda_1(x, y) = \langle y, \nabla U(x) \rangle_+$, $\lambda_2(x, y) = \lambda_c$, $Q_1((x, y), \mathbf{A} \times \{\mathbf{R}(x, y)\}) = \delta_x(\mathbf{A})$ and $Q_2((x, y), \cdot) = \delta_x \otimes \mu_v$, where \mathbf{R} is given by (2) with $g(x) = \nabla U(x)$. Note that (λ_1, Q_1) is the pure bounce mechanism associated with g , and (λ_2, Q_2) is a refreshment mechanism.

Variants of the BPS with randomized bounces have been recently introduced in [20, 26, 24].

3. ALTERNATIVE CONSTRUCTIONS

Consider PDMP characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$, an initial distribution $\mu_0 \in \mathcal{P}(\mathbf{M})$ and the associated process $(X_t)_{t \geq 0}$ defined in Section 2. The goal of this Section is to construct another process $(Y_t)_{t \geq 0}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the same distribution on $\mathbf{D}(\mathbb{R}_+, \mathbf{M} \cup \{\infty\})$ as $(X_t)_{t \geq 0}$.

Construction 2. Let W_0 be a random variable with distribution $\mu_0 \in \mathcal{P}(\mathbf{M})$ and $(E_{j,k}, U_{j,k})_{j \in \llbracket 1, \ell \rrbracket, k \in \mathbb{N}^*}$ be an i.i.d. family, independent of W_0 , such that for all $k \in \mathbb{N}$ and $j \in \llbracket 1, \ell \rrbracket$, $U_{j,k}$ is uniformly distributed on $[0, 1]$ and $E_{j,k}$ is an exponential random variable with parameter 1, independent of $U_{j,k}$.

Set $\tilde{S}_0 = 0$, $Y'_0 = W_0$, $\tilde{H}_{j,1} = E_{j,1}$ and $\tilde{N}_{j,0} = 1$ for $j \in \llbracket 1, \ell \rrbracket$. Suppose that $(Y'_k, \tilde{S}_k, (\tilde{H}_{j,k+1}, \tilde{N}_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket})$ and $(Y_t)_{t \leq S_k}$ have been defined for some $k \in \mathbb{N}$, with $Y'_k \in \mathbf{M}$ and $\tilde{S}_k \in \mathbb{R}_+$. Set

$$\tilde{S}_{j,k+1} = \inf \left\{ t > \tilde{S}_k : \tilde{H}_{j,k+1} < \int_{\tilde{S}_k}^t \lambda_j \left(s, \varphi_{\tilde{S}_k, s}(Y'_k) \right) ds \right\}, \quad \tilde{S}_{k+1} = \min_{j \in \llbracket 1, \ell \rrbracket} \tilde{S}_{j,k+1}.$$

- If $\tilde{S}_{k+1} = +\infty$, set $\tilde{S}_m = +\infty$, $Y'_m = \infty$, $\tilde{I}_m = 1$ for all $m > k$ and $Y_t = \varphi_{\tilde{S}_k, t}(Y'_k)$ for $t \geq \tilde{S}_k$.
- If $\tilde{S}_{k+1} < +\infty$, set

$$\begin{aligned} \tilde{I}_{k+1} &= \min\{j \in \llbracket 1, \ell \rrbracket, \tilde{S}_{j,k+1} = \tilde{S}_{k+1}\}, & \tilde{H}_{\tilde{I}_{k+1}, k+2} &= E_{\tilde{I}_{k+1}, \tilde{N}_{k+1}}, \\ Y'_{k+1} &= \mathbf{G}_{\tilde{I}_{k+1}}(\tilde{S}_{k+1}, \varphi_{\tilde{S}_k, \tilde{S}_{k+1}}(Y'_k), U_{\tilde{I}_{k+1}, \tilde{N}_{k+1}}) & \tilde{N}_{\tilde{I}_{k+1}, k+2} &= \tilde{N}_{\tilde{I}_{k+1}, k+1} + 1, \end{aligned}$$

and for $j \neq \tilde{I}_{k+1}$,

$$\tilde{H}_{j,k+2} = \tilde{H}_{j,k+1} - \int_{\tilde{S}_k}^{\tilde{S}_{k+1}} \lambda_j \left(s, \varphi_{\tilde{S}_k, s}(Y'_k) \right) ds.$$

Set $Y_t = \varphi_{\tilde{S}_k, t}(Y'_k)$ for $t \in (\tilde{S}_k, \tilde{S}_{k+1})$ and $Y_{\tilde{S}_{k+1}} = Y'_{k+1}$.

For $t \geq \sup_{k \in \mathbb{N}} \tilde{S}_k$, set $Y_t = \infty$.

We show in the following result that the two constructions we consider define the same distribution on $\mathbf{D}(\mathbb{R}_+, \mathbf{M} \cup \{\infty\})$.

Proposition 2. *The two Markov chains $(X'_k, S_k, I_k)_{k \in \mathbb{N}}$ and $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ have the same distribution on $((\mathbf{M} \cup \{\infty\}) \times (\mathbb{R}_+ \cup \{+\infty\}))^{\mathbb{N}}$. Therefore, $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ have the same distribution on $\mathbf{D}(\mathbb{R}_+, \mathbf{M} \cup \{\infty\})$.*

We preface the proof by a lemma. Denote by $(\tilde{\mathcal{F}}_k)_{k \in \mathbb{N}}$ and $(\tilde{\mathcal{F}}'_k)_{k \in \mathbb{N}}$ the filtration associated with the sequence of random variables $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ and $(Y'_k, \tilde{S}_k)_{k \in \mathbb{N}}$.

Lemma 3. *For all $k \in \mathbb{N}$, given $\{\tilde{S}_k < +\infty\}$, $(\tilde{H}_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket}$ are i.i.d. exponential random variables with parameter 1, independent of $\tilde{\mathcal{F}}'_k$. In addition, for all $i \in \llbracket 1, \ell \rrbracket$, given $\{\tilde{I}_k = i\} \cap \{\tilde{S}_k < +\infty\}$, $(\tilde{H}_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}}$ are i.i.d. exponential random variables with parameter 1, independent of $\tilde{\mathcal{F}}'_k$.*

Proof. Set for all $k \in \mathbb{N}^*$ and $j \in \llbracket 1, \ell \rrbracket$, $B_{j,k} = \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_k) \int_{\tilde{S}_{k-1}}^{\tilde{S}_k} \{\lambda_j(s, \varphi_{\tilde{S}_{k-1}, s}(Y'_{k-1}))\} ds$. Note that the second statement is equivalent to for all $k \in \mathbb{N}^*$ and $i \in \llbracket 1, \ell \rrbracket$, $(\tilde{H}_{j,k} - B_{j,k})_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}}$ are i.i.d. exponential random variables with parameter 1, independent of

$\sigma(\tilde{\mathcal{F}}'_{k-1}, \tilde{S}_{i,k})$ given $\{\tilde{S}_k = \tilde{S}_{i,k}\}$ since for all $A \in \tilde{\mathcal{F}}'_k$, $\{\tilde{S}_k = \tilde{S}_{i,k}\} \cap A \in \sigma(\tilde{\mathcal{F}}'_{k-1}, \tilde{S}_{i,k})$, which is the result that we will show.

The proof is by induction on $k \in \mathbb{N}^*$. For $k = 1$, by definition the first statement holds. The second part follows from the memoryless property of the exponential distribution and because for all $i \in \llbracket 1, \ell \rrbracket$, $(\tilde{H}_{1,j} = E_{1,j})_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}}$, is independent of $\sigma(\tilde{\mathcal{F}}'_0, \tilde{S}_{1,i})$.

Assume now that the result holds for $k \in \mathbb{N}^*$. Then for all $t_1, \dots, t_\ell \geq 0$, we have using the induction hypothesis, the definition of $(\tilde{H}_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket}$, the memoryless property of the exponential distribution and since given $\{\tilde{S}_k = \tilde{S}_{i,k}\}$, $E_{i, \tilde{N}_{k+1}}$ is independent of $\sigma(\tilde{\mathcal{F}}'_k, \tilde{S}_k, (\tilde{H}_{j,k})_{j \in \llbracket 1, \ell \rrbracket})$ for all $i \in \llbracket 1, \ell \rrbracket$,

$$\begin{aligned} & \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_k) \mathbb{P} \left(\bigcap_{j=1}^{\ell} \{\tilde{H}_{j,k+1} \geq t_j\} \middle| \tilde{\mathcal{F}}'_k \right) \\ &= \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_k) \sum_{i=1}^{\ell} \mathbb{P} \left(\bigcap_{j=1}^{\ell} \{\tilde{H}_{j,k+1} \geq t_j\} \cap \{\tilde{S}_k = \tilde{S}_{i,k}\} \middle| \tilde{\mathcal{F}}'_k \right) \\ &= \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_k) \sum_{i=1}^{\ell} \mathbb{P} \left(\bigcap_{j=1, j \neq i}^{\ell} \{\tilde{H}_{j,k+1} \geq t_j\} \cap \{\tilde{S}_k = \tilde{S}_{i,k}\} \cap \{\bar{E}_{i, \tilde{N}_{k+1}} \geq t_i\} \middle| \tilde{\mathcal{F}}'_k \right) \\ &= \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_k) \exp \left(- \sum_{j=1}^{\ell} t_j \right). \end{aligned}$$

which shows the first part of the statement. Finally we show the second statement of the induction. Note that $(\tilde{H}_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}}$ is an independent family of random variables, independent of $\tilde{\mathcal{F}}'_k$ and $\tilde{S}_{i,k+1}$, for $i \in \llbracket 1, \ell \rrbracket$ and therefore given $\{\tilde{I}_{k+1} = i\}$, $(\tilde{H}_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}}$ is independent of $(B_{j,k+1})_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}}$. Then, using the first statement and the memoryless property of the exponential distributions, we have for all $t_1, \dots, t_\ell \geq 0$,

$$\begin{aligned} & \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_{k+1}) \mathbb{P} \left(\bigcap_{j=1, j \neq i}^{\ell} \{\tilde{H}_{j,k+1} - B_{j,k+1} \geq t_j\} \cap \{\tilde{S}_{k+1} = \tilde{S}_{i,k+1}\} \middle| \sigma(\tilde{\mathcal{F}}'_k, \tilde{S}_{i,k+1}) \right) \\ &= \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_{k+1}) \mathbb{P} \left(\bigcap_{j=1, j \neq i}^{\ell} \left\{ \{\tilde{H}_{j,k+1} - B_{j,k+1} \geq t_j\} \cap \{\tilde{H}_{j,k+1} > B_{j,k+1}\} \right\} \middle| \sigma(\tilde{\mathcal{F}}'_k, \tilde{S}_{i,k+1}) \right) \\ &= \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_{k+1}) \exp \left(- \sum_{j=1, j \neq i}^{\ell} t_j \right). \end{aligned}$$

□

Proof of Proposition 2. We show that the two processes $(X'_k, S_k, I_k)_{k \in \mathbb{N}}$ and $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ have the same distribution. Note that, since (X'_0, S_0, I_0) and $(Y'_0, \tilde{S}_0, \tilde{I}_0)$ have the same distribution, this result is equivalent to show that $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ is also a Markov chain

with a Markov kernel characterized by (1). Let $k \in \mathbb{N}^*$, $\mathbf{A} \in \mathcal{B}(\mathbf{M})$, $t \geq \tilde{S}_k$, $i \in \llbracket 1, \ell \rrbracket$. By Lemma 3 and definition of $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$, since, given $\{\tilde{I}_{k+1} = i\} \cap \{\tilde{S}_{k+1} < +\infty\}$, $Y'_{k+1} = \mathbf{G}(\tilde{S}_{i,k+1}, \varphi_{\tilde{S}_k, \tilde{S}_{i,k+1}}(Y'_k), U_{i, N_{k+1}})$ and $U_{i, N_{k+1}}$ is independent of $\tilde{\mathcal{F}}_k$, then setting $\mathcal{G}_{k+1, i} = \sigma(\tilde{\mathcal{F}}_k, \tilde{H}_{i, k+1}, U_{i, N_{k+1}})$ and $B_{j, k+1} = \mathbb{1}_{\mathbb{R}_+}(\tilde{S}_{i, k+1}) \int_{\tilde{S}_k}^{\tilde{S}_{i, k+1}} \lambda_j(s, \varphi_{\tilde{S}_k, s}(Y'_k)) ds$, for $j \in \llbracket 1, \ell \rrbracket \setminus \{i\}$, we have

$$\begin{aligned} & \mathbb{P}\left(Y'_{k+1} \in \mathbf{A}, \tilde{S}_{k+1} \leq t, \tilde{I}_{k+1} = i \mid \tilde{\mathcal{F}}_k\right) \\ &= \mathbb{P}\left(Y'_{k+1} \in \mathbf{A}, \tilde{S}_{i, k+1} \leq t, \tilde{S}_{i, k+1} < \tilde{S}_{j, k+1}, \text{ for all } j \in \llbracket 1, \ell \rrbracket \setminus \{i\}, \tilde{I}_{k+1} = i \mid \tilde{\mathcal{F}}_k\right) \\ &= \mathbb{E}\left[\mathbb{1}_{\mathbf{A}}(Y'_{k+1}) \mathbb{1}_{[\tilde{S}_k, t]}(\tilde{S}_{i, k+1}) \mathbb{P}\left(\bigcap_{j \in \llbracket 1, \ell \rrbracket, j \neq i} \{\tilde{S}_{i, k+1} \leq \tilde{S}_{j, k+1}\} \mid \mathcal{G}_{k+1, i}\right) \mid \tilde{\mathcal{F}}_k\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\mathbf{A}}(Y'_{k+1}) \mathbb{1}_{[\tilde{S}_k, t]}(\tilde{S}_{i, k+1}) \mathbb{P}\left(\bigcap_{j \in \llbracket 1, \ell \rrbracket, j \neq i} \{B_{j, k+1} < \tilde{H}_{j, k+1}\} \mid \mathcal{G}_{k+1, i}\right) \mid \tilde{\mathcal{F}}_k\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\mathbf{A}}(Y'_{k+1}) \mathbb{1}_{[\tilde{S}_k, t]}(\tilde{S}_{i, k+1}) \exp\left\{-\sum_{j=1, j \neq i}^{\ell} \int_t^{\tilde{S}_{i, k+1}} \lambda_j(u, \varphi_{\tilde{S}_k, u}(Y'_k)) du\right\} \mid \tilde{\mathcal{F}}_k\right]. \end{aligned}$$

The proof then follows from the definition of $\tilde{S}_{i, k+1}$ and Y'_{k+1} , and Lemma 3. \square

For $k \geq 1$ with $S_k < \infty$, we say that, at time S_k , the process $(X_t)_{t \geq 0}$ given by Construction 1 has made a jump of type I_k , or equivalently that S_k is a jump time of type I_k . Denote

$$(3) \quad T^{(j)} = \inf\{S_k : k \geq 1, I_k = j\}$$

the first jump time of type j . Then, one example of application of Proposition 2 is the following result.

Proposition 4. *Let $\ell = \ell_1 + \ell_2$ with $\ell_1, \ell_2 \in \mathbb{N}^*$. Let $(X_t)_{t \geq 0}$ be a PDMP on \mathbf{M} with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ given by Construction 1 and initial distribution μ_0 . Define $T = \min\{T^{(i)} : i \in \llbracket \ell_1 + 1, \ell \rrbracket\}$, where $T^{(j)}$ is given by (3) for all $j \in \llbracket 1, \ell \rrbracket$. Then the cumulative distribution function of T is given for all $u \geq 0$ by*

$$(4) \quad \mathbb{P}(T \leq u) = \mathbb{P}\left(E < \sum_{i \in \llbracket \ell_1 + 1 \rrbracket}^{\ell} \int_0^{u \wedge \tau_\infty(Z)} \lambda_i(s, Z_s) ds\right),$$

where $(Z_t)_{t \geq 0}$ is a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell_1 \rrbracket})$ and initial distribution μ_0 , and E is a standard exponential random variable independent of $(Z_t)_{t \geq 0}$.

Proof. Let $(Y_t)_{t \geq 0}$ be a PDMP defined by Construction 2 with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and initial distribution μ_0 . Define similarly to $(X_t)_{t \geq 0}$ for all $j \in \llbracket 1, \ell \rrbracket$, $\tilde{T}^{(j)} = \inf\{\tilde{S}_k : k \geq 1, \tilde{I}_k = j\}$ and $\tilde{T} = \min\{\tilde{T}^{(i)} : i \in \llbracket \ell_1 + 1, \ell_2 \rrbracket\}$. Note that since by Proposition 2, $(X'_k, S_k, I_k)_{k \in \mathbb{N}}$ and $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ has the same distribution, T and \tilde{T} have the same distribution and it suffices to show that the cumulative distribution function of \tilde{T} is given by (4).

Let $(Z_t)_{t \geq 0}$ be a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell_1 \rrbracket})$ and initial distribution μ_0 defined by Construction 2 and based on the random variables $W_0, (E_{j,k}, U_{j,k})_{j \in \llbracket 1, \ell_1 \rrbracket, k \in \mathbb{N}}$, and let $(Z'_k, R_k, J_k)_{k \in \mathbb{N}}$ be the corresponding embedded chain. By construction, for all $t \leq \tilde{T} \wedge \tau_\infty(Z)$, $Y_t = Z_t$. In addition, define

$$N = \inf \left\{ k \in \mathbb{N}^* : \text{there exists } i \in \llbracket \ell_1 + 1, \ell \rrbracket, \tilde{I}_k = i \right\}.$$

By definition, $\tilde{T} = \tilde{S}_N$ on $\{\tilde{T} < +\infty\}$ and for all $k \in \mathbb{N}^*$, on $\{N = k\}$ for all $t \in [0, \tilde{S}_k]$, $Y_t = Z_t$, and for all $n \leq k$, $Y'_n = Z'_n$, $\tilde{S}_n = R_n$. Therefore, for all $k \in \mathbb{N}^*$, on $\{N \geq k\} \cap \{\tau_\infty(Z) \geq t\}$, for all $i \in \llbracket \ell_1 + 1, \ell \rrbracket$, we have by induction

$$\begin{aligned} \tilde{S}_{i,k} &= \inf \left\{ t > R_{k-1} : \tilde{H}_{i,k} < \int_{R_{k-1}}^t \lambda_i(s, \varphi_{R_{k-1}, s}(Z'_k)) ds \right\} \\ &= \inf \left\{ t > 0 : E_{i,0} < \int_0^t \lambda_i(s, Z_s) ds \right\}. \end{aligned}$$

Since $\{\tilde{T} < \tau_\infty(Z)\} \subset \{\tilde{T} < +\infty\}$, we thus obtain

$$\begin{aligned} \{\tilde{T} < \tau_\infty(Z)\} \cap \{\tilde{T} \geq t\} &= \{\tilde{T} < \tau_\infty(Z)\} \cap \left\{ \bigcap_{k \in \mathbb{N}^*, i \in \llbracket \ell_1 + 1, \ell \rrbracket} \{\tilde{S}_{i,k} \geq t\} \cap \{N = k\} \right\} \\ (5) \quad &= \{\tilde{T} < \tau_\infty(Z)\} \cap \left\{ \bigcap_{i \in \llbracket \ell_1 + 1, \ell \rrbracket} \left\{ E_{i,0} \geq \int_0^t \lambda_i(s, Z_s) ds \right\} \right\}. \end{aligned}$$

In addition, since $\{\tilde{T} = +\infty\} \subset \{\tau_\infty(Z) = \tau_\infty(Y)\} \subset \{\tilde{T} \geq \tau_\infty(Z)\}$, we have

$$\begin{aligned} \{\tilde{T} \geq \tau_\infty(Z) \vee t\} &= \{\tilde{T} = +\infty\} = \bigcap_{i \in \llbracket \ell_1 + 1, \ell \rrbracket, k \in \mathbb{N}^*} \{\tilde{S}_{i,k} \geq R_k\} \\ (6) \quad &= \bigcap_{i \in \llbracket \ell_1 + 1, \ell \rrbracket} \left\{ E_{i,0} \geq \int_0^{\tau_\infty(Z)} \lambda_i(s, Z_s) ds \right\}. \end{aligned}$$

Combining (5) and (6) and since $(E_{i,0})_{i \in \llbracket \ell_1 + 1, \ell_2 \rrbracket}$ is independent of $(Z_s)_{s \geq 0}$, we get

$$\begin{aligned} \mathbb{P}(\tilde{T} \geq t) &= \mathbb{P} \left(\bigcap_{i \in \llbracket \ell_1 + 1, \ell \rrbracket} \left\{ E_{i,0} \geq \int_0^{\tau_\infty(Z) \wedge t} \lambda_i(s, Z_s) ds \right\} \right) \\ &= \mathbb{E} \left[\exp \left(- \sum_{i=\ell_1+1}^{\ell} \int_0^{\tau_\infty(Z) \wedge t} \lambda_i(s, Z_s) ds \right) \right], \end{aligned}$$

which concludes the proof. \square

4. SUPERPOSITION AND SPLITTING OF JUMP MECHANISMS

We now introduce a tool to deal with PDMP: superposition and splitting of jump mechanisms.

Proposition 5. *Let $(\lambda_{i,1}, Q_{i,1})_{i \in \llbracket 1, \ell_1 \rrbracket}$ and $(\lambda_{i,2}, Q_{i,2})_{i \in \llbracket 1, \ell_2 \rrbracket}$ be two families of jump mechanisms on \mathbf{M} . Suppose that for all $(t, x, \mathbf{A}) \in \mathbb{R}_+ \times \mathbf{M} \times \mathcal{B}(\mathbf{M})$,*

$$(7) \quad \sum_{i=1}^{\ell_1} \lambda_{i,1}(t, x)(Q_{i,1}(t, x, \mathbf{A}) - \delta_x(\mathbf{A})) = \sum_{i=1}^{\ell_2} \lambda_{i,2}(t, x)(Q_{i,2}(t, x, \mathbf{A}) - \delta_x(\mathbf{A})).$$

Then, for all differential flow φ and initial distribution $\mu_0 \in \mathcal{P}(\mathbf{M})$,

$$\text{PDMP}(\varphi, (\lambda_{i,1}, Q_{i,1})_{i \in \llbracket 1, \ell_1 \rrbracket}, \mu_0) = \text{PDMP}(\varphi, (\lambda_{i,2}, Q_{i,2})_{i \in \llbracket 1, \ell_2 \rrbracket}, \mu_0).$$

If $(\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket}$ is a family of jump mechanisms, we define the associated minimal jump mechanism (λ_m, Q_m) and the associated total jump mechanism (λ_T, Q_T) for all $(t, x, \mathbf{A}) \in \mathbb{R}_+ \times \mathbf{M} \times \mathcal{B}(\mathbf{M})$, by

$$(8) \quad \begin{aligned} \lambda_m(t, x) &= \sum_{i=1}^{\ell} \lambda_i(t, x) Q_i(t, x, \mathbf{M} \setminus \{x\}) \\ Q_m(t, x, \mathbf{A}) &= \begin{cases} \lambda_m^{-1}(t, x) \sum_{i=1}^{\ell} \lambda_i(t, x) Q_i(t, x, \mathbf{A} \setminus \{x\}) & \text{if } \lambda_m(t, x) \neq 0, \\ \delta_x(\mathbf{A}) & \text{else.} \end{cases} \\ \lambda_T(t, x) &= \sum_{i=1}^{\ell} \lambda_i(t, x) \\ Q_T(t, x, \mathbf{A}) &= \begin{cases} \lambda_T^{-1}(t, x) \sum_{i=1}^{\ell} \lambda_i(t, x) Q_i(t, x, \mathbf{A}) & \text{if } \lambda_T(t, x) \neq 0, \\ \delta_x(\mathbf{A}) & \text{else.} \end{cases} \end{aligned}$$

The jump mechanism (λ_m, Q_m) is minimal in the sense that if $\lambda_m(t, x) \neq 0$, for $t \in \mathbb{R}_+$, $x \in \mathbf{M}$, then $Q_m(t, x, \{x\}) = 0$. As a consequence, if $(X_t)_{t \geq 0}$ is a PDMP with characteristics $(\varphi, \lambda_m, Q_m)$ and jump times S_k , $k \in \mathbb{N}$, then almost surely $X_{S_{k+1}} \neq \varphi_{S_k, S_{k+1}}(X_{S_k})$, and therefore $(X_t)_{t \geq 0}$ has no phantom jumps.

Since, for all $(t, x, \mathbf{A}) \in \mathbb{R}_+ \times \mathbf{M} \times \mathcal{B}(\mathbf{M})$ and $i \in \llbracket 1, n \rrbracket$,

$$Q_i(t, x, \mathbf{A} \setminus \{x\}) - \delta_x(\mathbf{A}) Q_i(t, x, \mathbf{M} \setminus \{x\}) = Q_i(t, x, \mathbf{A}) - \delta_x(\mathbf{A}),$$

the minimal jump mechanisms associated to $(\lambda_i^1, Q_i^1)_{i \in \llbracket 1, \ell_1 \rrbracket}$ and $(\lambda_i^2, Q_i^2)_{i \in \llbracket 1, \ell_2 \rrbracket}$ are equal if (7) holds. Therefore, the statement of Proposition 5 is equivalent to

$$\text{PDMP}(\varphi, (\lambda_i^1, Q_i^1)_{i \in \llbracket 1, \ell_1 \rrbracket}, \mu_0) = \text{PDMP}(\varphi, \lambda_T, Q_T, \mu_0) = \text{PDMP}(\varphi, \lambda_m, Q_m, \mu_0).$$

We first show the first equality in the following Lemma.

Lemma 6. *Let (λ_T, Q_T) be the total jump mechanism associated to $(\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket}$. Then, for all flow φ and $\mu_0 \in \mathcal{P}(\mathbf{M})$, $\text{PDMP}(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket}, \mu_0) = \text{PDMP}(\varphi, \lambda_T, Q_T, \mu_0)$.*

Proof. Let $(X_t)_{t \geq 0}$ be a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and initial distribution μ_0 defined by Construction 1, and $(X'_k, S_k)_{k \in \mathbb{N}}$ be its embedded chain. Since the process is completely determined by its embedded chain, and by the Markov property,

it is sufficient to prove that the Markov kernel of $(X'_k, S_k)_{k \in \mathbb{N}}$ is equal to the Markov kernel of the embedded chain associated to a PDMP with characteristics $(\varphi, \lambda_T, Q_T)$.

Summing out (1) over $j \in \llbracket 1, \ell \rrbracket$, we get for all $t \geq 0$ and $A \in \mathcal{B}(M)$

$$\begin{aligned} & \mathbb{P}(X'_{k+1} \in A, T_{k+1} \leq t \mid \mathcal{F}_k) \\ &= \mathbb{1}_M(X'_k) \int_{S_k} \sum_{j=1}^{\ell} Q_j(s, \varphi_{S_k, s}(X'_k), A) \lambda_j(s, \varphi_{S_k, s}(X'_k)) e^{-\sum_{i=1}^{\ell} \int_{S_k}^s \lambda_i(u, \varphi_{S_k, u}(X'_k)) du} ds \\ &= \mathbb{1}_M(X'_k) \int_{S_k}^t Q_T(s, \varphi_{S_k, s}(X'_k), A) \lambda_T(s, \varphi_{S_k, s}(X'_k)) e^{-\int_{S_k}^s \lambda_T(u, \varphi_{S_k, u}(X'_k)) du} ds, \end{aligned}$$

which concludes, since from (1) this is exactly the Markov kernel of the embedded chain associated to a PDMP with characteristics $(\varphi, \lambda_T, Q_T)$. \square

Before showing Proposition 5, we need the following technical lemma. In the following, we denote by Id the Identity Markov kernel, defined by $\text{Id}(t, x, A) = \delta_x(A)$ for all $t \geq 0$, $x \in M$, $A \in \mathcal{B}(M)$. The following lemma gives a rigorous proof of the intuitive idea that adding phantom jumps does not change the distribution of the process.

Lemma 7. *For any characteristics (φ, λ, Q) , jump rate $\lambda' : M \rightarrow \mathbb{R}_+$ and $\mu_0 \in \mathcal{P}(M)$,*

$$\text{PDMP}(\varphi, \lambda, Q, \mu_0) = \text{PDMP}(\varphi, \{(\lambda, Q), (\lambda', \text{Id})\}, \mu_0).$$

Proof. We consider $(Y_t)_{t \geq 0}$ a PDMP with characteristics $(\varphi, \{(\lambda, Q), (\lambda', \text{Id})\})$ and initial distribution μ_0 defined from random variables W_0 and $(E_{j,k}, U_{j,k})_{j \in \{1,2\}, k \in \mathbb{N}}$ by Construction 2, and its embedded chain $(Y'_k, \tilde{S}_k, j_k)_{k \in \mathbb{N}}$. Let $R_0 = 0$ and, for $k \geq 1$, let $R_k = \inf\{\tilde{S}_i > R_{k-1} : i \in \mathbb{N}, \tilde{I}_i = 1\}$ be the k^{th} jump of type 1 (i.e. associated with the jump mechanism (λ, Q)). For $i \geq 1$ such that $\tilde{I}_i = 2$, the i^{th} jump is a phantom one, i.e. $Y_{\tilde{S}_i} = \varphi_{\tilde{S}_{i-1}, \tilde{S}_i}(Y_{\tilde{S}_{i-1}})$. By the flow property $\varphi_{s,u} \circ \varphi_{t,s} = \varphi_{t,u}$, this implies that $Y_t = \varphi_{R_k, t}(Y_{R_k})$ for all $k \in \mathbb{N}$ and $t \in [R_k, R_{k+1} \wedge \tau_{\infty}(Y))$.

If $k \in \mathbb{N}$ is such that $R_k < \infty$, then $\{\varphi_{R_k, t}(Y_{R_k}) : t \in [R_k, (1 + R_k) \wedge R_{k+1}]\}$ is a compact set of M , on which λ' is bounded (as a locally bounded function). Hence, there cannot be an infinite number of jump of second type between times R_k and $(1 + R_k) \wedge R_{k+1}$. In particular, necessarily, $\sup\{R_k : k \in \mathbb{N}\} = \sup\{\tilde{S}_k : k \in \mathbb{N}\} = \tau_{\infty}(Y)$. As a consequence, $Y_t = \varphi_{R_k, t}(Y_{R_k})$ holds for all $k \in \mathbb{N}$ and $t \in [R_k, R_{k+1})$ on $\{R_k < +\infty\}$.

Define for all $k \in \mathbb{N}^*$, $N_k = \inf\{i > N_{k-1} : \tilde{I}_i = 1\}$, setting $N_0 = 0$. Construction 2 is such that, then, for all $k \in \mathbb{N}$, on $\{R_k < +\infty\} \cap \{N_{k+1} < +\infty\}$,

$$R_{k+1} = \inf \left\{ t \geq R_k : E_{1, k+1} < \int_{R_k}^t \lambda(s, \varphi_{R_k, s}(Y_{R_k})) ds \right\}.$$

In addition, for all $k \in \mathbb{N}$,

$$\begin{aligned} & \{R_k < +\infty\} \cap \{N_{k+1} = +\infty\} \cap \{R_k = +\infty\} \\ &= \{R_k < +\infty\} \cap \{N_{k+1} = +\infty\} \cap \left\{ \bigcap_{i \geq N_k} \left\{ \tilde{H}_{1, i+1} > \int_{S_i}^t \lambda(s, \varphi_{S_i, s}(Y_{S_i})) ds \right\} \right\} \\ &= \{R_k < +\infty\} \cap \{N_{k+1} = +\infty\} \cap \left\{ E_{1, k+1} \geq \int_{R_k}^{+\infty} \lambda(s, \varphi_{R_k, s}(Y_{R_k})) ds \right\}. \end{aligned}$$

Therefore, on $\{R_k < +\infty\} \cap \{N_{k+1} = +\infty\}$,

$$R_{k+1} = +\infty = \inf \left\{ t \geq R_k : E_{1,k+1} < \int_{R_k}^t \lambda(s, \varphi_{R_k,s}(Y_{R_k})) ds \right\},$$

and, if $R_{k+1} < \infty$, $Y_{R_{k+1}} = \mathbf{G}_1(R_{k+1}, \varphi_{R_k, R_{k+1}}(Y_{R_k}), U_{1,k})$.

Therefore, denoting $Z'_k = Y_{R_k}$ for all $k \in \mathbb{N}$, then (Z'_k, R_k) is the embedded chain associated to a PDMP $(Z_t)_{t \geq 0}$ with characteristics (φ, λ, Q) and constructed with the random variables W_0 and $(E_{1,k}, U_{1,k})_{k \geq 0}$ (through either Construction 1 or 2, since there is only one jump mechanism so that both coincides). For all $k \in \mathbb{N}$ and $t \in [R_k, R_{k+1})$, $Z_t = \varphi_{R_k,t}(Z'_k) = Y_t$, which concludes. \square

Proof of Proposition 5. As previously mentioned, to show Proposition 5, it is sufficient to prove that, for all differential flow φ and initial distribution μ_0 ,

$$\text{PDMP}(\varphi, (\lambda_i^1, Q_i^1)_{i \in \llbracket 1, \ell_1 \rrbracket}, \mu_0) = \text{PDMP}(\varphi, \lambda_{\mathbb{T}}, Q_{\mathbb{T}}, \mu_0) = \text{PDMP}(\varphi, \lambda_{\mathbb{m}}, Q_{\mathbb{m}}, \mu_0)$$

holds, where $(\lambda_{\mathbb{T}}, Q_{\mathbb{T}})$ and $(\lambda_{\mathbb{m}}, Q_{\mathbb{m}})$ are the total and minimal jump mechanism respectively associated to both $(\lambda_i^1, Q_i^1)_{i \in \llbracket 1, \ell_1 \rrbracket}$ and $(\lambda_i^2, Q_i^2)_{i \in \llbracket 1, \ell_2 \rrbracket}$ defined by (8). The first identity is given by Lemma 6, therefore it remains to show the second one.

By Lemma 7, since $\lambda_{\mathbb{T}} - \lambda_{\mathbb{m}}$ is by definition a jump rate (*i.e.* a positive and locally bounded measurable function), we get for all differential flow φ and initial distribution μ_0 that

$$\text{PDMP}(\varphi, \lambda_{\mathbb{m}}, Q_{\mathbb{m}}, \mu_0) = \text{PDMP}(\varphi, \{(\lambda_{\mathbb{m}}, Q_{\mathbb{m}}), (\lambda_{\mathbb{T}} - \lambda_{\mathbb{m}}, \text{Id})\}, \mu_0).$$

The proof is concluded upon noting that the total jump mechanism associated with $\{(\lambda_{\mathbb{m}}, Q_{\mathbb{m}}), (\lambda_{\mathbb{T}} - \lambda_{\mathbb{m}}, \text{Id})\}$ is equal to $(\lambda_{\mathbb{T}}, Q_{\mathbb{T}})$ and using Lemma 6 again. \square

Example - Bouncy Particle Sampler. By Proposition 5, the BPS process defined in Example 1 is a PDMP with characteristics (φ, λ, Q) , where for all $t \geq 0$, $(x, y) \in \mathbb{R}^d \times \mathbb{Y}$, and $A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{Y})$, $\varphi_t(x, y) = (x + ty, y)$,

$$(9) \quad \lambda(x, y) = \langle \nabla U(x), y \rangle_+ + \lambda_c,$$

and

$$Q((x, y), A) = \lambda^{-1}(x, y) \{ \langle \nabla U(x), y \rangle_+ \delta_{(x, R(x, y))}(A) + \lambda_c (\delta_x \otimes \mu_{\mathbb{V}})(A) \},$$

where R is defined in (2) with $g = \nabla U$.

5. NON-EXPLOSION

It is generally easier to prove that a given particular PDMP is non-explosive than to provide good general conditions that ensure non-explosion for PDMPs. Nevertheless, we give here two results on that topic that will prove useful in the rest of this work, and may be of interest in other situations.

Proposition 8. *Let $(X_t)_{t \geq 0}$ be a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and initial distribution μ_0 be given by Construction 1 for some random variables W_0 and $((E_{j,k})_{j \in \llbracket 1, \ell \rrbracket}, U_k)_{k \in \mathbb{N}}$. For all $M > 0$, let $(X_t^M)_{t \geq 0}$ be a PDMP with characteristics $(\varphi, (M \wedge \lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and initial distribution μ_0 be given by Construction 1 for the*

same random variables W_0 and $((E_{j,k})_{j \in \llbracket 1, \ell \rrbracket}, U_k)_{k \in \mathbb{N}}$ as $(X_t)_{t \geq 0}$. The process $(X_t)_{t \geq 0}$ is non-explosive if and only if, for all $t \geq 0$,

$$\lim_{M \rightarrow +\infty} \mathbb{P}(X_s = X_s^M, \text{ for all } s \in [0, t]) = 1.$$

Proof. Suppose that $(X_t)_{t \geq 0}$ is non-explosive. Then for almost all $\omega \in \Omega$, the process only jumps a finite number of time between times 0 and t , so that $\{X_s : s \in [0, t]\}$ is a compact set of \mathbb{M} . Since the rate jumps are locally bounded, $\lambda_*(\omega) = \sup\{\sum_{j=1}^{\ell} \lambda_j(s, X_s) : s \in [0, t]\}$ is finite for almost all $\omega \in \Omega$. We get for all $\omega \in \Omega$ and $M > \lambda_*(\omega)$, by definition that $\sup_{s \in [0, t]} |X_s - X_s^M| = 0$ and therefore, for almost all $\omega \in \Omega$, $\lim_{M \rightarrow +\infty} \mathbb{1}_{\{0\}}(\sup_{s \in [0, t]} |X_s - X_s^M|) = 1$. Thus, since $\{X_s = X_s^M, \text{ for all } s \in [0, t]\} = \{\sup_{s \in [0, t]} |X_s - X_s^M| = 0\}$, the proof is concluded using the Lebesgue dominated convergence theorem.

Now, to prove the converse, remark that $(X_t^M)_{t \geq 0}$ is non-explosive for all $M > 0$, since its jump rates are bounded. In particular, $X_t^M \in \mathbb{M}$ for all $t \geq 0$, so that

$$\mathbb{P}(\tau_\infty(X) > t) = \mathbb{P}(X_t \in \mathbb{M}) \geq \mathbb{P}(X_t = X_t^M)$$

for all $M > 0$. The conclusion then follows taking $M \rightarrow +\infty$. \square

In particular, Proposition 8 implies that, if $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ is non explosive, denoting by $(P_{s,t})_{t \geq s \geq 0}$ and $(P_{s,t}^M)_{t \geq s \geq 0}$ the Markov semi-group associated to characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and $(\varphi, (M \wedge \lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$, $M > 0$ then, since for all $\mu_0 \in \mathcal{P}(\mathbb{M})$ and all $t \geq 0$,

$$\|\mu_0 P_{0,t} - \mu_0 P_{0,t}^M\|_{\text{TV}} = \sup_{A \in \mathcal{B}(\mathbb{M})} |\mathbb{P}(X_t \in A) - \mathbb{P}(X_t^M \in A)| \leq 2\mathbb{P}(X_t \neq X_t^M),$$

we get $\lim_{M \rightarrow +\infty} \|\mu_0 P_{0,t} - \mu_0 P_{0,t}^M\|_{\text{TV}} = 0$.

The second result concerning non-explosion of PDMPs is the following:

Proposition 9. *Let $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ be homogeneous characteristics with $\ell > 1$. Assume that the characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell-1 \rrbracket})$ are non-explosive and $\|\lambda_\ell\|_\infty < +\infty$. Then $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ are non-explosive as well.*

Proof. Let $\mu_0 \in \mathcal{P}(\mathbb{M})$ and $(Y_t)_{t \geq 0}$ be a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell \rrbracket})$ and initial distribution μ_0 given by Construction 2 based on random variables W_0 and $(E_{j,k}, U_{j,k})_{j \in \llbracket 1, \ell \rrbracket, k \in \mathbb{N}}$. Let $(Y'_k, \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ be the corresponding embedded chain. First, consider the decomposition

$$(10) \quad \mathbb{P}\left(\sup_{n \in \mathbb{N}} \tilde{S}_n < +\infty\right) = \mathbb{P}\left(\sup_{n \in \mathbb{N}} \tilde{S}_n < +\infty, \sup_{n \in \mathbb{N}^*} \tilde{N}_{\ell, n} = +\infty\right) \\ + \mathbb{P}\left(\sup_{n \in \mathbb{N}} \tilde{S}_n < +\infty, \sup_{n \in \mathbb{N}^*} \tilde{N}_{\ell, n} < +\infty\right).$$

Let us show that both terms of the right-hand-side are equal to 0.

Define recursively $(N_n^{(\ell)})_{n \in \mathbb{N}}$ by $N_0^{(\ell)} = 0$ and for all $n \in \mathbb{N}$,

$$N_{n+1}^{(\ell)} = \inf \left\{ k > N_n^{(\ell)} : \tilde{I}_k = \ell \right\}.$$

Note that

$$(11) \quad N_n^{(\ell)} = k \text{ if and only if } \tilde{N}_{\ell,k} = n .$$

Then, on $\{\sup_{n \in \mathbb{N}^*} \tilde{N}_{\ell,n} = +\infty\}$, for all $n \in \mathbb{N}$, $N_n^{(\ell)} < +\infty$ almost surely. Hence, on $\{\sup_{n \in \mathbb{N}^*} \tilde{N}_{n,\ell} = +\infty\}$, for all $n \in \mathbb{N}^*$, by definition we have almost surely

$$\sup_{k \in \mathbb{N}^*} \tilde{S}_k \geq \sum_{k=1}^n \left\{ \tilde{S}_{N_k^{(\ell)}} - \tilde{S}_{N_{k-1}^{(\ell)}} \right\} \geq \sum_{i=1}^n (E_{\ell,i} / \|\lambda_\ell\|_\infty) ,$$

where the last inequality follows from the bound on $\{N_k^{(\ell)} < +\infty\}$,

$$\begin{aligned} \tilde{S}_{N_k^{(\ell)}} &= \tilde{S}_{\ell, N_k^{(\ell)}} = \inf \left\{ t > \tilde{S}_{N_{k-1}^{(\ell)}} : H_{\ell, N_k^{(\ell)}} < \int_{\tilde{S}_{N_{k-1}^{(\ell)}}}^t \lambda_\ell(X_s) ds \right\} \\ &= \inf \left\{ t > \tilde{S}_{N_{k-1}^{(\ell)}} : E_{\ell,k} < \int_{\tilde{S}_{N_{k-1}^{(\ell)}}}^t \lambda_\ell(X_s) ds \right\} \geq \tilde{S}_{N_{k-1}^{(\ell)}} + E_{\ell,k} / \|\lambda_\ell\|_\infty . \end{aligned}$$

Therefore by the law of large number,

$$\mathbb{P} \left(\sup_{n \in \mathbb{N}} \tilde{S}_n < +\infty, \sup_{n \in \mathbb{N}^*} \tilde{N}_{\ell,n} = +\infty \right) \leq \inf_{n \in \mathbb{N}^*} \mathbb{P} \left(\sum_{i=1}^n (E_{\ell,i} / \|\lambda_\ell\|_\infty) < +\infty \right) = 0 .$$

We bound now the second term in (10). Let $k \in \mathbb{N}^*$. Note that by Construction 2, on $\{\sup_{n \in \mathbb{N}} \tilde{N}_{\ell,n} = k\}$, for all $i \in \llbracket 1, k-1 \rrbracket$, $\{X_t : t \in [\tilde{S}_{N_i^{(\ell)}}, \tilde{S}_{N_{i+1}^{(\ell)}}]\}$ is a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell-1 \rrbracket})$ with initial data $X_{\tilde{S}_{N_i^{(\ell)}}}$. Then, by (11), the Markov property and an immediate induction using that $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell-1 \rrbracket})$ is non explosive, we have that on $\{\sup_{n \in \mathbb{N}^*} \tilde{N}_{\ell,n} = k\}$, almost surely $X_{\tilde{S}_{N_i^{(\ell)}}} \in \mathbb{M}$, for $i \in \llbracket 1, k \rrbracket$. The proof is then concluded since, using that $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell-1 \rrbracket})$ is non explosive, Construction 2 and the Markov property again on $\{\sup_{n \in \mathbb{N}} \tilde{N}_{\ell,n} = k\}$, $\{X_t : t \in [\tilde{S}_{N_k^{(\ell)}}, +\infty)\}$ is a PDMP with characteristics $(\varphi, (\lambda_i, Q_i)_{i \in \llbracket 1, \ell-1 \rrbracket})$ started at $X_{\tilde{S}_{N_k^{(\ell)}}}$. \square

Let us come back to our main example.

Example - Bouncy Particle Sampler.

Proposition 10. *The BPS process defined in Example 1 is non-explosive for any initial distribution.*

Proof. Using the notations of Example 1, consider $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ the PDMP on $\mathbb{R}^d \times \mathbb{Y}$ with characteristics $(\varphi, \lambda_1, Q_1)$ and initial condition $(\bar{X}_0, \bar{Y}_0) = (x, y) \in \mathbb{M}$ defined by Construction 1 from an i.i.d. sequence of random variables $(E_k, U_k)_{k \in \mathbb{N}}$, associated with the sequence of jump times $(S_k)_{k \in \mathbb{N}}$. Note that almost surely $\|\bar{Y}_t\| = \|y\|$ for all $t \in [0, \sup_{n \in \mathbb{N}} S_n)$, so that $\|\bar{X}_t - x\| \leq t \|y\|$ and

$$\lambda_1(\bar{X}_t, \bar{Y}_t) \leq C(t) = \|y\| \sup \left\{ \|\nabla U(x')\| : x' \in \mathbb{R}^d, \|x' - x\| \leq t \|y\| \right\} .$$

Therefore on $\{\sup_{n \in \mathbb{N}} S_n < +\infty\}$, by Construction 1 almost surely there exists $C \geq 0$ such that

$$S_{n+1} - S_n \geq E_n / (C + 1) \text{ for all } n \in \mathbb{N}^* .$$

Then, we have on $\{\sup_{n \in \mathbb{N}} S_n < +\infty\}$, that almost surely there exists $C \geq 0$ such that

$$+\infty > \sup_{n \in \mathbb{N}} S_n = \sum_{n \in \mathbb{N}} \{S_{n+1} - S_n\} \geq \sum_{n \in \mathbb{N}^*} \{E_n / (C + 1)\} .$$

As a result,

$$\mathbb{P} \left(\sup_{n \in \mathbb{N}} S_n < +\infty \right) \leq \mathbb{P} \left(\bigcup_{\ell \in \mathbb{N}^*} \left\{ \sum_{n \in \mathbb{N}^*} E_n / (\ell + 1) < +\infty \right\} \right) \leq 0 .$$

It follows that the PDMP with characteristics $(\varphi, \lambda_1, Q_1)$ is non-explosive. By Proposition 9, the BPS is non-explosive. \square

6. COMPARISON OF PDMP VIA SYNCHRONOUS COUPLING

The result of this section is crucial in many aspects: first, it gives stability estimates with respect to the jump rates and the underlying Markov kernel for a modification of a PDMP for example for an approximate thinning procedure. Second, it will be an essential tools to verify assumptions for the BPS in the following sections. The main goal of this section is to prove the following:

Proposition 11. *Let $(P_{s,t}^1)_{t \geq s \geq 0}$ and $(P_{s,t}^2)_{t \geq s \geq 0}$ be two non-explosive PDMP semigroups with characteristics $(\varphi, \lambda_1, Q_1)$ and $(\varphi, \lambda_2, Q_2)$ respectively. Suppose that there exists a measurable $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying for all $t \geq 0$,*

$$(12) \quad g(t) \geq \sup_{\substack{x \in \mathbb{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbb{M})}} \{ \lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})| \} + \sup_{x \in \mathbb{M}} | \lambda_1(t, x) - \lambda_2(t, x) |$$

or alternatively

$$(13) \quad g(t) \geq \sup_{\substack{x \in \mathbb{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbb{M})}} | \lambda_1(t, x)(Q_1(t, x, \mathbf{A}) - \delta_x(\mathbf{A})) - \lambda_2(t, x)(Q_2(t, x, \mathbf{A}) - \delta_x(\mathbf{A})) | .$$

Then for all $t \geq 0$ and $x \in \mathbb{M}$,

$$\| \delta_x P_{0,t}^1 - \delta_x P_{0,t}^2 \|_{\text{TV}} \leq 2 \left\{ 1 - \exp \left(- \int_0^t g(s) ds \right) \right\} .$$

Remark 12. *Note that if $(\lambda_i)_{i=1,2}$ and $(Q_i)_{i=1,2}$ are two locally bounded jump rates and Markov kernels respectively, then*

$$(14) \quad \sup_{\substack{x \in \mathbb{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbb{M})}} \{ \lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})| \} \\ \leq \sup_{\substack{x \in \mathbb{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbb{M})}} \{ | \lambda_1(t, x) Q_1(t, x, \mathbf{A}) - \lambda_2(t, x) Q_2(t, x, \mathbf{A}) | \} .$$

Indeed, let $x \in \mathbf{M}$ and $\mathbf{A} \in \mathcal{B}(\mathbf{M})$. Without loss of generality, we can assume that $\lambda_1(t, x) \geq \lambda_2(t, x)$. If $Q_1(t, x, \mathbf{A}) \geq Q_2(t, x, \mathbf{A})$, then we have

$$\begin{aligned} & \lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})| \\ & \leq \lambda_1(t, x)Q_1(t, x, \mathbf{A}) - \lambda_2(t, x)Q_2(t, x, \mathbf{A}) = |\lambda_1(t, x)Q_1(t, x, \mathbf{A}) - \lambda_2(t, x)Q_2(t, x, \mathbf{A})| . \end{aligned}$$

Otherwise $Q_1(t, x, \mathbf{A}^c) \geq Q_2(t, x, \mathbf{A}^c)$ and we get

$$\lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})| \leq |\lambda_1(t, x)Q_1(t, x, \mathbf{A}^c) - \lambda_2(t, x)Q_2(t, x, \mathbf{A}^c)| .$$

Therefore, for all $x \in \mathbf{M}$ and $\mathbf{A} \in \mathcal{B}(\mathbf{M})$,

$$\begin{aligned} & \lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})| \\ & \leq \sup_{\tilde{\mathbf{A}} \in \mathcal{B}(\mathbf{M})} \left| \lambda_1(t, x)Q_1(t, x, \tilde{\mathbf{A}}) - \lambda_2(t, x)Q_2(t, x, \tilde{\mathbf{A}}) \right| , \end{aligned}$$

which implies (14). Therefore, to establish that (12), it is sufficient to show that there exists a measurable function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$,

$$(15) \quad g(t) \geq 2 \sup_{\substack{x \in \mathbf{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbf{M})}} \{|\lambda_1(t, x)Q_1(t, x, \mathbf{A}) - \lambda_2(t, x)Q_2(t, x, \mathbf{A})|\} ,$$

Conversely, we easily get for all $t \in \mathbb{R}_+$,

$$(16) \quad \begin{aligned} & \sup_{\substack{x \in \mathbf{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbf{M})}} \{|\lambda_1(t, x)Q_1(t, x, \mathbf{A}) - \lambda_2(t, x)Q_2(t, x, \mathbf{A})|\} \\ & \leq \sup_{\substack{x \in \mathbf{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbf{M})}} \{\lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})|\} + \sup_{x \in \mathbf{M}} |\lambda_1(t, x) - \lambda_2(t, x)| . \end{aligned}$$

Therefore (12) and (15) are essentially equivalent up to a factor 2.

The proof of Proposition 11 relies on the construction of a Markovian synchronous coupling between $(P_{s,t}^1)_{t \geq s \geq 0}$ and $(P_{s,t}^2)_{t \geq s \geq 0}$. More precisely, we want to construct a PDMP $(X_t, Y_t)_{t \geq 0}$ on \mathbf{M}^2 starting from $(x, y) \in \mathbf{M}^2$ such that the distributions of $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are PDMP $(\varphi, \lambda^1, Q^1, \delta_x)$ and PDMP $(\varphi, \lambda^2, Q^2, \delta_y)$ respectively. In the case where $x = y$, the synchronous coupling attempts to keep $X_s = Y_s$ for all $s \in [0, t]$ by ensuring that, as much as possible, both processes jump at the same time and, when they do, jump as much as possible to the same point. Let us give its precise definition.

First, by [25, Corollary 5.22], there exists a jump kernel K_0 on \mathbf{M}^2 , such that for all $t \in \mathbb{R}_+$ and $(x, y) \in \mathbf{M}^2$, $K_0(t, (x, y), \cdot)$ is an optimal transference plane of $Q_1(t, x, \cdot)$ and $Q_2(t, y, \cdot)$, where optimal means that

$$(17) \quad 2K_0(t, (x, y), \Delta_{\mathbf{M}}^c) = \|Q_1(t, x, \cdot) - Q_2(t, y, \cdot)\|_{\text{TV}} .$$

Define, for $i = 0, 1, 2$ and $j = 1, 2$ the jump rate r_i and the jump kernel K_j on \mathbf{M}^2 as follows: for $t \in \mathbb{R}_+$, $(x, y) \in \mathbf{M}^2$ and $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathbf{M})$,

$$(18) \quad \begin{aligned} & r_0(t, (x, y)) = \lambda_1(t, x) \wedge \lambda_2(t, y) , \\ & r_1(t, (x, y)) = (\lambda_1(t, x) - \lambda_2(t, y))_+ , \quad K_1(t, (x, y), \mathbf{A} \times \mathbf{B}) = Q_1(t, x, \mathbf{A})\delta_y(\mathbf{B}) , \\ & r_2(t, (x, y)) = (\lambda_2(t, x) - \lambda_1(t, y))_+ , \quad K_2(t, (x, y), \mathbf{A} \times \mathbf{B}) = \delta_x(\mathbf{A})Q_2(t, y, \mathbf{B}) . \end{aligned}$$

Let φ^\otimes be the flow on M^2 defined for all $t \in \mathbb{R}_+$ and $(x, y) \in M^2$ by

$$\varphi^\otimes(t, (x, y)) = (\varphi(t, x), \varphi(t, y)) .$$

Lemma 13. *Let $(x, y) \in M^2$ and $(X_t, Y_t)_{t \geq 0}$ be a PDMP on M^2 with initial distribution $\delta_{(x,y)}$ and characteristics $(\varphi^\otimes, (r_i, K_i)_{i \in \llbracket 0, 2 \rrbracket})$. Suppose that it is non explosive. Then $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ have distributions PDMP($\varphi, \lambda^1, Q^1, \delta_x$) and PDMP($\varphi, \lambda^2, Q^2, \delta_y$) respectively.*

As a consequence, $(X_t, Y_t)_{t \geq 0}$ is referred to as a synchronous coupling of $(\delta_x P_{0,t}^1)_{t \geq 0}$ and $(\delta_x P_{0,t}^2)_{t \geq 0}$.

Proof. We only show the result for $(X_t)_{t \geq 0}$, the case for $(Y_t)_{t \geq 0}$ being similar. Consider the Markov kernel on $M^2 \times \mathcal{B}(M^2)$ defined for all $(x, y) \in M^2$ and $A \in \mathcal{B}(M^2)$ by

$$\tilde{K}((x, y), A) = \begin{cases} \frac{r_0(x,y)K_0((x,y),A)+r_1(x,y)K_1((x,y),A)}{r_0(x,y)+r_1(x,y)} & \text{if } r_0(x, y) + r_1(x, y) \neq 0 \\ \delta_{(x,y)}(A) & \text{otherwise .} \end{cases}$$

and let $(X_t, Y_t)_{t \geq 0}$ be a PDMP with characteristics $(\varphi^\otimes, r_0 + r_1, \tilde{K}, r_2, K_2)$ with initial distribution $\delta_{(x,y)}$ defined by Construction 2 based on some random variables $(E_{j,k}, U_{j,k})_{j \in \llbracket 1, 2 \rrbracket, k \in \mathbb{N}}$. By Proposition 5, PDMP($\varphi^\otimes, r_0 + r_1, \tilde{K}, r_2, K_2, \delta_x \otimes \delta_y$) = PDMP($\varphi^\otimes, (r_i, K_i)_{i \in \llbracket 0, 2 \rrbracket}, \delta_x \otimes \delta_y$). Therefore, by definition, it suffices to show that $(X_t)_{t \geq 0}$ is distributed according to PDMP($\varphi, \lambda^1, Q^1, \delta_x$). Let $((X'_k, Y'_k), \tilde{S}_k, \tilde{I}_k)_{k \in \mathbb{N}}$ be the embedded chain associated with $(X_t, Y_t)_{t \geq 0}$. Set $R_0 = 0$ and, for $k \in \mathbb{N}$,

$$R_{k+1} = \inf \left\{ \tilde{S}_i > R_k : i \in \mathbb{N}, \tilde{I}_i = 1 \right\} .$$

The process being non-explosive, $\sup_{n \in \mathbb{N}} \tilde{S}_n = +\infty$, so that $\sup_{n \in \mathbb{N}} R_n = +\infty$. For all $k \in \mathbb{N}$, set $\bar{X}_k = X_{R_k}$ if $R_k < \infty$ and $\bar{X}_k = \infty$ otherwise.

By definition of \tilde{K}_2 and φ^\otimes , for all $k \in \mathbb{N}$ such that $R_k < \infty$ and all $t \in [R_k, R_{k+1})$, then $X_t = \varphi_{R_k, t}(\bar{X}_k)$. Moreover, similarly to the proof of Lemma 7, for all $t \geq 0$ and $(x, y) \in M^2$, $r_0(t, (x, y)) + r_1(t, (x, y)) = \lambda_1(t, x)$, so that

$$(19) \quad R_{k+1} = \inf \left\{ t \geq R_k : E_{1, k+1} < \int_{R_k}^t \lambda_1(s, \varphi_{R_k, s}(\bar{X}_k)) ds \right\} ,$$

and, if $R_{k+1} < \infty$, denoting \mathbf{G}_1 the representation of \tilde{K} used in the construction of the process,

$$(20) \quad (X_{R_{k+1}}, Y_{R_{k+1}}) = \mathbf{G}_1(R_{k+1}, (\varphi_{R_k, R_{k+1}}(\bar{X}_k), \bar{Y}_{k+1}), U_{1, k+1}) ,$$

where $\bar{Y}_{k+1} = \varphi_{\tilde{S}_{i-1}, \tilde{S}_i}(Y_{\tilde{S}_{i-1}})$ with $i \in \mathbb{N}^*$ such that $R_{k+1} = \tilde{S}_i$. Now, for all $k \in \mathbb{N}$, $t \geq 0$, $(x, y) \in M^2$ and $A \in \mathcal{B}(M)$,

$$(21) \quad \begin{aligned} \lambda_1(t, x) \mathbb{P}(\mathbf{G}_1(t, (x, y), U_{1, k+1}) \in A \times M) \\ &= r_0(t, (x, y)) K_0(t, (x, y), A \times M) + r_1(t, (x, y)) K_1(t, (x, y), A \times M) \\ &= (r_0(t, (x, y)) + r_1(t, (x, y))) Q_1(t, x, A) = \lambda_1(t, x) Q_1(t, x, A) . \end{aligned}$$

Let $(\mathcal{F}_k)_{k \in \mathbb{N}}$ be the filtration associated with $(\bar{X}_k, R_k)_{k \in \mathbb{N}}$. In particular, $E_{1,k+1}$ and $U_{1,k+1}$ are independent of \mathcal{F}_k , so that (19), (20) and (21) yield, for all $k \in \mathbb{N}$, $t \geq R_k$ and $\mathbf{A} \in \mathcal{B}(\mathbf{M})$,

$$\begin{aligned} & \mathbb{P}(\bar{X}_{k+1} \in \mathbf{A}, R_{k+1} \leq t \mid \mathcal{F}_k) \\ &= \mathbb{1}_{\mathbf{M}}(\bar{X}_k) \int_{R_k}^t \mathbb{P}(\mathbf{G}_1(s, (\bar{X}_{k+1}, \bar{Y}_{k+1}), U_{1,k+1}) \in \mathbf{A} \mid \mathcal{F}_k \vee \sigma(U_{1,k+1})) \\ & \quad \lambda_1(s, \varphi_{R_k, s}(\bar{X}_k)) \exp\left\{-\int_{R_k}^s \lambda_1(u, \varphi_{R_k, u}(\bar{X}_k)) du\right\} ds \\ &= \mathbb{1}_{\mathbf{M}}(\bar{X}_k) \int_{R_k}^t Q_1(s, x, \mathbf{A}) \lambda_1(s, \varphi_{R_k, s}(\bar{X}_k)) \exp\left\{-\int_{R_k}^s \lambda_1(u, \varphi_{R_k, u}(\bar{X}_k)) du\right\} ds. \end{aligned}$$

As a consequence, $(\bar{X}_k, R_k)_{k \in \mathbb{N}}$ is the embedded chain associated to a PDMP with characteristics $(\varphi, \lambda_1, Q_1)$. The fact that $X_t = \varphi_{R_k, t}(\bar{X}_k)$ for all $k \in \mathbb{N}$ such that $R_k < \infty$ and all $t \in [R_k, R_{k+1}[$ concludes the proof. \square

Proof of Proposition 11. The proof is divided in two main steps. In the first one, we assume that λ^1 and λ^2 are uniformly bounded and the second one is the extension of the first result in the case where the jump rates are not bounded.

(1) Assume for the moment that $\|\lambda^i\|_\infty < M$, for $i = 1, 2$ and some $M > 0$. Let $(P_{s,t}^1)_{t \geq s \geq 0}$ and $(P_{s,t}^2)_{t \geq s \geq 0}$ be two PDMPs semigroups with characteristics $(\varphi, Q_1, \lambda_1)$ and $(\varphi, Q_2, \lambda_2)$ respectively. Since λ^1 and λ^2 are uniformly bounded, the synchronous characteristics $(\varphi, (r_i, K_i)_{i \in [0, 2]})$, where $(r_i, K_i)_{i \in [0, 2]}$ is defined in (17) and (18), are non explosive. From Lemma 13, the synchronous coupling defined above is a Markov coupling between these two semigroups. Then, by characterisation of the total variation distance by couplings, to get an estimate on $\|\delta_x P_{0,t}^1 - \delta_x P_{0,t}^2\|_{\text{TV}}$ for $t \geq 0$, we just need to bound the probability that this coupling stay equal on $[0, t]$ if it starts from $(x, x) \in \mathbf{M}^2$. However, to do so, we consider different characteristics from $(\varphi^\otimes, (r_i, K_i)_{i \in [0, 2]})$.

Define the Markov kernels, for all $(x, y) \in \mathbf{M}^2$ and $(\mathbf{A}, \mathbf{B}) \in \mathcal{B}(\mathbf{M})^2$, dropping the subscript \mathbf{M} for the diagonal Δ

$$\begin{aligned} K_{0, \Delta}((x, y), \mathbf{A} \times \mathbf{B}) &= \begin{cases} \frac{K_0((x, y), \mathbf{A} \times \mathbf{B} \cap \Delta)}{K_0((x, y), \Delta)} & \text{if } K_0((x, y), \Delta) \neq 0 \\ \delta_{(x, y)}(\mathbf{A} \times \mathbf{B}) & \text{otherwise,} \end{cases} \\ K_{0, \neq}((x, y), \mathbf{A} \times \mathbf{B}) &= \begin{cases} \frac{K_0((x, y), \mathbf{A} \times \mathbf{B} \cap \Delta^c)}{K_0((x, y), \Delta^c)} & \text{if } K_0((x, y), \Delta^c) \neq 0 \\ \delta_{(x, y)}(\mathbf{A} \times \mathbf{B}) & \text{otherwise,} \end{cases} \\ K_{1, \Delta}((x, y), \mathbf{A} \times \mathbf{B}) &= \begin{cases} \frac{K_1((x, y), \mathbf{A} \cap \{x\} \times \mathbf{B})}{K_1((x, y), \{x\} \times \mathbf{M})} & \text{if } K_1((x, y), \{x\} \times \mathbf{M}) \neq 0 \\ \delta_{(x, y)}(\mathbf{A} \times \mathbf{B}) & \text{otherwise,} \end{cases} \\ K_{1, \neq}((x, y), \mathbf{A} \times \mathbf{B}) &= \begin{cases} \frac{K_1((x, y), \mathbf{A} \setminus \{x\} \times \mathbf{M})}{K_1((x, y), \mathbf{M} \setminus \{x\} \times \mathbf{M})} & \text{if } K_1((x, y), \mathbf{M} \setminus \{x\} \times \mathbf{M}) \neq 0 \\ \delta_{(x, y)}(\mathbf{A} \times \mathbf{B}) & \text{otherwise,} \end{cases} \end{aligned}$$

$$K_{2,\Delta}((x, y), A \times B) = \begin{cases} \frac{K_2((x, y), A \times B \cap \{x\})}{K_2((x, y), M \times \{x\})} & \text{if } K_2((x, y), M \times \{x\}) \neq 0 \\ \delta_{(x,y)}(A \times B) & \text{otherwise,} \end{cases}$$

$$K_{2,\neq}((x, y), A \times B) = \begin{cases} \frac{K_2((x, y), A \times B \setminus \{x\})}{K_2((x, y), M \times M \setminus \{x\})} & \text{if } K_2((x, y), M \times M \setminus \{x\}) \neq 0 \\ \delta_{(x,y)}(A \times B) & \text{otherwise.} \end{cases}$$

Define also the rate jumps for all $(x, y) \in M^2$ by

$$\begin{aligned} r_{0,\Delta}(x, y) &= K_0((x, y), \Delta)r_0(x, y), & r_{0,\neq}(x, y) &= K_0((x, y), \Delta^c)r_0(x, y), \\ r_{1,\Delta}(x) &= K_1((x, y), \{x\} \times M)r_1(x), & r_{1,\neq}(x) &= K_1((x, y), M \setminus \{x\} \times M)r_1(x), \\ r_{2,\Delta}(x) &= K_2((x, y), \{y\} \times M)r_2(x), & r_{2,\neq}(x) &= K_2((x, y), M \times M \setminus \{y\})r_2(x). \end{aligned}$$

By Proposition 5, for any initial distribution μ_0 on M^2 ,

$$\text{PDMP}(\varphi^\otimes, (r_i, K_i)_{i \in [0,2]}, \mu_0) = \text{PDMP}(\varphi^\otimes, (r_{i,\circ}, K_{i,\circ})_{i \in [0,2], \circ \in \{\Delta, \neq\}}, \mu_0).$$

Let $(X_t, Y_t)_{t \geq 0}$ be a PDMP associated with the characteristics $(\varphi, (r_{i,\circ}, K_{i,\circ})_{i \in [0,2], \circ \in \{\Delta, \neq\}})$ and let $(S_n^{\neq, i})_{n \in \mathbb{N}, i \in [0,2]}$ be the jump times associated with the jump rates $r_{0,\neq}, r_{1,\neq}, r_{2,\neq}$ respectively. By Lemma 13 since $(X_t, Y_t)_{t \geq 0}$ is non-explosive and Proposition 4, we get for all $t \geq 0$

$$\begin{aligned} \mathbb{P}(X_t \neq Y_t) &\leq 1 - \mathbb{P}\left(\min_{i \in [0,2]} S_1^{\neq, i} \geq t, X_s = Y_s \text{ for all } s \in [0, t]\right) \\ &\leq 1 - \mathbb{E}\left[\exp\left[-\int_0^t \{r_{0,\neq}(s, (\bar{X}_s, \bar{X}_s)) + r_{1,\neq}(s, (\bar{X}_s, \bar{X}_s)) + r_{2,\neq}(s, (\bar{X}_s, \bar{X}_s))\} ds\right]\right], \end{aligned}$$

where $(\bar{X}_s, \bar{X}_s)_{s \geq 0}$ is a PDMP with characteristics $(\tilde{\varphi}, (r_{i,\Delta}, K_{i,\Delta})_{i \in [0,2]})$ starting at (x, x) . This result concludes the proof since by definition, (17), (18) and (12), for all $y \in M$ and $s \geq 0$, we have

$$r_{0,\neq}(s, (y, y)) + r_{1,\neq}(s, (y, y)) + r_{2,\neq}(s, (y, y)) \leq g(s).$$

(2) In the case where λ^1 and λ^2 are not uniformly bounded, consider for all $M > 0$ the two semi-group $(P_{s,t}^{1,M})_{t \geq s \geq 0}$ and $(P_{s,t}^{2,M})_{t \geq s \geq 0}$ associated with the characteristics $(\varphi, \lambda^1 \wedge M, Q^1)$ and $(\varphi, \lambda^2 \wedge M, Q^2)$ respectively. Then for all $M > 0$ the triangle inequality yields

$$\|\delta_x P_{0,t}^1 - \delta_x P_{0,t}^2\|_{\text{TV}} \leq \|\delta_x P_{0,t}^1 - \delta_x P_{0,t}^{1,M}\|_{\text{TV}} + \|\delta_x P_{0,t}^{1,M} - \delta_x P_{0,t}^{2,M}\|_{\text{TV}} + \|\delta_x P_{0,t}^{2,M} - \delta_x P_{0,t}^2\|_{\text{TV}}.$$

Using Proposition 8 and the assumption that the semi-groups we consider are non-explosive,

$$(22) \quad \|\delta_x P_{0,t}^1 - \delta_x P_{0,t}^2\|_{\text{TV}} \leq \limsup_{M \rightarrow +\infty} \|\delta_x P_{0,t}^{1,M} - \delta_x P_{0,t}^{2,M}\|_{\text{TV}}.$$

On the other hand, by the first part of the proof for all $M > 0$,

$$(23) \quad \|\delta_x P_{0,t}^{1,M} - \delta_x P_{0,t}^{2,M}\|_{\text{TV}} \leq 2 \left\{ 1 - \exp\left(-\int_0^t g(s) ds\right) \right\},$$

where g satisfies (12) since for all $t \in \mathbb{R}_+$ and $M > 0$,

$$g(t) \geq \sup_{\substack{x \in \mathbf{M}, \\ \mathbf{A} \in \mathcal{B}(\mathbf{M})}} \{M \wedge \lambda_1(t, x) \wedge \lambda_2(t, x) |Q_1(t, x, \mathbf{A}) - Q_2(t, x, \mathbf{A})|\} \\ + \sup_{x \in \mathbf{M}} |M \wedge \lambda_1(t, x) - M \wedge \lambda_2(t, x)| .$$

Combining (22) and (23) concludes the proof.

(3) Let us finish the proof by establishing (13) via our results on superposition. Indeed by Proposition 5, for all initial distribution $\mu_0 \in \mathcal{P}(\mathbf{M})$, $\text{PDMP}(\varphi, Q^1, \lambda^1, \mu_0) = \text{PDMP}(\varphi, \tilde{Q}^1, \tilde{\lambda}^1, \mu_0)$ and $\text{PDMP}(\varphi, Q^2, \lambda^2, \mu_0) = \text{PDMP}(\varphi, \tilde{Q}^2, \tilde{\lambda}^2, \mu_0)$ where $\tilde{\lambda}^1 = \tilde{\lambda}^2 = \lambda^1 \vee \lambda^2$ and for $t \in \mathbb{R}_+$, $x \in \mathbf{M}$, $\mathbf{A} \in \mathcal{B}(\mathbf{M})$ and $i = 1, 2$

$$\tilde{Q}^i(t, x, \mathbf{A}) = \frac{\lambda^i(t, x)}{\lambda^1(t, x) \vee \lambda^2(t, x)} Q^i(t, x, \mathbf{A}) + \left(1 - \frac{\lambda^i(t, x)}{\lambda^1(t, x) \vee \lambda^2(t, x)}\right) \delta_x(\mathbf{A}) .$$

Therefore, $(P_{s,t}^1)_{t \geq s \geq 0}$ and $(P_{s,t}^2)_{t \geq s \geq 0}$ are also associated with the characteristics $(\varphi, \tilde{Q}^1, \tilde{\lambda}^1)$ and $(\varphi, \tilde{Q}^2, \tilde{\lambda}^2)$. Applying the case where g is given by Equation (12) to these characteristics concludes. \square

7. GENERATOR

From this section, only homogeneous processes are considered. Nevertheless, some results below can be applied to inhomogeneous PDMP since if $(X_t)_{t \geq 0}$ is such a process on \mathbf{M} with characteristics (φ, Q, λ) , then the process $(X_t, t)_{t \geq 0}$ is a homogeneous PDMP on $(\mathbf{M} \times \mathbb{R}_+, \mathcal{B}(\mathbf{M} \times \mathbb{R}_+))$ with characteristics $(\bar{\varphi}, \bar{Q}, \lambda)$ defined for all $t, s \in \mathbb{R}_+$, $x \in \mathbf{M}$ and $\mathbf{A} \in \mathcal{B}(\mathbf{M} \times \mathbb{R}_+)$ by

$$\bar{\varphi}_s((x, t)) = (\varphi_{t, t+s}(x), t+s) , \quad \bar{Q}((x, t), \mathbf{A}) = \int_{\mathbf{M} \times \mathbb{R}_+} \mathbb{1}_{\mathbf{A}}((y, u)) Q(x, dy) \otimes \delta_t(du) .$$

This section is devoted to the introduction of the strong and extended generator of a non-explosive PDMP.

Consider a homogeneous PDMP semigroup $(P_t)_{t \geq 0}$ with non-explosive characteristics (φ, λ, Q) . Note that $(P_t)_{t \geq 0}$ is a contraction semigroup on $\mathbf{B}(\mathbf{M})$, *i.e.* for all $s, t \in \mathbb{R}_+$, $P_{s+t} = P_t P_s$ and for all function $f \in \mathbf{B}(\mathbf{M})$, $\|P_t f\|_\infty \leq \|f\|_\infty$. In addition, define the subset $\mathbf{B}_0(\mathbf{M}) \subset \mathbf{B}(\mathbf{M})$ by

$$\mathbf{B}_0(\mathbf{M}) = \left\{ f \in \mathbf{B}(\mathbf{M}) : \lim_{t \rightarrow 0} \|P_t f - f\|_\infty = 0 \right\} .$$

By [9, p.28-29], $\mathbf{B}_0(\mathbf{M})$ is a closed subspace of $\mathbf{B}(\mathbf{M})$ and a Banach space for the uniform norm. Then by definition, $(P_t)_{t \geq 0}$ is a strongly continuous semigroup on $\mathbf{B}_0(\mathbf{M})$, *i.e.* for all $f \in \mathbf{B}_0(\mathbf{M})$, $\lim_{t \rightarrow 0} \|P_t f - f\|_\infty = 0$.

Define $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$ the strong generator of $(P_t)_{t \geq 0}$ by

$$D(\bar{\mathcal{A}}) = \left\{ f \in \mathbf{B}_0(\mathbf{M}) : \text{there exists } g : \mathbf{M} \rightarrow \mathbb{R} \lim_{t \rightarrow 0} \|t^{-1}(P_t f - f) - g\|_\infty = 0 \right\} , \\ \bar{\mathcal{A}}f = g \text{ for all } f \in D(\bar{\mathcal{A}}) .$$

A subset $D \subset D(\bar{\mathcal{A}})$ is a core of $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$ if the closure of the restriction of $\bar{\mathcal{A}}$ to D is equal to $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$. One use of the strong generator of $(P_t)_{t \geq 0}$ is to show that a given

measure on $(M, \mathbf{B}M)$ is an invariant measure for $(P_t)_{t \geq 0}$. Indeed by [13, Proposition 9.2], μ is invariant measure for $(P_t)_{t \geq 0}$ if and only if for all $f \in \mathcal{D}$, where \mathcal{D} is a core for $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$, $\int_M \bar{\mathcal{A}}f(x)\mu(dx) = 0$. Therefore, the strong generator $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$ is an essential tool to study $(P_t)_{t \geq 0}$. Unfortunately, characterizing the domain $D(\bar{\mathcal{A}})$ is in general not possible. In addition, while it would be possible to only use a core of $(\bar{\mathcal{A}}, D(\bar{\mathcal{A}}))$, there is very few results giving such a subset for PDMPs contrary to diffusion processes (see e.g. [13, Chapter 8]). However, we will see in this section that for a class of PDMPs, to show that a measure μ is invariant, it is sufficient to show that for all $f \in C_c^1(M)$, $\int_M \mathcal{A}f(x)d\mu(x)$, where $(\mathcal{A}, D(\mathcal{A}))$ is the extended generator of $(P_t)_{t \geq 0}$, defined as follows.

For $x \in M$, denote \mathbb{P}_x the distribution PDMP $(\varphi, \lambda, Q, \delta_x)$ on $D(\mathbb{R}_+, M)$ and \mathbb{E}_x the corresponding expectation. Let $(\bar{X}_t)_{t \geq 0}$ be the canonical process on $D(\mathbb{R}_+, M)$, defined by $(\bar{X}_t)_{t \geq 0}(\omega) = \omega$ for all $\omega \in D(\mathbb{R}_+, M)$, and let $(\mathcal{F}_t)_{t \geq 0}$ be its associated filtration. Let $\bar{S}_0 = 0$ and, for $k \in \mathbb{N}$, $\bar{S}_{k+1} = \inf\{t > \bar{S}_k : \bar{X}_t \neq \varphi_{t-\bar{S}_k}(\bar{X}_{\bar{S}_k})\}$ be its true jump times. Define for all $t \in \mathbb{R}_+$, $\bar{N}_t = \sum_{k \in \mathbb{N}^*} \mathbb{1}_{[0, t]}(\bar{S}_k)$ and consider the following assumption

A1. For all $x \in M$ and $t \in \mathbb{R}_+$, $\mathbb{E}_x [\bar{N}_t] < +\infty$.

For all $t \geq 0$ and for all measurable functions $f, g : M \rightarrow \mathbb{R}$, such that, for all $x \in M$, $s \mapsto g(\bar{X}_s)$ is \mathbb{P}_x -almost surely locally integrable, denote

$$M_t^{f,g} = f(\bar{X}_t) - f(\bar{X}_0) - \int_0^t g(\bar{X}_s)ds .$$

The (extended) generator and its domain $(\mathcal{A}, D(\mathcal{A}))$ associated with the semi-group $(P_t)_{t \geq 0}$ are defined as follows: $f \in D(\mathcal{A})$ if there exists a measurable function $g : M \rightarrow \mathbb{R}$ such that $(M_t^{f,g})_{t \geq 0}$ is a local martingale under \mathbb{P}_x for all $x \in M$ and, for such a function, $\mathcal{A}f = g$. Despite its very formal definition, $(\mathcal{A}, D(\mathcal{A}))$ associated with $(P_t)_{t \geq 0}$ can be easily described. Indeed, under **A1**, [9, Theorem 26.14] shows that $D(\mathcal{A}) = E_1 \cap E_2$ where

$$E_1 = \{f \in \mathbb{M}(M) : t \mapsto f(\varphi_t(x)) \text{ is absolutely continuous on } \mathbb{R}_+ \text{ for all } x \in M\} ,$$

and E_2 is the set of measurable functions $f : M \rightarrow \mathbb{R}$ such that there exists an increasing sequence of $(\mathcal{F}_t)_{t \geq 0}$ -stopping time $(\sigma_n)_{n \geq 0}$, such that for all $x \in M$, $\lim_{n \rightarrow +\infty} \sigma_n = +\infty$ \mathbb{P}_x -almost surely and for all $n \in \mathbb{N}$,

$$\mathbb{E}_x \left[\sum_{k=0}^{+\infty} \mathbb{1}_{\{\bar{S}_{k+1} \leq \sigma_n\}} \left| f(\bar{X}_{\bar{S}_{k+1}}) - f(\varphi_{t-\bar{S}_k}(\bar{X}_{\bar{S}_k})) \right| \right] < +\infty .$$

Then, for all $f \in D(\mathcal{A})$ and $x \in M$,

$$(24) \quad \mathcal{A}f(x) = D_\varphi f(x) + \lambda(x) (Qf(x) - f(x)) ,$$

where

$$D_\varphi f(x) = \begin{cases} \lim_{t \rightarrow 0} \frac{f(\varphi_t(x)) - f(x)}{t} , & \text{if this limit exists} \\ 0 & \text{otherwise .} \end{cases}$$

In fact, in [9, Theorem 26.14], Q is required to satisfy $Q(x, \{x\}) = 0$ for all $x \in M$, but if it is not the case, from Proposition 5, we can apply this result with the minimal jump rate associated to (λ, Q) such as introduced in Section 4.

Note that $D(\bar{\mathcal{A}}) \subset D(\mathcal{A})$ and for all $f \in D(\bar{\mathcal{A}})$, $\mathcal{A}f = \bar{\mathcal{A}}f$, since by [9, Proposition 14.13], for all $f \in D(\bar{\mathcal{A}})$, $(M_t^{f, \bar{\mathcal{A}}f})_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale.

In addition, $C^1(\mathbb{M}) \subset D(\mathcal{A})$ and, if $f \in C_c^1(\mathbb{M})$, then $\mathcal{A}f$ is bounded, therefore $(M_t^{f, \mathcal{A}f})_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale. Moreover, since we supposed that $b(x) = (\partial_t)_{t=0} \varphi_t(x)$ exists for all $x \in \mathbb{M}$, then $D_\varphi f(x) = \langle b(x), \nabla f(x) \rangle$ for all $f \in C^1(\mathbb{M})$ and $x \in \mathbb{M}$. However, we need some conditions on λ and Q to show that $C_c^1(\mathbb{M}) \subset D(\bar{\mathcal{A}})$.

A 2. Let $(P_t)_{t \geq 0}$ be a non explosive PDMP semi-group with characteristics (φ, λ, Q) . Assume that for all $T \geq 0$, there exists $M \geq 0$ such that for all $x \in \mathbb{M}$ and $t \in [0, T]$, $\text{supp}\{P_t(x, \cdot)\} \subset \bar{\mathbb{B}}(x, M)$.

Lemma 14. Assume **A 2**.

- (a) For all $f \in C_c(\mathbb{M})$, $T \in \mathbb{R}_+$, there exists a bounded set \mathbf{A} such that $P_t f(x) = 0$, for all $x \notin \mathbf{A}$ and $t \in [0, T]$.
- (b) Condition **A 1** is satisfied.

Proof. (a) Let $f \in C_c(\mathbb{M})$, $T \in \mathbb{R}_+$. By assumption, there exist $M_f, M_T \in \mathbb{R}_+$ such that $\text{supp}(f) \subset \mathbb{B}(x_0, M_f)$, $x_0 \in \mathbb{M}$, and $\text{supp}\{P_t(x, \cdot)\} \subset \bar{\mathbb{B}}(x, M)$ for all $t \in [0, T]$, $x \in \mathbb{M}$. Therefore, we get that, for all $x \notin \bar{\mathbb{B}}(x_0, M_T + M_f + 1)$,

$$P_t f(x) = \int_{\mathbb{M}} \mathbb{1}_{\bar{\mathbb{B}}(x, M_T) \cap \mathbb{B}(x_0, M_f)}(y) f(y) P_t(x, dy) = 0.$$

Indeed, by construction $\bar{\mathbb{B}}(x, M_T) \cap \mathbb{B}(x_0, M_f) = \emptyset$ since by the triangle inequality, $\text{dist}(x, y) \leq M_T$ implies that $\text{dist}(x_0, y) \geq M_f + 1$.

(b) Let $(X_t)_{t \geq 0}$ be a PDMP process with characteristics (φ, λ, Q) started from $x \in \mathbb{M}$, given by Construction 1, with jump times $(S_k)_{k \in \mathbb{N}}$. Note that by definition, for all $k \in \mathbb{N}$, $S_k \leq \bar{S}_k$, where $(\bar{S}_k)_{k \in \mathbb{N}}$ is the true jump times of the process. Therefore, defining $N_t = \sum_{k=1}^{+\infty} \mathbb{1}_{[0, t]}(S_k)$, we have for all $t \geq 0$, $\bar{N}_t \leq N_t$, and to show that **A 1** holds, it suffices to show that $\mathbb{E}[N_T] < +\infty$ for all $T \in \mathbb{R}_+$ and $x \in \mathbb{M}$.

Let $T \geq 0$ and $M \geq 0$ be such that for all $t \in [0, T]$ and $y \in \mathbb{M}$, $\text{supp}\{P_t(y, \cdot)\} \subset \mathbb{B}(0, M)$. Then, since $(X_t)_{t \in \mathbb{R}_+}$ is a càdlàg process, almost surely, for all $t \in [0, T]$, $X_t \in \bar{\mathbb{B}}(x, M)$. Therefore, for all $k \in \mathbb{N}$,

$$\mathbb{1}_{[0, t]}(S_{k+1})(S_{k+1} - S_k) \leq E_{k+1} / (1 + \|\lambda\|_{\infty, \bar{\mathbb{B}}(x, M)}).$$

Then, for all $t \in [0, T]$, N_t is bounded by $\sum_{k=1}^{+\infty} \mathbb{1}_{[0, t]}(E_{k+1} / (1 + \|\lambda\|_{\infty, \mathbb{B}(x, M)}))$, which is a Poisson process with rate $1 + \|\lambda\|_{\infty, \mathbb{B}(x, M)}$. Therefore, for all $t \in [0, T]$, $\mathbb{E}[N_t] < +\infty$. \square

Proposition 15. Let $(P_t)_{t \geq 0}$ be a non explosive PDMP semigroup with characteristics (φ, λ, Q) . Assume **A 2**, $(t, x) \mapsto \varphi_t(x) \in C^1(\mathbb{R}_+ \times \mathbb{M})$, $\lambda \in C(\mathbb{M})$ and for all $f \in C_c(\mathbb{M})$, $\lambda Q f \in C_c(\mathbb{M})$. Then $C_c^1(\mathbb{M}) \subset D(\bar{\mathcal{A}})$.

Proof. Let $f \in C_c^1(\mathbb{M})$. By Lemma 14 and since $\lambda Q f \in C_c(\mathbb{M})$, there exists a compact set $\mathbb{K} \subset \mathbb{M}$, such that for all $x \notin \mathbb{K}$, $P_t f(x) = 0$, for all $t \in [0, 1]$, $\lambda(x) Q f(x) = 0$ and $f(x) = 0$. Therefore, for all $t \in [0, 1]$,

$$(25) \quad \left\| t^{-1}(P_t f(x) - f(x)) - \mathcal{A}f(x) \right\|_{\infty} = \left\| t^{-1}(P_t f(x) - f(x)) - \mathcal{A}f(x) \right\|_{\infty, \mathbb{K}}.$$

As seen above, $(M_t^{f, \mathcal{A}f})_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale. Therefore, for all $x \in \mathbf{M}$,

$$t^{-1} \{P_t f(x) - f(x)\} - \mathcal{A}f(x) = t^{-1} \mathbb{E}_x \left[\int_0^t \{\mathcal{A}f(\bar{X}_s) - \mathcal{A}f(x)\} ds \right].$$

Then since $f \in C_c^1(\mathbf{M})$, $(t, x) \mapsto \varphi_t(x)$ is continuously differentiable, λ is locally bounded and λQf is bounded, there exists $C_1 \geq 0$ such that for all $t > 0$ and $x \in \mathbf{M}$, we have

$$(26) \quad |t^{-1} \{P_t f(x) - f(x)\} - \mathcal{A}f(x)| \leq C_1 \mathbb{P}_x(\bar{S}_1 \leq t) + t^{-1} \mathbb{E}_x \left[\mathbb{1}_{[0, t)}(\bar{S}_1) \int_0^t \{A_s^1 + A_s^2 + A_s^3\} ds \right],$$

where

$$\begin{aligned} A_s^1 &= \langle (\partial_u)_{u=0} \{\varphi_u\}(\varphi_s(x)), \nabla f(\varphi_s(x)) \rangle - \langle (\partial_u)_{u=0} \{\varphi_u\}(x), \nabla f(x) \rangle \\ A_s^2 &= \lambda(\varphi_s(x)) Qf(\varphi_s(x)) - \lambda(x) Qf(x), \quad A_s^3 = -\lambda(\varphi_s(x)) f(\varphi_s(x)) - \lambda(x) f(x). \end{aligned}$$

Using that $(x, s) \mapsto \varphi_s(x)$ is locally bounded on $\mathbb{R}_+ \times \mathbf{M}$ and λ on \mathbf{M} , there exists C_2 such that for all $t \in [0, 1]$ and $x \in \mathbf{K}$,

$$(27) \quad \mathbb{P}_x(\bar{S}_1 \leq t) \leq \int_0^t ds \lambda(\varphi_s(x)) \exp\left(-\int_0^s du \lambda(\varphi_u(x))\right) \leq C_2 t.$$

In addition, using that $(t, x) \mapsto \varphi_t(x) \in C^1(\mathbf{M})$, there exists $C_3 \geq 0$ such that for all $t \in [0, 1]$ and $x \in \mathbf{K}$, $\text{dist}(\varphi_t(x), x) \leq tC_3$. Then, since $\langle (\partial_u)_{u=0} \varphi_u, \nabla f \rangle$, λQf and λf are continuous, they are uniformly continuous on $\mathbf{K} + \bar{\mathbf{B}}(0, C_3)$ and therefore for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $x \in \mathbf{K}$ and $s \in [0, \eta]$, $|A_s^i| \leq \varepsilon$, $i = 1, 2, 3$. Combining this result and (27) in (26), we get for all $x \in \mathbf{K}$, $\varepsilon > 0$ and $t \in [0, \eta \wedge 1]$,

$$|t^{-1} \{P_t f(x) - f(x)\} - \mathcal{A}f(x)| \leq C_1 C_2 t + 3\varepsilon.$$

Therefore, by (25) we get for all $\varepsilon > 0$,

$$\limsup_{t \rightarrow 0} \|t^{-1} \{P_t f(x) - f(x)\} - \mathcal{A}f(x)\|_\infty \leq 3\varepsilon.$$

Taking $\varepsilon \rightarrow 0$ concludes the proof. \square

Let us finish by our running example.

Example - Bouncy Particle Sampler. Consider the BPS process defined in Example 1 and suppose that Y is bounded. It is then easy to verify that **A2** is verified, and the generator is given for all $f \in C_c^1(\mathbf{M})$ by

$$\begin{aligned} \mathcal{A}f(x, y) &= \langle y, \nabla f(x, y) \rangle + (\langle y, \nabla U(x) \rangle)_+ (f(x, R(x, y)) - f(x, y)) \\ &\quad + \lambda_c \left(\int_Y f(x, w) d\mu_v(w) - f(x, y) \right). \end{aligned}$$

8. REGULARITY ESTIMATES FOR PDMP SEMIGROUPS

The main goal of this section is to provide a modest alternative to the (hypo-)elliptic regularity theory of diffusions for PDMP (see also [2] on that topic). To do so, we need the following definition.

Definition 16. *We say that a differential flow φ on M and a Markov kernel Q are compactly compatible if for all compact set $K \subset M$ and $T \geq 0$, there exists a compact set $\tilde{K} \subset M$ satisfying: for all $n \in \mathbb{N}^*$, $(t_i)_{i \in \llbracket 1, n \rrbracket} \in \mathbb{R}_+^n$, $\sum_{i=1}^n t_i \leq T$, there exists a sequence $(K_i)_{i \in \llbracket 1, n \rrbracket}$ of compact sets of M such that, setting $K_0 = K$,*

- (i) *for all $i \in \llbracket 1, n \rrbracket$, K_i only depends on $(t_j)_{j \in \llbracket 1, i \rrbracket}$ and $\cup_{i=0}^n K_i \subset \tilde{K}$;*
- (ii) *for all $i \in \llbracket 0, n-1 \rrbracket$, $s_{i+1} \in [0, t_{i+1}]$ and $s_{n+1} \in [0, T - \sum_{j=1}^n t_j]$,*

$$\bigcup_{x \in K_i} \text{supp}\{Q(\varphi_{t_{i+1}}(x), \cdot)\} \subset K_{i+1}, \quad \varphi_{s_{i+1}}(K_i) \subset \tilde{K}, \quad \varphi_{s_{n+1}}(K_n) \subset \tilde{K}$$

Note that by definition, if φ and Q are compactly compatible and the PDMP semigroup with characteristics (φ, λ, Q) is non explosive, for all $T \geq 0$ and all compact set $K \subset M$, there exists a compact set $\tilde{K} \subset M$, such that $\mathbb{P}(X_t \in \tilde{K}, \text{ for all } t \in [0, T]) = 1$, where $(X_t)_{t \geq 0}$ is a PDMP process with characteristics (φ, λ, Q) and starting from $X_0 \in K$.

A3. *The characteristics (φ, λ, Q) satisfy*

- (i) *the flow φ and the Markov kernel Q are compactly compatible;*
- (ii) *$\lambda \in C^1(M)$ and for all $f \in C^1(M)$, $\lambda Qf \in C^1(M)$ and there exists a locally bounded function $\Psi : M \rightarrow \mathbb{R}_+$ such that for all $x \in K$,*

$$\|\nabla(\lambda Qf)(x)\| \leq \|\Psi\|_{\infty, K} \sup\{|f(y)| + \|\nabla f(y)\| : y \in \text{supp}\{Q(x, \cdot)\}\};$$

- (iii) *for all compact $K \subset M$,*

$$\sup\{\|\nabla \varphi_s(x)\| : s \in [0, t], x \in K\} < +\infty.$$

Lemma 17. *Let $(P_t)_{t \geq 0}$ be a non explosive PDMP semigroup on M with corresponding characteristics (φ, λ, Q) satisfying **A3**. Then for all $f \in C^1(M)$, $T \in \mathbb{R}_+$, $P_T f \in C^1(M)$ and for all compact set $K \subset M$, there exists $C \geq 0$ such that for all $t \in [0, T]$,*

$$(28) \quad \sup_{x \in K} \{|P_t f|(x) + \|\nabla(P_t f)(x)\|\} \leq C.$$

Proof. For all $x \in M$ denote by $(X_t^x)_{t \geq 0}$ a PDMP starting from x associated with the characteristics (φ, λ, Q) and defined by Construction 1. Let $(S_n^x)_{n \in \mathbb{N}}$ be the jump times of $(X_t^x)_{t \geq 0}$ for all $x \in M$ and $(\mathcal{F}_t)_{t \geq 0}$ the associated filtration. Let $f \in C^1(M)$, $T \geq 0$ and a compact set $K \subset M$. For $T = 0$, the result is straightforward so we consider $T > 0$. Let \tilde{K} satisfying for all $n \in \mathbb{N}$, $(t_i)_{i \in \llbracket 1, n+1 \rrbracket} \in \mathbb{R}_+^{n+1}$, $\sum_{i=1}^{n+1} t_i \leq T$, (i)-(ii) in Definition 16.

Since for all $x \in K$, $\mathbb{P}(X_t^x \in \tilde{K}, \text{ for all } t \in [0, T]) = 1$, for all $t \in [0, T]$ and $x \in M$,

$$(29) \quad |P_t f(x)| = |\mathbb{E}[f(X_t)]| \leq \sup_{y \in \tilde{K}} |f(y)|.$$

Furthermore, $(P_t)_{t \geq 0}$ is assumed to be non explosive. Therefore $\sup_{n \in \mathbb{N}} S_n^x = +\infty$ and we can consider the following decomposition for all $t \in [0, T]$ and $x \in \mathbb{K}$

$$(30) \quad P_t f(x) = \sum_{n=0}^{+\infty} \mathbb{E} \left[\mathbb{1}_{[S_n^x, S_{n+1}^x)}(t) f(X_t^x) \right] .$$

We show that for all $n \in \mathbb{N}$,

$$F_{n,t} : x \mapsto \mathbb{E} \left[\mathbb{1}_{[S_n^x, S_{n+1}^x)}(t) f(X_t^x) \right] ,$$

is continuously differentiable and in addition there exists $C \geq 0$ such that for all $n \in \mathbb{N}$, $t \in [0, T]$

$$(31) \quad \sup_{x \in \mathbb{K}} \|\nabla F_{n,t}(x)\| \leq C^n/n! .$$

Assume for the moment that this result holds. Then, we have for all $t \in [0, T]$,

$$\lim_{N \rightarrow +\infty} \sum_{n=N}^{\infty} \sup_{x \in \mathbb{K}} \|\nabla F_{n,t}(x)\| = 0 .$$

By (30), it implies that $x \mapsto P_T f(x)$ is continuously differentiable. In addition, for all compact set $\mathbb{K} \subset \mathbb{M}$, there exists $C \geq 0$ such that for all $t \in [0, T]$, $\|\nabla P_t f(x)\|_{\mathbb{K}} \leq C$. This result and (29) imply (28).

We now turn in showing that for all $n \in \mathbb{N}$, F_n is continuously differentiable and (31) holds. We first show this result for $n = 0$. In a second time, we make an induction on $n \in \mathbb{N}^*$.

For all $x \in \mathbb{K}$ and $t \in [0, T]$, we have

$$F_{0,t}(x) = f(\phi_t(x)) \exp \left(- \int_0^t \lambda(\phi_s(x)) ds \right) .$$

Therefore, for all $x \in \mathbb{K}$ and $t \in [0, T]$, we obtain by **A3-(ii)-(iii)**

$$\begin{aligned} \nabla F_{0,t}(x) &= \{ \nabla f(\varphi_t(x)) \} \nabla(\varphi_t)(x) \exp \left(- \int_0^t \lambda(\varphi_s(x)) ds \right) \\ &\quad + f(\varphi_t(x)) \left[\int_0^t \{ \nabla \lambda(\varphi_s(x)) \cdot \nabla(\varphi_s)(x) \} ds \right] \exp \left(- \int_0^t \lambda(\varphi_s(x)) ds \right) . \end{aligned}$$

Since for all $x \in \mathbb{K}$ and $t \in [0, T]$, $\varphi_t(x) \in \tilde{\mathbb{K}}$, $f \in C^1(\mathbb{M})$ and using **A3-(ii)-(iii)**, we get there exists $C_0 \geq 0$ such that for all $t \in [0, T]$,

$$(32) \quad \sup_{x \in \mathbb{K}} [|F_0(x)| + \|\nabla F_0(x)\|] \leq C_0 .$$

We now show the result for $n \in \mathbb{N}^*$. We give first an explicit expression of F_n for all $n \in \mathbb{N}^*$. Indeed, we have conditioning successively on $\mathcal{F}_{S_{n+1}^x}, \dots, \mathcal{F}_{S_1^x}$, for all $x \in \mathbb{K}$ and $t \in [0, T]$

$$\begin{aligned} F_{n,t}(x) &= \int_0^t dt_1 \exp \left(- \int_0^{t_1} \lambda \{ \varphi_{s_1}(x) \} ds_1 \right) \int_{\mathbb{M}} \mathbb{K}(\varphi_{t_1}(x), dx_1) \\ &\quad \int_0^{t-t_1} dt_2 \exp \left(- \int_0^{t_2} \lambda \{ \varphi_{s_2}(x_1) \} ds_2 \right) \int_{\mathbb{M}} \mathbb{K}(\varphi_{t_2}(x_1), dx_2) \end{aligned}$$

$$\begin{aligned} & \cdots \int_0^{t-\sum_{i=1}^{n-1} t_i} dt_n \exp\left(-\int_0^{t_n} \lambda\{\varphi_{s_n}(x_{n-1})\} ds_n\right) \\ & \int_{\mathbf{M}} \mathbf{K}(\varphi_{t_n}(x_{n-1}), dx_n) f(\varphi_{t-\sum_{i=1}^n t_i}(x_n)) \\ & \exp\left(-\int_0^{t-\sum_{i=1}^n t_i} \lambda\{\varphi_{s_{n+1}}(x_n)\} ds_{n+1}\right), \end{aligned}$$

where \mathbf{K} is the kernel defined on $(\mathbf{M}, \mathcal{B}(\mathbf{M}))$ for all $x \in \mathbf{M}$ and $\mathbf{A} \in \mathcal{B}(\mathbf{M})$ by

$$\mathbf{K}(x, \mathbf{A}) = \lambda(x)Q(x, \mathbf{A}).$$

We introduce a sequence of operator $(Q^{(n)})_{n \in \mathbb{N}^*}$, defined for all $g : \mathbb{R}_+ \times \mathbf{M} \rightarrow \mathbb{R}$, bounded on all compact of $[0, T] \times \mathbf{M}$ and measurable, $t \in [0, T]$ and $x \in \mathbf{M}$ by

$$\begin{aligned} Q^{(n)}g(t, x) &= \int_0^t dt_1 \exp\left(-\int_0^{t_1} \lambda\{\varphi_{s_1}(x)\} ds_1\right) \int_{\mathbf{M}} \mathbf{K}(\varphi_{t_1}(x), dx_1) \\ & \cdots \int_0^{t-\sum_{i=1}^{n-1} t_i} dt_n \exp\left(-\int_0^{t_n} \lambda\{\varphi_{s_n}(x_{n-1})\} ds_n\right) \\ & \int_{\mathbf{M}} \mathbf{K}(\varphi_{t_n}(x_{n-1}), dx_n) g\left(t - \sum_{i=1}^n t_i, x_n\right). \end{aligned}$$

Taking for g the function $g_F : (s, y) \mapsto f(\varphi_s(y)) \exp(-\int_0^s du \lambda(\varphi_u(y)))$, we have $F_{n,t} = Q^{(n)}g_F(t, \cdot)$. Since $f \in C^1(\mathbf{M})$ and by **A 3-(iii)**, g_F is measurable, for all $s \in [0, T]$, $y \mapsto g_F(s, y)$ is continuously differentiable on \mathbf{M} and satisfies for all $T' \in [0, T]$, $\mathbf{K}' \subset \mathbf{M}$ compact, $\sup_{s \in [0, T'], y \in \mathbf{K}'} \{|g_F(s, y)| + \|\nabla_x g_F(s, y)\|\} < +\infty$. Therefore if we show that for all measurable function $g : \mathbb{R}_+ \times \mathbf{M} \rightarrow \mathbb{R}_+$ such that for all $s \in [0, T]$, $y \mapsto g(s, y)$ is continuously differentiable on \mathbf{M} satisfying for all $T' \in [0, T]$, $\mathbf{K}' \subset \mathbf{M}$ compact, $\sup_{s \in [0, T'], y \in \mathbf{K}'} |g(s, y)| + \|\nabla_x g(s, y)\| < +\infty$: 1) for all $t \in [0, T]$ and $x \in \mathbf{K}$, $x \mapsto Q^{(n)}g(t, x)$ is differentiable at x 2) there exists $C \geq 0$ such that for all $n \in \mathbb{N}^*$, $x \in \mathbf{K}$ and $t \in [0, T]$,

$$(33) \quad \left| Q^{(n)}g(t, x) \right| + \left\| \nabla_x Q^{(n)}g(t, x) \right\| \leq C^n / (n!).$$

This result and (32) show that (31) holds and the proof is finished. Denote by $\mathbf{G}([0, T] \times \mathbf{M})$ the set of function g satisfying the assumptions above.

Note the following relation between $Q^{(n-1)}$ and $Q^{(1)}$ which will be essential in our next reasonings: for all $g : \mathbb{R}_+ \times \mathbf{M} \rightarrow \mathbb{R}$, bounded on all compact of $\mathbb{R}_+ \times \mathbf{M}$ and measurable, $t \in [0, T]$ and $x \in \mathbf{M}$

$$(34) \quad \begin{aligned} Q^{(n)}g(t, x) &= \int_0^t dt_1 \exp\left(-\int_0^{t_1} \lambda\{\varphi_{s_1}(x)\} ds_1\right) \int_{\mathbf{M}} \mathbf{K}(\varphi_{t_1}(x), dx_1) Q^{(n-1)}g(t - t_1, x_1) \\ &= Q^{(1)}[Q^{(n-1)}g](t, x). \end{aligned}$$

First, we make an induction on $i \in \llbracket 1, n \rrbracket$ to show that for all $i \in \llbracket 1, n \rrbracket$ and $g \in \mathbf{G}$, that $(s, x) \mapsto Q^{(i)}g(s, x) \in \mathbf{G}([0, T] \times \mathbf{M})$, which will show that for all $t \in [0, T]$, $y \mapsto Q^n g(t, y)$

is continuously differentiable. For $i = 1$, note that for all $s \in [0, T]$ and $y \in M$,

$$(35) \quad Q^{(1)}g(s, y) = \int_0^s dt_1 \exp\left(-\int_0^{t_1} \lambda\{\varphi_{s_1}(y)\} ds_1\right) \int_M K(\varphi_{t_1}(y), dy_1)g(s - t_1, y_1).$$

Let $T' \in [0, T]$, $K' \subset M$ be compact and \tilde{K}' given by Definition 16 associated with K' and T' . Then for all $s \in [0, T']$, $t_1 \in [0, s]$, $\varphi_{t_1}(y) \in \tilde{K}'$ for all $y \in K'$. Therefore we have by assumption on g , **A 3-(ii)-(iii)**, $y \mapsto \int_M K(\varphi_{t_1}(y), dy_1)g(s - t_1, y_1)$ for all $s \in [0, T']$, $t_1 \in [0, s]$, is differentiable and there exists $C \geq 0$ such that for all $s \in [0, T']$, $t_1 \in [0, s]$

$$\sup_{y \in K'} \{ |Kg(s - t_1 \cdot)(\varphi_{t_1}(y))| + \|\nabla_x Kg(s - t_1 \cdot)(\varphi_{t_1}(y))\| \} < C$$

By (35), we get then the result for $i = 1$. The result for $i \in [2, n]$ is then a straightforward consequence of (34) and the case $i = 1$.

We now show that for all $g \in G([0, T] \times M)$ that there exists $C \geq 0$ such that for all $x \in K$ and $t \in [0, T]$, (33) holds. By an induction on $N \in [1, n]$, we show that for all $t \in [0, T]$, $(t_i)_{i \in [1, n-N]} \in \mathbb{R}_+^{n-N}$, $\sum_{j=1}^{n-N} t_j \leq t$, there exists $(K_i)_{i \in [0, n-N]}$ satisfying (i)-(ii) in Definition 16 with respect to K , T , \tilde{K} , $(t_i)_{i \in [1, n-N]}$ and the following bound holds

$$(36) \quad \sup_{x_{n-N} \in K_{n-N}} \left| Q^N g \left(t - \sum_{j=1}^{n-N} t_j, x_{n-N} \right) \right| + \left\| \nabla Q^N g \left(t - \sum_{j=1}^{n-N} t_j, x_{n-N} \right) \right\| \\ \leq C_1^N \left\{ \sup_{s \in [0, T], y \in \tilde{K}} |g(s, y)| + \sup_{s \in [0, T], y \in \tilde{K}} \|\nabla_x g(s, y)\| \right\} \left(t - \sum_{j=1}^{n-N} t_j \right)^N / (N!),$$

where

$$(37) \quad C_1 = \|\lambda\|_{\infty, \tilde{K}} + C_2, \quad C_2 = \sup_{s \in [0, T]} \|\nabla_x \varphi_s(x)\|_{\infty, \tilde{K}} (T \|\nabla \lambda\|_{\infty, \tilde{K}} + \|\Psi\|_{\infty, \tilde{K}}).$$

Then, the result for $N = n$ will conclude the proof.

For $N = 1$, let $t \in [0, T]$, $(t_i)_{i \in [1, n-1]} \in \mathbb{R}_+^{n-1}$, $\sum_{j=1}^{n-1} t_j \leq t$. Note that for all $y \in M$, setting $u_{n-1} = \sum_{i=1}^{n-1} t_i$,

$$Q^{(1)}g(t - u_{n-1}, y) = \int_0^{t - u_{n-1}} dt_n e^{-\int_0^{t_n} \lambda\{\varphi_{s_n}(y)\} ds_n} \int_M K(\varphi_{t_n}(y), dy_n)g(t - u_{n-1} - t_n, y_n).$$

For all $t_n \in \mathbb{R}_+$, such that $\sum_{i=1}^n t_i < t$, by **A 3-(i)**, there exists $(K_i)_{i \in [0, n]}$ satisfying (i)-(ii) in Definition 16. In particular, $(K_i)_{i \in [0, n-1]}$ only depends on $(t_i)_{i \in [1, n-1]}$. Then, using **A 3**, we get for all $x_{n-1} \in K_{n-1}$,

$$\left| Q^{(1)}g(t - u_{n-1}, x_{n-1}) \right| \\ \leq \|\lambda\|_{\infty, \tilde{K}} \int_0^{t - u_{n-1}} dt_n \sup \{ |g|(t - u_{n-1} - t_n, y_n) : y_n \in \text{supp}\{Q(\varphi_{t_n}(x_{n-1}), dy_n)\} \} \\ \leq \|\lambda\|_{\infty, \tilde{K}} \int_0^{t - u_{n-1}} dt_n \sup \{ |g|(t - u_{n-1} - t_n, y_n) : y_n \in K_n \},$$

and

$$\begin{aligned}
& \left\| \nabla_x Q^{(1)} g(t - u_{n-1}, x_{n-1}) \right\| \\
& \leq C_2 \int_0^{t-u_{n-1}} dt_n \sup \{ |g|(t - u_{n-1} - t_n, y_n) : y_n \in \text{supp}\{Q(\varphi_{t_n}(x_{n-1}), dy_n)\} \} \\
& + C_2 \int_0^{t-u_{n-1}} dt_n \sup \{ \|\nabla_x g(t - u_{n-1} - t_n, y_n)\| : y_n \in \text{supp}\{Q(\varphi_{t_n}(x_{n-1}), dy_n)\} \} \\
& \leq C_2 \int_0^{t-u_{n-1}} dt_n \sup \{ |g|(t - u_{n-1} - t_n, y_n^1) + \|\nabla_x g(t - u_{n-1} - t_n, y_n^2)\| : y_n^1, y_n^2 \in \mathbb{K}_n \} ,
\end{aligned}$$

where C_2 is given by (37). Combining these two results and using that $\mathbb{K}_n \subset \tilde{\mathbb{K}}$ for all t_n , $\sum_{i=1}^n t_i < t$ give (36) for $N = 1$.

Now assume that the result holds for $N \in \llbracket 1, n-1 \rrbracket$ and let $(t_i)_{i \in \llbracket 1, n-N-1 \rrbracket} \in \mathbb{R}_+^{n-N-1}$. By induction hypothesis, for all $t_{n-N} \in \mathbb{R}_+$, such that $\sum_{i=1}^{n-N} t_i < t$, there exists $(\mathbb{K}_i)_{i \in \llbracket 0, n-N \rrbracket}$ satisfying (i)-(ii) in Definition 16 with respect to \mathbb{K} , T , $\tilde{\mathbb{K}}$, $(t_i)_{i \in \llbracket 1, n-N \rrbracket}$. Then, using A3 and (34), we get for all $x_{n-N-1} \in \mathbb{K}_{n-N-1}$, setting $u_{n-N-1} = \sum_{i=1}^{n-N-1} t_i$ and $\mathbb{A}_{n-N} = \text{supp}\{Q(\varphi_{t_{n-N}}(x_{n-N-1}), dy_{n-N})\}$

$$\begin{aligned}
& \left| Q^{(N)} g(t - u_{n-N-1}, x_{n-N-1}) \right| \\
& \leq \|\lambda\|_{\infty, \tilde{\mathbb{K}}} \int_0^{t-u_{n-N-1}} dt_{n-N} \sup \left\{ \left| Q^{(N-1)} g \right|(t - u_{n-N-1} - t_{n-N}, y_{n-N}) : y_{n-N} \in \mathbb{A}_{n-N} \right\} \\
& \leq \|\lambda\|_{\infty, \tilde{\mathbb{K}}} \int_0^{t-u_{n-N-1}} dt_{n-N} \sup \left\{ \left| Q^{(N-1)} g \right|(t - u_{n-N-1} - t_{n-N}, y_{n-N}) : y_{n-N} \in \mathbb{K}_{n-N} \right\} ,
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \nabla_x Q^{(N)} g(t - u_{n-N-1}, x_{n-N-1}) \right\| \\
& \leq C_2 \int_0^{t-u_{n-N-1}} dt_{n-N} \sup \left\{ \left| Q^{(N-1)} g \right|(t - u_{n-N-1} - t_{n-N}, y_{n-N}) : y_{n-N} \in \mathbb{A}_{n-N} \right\} \\
& + C_2 \int_0^{t-u_{n-N-1}} dt_{n-N} \sup \left\{ \left\| \nabla_x Q^{(N-1)} g(t - u_{n-N-1} - t_{n-N}, y_{n-N}) \right\| : y_{n-N} \in \mathbb{A}_{n-N} \right\} \\
& \leq C_2 \int_0^{t-u_{n-N-1}} dt_{n-N} \sup \left\{ \left| Q^{(N-1)} g \right|(t - u_{n-N-1} - t_{n-N}, y_{n-N}) : y_{n-N} \in \mathbb{K}_{n-N} \right\} \\
& + C_2 \int_0^{t-u_{n-N-1}} dt_{n-N} \sup \left\{ \left\| \nabla_x Q^{(N-1)} g(t - u_{n-N-1} - t_{n-N}, y_{n-N}) \right\| : y_{n-N} \in \mathbb{K}_{n-N} \right\} ,
\end{aligned}$$

where C_2 is given by (37). Combining these two results using $\mathbb{K}_{n-N} \subset \tilde{\mathbb{K}}$ for all t_{n-N} , $\sum_{i=1}^{n-N} t_i < t$ and the induction hypothesis conclude the proof of (36). \square

Remark 18. Lemma 17 can be generalized under the condition that for some $k \in \mathbb{N}^*$, the characteristics (φ, λ, Q) satisfy

- (i) the flow φ and the Markov kernel Q are compactly compatible;

(ii) $\lambda \in C^k(\mathbf{M})$ and for all $f \in C^k(\mathbf{M})$, $\lambda Qf \in C^k(\mathbf{M})$ and there exists a locally bounded function $\Psi : \mathbf{M} \rightarrow \mathbb{R}_+$ such that for all $x \in \mathbf{K}$, $i \in \llbracket 1, k \rrbracket$,

$$\|D^i(\lambda Qf)(x)\| \leq \|\Psi\|_{\infty, \mathbf{K}} \sup \{ \|D^i f(y)\| : y \in \text{supp}\{Q(x, \cdot)\}, i \in \llbracket 1, k \rrbracket \};$$

(iii) for all $t \in \mathbb{R}_+$, we have $x \mapsto \varphi_t(x)$ is k -times continuously differentiable and for all compact $\mathbf{K} \subset \mathbf{M}$,

$$\sup \{ \|D^i \varphi_s(x)\| : s \in [0, t], x \in \mathbf{K}, i \in \llbracket 1, k \rrbracket \} < +\infty.$$

Then for all function $f \in C^k(\mathbf{M})$, and $T \in \mathbb{R}_+$, $P_T f \in C^k(\mathbf{M})$. In addition, for all compact set $\mathbf{K} \subset \mathbf{M}$, and $T \in \mathbb{R}_+$, there exists $C \geq 0$ such that for all $t \in [0, T]$, $\sup_{x \in \mathbf{K}} |P_t f|(x) + \sup_{x \in \mathbf{K}, i \in \llbracket 1, k \rrbracket} \|D^i P_t f(x)\| \leq C$.

Combining Lemma 14-Proposition 15-Lemma 17 and [13, Proposition 3.3, Chapter 1], we get the following result.

Corollary 19. *Assume the PDMP characteristics (φ, λ, Q) satisfies **A2-A3** and $(t, x) \mapsto \varphi_t(x) \in C^1(\mathbf{M})$. Then, $C_c^1(\mathbf{M})$ is a core for the strong generator of $(P_t)_{t \geq 0}$.*

Example - Bouncy Particle Sampler. For the BPS process, λ is not in $C^1(\mathbf{M})$ so that we cannot apply the previous theory. One aim of the following section will then be to introduce a framework where we may overpass this limitation.

9. INVARIANT MEASURES OF PDMP

The main purpose of this Section is to provide a practical conditions on characteristics (φ, λ, Q) such that if a probability measure μ on $(\mathbf{M}, \mathcal{B}(\mathbf{M}))$ satisfies for all $f \in C_c^1(\mathbf{M})$, $\int_{\mathbf{M}} \mathcal{A}f(x) d\mu(x) = 0$, where \mathcal{A} is the weak generator of the semigroup $(P_t)_{t \geq 0}$ associated with (φ, λ, Q) , then μ is invariant for $(P_t)_{t \geq 0}$.

Definition 20. *We say that the PDMP semi-group $(P_t)_{t \geq 0}$ with characteristics (φ, λ, Q) is smoothly and compactly approximable if for all $\varepsilon > 0$ there exist characteristics $(\varphi, \lambda^\varepsilon, Q^\varepsilon)$ satisfying **A2-A3** and*

$$(38) \quad \sup_{\substack{x \in \mathbf{M} \\ \mathbf{A} \in \mathcal{B}(\mathbf{M})}} \{ \lambda^\varepsilon(x) \wedge \lambda(x) |Q^\varepsilon(x, \mathbf{A}) - Q(x, \mathbf{A})| + |\lambda^\varepsilon(x) - \lambda(x)| \} \leq \varepsilon.$$

We may now give the main result of this section.

Proposition 21. *Let $(P_t)_{t \geq 0}$ be a non explosive PDMP semigroup with characteristics (φ, λ, Q) such that **A1** holds. Assume that $(P_t)_{t \geq 0}$ is smoothly and compactly approximable and $\int_{\mathbf{M}} |\lambda|(x) \mu(dx) < +\infty$. In addition, assume that the generator \mathcal{A} associated with $(P_t)_{t \geq 0}$ satisfies for all $f \in C_c^1(\mathbf{M})$,*

$$(39) \quad \int_{\mathbf{M}} \mathcal{A}f(x) \mu(dx) = 0.$$

Then μ is invariant for $(P_t)_{t \geq 0}$.

Before proceeding to the proof, let us present this technical lemma.

Lemma 22. *Let $(P_t^1)_{t \geq 0}$ and $(P_t^2)_{t \geq 0}$ be non explosive PDMP semigroups with characteristics $(\varphi, \lambda^1, Q^1)$ and $(\varphi, \lambda^2, Q^2)$ respectively such that **A 1** holds. Assume that there exists $\eta \geq 0$, such that*

$$\sup_{\substack{x \in \mathbf{M} \\ \mathbf{A} \in \mathcal{B}(\mathbf{M})}} |\lambda^1(x)Q^1(x, \mathbf{A}) - \lambda^2(x)Q^2(x, \mathbf{A})| \leq \eta .$$

Then for all $f \in C_c^1(\mathbf{M})$,

$$\|\mathcal{A}^1 f - \mathcal{A}^2 f\|_\infty \leq 2\eta \|f\|_\infty ,$$

where $\mathcal{A}^1, \mathcal{A}^2$ are the extended generators of $(P_t^1)_{t \geq 0}, (P_t^2)_{t \geq 0}$ respectively defined by (24).

Proof. Let $\varepsilon > 0$ and $f \in C_c^1(\mathbf{M})$. Since $C_c^1(\mathbf{M})$ belongs to the domain of \mathcal{A}^1 and \mathcal{A}^2 , we have for all $x \in \mathbf{M}$ by [11, Section III.1]

$$\begin{aligned} |\mathcal{A}^1 f(x) - \mathcal{A}^2 f(x)| &= |\lambda^1(x)Q^1 f(x) - \lambda^1(x)f(x) - [\lambda^2(x)Q^2 f(x) - \lambda^2(x)f(x)]| \\ &\leq 2\|f\|_\infty \sup_{\mathbf{A} \in \mathcal{B}(\mathbf{M})} |\lambda^1(x)Q^1(x, \mathbf{A}) - \lambda^2(x)Q^2(x, \mathbf{A})| \leq 2\eta \|f\|_\infty , \end{aligned}$$

which concludes the proof. \square

Proof of Proposition 21. Since $C_c^1(\mathbf{M})$ is dense in $L^\infty(\mu)$, where $L^\infty(\mu)$ is the space of measurable function with bounded essential supremum with respect to μ , it is sufficient to show that

$$\int_{\mathbf{M}} P_t f(x) \mu(dx) = \int_{\mathbf{M}} f(x) \mu(dx)$$

for all $t \geq 0$ and $f \in C_c^1(\mathbf{M})$. Consider $\varepsilon > 0$, a PDMP semi-group $(P_s^\varepsilon)_{s \geq 0}$ with characteristics $(\varphi, \lambda^\varepsilon, Q^\varepsilon)$ satisfying **A2-A3** and (38), and $t \geq 0$. By Proposition 11, we have for all $x \in \mathbf{M}$, $\|\delta_x P_t - \delta_x P_t^\varepsilon\|_{\text{TV}} \leq 2(1 - e^{-t\varepsilon})$, therefore we get for all $f \in C_c^1(\mathbf{M})$

$$\begin{aligned} \left| \int_{\mathbf{M}} P_t f(x) \mu(dx) - \int_{\mathbf{M}} f(x) \mu(dx) \right| &\leq \left| \int_{\mathbf{M}} \{P_t f(x) - P_t^\varepsilon f(x)\} \mu(dx) \right| \\ &\quad + \left| \int_{\mathbf{M}} \{P_t^\varepsilon f(x) - f(x)\} \mu(dx) \right| \\ (40) \qquad \qquad \qquad &\leq 2(1 - e^{-\varepsilon t}) \|f\|_\infty + \left| \int_{\mathbf{M}} \{P_t^\varepsilon f(x) - f(x)\} \mu(dx) \right| . \end{aligned}$$

Furthermore, by Corollary 19, $C_c^1(\mathbf{M})$ is a core for the strong generator associated with $(P_s^\varepsilon)_{s \geq 0}$. Thus, it belongs to the domain of the strong generator of $(P_t^\varepsilon)_{t \geq 0}$ and by Dynkin formula [13, Proposition 1.5-(c)], we get

$$\int_{\mathbf{M}} P_t^\varepsilon f(x) \mu(dx) = \int_{\mathbf{M}} f(x) \mu(dx) - \int_{\mathbf{M}} \int_0^t \mathcal{A}^\varepsilon P_s^\varepsilon f(x) ds \mu(dx) .$$

Therefore,

$$\begin{aligned} (41) \quad \left| \int_{\mathbf{M}} P_t^\varepsilon f(x) \mu(dx) - \int_{\mathbf{M}} f(x) \mu(dx) \right| \\ \leq \left| \int_{\mathbf{M}} \int_0^t \{\mathcal{A}^\varepsilon - \mathcal{A}\} P_s^\varepsilon f(x) ds \mu(dx) \right| + \left| \int_{\mathbf{M}} \int_0^t \mathcal{A} P_s^\varepsilon f(x) ds \mu(dx) \right| . \end{aligned}$$

In addition, by (28) in Lemma 17, A3-(iii) and Lemma 14, $\int_{\mathbb{M}} \int_0^t |\mathcal{A}P_s^\varepsilon f(x)| ds \mu(dx) < +\infty$. Therefore, by Fubini's theorem and (39), we get $\int_{\mathbb{M}} \int_0^t \mathcal{A}P_s^\varepsilon f(x) ds \mu(dx) = 0$. Combining this result, (41) becomes

$$(42) \quad \left| \int_{\mathbb{M}} P_t^\varepsilon f(x) \mu(dx) - \int_{\mathbb{M}} f(x) \mu(dx) \right| \leq \left| \int_{\mathbb{M}} \int_0^t \{\mathcal{A}^\varepsilon - \mathcal{A}\} P_s^\varepsilon f(x) ds \mu(dx) \right|.$$

Note that by (16), (38) and Lemma 22, for all $f \in C^1(\mathbb{M})$

$$\|\mathcal{A}f - \mathcal{A}^\varepsilon f\|_\infty \leq \varepsilon \|f\|_\infty.$$

Using this result in (42), we obtain

$$(43) \quad \left| \int_{\mathbb{M}} P_t^\varepsilon f(x) \mu(dx) - \int_{\mathbb{M}} f(x) \mu(dx) \right| \leq \varepsilon t \sup_{\substack{x \in \mathbb{M} \\ s \in [0, t]}} |P_s f(x)| \leq \varepsilon t \|f\|_\infty.$$

Plugging (43) in (40), we finally obtain

$$\left| \int_{\mathbb{M}} P_t f(x) \mu(dx) - \int_{\mathbb{M}} f(x) \mu(dx) \right| \leq (2(1 - e^{-\varepsilon t}) + \varepsilon t) \|f\|_\infty.$$

Taking the limit as ε goes to 0 concludes the proof. \square

With this notion of smoothly approximable semigroup, we will be able to consider the BPS process.

Example - Bouncy Particle Sampler.

Proposition 23. *Let $U \in C^2(\mathbb{R}^d)$, $\lambda_c > 0$ and $\mu_\nu \in \mathcal{P}(\mathbb{Y})$ with $\mathbb{Y} \subset \mathbb{R}^d$ bounded. Then the associated BPS on $\mathbb{R}^d \times \mathbb{Y}$, given by Example 1, is smoothly and compactly approximable.*

Proof. Let (φ, λ, Q) be the characteristics of the BPS given by Section 4. Let $\varepsilon > 0$, $\lambda^\varepsilon : \mathbb{M} \rightarrow \mathbb{R}_+$ and Q^ε be a Markov kernel on $(\mathbb{R}^d \times \mathbb{Y}, \mathcal{B}(\mathbb{R}^d \times \mathbb{Y}))$ defined for all $(x, y) \in \mathbb{R}^d \times \mathbb{Y}$ and $\mathbf{A} \in \mathcal{B}(\mathbb{R}^d)$ by

$$\lambda^\varepsilon(x, y) = \lambda_1^\varepsilon(x, y) + \bar{\lambda}, \quad \lambda_1^\varepsilon(x, y) = (\langle y, \nabla U(x) \rangle - \varepsilon)_+^2 / (\varepsilon + (\langle y, \nabla U(x) \rangle - \varepsilon)_+),$$

$$Q^\varepsilon((x, y), \mathbf{A}) = (1/\lambda^\varepsilon(x, y)) \{ \lambda_1^\varepsilon \delta_{(x, R(x, y))}(\mathbf{A}) + \lambda_c (\delta_x \otimes \mu_\nu)(\mathbf{A}) \},$$

where R is defined by (2) for $g = \nabla U$. Then, similarly to the BPS process, $(\varphi, \lambda^\varepsilon, Q^\varepsilon)$ defines a non explosive semi-group $(P_t^\varepsilon)_{t \geq 0}$ on $(\mathbb{R}^d \times \mathbb{Y}) \times \mathcal{B}(\mathbb{R}^d \times \mathbb{Y})$. In addition we have

$$\sup_{(x, y) \in \mathbb{R}^d \times \mathbb{Y}} \left| \lambda^{(\varepsilon)}(x, y) - \lambda(x, y) \right| \leq 2\varepsilon.$$

Therefore, using Remark 12, we get

$$\sup_{\substack{(x, y) \in \mathbb{R}^d \times \mathbb{Y} \\ \mathbf{A} \in \mathcal{B}(\mathbb{R}^d \times \mathbb{Y})}} \{ \lambda^\varepsilon(x, y) \wedge \lambda(x, y) |Q^\varepsilon((x, y), \mathbf{A}) - Q((x, y), \mathbf{A})| + |\lambda^\varepsilon(x, y) - \lambda(x, y)| \}$$

$$\leq 2 \sup_{\substack{(x, y) \in \mathbb{R}^d \times \mathbb{Y} \\ \mathbf{A} \in \mathcal{B}(\mathbb{R}^d \times \mathbb{Y})}} |\lambda^\varepsilon(x, y) Q^\varepsilon((x, y), \mathbf{A}) - \lambda(x, y) Q((x, y), \mathbf{A})| \leq 4\varepsilon,$$

which shows (38).

Since Y is assumed to be bounded, $Y \subset \overline{B}(0, M_Y)$, with $M_Y \in \mathbb{R}_+$. Therefore by definition, for all $t \in \mathbb{R}_+$ and $(x, y) \in \mathbb{R}^d \times Y$, we have $P_t^\varepsilon((x, y), \overline{B}(x, tM_Y) \times Y) = 1$ and P_t^ε satisfies **A2**.

Finally, we show that **A3** is satisfied. **A3-(iii)** trivially holds by definition of φ .

For all closed ball, $\overline{B}(0, M) \subset \mathbb{R}^d$, $M \in \mathbb{R}_+$, $Q^\varepsilon((x, y), \overline{B}(0, M) \times Y) = 1$ for all $x \in \overline{B}(0, M)$. For all compact set $K \subset B(0, M) \times Y \subset \mathbb{R}^d \times Y$, $M \geq 0$, and $T \in \mathbb{R}_+$, define $\tilde{K} = B(0, M + TR) \times Y$, with $R = \sup_{y \in Y} \|y\|$. Then, for all $n \in \mathbb{N}^*$, $(t_i)_{i \in \llbracket 1, n \rrbracket} \in \mathbb{R}_+^n$, $\sum_{i=1}^n t_i \leq T$, conditions **(i)-(ii)** of Definition 16 are satisfied with \tilde{K} , and $(K_i)_{i \in \llbracket 1, n \rrbracket}$ given by

$$K_i = \overline{B}(0, R_i) \times Y, \quad R_i = M + R \sum_{j=0}^i t_j.$$

Thus, φ and Q^ε are compactly compatible.

Then note that for all $\varepsilon > 0$, λ^ε is continuously differentiable on M since $t \mapsto (t - \varepsilon)_+^2 / (\varepsilon + (t - \varepsilon)_+)$ is on \mathbb{R} ; and its gradient is given for all $x \in M$ by

$$\nabla(\lambda^\varepsilon)(x, y) = \begin{cases} \frac{(\langle y, \nabla U(x) \rangle - \varepsilon)(\langle y, \nabla U(x) \rangle + \varepsilon)}{\langle y, \nabla U(x) \rangle^2} (\nabla^2 U(x)y, \nabla U(x)) & \text{if } \langle y, \nabla U(x) \rangle \geq \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

In addition, for all continuously differentiable function $f : \mathbb{R}^d \times Y \rightarrow \mathbb{R}$, $(x, y) \in \mathbb{R}^d \times Y$, we have $\lambda^\varepsilon(x, y)Q^\varepsilon f(x, y) = A_1(x, y) + A_2(x, y)$ where

$$A_1(x, y) = \lambda_1^\varepsilon(x, y)f(x, R(x, y)), \quad A_2(x, y) = \lambda_c \int_Y f(x, \tilde{y}) \mu_\nu(d\tilde{y}).$$

We show that A_1 and A_2 are continuously differentiable and satisfy for $i = 1, 2$, for all compact set $K \in \mathbb{R}^d \times Y$, for all $(x, y) \in K$,

$$(44) \quad \|\nabla A_i(x, y)\| \leq \sup_{(w, z) \in K} \{\Psi_i(w, z)\} \sup \{|f|(x, \tilde{y}) + \|\nabla f(x, \tilde{y})\| : \tilde{y} \in Y\},$$

where $\Psi_i : \mathbb{R}^d \times Y \rightarrow \mathbb{R}_+$, $i = 1, 2$, are bounded on compact sets of $\mathbb{R}^d \times Y$. Note that if we show (44), since for all $(x, y) \in \mathbb{R}^d \times Y$, $\text{supp}\{Q((x, y), \cdot)\} = \{x\} \times Y$, this result concludes the proof that **A3-(ii)** holds.

First, for all $(x, y) \in \mathbb{R}^d \times Y$, R is continuously differentiable at (x, y) is $\langle y, \nabla U(x) \rangle \neq 0$. Since f , λ_1^ε are continuously differentiable and $\lambda_1^\varepsilon(x, y) = 0$ if $\langle y, \nabla U(x) \rangle \leq \varepsilon$, A_1 is continuously differentiable and satisfies for all $(x, y) \in \mathbb{R}^d \times Y$, $\langle y, \nabla U(x) \rangle \geq \varepsilon$

$$(45) \quad \|\nabla A_1(x, y)\| \leq \|\nabla \lambda_1^\varepsilon(x, y)\| |f|(x, R(x, y)) + \lambda_1^\varepsilon(x, y)(1 + \|\nabla R(x, y)\|) \|\nabla f(x, R(x, y))\|.$$

Regarding A_2 , we have for all $(x, y) \in \mathbb{R}^d \times Y$, since ∇f is bounded on all compact sets of $\mathbb{R}^d \times Y$ and the Lebesgue dominated convergence theorem,

$$\|\nabla A_2(x, y)\| \leq \lambda_c \int_Y \|\nabla_x f(x, \tilde{y})\| \mu_\nu(d\tilde{y}) \leq \lambda_c \sup_{y \in Y} \|\nabla f(x, \tilde{y})\|,$$

where ∇_x is the differential operator with respect to the x -variable. Combining this result and (45), we get that (44) holds and therefore **A3-(ii)** as well. \square

Proposition 24. *Consider the BPS characteristics $(\varphi, \lambda_1, Q_1, \lambda_2, Q_2)$ defined in Example 1, and let $(P_t)_{t \geq 0}$ be the corresponding semigroup. Assume that μ_ν is rotation invariant, i.e. for all $O \in \mathbb{R}^{d \times d}$, $O^T O = \text{Id}$, $\mu_\nu(OA) = \mu_\nu(A)$, for all $A \in \mathcal{B}(Y)$. In addition, suppose that*

$$(46) \quad \int_{\mathbb{R}^d} (1 + \|\nabla U(x)\|) e^{-U(x)} dx < \infty, \quad \int_Y \|y\| \mu_\nu(dy) < \infty.$$

Then $\tilde{\pi} = \pi \otimes \mu_\nu$ is invariant for $(P_t)_{t \geq 0}$, where π is the probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with density with respect to the Lebesgue measure proportional to $x \mapsto e^{-U(x)}$.

Proof. Consider first the case where Y is bounded. Then, by Proposition 23, $(P_t)_{t \geq 0}$ is smoothly and compactly approximable and the other assumptions of Proposition 21 are easily checked. In particular, for all $f \in C_c^1(\mathbb{R}^d \times Y)$, $\int_{\mathbb{R}^d} \mathcal{A}f(x) \pi(dx) = 0$ follows from an integration by parts and because Y and μ_ν is rotation invariant, see e.g. [21] for more details.

Now, in the general case where Y may be unbounded, consider the conditional distribution associated with μ_ν defined for all $A \in \mathcal{B}(Y)$,

$$\mu_\nu^R(A) = \mu_\nu(B(0, R) \cap A) / \mu_\nu(B(0, R)),$$

for R large enough such that $\mu_\nu(B(0, R)) \neq 0$. Consider a BPS semi-group $(P_t^R)_{t \geq 0}$ associated with U and μ_ν^R , i.e. with characteristics (φ, λ, Q^R) with φ given by (2), λ by (9) and Q^R is defined for all $(x, y) \in \mathbb{R}^d \times Y$, $A \in \mathcal{B}(\mathbb{R}^d \times Y)$ by

$$Q^R((x, y), A) = \lambda^{-1}(x, y) \{ \langle \nabla U(x), y \rangle \delta_{(x, R(x, y))}(A) + \lambda_c(\delta_x \otimes \mu_\nu^R)(A) \}.$$

Note that since $Y \cap B(0, R)$ is bounded, $\tilde{\pi}^R = \pi \otimes \mu_\nu^R$ is an invariant measure for $(P_t^R)_{t \geq 0}$. Therefore for any $f : \mathbb{R}^d \times Y \rightarrow \mathbb{R}$, bounded and measurable, $t \geq 0$, we have for all $R \geq 0$ large enough,

$$\begin{aligned} & \left| \int_{\mathbb{R}^d \times Y} P_t f(x, y) d\tilde{\pi}(x, y) - \int_{\mathbb{R}^d \times Y} f(x, y) d\tilde{\pi}(x, y) \right| \\ & \leq \left| \int_{\mathbb{R}^d \times Y} P_t f(x, y) d\tilde{\pi}(x, y) - \int_{\mathbb{R}^d \times Y} P_t f(x, y) d\tilde{\pi}^R(x, y) \right| \\ & \quad + \left| \int_{\mathbb{R}^d \times Y} P_t f(x, y) d\tilde{\pi}^R(x, y) - \int_{\mathbb{R}^d \times Y} P_t^R f(x, y) d\tilde{\pi}^R(x, y) \right| \\ & \quad + \left| \int_{\mathbb{R}^d \times Y} f(x, y) d\tilde{\pi}^R(x, y) - \int_{\mathbb{R}^d \times Y} f(x, y) d\tilde{\pi}(x, y) \right| \\ & \leq 2 \|f\|_\infty \mu_\nu(\mathbb{R}^d \setminus B(0, R)) + \left| \int_{\mathbb{R}^d \times Y} P_t f(x, y) d\tilde{\pi}^R(x, y) - \int_{\mathbb{R}^d \times Y} P_t^R f(x, y) d\tilde{\pi}^R(x, y) \right|. \end{aligned}$$

Since $\lim_{R \rightarrow +\infty} \mu_\nu(\mathbb{R}^d \setminus B(0, R)) = 0$, it remains to show that the last term in the right-hand side goes to 0 as $R \rightarrow +\infty$ and the proof will be finished. Besides, note that this result holds if for $t > 0$, we show that for all $(x, y) \in \mathbb{R}^d \times Y$,

$$(47) \quad \lim_{R \rightarrow +\infty} \|\delta_{(x, y)} P_t - \delta_{(x, y)} P_t^R\|_{\text{TV}} = 0.$$

But by Proposition 11 and definition of characteristics of $(P_s)_{s \geq 0}$ and $(P_s^R)_{s \geq 0}$, we get for all $t \geq 0$, $(x, y) \in \mathbb{R}^d \times Y$,

$$\|\delta_{(x,y)}P_t - \delta_{(x,y)}P_t^R\|_{\text{TV}} \leq 2(1 - \exp(-\lambda_c \mu(Y \setminus B(0, R)))) ,$$

which shows that (47) holds. \square

10. STABILITY OF INVARIANT MEASURE AND JUMP RATE

We conclude this work with an asymptotic counterpart of the comparison theorems established in Section 6. For a measurable function $V : M \rightarrow [1, +\infty)$ and $\nu_1, \nu_2 \in \mathcal{P}(M)$, $\nu_1(V), \nu_2(V) < +\infty$, define the V -norm between ν_1 and ν_2 by

$$\|\nu_1 - \nu_2\|_V = \sup \left\{ \left| \int_M f d\nu_1 - \int_M f d\nu_2 \right|, \|f/V\|_\infty \leq 1 \right\} .$$

We say that a semi-group on M with invariant probability measure μ is V -uniformly geometrically ergodic with constants $C, \rho > 0$ if for all $\nu \in \mathcal{P}(M)$, $\nu(V) < +\infty$ and $t \geq 0$,

$$\|\mu - \nu P_t\|_V \leq C e^{-\rho t} \nu(V) .$$

Proposition 25. *Let $(P_t^1)_{t \geq 0}$ and $(P_t^2)_{t \geq 0}$ be two non-explosive homogeneous PDMP semi-group with characteristics $(\varphi, \lambda_1, Q_1)$ and $(\varphi, \lambda_2, Q_2)$ respectively. Let $\mu_1, \mu_2 \in \mathcal{P}(M)$ be invariant for $(P_t^1)_{t \geq 0}$ and $(P_t^2)_{t \geq 0}$ respectively. Suppose that $(P_t^1)_{t \geq 0}$ is V -uniformly geometrically ergodic with constants $C, \rho \in \mathbb{R}_+^*$ for a function $V : M \rightarrow [1, +\infty)$ such that $\mu_2(V) < +\infty$. Assume in addition that*

$$C_c^1(M) \subset B_0^1(M) = \left\{ f \in B(M) : \lim_{t \rightarrow 0} \|P_t^1 f - f\|_\infty = 0 \right\} ,$$

and that for all $t \in \mathbb{R}_+$ and $f \in C_c^1(M)$, $x \mapsto \int_0^t P_s^1 f(x) ds \in D(\bar{\mathcal{A}}_2)$, where $(\bar{\mathcal{A}}_2, D(\bar{\mathcal{A}}_2))$ is the strong generator of $(P_t^2)_{t \geq 0}$. Then

$$\|\mu_1 - \mu_2\|_V \leq C \rho^{-1} \sup \left\{ \int_M |\lambda_1 Q_1 h - \lambda_2 Q_2 h + (\lambda_2 - \lambda_1) h| d\mu_2 : \|h/V\|_\infty \leq 1 \right\} .$$

Proof. By density, it is sufficient to bound $|\mu_1(f) - \mu_2(f)|$ for all $f \in C_c^1(M)$ with $\|f/V\|_\infty \leq 1$. Let $f \in C_c^1(M)$ with $\|f/V\|_\infty \leq 1$ and, for $t \geq 0$, let $g_t = \int_0^t P_s^1 (f - \mu_1(f)) ds$. According to [9, Proposition 14.10], for all $t \geq 0$, $g_t \in D(\bar{\mathcal{A}}_1)$ as a sum of a constant function and of $\int_0^t P_s f ds$ with $f \in B_0^1(M)$, where $(\bar{\mathcal{A}}_1, D(\bar{\mathcal{A}}_1))$ is the strong generator of $(P_t^1)_{t \geq 0}$; moreover it holds

$$\bar{\mathcal{A}}_1 g_t = \bar{\mathcal{A}}_1 \int_0^t P_s f ds = P_t f - f .$$

Using that $(P_t^1)_{t \geq 0}$ is V -uniformly geometrically ergodic and that $g_t \in D(\bar{\mathcal{A}}_2)$, we obtain for all $t \geq 0$ and $x \in M$ that

$$\begin{aligned} |\mu_1(f) - f(x) - \bar{\mathcal{A}}_2 g_t(x)| &\leq \|\delta_x P_t - \mu_1\|_V + |P_t f(x) - f(x) - \bar{\mathcal{A}}_2 g_t(x)| \\ &\leq C e^{-\rho t} V(x) + |\bar{\mathcal{A}}_1 g_t(x) - \bar{\mathcal{A}}_2 g_t(x)| . \end{aligned}$$

In addition, by definition of $(\bar{\mathcal{A}}_2, D(\bar{\mathcal{A}}_2))$ and since μ_2 is invariant for $(P_t^2)_{t \geq 0}$, then $\mu_2(\bar{\mathcal{A}}_2 g_t) = 0$ for all $t \geq 0$, so that

$$|\mu_1(f) - \mu_2(f)| = |\mu_2\{\mu_1(f) - f - \bar{\mathcal{A}}_2 g_t\}| \leq \mu_2(|(\bar{\mathcal{A}}_1 - \bar{\mathcal{A}}_2)g_t|) + Ce^{-\rho t} \mu_2(V).$$

Since $D(\bar{\mathcal{A}}_i) \subset D(\mathcal{A}_i)$, $i = 1, 2$, for all $t \geq 0$, $(\bar{\mathcal{A}}_1 - \bar{\mathcal{A}}_2)g_t = (\mathcal{A}_1 - \mathcal{A}_2)g_t$. Finally, $(P_t^1)_{t \geq 0}$ being V -uniformly geometrically ergodic, then for all $x \in M$, $g_t(x) \leq (C/\rho)V(x)$. The proof is then concluded taking $t \rightarrow +\infty$. \square

Example - Bouncy Particle Sampler. Let us apply this result in the case of the Bouncy Particle Sampler. Ergodicity of the BPS is studied in [12], to which we will refer for details on this matter in the following. For the sake of simplicity, we will work under restrictive conditions.

Proposition 26. *Consider the BPS with characteristics $(\varphi, \lambda_1, Q_1, \lambda_2, Q_2)$ defined in Example 1, with $U \in C^2(M)$ and $Y \subset \bar{B}(0, 1)$, and $(P_t)_{t \geq 0}$ the corresponding semi-group. For $M > 0$, let $(P_t^M)_{t \geq 0}$ be the PDMP semi-group with characteristics $(\varphi, \lambda_1 \wedge M, Q_1, \lambda_2, Q_2)$.*

Assume that μ_ν is rotation invariant and (46) holds. In addition, assume that there exist $R > 0$, $W \in C^2(\mathbb{R}^d)$ and $F \in C^2(\mathbb{R})$ such that $U(x) = F(W(x))$ for $x \notin B(0, R)$, $\|\nabla W\|_\infty + \|\nabla^2 W\|_\infty < +\infty$, $\int_M \exp(-W(x))dx < \infty$, $\lim_{\|x\| \rightarrow +\infty} W(x) = +\infty$ and $\lim_{w \rightarrow +\infty} F'(w) = +\infty$. Then there exists $C > 0$ and $\tilde{M} > 0$ such that for all $M > \tilde{M}$, $(P_t^M)_{t \geq 0}$ admits a unique invariant measure $\tilde{\pi}_M$ that satisfies

$$\|\tilde{\pi} - \tilde{\pi}_M\|_{e^W} \leq C \int_M (\|\nabla U(x)\| - M)_+ e^{W(x) - U(x)} dx,$$

where $\tilde{\pi} = \pi \otimes \mu_\nu$ where π is the probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with density with respect to the Lebesgue measure proportional to $x \mapsto e^{-U(x)}$.

For example, if $U(x) = \langle x, Ax \rangle$ for some definite positive matrix A outside a ball, then these conditions are satisfied by $W(x) = \sqrt{1 + U(x)}$ and $F(w) = w^2 - 1$. In this case, Proposition 26 implies that there exist $M_0, C, c > 0$ such that for all $M > M_0$,

$$\|\tilde{\pi} - \tilde{\pi}_M\|_{e^W} \leq Ce^{-cM^2}.$$

Note that for all $M > 0$, $(P_t^M)_{t \geq 0}$ can be sampled by thinning procedures, see [17, 16].

Proof. For all $\varepsilon > 0$, in a similar way as in Proposition 23, we can construct a semi-group $(P_t^{M, \varepsilon})_{t \geq 0}$ with characteristics $(\varphi, \lambda_{M, \varepsilon}, Q_1, \lambda_2, Q_2)$ where $\lambda_{M, \varepsilon}$ is such that $(P_t^{M, \varepsilon})_{t \geq 0}$ satisfies A2-A3 and that

$$(48) \quad \sup_{(x, y) \in \mathbb{R}^d \times Y} |\lambda_1(x, y) \wedge M - \lambda_{M, \varepsilon}(x, y)| \leq \varepsilon.$$

Let us show that we can apply Proposition 25 twice, with each time $P_t^1 = P_t^{M, \varepsilon}$ and P_t^2 equal either to P_t^M or P_t . Consider the Lyapunov function defined for all $(x, y) \in \mathbb{R}^d \times Y$ by $V(x, y) = \exp(W(x))\phi(\langle y, \nabla W(x) \rangle)$, where $\phi \in C^2(\mathbb{R})$ is an increasing function, with $\phi(r) = 1$ for $r \leq -2$ and $\phi(r) \leq 3$ for $r \geq 1$.

We show first that $(P_t)_{t \geq 0}$, $(P_t^M)_{t \geq 0}$ and $(P_t^{M, \varepsilon})_{t \geq 0}$ are V -uniformly geometrically ergodic, which will imply that all these semi-groups admit a unique stationary measure for which V is integrable. For $h > 0$, let \mathcal{A}_{hW} be the generator of the BPS semi-group

with potential hW , refreshment rate $\lambda_c > 0$ and refreshment law μ_ν . Following [12, Section 3.2], there exist $\phi : \mathbb{R} \rightarrow [1, 3]$ and $h_0 > 0$ such that there exist $\alpha, C > 0$ satisfying

$$\mathcal{A}_{h_0W}V \leq -\alpha V + C.$$

Moreover, for all $M > 0$, denoting \mathcal{A}_M the generator of $(P_t^M)_{t \geq 0}$,

$$\begin{aligned} (\mathcal{A}_M - \mathcal{A}_{h_0W})V(x, y) &= (\lambda_1(x, y) \wedge M - h_0 \langle y, \nabla W(x) \rangle_+) \\ &\quad \times (\phi(-\langle y, \nabla W(x) \rangle) - \phi(\langle y, \nabla W(x) \rangle)) e^{W(x)}. \end{aligned}$$

Note that, since $Y \subset \overline{B}(0, 1)$, W is Lipschitz and F' and W going to infinity at infinity, then for $M_0 = h_0 \|\nabla W\|_\infty$ and some $R_0 > R$ large enough, for all $M \geq M_0$, $x \notin B(0, R_0)$ and $y \in Y$,

$$\begin{aligned} \lambda_1(x, y) \wedge M - h_0 \langle y, \nabla W(x) \rangle_+ &\geq ((F'(W(x)) - h_0) \langle y, \nabla W(x) \rangle_+) \wedge (M - h_0 \|\nabla W\|_\infty) \\ &\geq 0. \end{aligned}$$

Besides, ϕ being increasing, $\phi(-r) - \phi(r) \leq 0$ for all $r \geq 0$, so that for all $M \geq M_0$, $x \notin B(0, R_0)$ and $y \in Y$,

$$\mathcal{A}_M V(x, y) \leq \mathcal{A}_{h_0W} V(x, y) \leq -\alpha V(x, y) + C.$$

Hence, for $M \geq M_0$ and all $(x, y) \in \mathbb{R}^d \times Y$,

$$\mathcal{A}_M V(x, y) \leq -\alpha V(x, y) + C + \sup_{(x, y) \in B(0, R_0) \times Y} |\mathcal{A}_M V(x, y) + \alpha V(x, y)| \leq -\alpha V(x, y) + C',$$

for some C' that does not depend on $M > M_0$, since λ_1 is bounded on $B(0, R_0) \times Y$. By a similar argument, denoting $\mathcal{A}_{M, \varepsilon}$ the generator of $(P_t^{M, \varepsilon})_{t \geq 0}$,

$$\mathcal{A}_{M, \varepsilon} V = \mathcal{A}_M V + (\mathcal{A}_{M, \varepsilon} - \mathcal{A}_M) V \leq -\alpha V + 6\varepsilon e^W + C \leq -(\alpha - 12\varepsilon)V + C.$$

Hence, we have obtained $M_0, \varepsilon_0, \alpha', C' > 0$ such that for all $M > M_0$, all $\varepsilon \in [0, \varepsilon_0]$ then

$$(49) \quad \mathcal{A}_{M, \varepsilon} V \leq -\alpha' V + C'.$$

Moreover, following [12, Section 3.3], for all compact set $K \subset \mathbb{R}^d \times Y$, there exist $\eta, t_0 > 0$ such that, for all $M > M_0$ and $\varepsilon \in [0, \varepsilon_0]$, for all $(x, y), (x', y') \in K$ and all $t \geq t_0$,

$$(50) \quad \|\delta_{x, y} P_t^{M, \varepsilon} - \delta_{x', y'} P_t^{M, \varepsilon}\|_{TV} \leq 2(1 - \eta).$$

Note that, indeed, η and t_0 do not depend on M or ε since their construction only involves the supremum of $\lambda_{M, \varepsilon}$ over some compact set, which is smaller than ε_0 plus the supremum of λ_1 over the same compact (see [12, Section 3.3] for details).

By [19, Theorem 6.1], (49) together with (50) implies that $(P_t^{M, \varepsilon})_{t \geq 0}$ admits a unique invariant measure $\tilde{\pi}_{M, \varepsilon}$, $\tilde{\pi}_{M, \varepsilon}(V) < +\infty$ and is V -uniformly ergodic for all $M > M_0$ and $\varepsilon \in [0, \varepsilon_0]$ with some constants that does not depend on M nor ε (see [12, Section 3] for details). More precisely for all $M > M_0$ and $\varepsilon \in [0, \varepsilon_0]$, there exists $C \geq 0$ and $\rho > 0$ such that for all initial distribution ν_0 , $\nu_0(V) < +\infty$,

$$(51) \quad \|\nu_0 P_t^{M, \varepsilon} - \tilde{\pi}_{M, \varepsilon}\|_{TV} \leq C \rho^t \nu_0(V).$$

The case $\varepsilon = 0$ in particular implies that $(P_t^M)_{t \geq 0}$ admits an invariant measure $\tilde{\pi}_M$ that satisfies $\tilde{\pi}_M(V) < +\infty$. Besides, note that for $R_1 > R_0$ large enough to ensure that $F(W(x)) \geq 2W(x)$ for all $x \notin B(0, R_1)$,

$$\int_{\mathbb{M}} e^{W(x)-U(x)} dx \leq \int_{B(0, R_1)} e^{W(x)-U(x)} dx + \int_{\mathbb{R}^d \setminus B(0, R_1)} e^{-W(x)} dx < +\infty,$$

so that $\tilde{\pi}(V) < +\infty$. Finally, note that $V \leq e^W \leq 2V$, so that the associated V -norms are equivalent.

In order to apply Proposition 25, it remains to check the regularity conditions. Since $(P_t^{M, \varepsilon})_{t \geq 0}$ satisfies A2-A3, by Lemma 17-Lemma 14, $\int_0^t P_s^{M, \varepsilon} f ds \in C_c^1(\mathbb{R}^d \times \mathbb{Y})$ for all $f \in C_c^1(\mathbb{R}^d \times \mathbb{Y})$, $t \geq 0$, $M, \varepsilon > 0$. By Proposition 15, for all $M, \varepsilon > 0$, $C_c^1(\mathbb{R}^d \times \mathbb{Y}) \subset D(\bar{\mathcal{A}}_{M, \varepsilon}) \cap D(\bar{\mathcal{A}}_M) \cap D(\bar{\mathcal{A}})$, where $\bar{\mathcal{A}}$ is the strong generator of $(P_t)_{t \geq 0}$. As a consequence, we can apply twice Proposition 25 which implies, combining with (51), that there exists $C_1 \geq 0$ satisfying for any $M > M_0$ and $\varepsilon \in [0, \varepsilon_0]$,

$$\begin{aligned} \|\tilde{\pi} - \tilde{\pi}_M\|_{e^W} &\leq \|\tilde{\pi} - \tilde{\pi}_{M, \varepsilon}\|_{e^W} + \|\tilde{\pi}_{M, \varepsilon} - \tilde{\pi}_M\|_{e^W} \\ &\leq C_1 \sup \left\{ \int_{\mathbb{R}^d \times \mathbb{Y}} |\lambda_1 - \lambda_{M, \varepsilon}| (|Q_1 h| + |h|) d\tilde{\pi} : \|he^{-W}\|_\infty \leq 1 \right\} \\ &\quad + C_1 \sup \left\{ \int_{\mathbb{R}^d \times \mathbb{Y}} |\lambda_1 \wedge M - \lambda_{M, \varepsilon}| (|Q_1 h| + |h|) d\tilde{\pi}_M : \|he^{-W}\|_\infty \leq 1 \right\}, \end{aligned}$$

Using that $Q_1 e^W = e^W$, (48), $\tilde{\pi}(e^W)$ and $\tilde{\pi}_M(e^W) < +\infty$, we obtain that there exists $C_2 \geq 0$ such that

$$\|\tilde{\pi} - \tilde{\pi}_M\|_{e^W} \leq C_2 \left[\int_{\mathbb{R}^d \times \mathbb{Y}} \{\lambda_1 - \lambda_1 \wedge M\} e^W d\tilde{\pi} + \varepsilon \right].$$

Finally, the proof is concluded taking $\varepsilon \rightarrow 0$ and upon noting that for all $(x, y) \in \mathbb{R}^d \times \mathbb{Y}$,

$$\begin{aligned} \lambda_1(x, y) - \lambda_1(x, y) \wedge M &= (\langle y, \nabla U(x) \rangle_+ - M) \mathbb{1}_{[M, +\infty)}(\langle y, \nabla U(x) \rangle_+) \\ &\leq (\|\nabla U(x)\| - M) \mathbb{1}_{[M, +\infty)}(\langle y, \nabla U(x) \rangle_+) \leq (\|\nabla U(x)\| - M)_+. \end{aligned}$$

□

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