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WEAK CONVERGENCE OF EMPIRICAL WASSERSTEIN TYPE DISTANCES

A PREPRINT

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ABSTRACT

We estimate contrasts $\int_0^1 \rho(F^{-1}(u) - G^{-1}(u))du$ between two continuous distributions F and G on \mathbb{R} such that the set $\{F = G\}$ is a finite union of intervals, possibly empty or \mathbb{R} . The non-negative convex cost function ρ is not necessarily symmetric and the sample may come from any joint distribution H on \mathbb{R}^2 with marginals F and G having light enough tails with respect to ρ . The rates of weak convergence and the limiting distributions are derived in a wide class of situations including the classical Wasserstein distances W_1 and W_2 . The new phenomenon we describe in the case $F = G$ involves the behavior of ρ near 0, which we assume to be regularly varying with index ranging from 1 to 2 and to satisfy a key relation with the behavior of ρ near ∞ through the common tails. Rates are then also regularly varying with powers ranging from $1/2$ to 1 also affecting the limiting distribution, in addition to H .

Central limit theorems, Generalized Wasserstein distances, Empirical processes, Strong approximation, Dependent samples, Non-parametric statistics, Goodness-of-fit tests.
62G30, 62G20, 60F05, 60F17

1 Introduction

1.1 Motivation

In [3] we addressed the problem of estimating the distance between two asymptotically well separated and continuous distributions on the real line \mathbb{R} , with respect to a large class of generalized Wasserstein costs. The framework was the same as in [12] and is very simple. A sequence of independent and identically distributed (*i.i.d.*) random variables (*r.v.*) taking values in \mathbb{R}^2 is available. The marginals have distinct continuous cumulative distribution function (*c.d.f.*) F and G . For instance, each couple may result from simultaneous experiments. We estimated contrasts $\int_0^1 c(F^{-1}(u), G^{-1}(u))du$ between F and G by the natural and easily computed non-parametric plug-in estimator $\int_0^1 c(\mathbb{F}_n^{-1}(u), \mathbb{G}_n^{-1}(u))du$. Here F^{-1} is the generalized inverse of F , \mathbb{F}_n is the empirical *c.d.f.*, and c is a non-negative cost. The almost sure (*a.s.*) consistency of this estimator being easily established under minimal assumptions we mainly developed a sharp method of proof of the Central Limit Theorem (CLT) assuming that the tails of F and G are distinct enough and compatible with the cost c . The most original contribution in [3] was to investigate rather deeply the latter relationship in the untrimmed case and for dependent samples. This showed that the problem can not be reduced to the study of each marginal $\int_0^1 c(\mathbb{F}_n^{-1}(u), F^{-1}(u))du$ and instead requires crossed assumptions on tails, costs and densities beyond moments. However the special case of the distance W_1 was not captured, asymptotically non-symmetric costs or asymptotically too close marginals were not allowed, the case $F = G$ and the one marginal case were not considered.

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In the present paper – the first version of this preprint is [2] – the general setting remains exactly the same, but we investigate the most important situations for statistical applications, among which the goodness-of-fit hypothesis $F = G$, the alternative hypothesis where $F \neq G$ on \mathbb{R} and may have arbitrarily close tails, and the intermediate hypothesis where the two situations $F = G$ and $F \neq G$ are encountered, but alternate along a finite number of intervals. The distance W_1 and non-symmetric costs are now allowed provided that they are regularly varying at both sides of 0. We focus on the new difficulties, however we often refer to [3] to borrow some long arguments and apply already developed tools. New assumptions arise that again illustrate how delicate tail integrals of transforms of empirical quantile functions can be for heavy-tailed distributions.

The method of proofs relies on a careful subdivisions of the integrals and events, and a joint approximation of the quantile processes $\sqrt{n}(\mathbb{F}_n^{-1}(u) - F^{-1}(u))$, $u \in (0, 1)$ by properly scaled Brownian bridges on an appropriate sub-interval. As a matter of fact, it is not possible to directly apply a functional delta-method since the Hadamard differentiability of $F \rightarrow F^{-1}$ can not be extended to encompass distribution with densities arbitrarily close to 0 and in particular with unbounded supports. Moreover the Brownian approximation - weak or strong - of the quantile processes suffer many problems near 0 and 1 due to extreme values. Lastly, the general costs we use - even the simple Wasserstein costs - make the problem more difficult to handle and shows up to be determinant for both rates and limits in the case $F = G$.

Let us mention related results in the framework of univariate probability distributions. The commonly used p -Wasserstein distance $W_p(F, G)$ is

$$W_p^p(F, G) = \int_0^1 |F^{-1}(u) - G^{-1}(u)|^p du. \quad (1)$$

Many authors were interested in the convergence of $W_p^p(\mathbb{F}_n, F)$, see *e.g.* the survey paper [4] or [8, 9, 1]. Up to our knowledge there are only two recent works studying the convergence of $W_2^2(\mathbb{F}_n, \mathbb{G}_n)$ [10, 14], for independent samples. The results of [10] are valid in any finite dimension with the drawback that the estimator is not explicit from the data and the centering in the central limit theorem (CLT) is the biased $\mathbb{E}W_2^2(\mathbb{F}_n, \mathbb{G}_n)$ rather than $W_2^2(F, G)$ itself, moreover the limiting variance has no closed form expression and seems not easy to estimate. In [14] the estimator is the same as our's, however only discrete distributions and W_2 distance are considered. Notice also that in the early work [13] a trimmed version of the Mallows distance $W_2^2(\mathbb{F}_n, \mathbb{G}_n)$ is studied, however under an implicit assumption on the level of trimming which has to hold in probability. Moreover in the case of dependent samples, a trimmed version of $W_2^2(\mathbb{F}_n, \mathbb{G}_n)$ is studied in [12].

We investigate below a larger class of convex costs, even larger than in [3]. The samples are possibly not independent, and the conditions relating the tails of F and G to the cost function c are easily checked. Combined to our technique of proof they allow to control the critical parts of the untrimmed integrals in a weaker sense than in probability, hence our explicit sufficient conditions are lighter than the above mentioned implicit ones. We obtain a general CLT for $W_c(\mathbb{F}_n, \mathbb{G}_n)$ when $F = G$ are continuous, thus providing a new class of goodness-of-fit and comparison tests with exact rates and non-degenerate limits. In order to evaluate the power of these tests we study the weak convergence under many alternatives $F \neq G$ among which the case where $F = G$ on large intervals.

1.2 Setting

The p -Wasserstein distance between two *c.d.f.* F and G on \mathbb{R} is defined by

$$W_p^p(F, G) = \min_{X \sim F, Y \sim G} \mathbb{E}|X - Y|^p \quad (2)$$

where $X \sim F, Y \sim G$ means that X and Y are joint real *r.v.* having *c.d.f.* F and G respectively. The minimum in (2) is (1). To any non negative function $c(x, y)$ from \mathbb{R}^2 to \mathbb{R} let associate the Wasserstein type cost

$$W_c(F, G) = \min_{X \sim F, Y \sim G} \mathbb{E}c(X, Y). \quad (3)$$

We are interested in triplets (c, F, G) such that $W_c(F, G)$ is finite and can be estimated by using an explicit CLT. To guaranty that an analogue of (1) exists we consider cost functions defining a negative measure on \mathbb{R}^2 , hence satisfying

$$c(x', y') - c(x', y) - c(x, y') + c(x, y) \leq 0, \quad x \leq x', y \leq y'. \quad (4)$$

If c satisfies (4) then for any functions a and b , $a(x) + b(y) + c(x, y)$ satisfies (4). In particular $c(x, y) = -xy$ and $(x - y)^2 = x^2 + y^2 - 2xy$ satisfy (4). More generally if ρ is a convex real function then $c(x, y) = \rho(x - y)$ satisfies (4). Two important special cases are the symmetric power functions $|x - y|^p$, $p \geq 1$, associated to W_p and the non-symmetric contrast functions $c(x, y) = (x - y)(\alpha - \mathbf{1}_{x-y < 0})$ associated to the α^{th} quantile, $0 < \alpha < 1$. The following result yields the minimum in (3) in a closed form analogous to (1).

Theorem 1 (Cambanis, Simon, Stout [5]) *If c satisfies (4) then*

$$W_c(F, G) = \int_0^1 c(F^{-1}(u), G^{-1}(u)) du.$$

Let $(X_i, Y_i)_{1 \leq i \leq n}$ be an *i.i.d.* sample of a random vector with joint *c.d.f.* H on \mathbb{R}^2 and marginal *c.d.f.* F and G on \mathbb{R} . Write \mathbb{F}_n and \mathbb{G}_n the random empirical *c.d.f.* built from the two marginal samples. Thus \mathbb{F}_n and \mathbb{G}_n are not independent in general. Consider a cost function c satisfying (4). Let $X_{(i)}$ (resp. $Y_{(i)}$) denote the i^{th} order statistic of the sample $(X_i)_{1 \leq i \leq n}$ (resp. $(Y_i)_{1 \leq i \leq n}$), i.e. $X_{(1)} \leq \dots \leq X_{(n)}$. By Theorem 1, the non-parametric statistic

$$W_c(\mathbb{F}_n, \mathbb{G}_n) = \frac{1}{n} \sum_{i=1}^n c(X_{(i)}, Y_{(i)}) \quad (5)$$

is a natural estimator of $W_c(F, G)$. Now, the $c(X_{(i)}, Y_{(i)})$'s being neither independent nor with identical distributions the statistic (5) is not classical - such as *i.i.d.* mean, L-statistic, U-statistic etc. Notice also that $W_c(F, G)$ does not depend on the generally unknown H whereas the *r.v.* $W_c(\mathbb{F}_n, \mathbb{G}_n)$ strongly depends on H through its distribution. In [3] we established the CLT

$$\sqrt{n} (W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) \rightarrow_{weak} \mathcal{N}(0, \sigma^2)$$

whenever the tails of F and G differ from at least $\tau > 0$ and $c(x, y)$ is asymptotically $\rho(x - y)$ with ρ non-negative, symmetric, convex. The influence of H only appeared in the limiting variance $\sigma^2 = \sigma^2(H, c)$ together with c . The sufficient conditions relating explicitly c , F and G were designed to carefully control the extremes, define sharply the truncation level and approximate the underlying joint quantile processes. We now intend to complete the picture by extending this CLT to other important cases, in particular $\tau = 0$ and non symmetric costs ρ .

1.3 Overview

Hereafter we consider a cost $c(x, y) = \rho_c(x - y)$ where ρ_c is a non-negative real convex function such that $\rho_c(0) = 0$, and is not assumed to be symmetric. In the spirit of [3] we separate out three sets of assumptions, labeled (FG) , (C) and (CFG) respectively.

First, (FG) concerns the regularity and tails of F and G , and especially their density-quantile function. Conditions (FG) are satisfied by distributions having regular tails, among which all classical probability distributions.

Second, (C) restricts the rate of increase at infinity of ρ_c and the regular variation at 0 of ρ_c , without even assuming differentiability at 0. Conditions (C) encompass a large class of Wasserstein type costs c and the distance W_1 is now allowed, together with non-symmetric variants of Wasserstein distances W_p^p , $p \geq 1$, possibly with slowly varying factors – a non trivial extension – or exponential and over-exponential costs.

The conditions (FG) and (C) are thus designed to separately select a class of probability distributions and admissible costs.

The third set (CFG) aims at mixing the requirements on c , F and G making them compatible. We distinguish between (CFG_E) , (CFG_D) and (CFG_{ED}) depending on the situations $\{F = G\} = \mathbb{R}$ or $\{F \neq G\} = \mathbb{R}$ or $\{F = G\} \neq \mathbb{R}$ and $\{F \neq G\} \neq \mathbb{R}$, respectively. The joint distribution H of the couples is not restricted and again only affects the limiting distributions. In order to exhibit an exact rate of convergence it shows up that the tail constraints on F and G that naturally depend on ρ at ∞ also strongly depend on the exact regular variation of ρ_c at 0 whenever $F = G$ in tails, that is the key requirement of (CFG_E) and (CFG_{ED}) .

When dealing with empirical Wasserstein type integrals, to adapt the functional delta method one would need to truncate and then to assume a convergence in probability of the extremal parts. This would be a restriction excluding many distributions F and G , depending on where the integral is non-adaptively trimmed. Moreover, proving the validity of the assumed convergence of the truncated parts would require variants of Steps 1, 2, 3 of our proofs. In contrast, (CFG_E) and (CFG_D) explicitly relate the tails to the cost in such a way that the implicit truncation levels can be defined appropriately.

Before entering the mathematical details of these assumptions let us present two consequences of our results. The regular variation of tails is in the sense of (i) in Section 2.2 below and \rightarrow_{weak} denotes the convergence in distribution.

Proposition 2 *Consider the Wasserstein distance W_p^p for $1 < p < 2$. Assume that $F = G$ is two times differentiable, $\log F(x)$ and $\log(1 - F(x))$ are regularly varying as $|x| \rightarrow \infty$, and $F(x)(1 - F(x)) \leq C|x|^{-\left(\frac{2(p+2)}{2-p} + \varepsilon\right)}$ for some*

$\varepsilon > 0$, $C > 0$ and all $|x|$ large enough. Then it holds

$$n^{\frac{p}{2}} W_p^p(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{weak} \int_0^1 |\mathbb{B}(u)|^p du,$$

where \mathbb{B} is an explicit centered Gaussian process and the limiting r.v. is positive and finite.

The restriction $p < 2$ is not surprising since when X and Y are Gaussian and the two samples are independent, the limiting random integral is *a.s.* infinite. More precisely, in the case $p = 2$ we establish the weak convergence of $nW_2^2(\mathbb{F}_n, \mathbb{G}_n)$ by requiring F to be sub-Gaussian, as in [9] for $nW_2^2(\mathbb{F}_n, F)$.

In the case $p = 1$ we get, with the same Gaussian process \mathbb{B} as above, the following result, which seems new for $W_1(\mathbb{F}_n, \mathbb{G}_n) = \|\mathbb{F}_n - \mathbb{G}_n\|_1$.

Proposition 3 *Assume that the set $\{F = G\}$ is a finite union of non empty intervals of \mathbb{R} , that F, G are two times differentiable and that $\log F(x)$, $\log G(x)$, $\log(1 - F(x))$ and $\log(1 - G(x))$ are regularly varying as $|x| \rightarrow \infty$. Let $r = 2$ if $\{F = G\}$ is compact, and $r = 6$ otherwise. Assume that $\max(F(x)(1 - F(x)), G(x)(1 - G(x))) \leq C|x|^{-(r+\varepsilon)}$ for some $\varepsilon > 0$, $C > 0$ and all $|x|$ large enough. Then it holds*

$$\sqrt{n} (W_1(\mathbb{F}_n, \mathbb{G}_n) - W_1(F, G)) \rightarrow_{weak} \int_{F^{-1} \neq G^{-1}} \mathbb{B}(u) du + \int_{F^{-1} = G^{-1}} |\mathbb{B}(u)| du$$

and the limiting r.v. is finite.

As can be seen in the two previous results this paper focuses on the probability distributions with infinite support. Nevertheless our results also hold for compactly supported probability distributions with derivable densities. At the end of Section 3 we provide simplified sufficient assumptions in the compactly supported case.

The paper is organized as follows. Assumptions are discussed in Section 2. In Section 3 we state our main results in the form of CLT's for $W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)$ at various rates. We propose a few perspectives for applications in Section 4. All the results are proved in Section 5.

2 Assumptions

2.1 Assumptions (FG)

Consider a sequence $(X_n, Y_n) \in \mathbb{R}^2$ of independent random vectors having the same *c.d.f.* H as (X, Y) . The distribution H may have a density or not. However we assume that the marginal *c.d.f.*'s F of X and G of Y have support \mathbb{R} and positive densities $f = F'$ and $g = G'$. Let (E, D) be the partition of $(0, 1)$ defined by

$$E = \{u : F^{-1}(u) = G^{-1}(u)\}, \quad D = \{u : F^{-1}(u) \neq G^{-1}(u)\}. \quad (6)$$

If u shifts infinitely many times between E and D it becomes difficult to control the stochastic integral $W_c(\mathbb{F}_n, \mathbb{G}_n)$. The case where $|F^{-1}(u) - G^{-1}(u)| > \tau > 0$ as $u \rightarrow 1$ and $u \rightarrow 0$ has been treated in details in [3]. We allow the diagonal $|F^{-1}(u) - G^{-1}(u)| \leq \tau$ and thus encompass the case $E = (0, 1)$ together with some tractable situations where $E \neq \emptyset$ and $D \neq \emptyset$. Let assume that there exists a finite integer $\kappa \geq 2$ and $0 = u_0 < u_1 < \dots < u_\kappa = 1$ such that, writing $A_k = (u_{k-1}, u_k)$,

$$(FG0) \quad F^{-1}(u_k) = G^{-1}(u_k) \text{ and } A_k \subset E \text{ or } A_k \subset D, \text{ for } k = 1, \dots, \kappa.$$

This covers three generic cases, namely $E = (0, 1)$, $D = (0, 1)$ and when $D \neq \emptyset$, $E \neq \emptyset$ are finite unions of intervals. The exponential rate of decrease of the right and left tails of F and G are defined to be, for $x \in \mathbb{R}_+$,

$$\begin{aligned} \psi_X^+(x) &= -\log \mathbb{P}(X > x), & \psi_Y^+(x) &= -\log \mathbb{P}(Y > x), \\ \psi_X^-(x) &= -\log \mathbb{P}(X < -x), & \psi_Y^-(x) &= -\log \mathbb{P}(Y < -x). \end{aligned}$$

Only ψ_X^+ and ψ_Y^+ will be considered in subsequent proofs where arguments given for the right hand tail $u \rightarrow 1$ in the integrals $W_c(F, G)$ and $W_c(\mathbb{F}_n, \mathbb{G}_n)$ work similarly for the left hand tail $u \rightarrow 0$. Define the density quantile functions $h_X = f \circ F^{-1}$ and $h_Y = g \circ G^{-1}$ then assume

$$(FG1) \quad F, G \in \mathcal{C}_2(\mathbb{R}), \quad f, g > 0 \text{ on } \mathbb{R}.$$

$$(FG2) \quad \sup_{0 < u < 1} \min(u, 1 - u) |(\log h(u))'| < +\infty \quad \text{for } h = h_X, h_Y.$$

$$(FG3) \quad \sup_{0 < u < 1} \frac{\min(u, 1 - u)}{(|\Gamma^{-1}(u)| + 1) h(u)} < +\infty \quad \text{for } (h, \Gamma) = (h_X, F) \text{ or } (h_Y, G).$$

Observe that (FG1) and (FG2) are classical in the context of approximation of quantile processes – see e.g. [6].

Remark 4 Rewriting (FG2) and (FG3) we get

$$\sup_{x \in \mathbb{R}} \frac{\min(F(x), 1 - F(x))}{f(x)} \left(\frac{1}{|x| + 1} + \frac{|f'(x)|}{f(x)} \right) < +\infty,$$

$$\sup_{x \in \mathbb{R}} \frac{\min(G(x), 1 - G(x))}{g(x)} \left(\frac{1}{|x| + 1} + \frac{|g'(x)|}{g(x)} \right) < +\infty.$$

In Proposition 5 of [3] we provided a simple sufficient condition for (FG1), (FG2), (FG3) based on the regular variation of ψ_X^\pm and ψ_Y^\pm . All classical tail distributions satisfy the conditions (FG).

2.2 Notation for regularity

To specify the allowed cost functions $c(x, y)$ the following definitions are required. As usual for $k \in \mathbb{N}_*$ and $I \subset \mathbb{R}$ let $\mathcal{C}_k(I)$ denote the set of functions that are k times continuously differentiable on I and $\mathcal{C}_0(I)$ the set of continuous functions on I . In forthcoming assumptions and proofs we consider functions defined either on $(0, x_0)$ or on $(y_0, +\infty)$ for some $0 < x_0 < y_0$. We distinguish the two domains by using a variable $x \rightarrow 0$ and a variable $y \rightarrow +\infty$. In [3] only large values $y \in (y_0, +\infty)$ played a role in terms of regular variation, so that we keep the same setting in (i) below. Unexpectedly, it turns out that the two domains interfere when $|F - G|$ is arbitrarily small, and we need (ii).

(i) Regularity on $(y_0, +\infty)$. Let $\mathcal{M}_2((y_0, +\infty))$ be the subset of functions $l \in \mathcal{C}_2((y_0, +\infty))$ such that l'' is monotone on $(y_0, +\infty)$. Write $RV(+\infty, \gamma)$ the set of regularly varying functions at $+\infty$ with index $\gamma \geq 0$. If $\gamma = 0$ we restrict ourselves to slowly varying functions L at $+\infty$ such that

$$L'(y) = \frac{\varepsilon(y)L(y)}{y}, \quad \lim_{y \rightarrow +\infty} \varepsilon(y) = 0. \quad (7)$$

This weak restriction is explained at Section 6 of [3]. In order to find distributions F and G compatible with the cost c we further impose

$$L'(y) \geq \frac{l_1}{y}, \quad l_1 \geq 1, \quad y \geq y_0. \quad (8)$$

For $\gamma = 0$, introduce

$$RV_2(+\infty, 0) = \{L : L \in \mathcal{M}_2((y_0, +\infty)) \text{ such that (7), (8) hold}\}$$

and for $\gamma > 0$,

$$RV_2(+\infty, \gamma) = \{l : l \in \mathcal{M}_2((y_0, +\infty)), l(y) = y^\gamma L(y) \text{ such that } L' \text{ obeys (7)}\}.$$

(ii) Regularity on $(0, x_0)$. We consider positive slowly varying functions L at 0,

$$\lim_{x \searrow 0} \frac{L(\theta x)}{L(x)} = 1 \text{ for any } \theta > 0. \quad (9)$$

For $b > 1$ let introduce

$$RV_2(0, b) = \{\rho : L \in \mathcal{C}_2((0, x_0)), \rho(x) = x^b L(x) \text{ such that } L \text{ satisfies (9)}\}.$$

For $b = 1$ let define

$$RV_2(0, 1) = \{\rho : L \in \mathcal{C}_2((0, x_0)), \rho(x) = xL(x) \text{ such that } L \text{ satisfies (9), (10)}\}$$

where we impose the following finite limit

$$\lim_{x \searrow 0} L(x) = L(0) \in \mathbb{R}_+. \quad (10)$$

2.3 Assumptions (C)

We consider costs such that, for some $0 < x_0 < y_0 < +\infty$,

- (C0) $c(z, z') = \rho_c(z - z') \geq 0$, $z, z' \in \mathbb{R}$, $c(0, 0) = 0$, ρ_c is convex.
- (C1) $\rho_c(x) = \rho_-(x)1_{x \leq 0} + \rho_+(x)1_{x \geq 0}$, $x \in \mathbb{R}$, $\rho_\pm \in \mathcal{C}_2((0, +\infty))$.
- (C2) $\rho_+(x) = x^{b_+} L_+(x) > 0$, $0 < x \leq x_0$, $\rho_+ \in RV_2(0, b_+)$, $b_+ \geq 1$,
 $\rho_-(x) = x^{b_-} L_-(x) > 0$, $0 < x \leq x_0$, $\rho_- \in RV_2(0, b_-)$, $b_- \geq 1$.
- (C3) $\rho_+(y) = \exp(l_+(y))$, $y \geq y_0$, $l_+ \in RV_2(+\infty, \gamma_+)$, $\gamma_+ \geq 0$,
 $\rho_-(y) = \exp(l_-(y))$, $y \geq y_0$, $l_- \in RV_2(+\infty, \gamma_-)$, $\gamma_- \geq 0$.

Notice that $\rho_{\pm}(0) = 0$ and ρ_{\pm} are positive, continuous, convex and increasing on \mathbb{R}_+ . Define $\rho(x) = \max(\rho_+(x), \rho_-(x))$ and $b = \min(b_+, b_-)$. For $0 \leq x \leq x_0$ it holds

$$\rho(x) = x^b L(x), \quad L(x) = \begin{cases} L_+(x) & \text{if } b_+ < b_-, \\ L_-(x) & \text{if } b_- < b_+, \\ \max(L_+(x), L_-(x)) & \text{if } b_+ = b_-. \end{cases} \quad (11)$$

Further assume that

$$(C4) \quad \lim_{x \rightarrow 0} \frac{\rho_+(x)}{\rho(x)} = \pi_+, \quad \lim_{x \rightarrow 0} \frac{\rho_-(x)}{\rho(x)} \rightarrow \pi_-, \quad \pi_+, \pi_- \in [0, 1].$$

Typical costs satisfying the conditions (C) are the following.

Example 5 Let $a = (a_-, a_+)$ be such that $a_{\pm} > 0$ and $b = (b_-, b_+)$ be such that $b_{\pm} \geq 1$. Then

$$c_{a,b}(z, z') = a_- (z' - z)^{b_-} 1_{z < z'} + a_+ (z - z')^{b_+} 1_{z' < z}$$

satisfies (C) with $\gamma_- = \gamma_+ = 0$ and $\varepsilon(y) = O(1/\log y)$. This includes the Wasserstein distance W_p^p , $p \geq 1$, by taking $a = (1, 1)$ and $b = (p, p)$. It is possible to define costs mixing the Wasserstein distance W_p^p , $p \geq 1$ near 0 and W_q^q , $q \geq 1$ away from 0. Note that de facto when E is not compact we will restrict to $p < 2$ near 0 in order to include at least the Gaussian distributions in (CFG_E) and (CFG_{ED}) below. For instance the cost $\rho(x) = |x|(1 + |x|)$ is well suited for distributions with heavier tails than Gaussian.

2.4 Assumptions (CFG)

The joint influence of l_{\pm} , L_{\pm} and b_{\pm} on the allowed tails F^{-1} and G^{-1} is expressed as follows. Remind the sets E and D from (6). We need three different assumptions, each corresponding to the generic cases $E = (0, 1)$, $D = (0, 1)$ and when at least one interval is included in E and one in D .

Studying the case $E = (0, 1)$ we worked out the following conditions (CFG_E) . They only deal with the behavior of F, G, ρ_c at infinity but also involve the orders $b_{\pm} \geq 1$ of the local regular variation (C2) near zero that indeed rule the CLT rate. The case $b_- = 2$ or $b_+ = 2$, which is restricted to sub-Gaussian distributions, is treated separately at Theorem 11.

Assumption (CFG_E) . Assume that $b_- < 2$ and $b_+ < 2$. Assume that for some $\theta_2 > 0$ and

$$(l, \psi) \in \{(l_+, \psi_X^+), (l_-, \psi_X^+), (l_-, \psi_X^-), (l_+, \psi_X^-)\} \quad (12)$$

we have,

(i) if $1 < b < 2$, for all $y > y_0$,

$$l \circ \psi^{-1}(y) \leq \left(1 - \frac{b}{2}\right) y + \log L(\exp(-y/2)) - 2 \log \psi^{-1}(y) - \theta_2 \log y, \quad (13)$$

and,

(ii) if $b = 1$, for all $y > y_0$,

$$l \circ \psi^{-1}(y) \leq \frac{y}{2} - 2 \log \psi^{-1}(y) - \theta_2 \log y. \quad (14)$$

From the study of the case $D = (0, 1)$ the conditions (CFG_D) that comes out only deal with the behavior of F, G, ρ_c at infinity and the CLT rate is standard. The special case where $|F^{-1}(u) - G^{-1}(u)| > \tau > 0$ as $u \rightarrow 1$ and $u \rightarrow 0$ under (C2) with $b > 1$ is already covered by [3]. In order to cover more cases we further impose (16) and allow $b = 1$. Therefore (CFG_D) extends the condition (CFG) in [3].

Assumption (CFG_D) . Let θ_-, θ_+ be the parameter $\theta > 1$ of condition (CFG) in [3] for the left and right tails respectively.

(i) For any (l, ψ) from (12) and $\theta = \theta_+$ if $l = l_+$ or $\theta = \theta_-$ if $l = l_-$ we have

$$(\psi \circ l^{-1})'(y) \geq 2 + \frac{2\theta}{y}, \quad y > y_0. \quad (15)$$

(ii) If

$$\liminf_{u \rightarrow 1} |F^{-1}(u) - G^{-1}(u)| = 0 \quad \text{or} \quad \liminf_{u \rightarrow 0} |F^{-1}(u) - G^{-1}(u)| = 0$$

and for $(l, \psi) = (l_+, \psi_X^+), (l_-, \psi_Y^+)$ or $(l, \psi) = (l_-, \psi_X^-), (l_+, \psi_Y^-)$ respectively, assume that for some $\theta_2 > 0$ it holds

$$l \circ \psi^{-1}(y) \leq \frac{y}{2} - 2 \log \psi^{-1}(y) - \theta_2 \log y, \quad y > y_0. \quad (16)$$

When $D \neq \emptyset$ and $E \neq \emptyset$, two situations arise. Firstly, if E is compact in $(0, 1)$, that is $(A_1 \cup A_\kappa) \subset D$ we only need (CFG_D) . Secondly if at least one among A_1 or A_κ is included in E , which means that $F = G$ on an infinite interval, then we need to also impose (CFG_E) on the involved intervals.

Assumption (CFG_{ED}) . Assume (CFG_D) . If $A_1 \subset E$ then assume (CFG_E) for $(l, \psi) = (l_-, \psi_X^-), (l_+, \psi_X^-)$. If $A_\kappa \subset E$ then assume (CFG_E) for $(l, \psi) = (l_-, \psi_X^+), (l_+, \psi_X^+)$.

Remark 6 If $\gamma_\pm > 0$ we have $\theta_\pm > 2$ and, if $\gamma_\pm = 0$ we have, as in [3],

$$\theta^\pm > 2 - \liminf_{y \rightarrow +\infty} \frac{\log(1/\varepsilon_\pm(y))}{\log l_\pm(y)}$$

where $\varepsilon_\pm(y)$ corresponds to the function $\varepsilon(y)$ of (7) applied to $L(y) = l_\pm(y)$.

Remark 7 As will be seen in the proofs, (CFG_E) and $(FG3)$ imply that we can find b' such that $1 \leq b < b' < 2$ and

$$\int_0^1 \left(\frac{\sqrt{u(1-u)}}{h_X(u)} \right)^b du \leq \int_0^1 \left(\frac{|F^{-1}(u)|}{\sqrt{u(1-u)}} \right)^{b'} du < +\infty \quad (17)$$

which is a little stronger than the necessary condition that the left hand integral is finite. By using $F^{-1}(u) = \psi^{-1}(\log(1/(1-u)))$, (13) also reads

$$(F^{-1}(u))^2 \rho(F^{-1}(u)) \leq \frac{L(\sqrt{1-u})}{(1-u)^{1-b/2} (\log(1/(1-u)))^{\theta_2}}, \quad u > u_0.$$

In particular, if $L(x) = 1$ we deduce that (FG) and (CFG_E) imply

$$\mathbb{P}(X > y) \leq \left(\frac{1}{y^2 \rho(y)} \right)^{2/(2-b)}, \quad y > y_0.$$

This induces the moment conditions of Propositions 2 and 3.

Example 8 For light tails of Weibull type, $\psi(y) = y^w$, $w > 0$, (17) is true and (CFG_E) requires that $l(y^{1/w}) < Cy$ as $y \rightarrow +\infty$ and hence a cost of type $l(y) = y^\gamma$, $y > y_0$ and $l(x) = x^b$, $x < x_0$, is allowed provided that $\gamma < w$ and $1 \leq b < 2$. For heavy tailed distributions such as Pareto, $\psi(y) = p \log y$ with index $p > 2$, the conditions (CFG_E) , (CFG_D) and (CFG_{ED}) induce more constraints. For instance (CFG_E) applied with $\rho_c(x) = x^b$, $x < x_0$, and $l(y) = \alpha \log y$, $y > y_0$, implies that $p > 4/(2-b)$ and $1 \leq \alpha < p(1-b/2-2/p)$, hence the minimal requirement on p is $p \geq 6/(2-b)$. Choosing $\rho_c(x) = x^b$ on \mathbb{R}_+ we have $\alpha = b$ and the last constraint becomes $p > 2(b+2)/(2-b)$.

3 Statement of the results

Consider the joint Gaussian process $\mathcal{G} = \{(\mathbb{B}^X(u), \mathbb{B}^Y(u)) : u \in (0, 1)\}$ with

$$\mathbb{B}^X(u) = \frac{B^X(u)}{h_X(u)}, \quad \mathbb{B}^Y(u) = \frac{B^Y(u)}{h_Y(u)}, \quad (18)$$

where (B^X, B^Y) are two standard Brownian bridges with covariance

$$\text{cov}(B^X(u), B^X(v)) = \text{cov}(B^Y(u), B^Y(v)) = \min(u, v) - uv, \quad u, v \in (0, 1),$$

and cross covariance

$$\text{cov}(B^X(u), B^Y(v)) = H(F^{-1}(u), G^{-1}(v)) - uv, \quad u, v \in (0, 1).$$

The existence of \mathcal{G} is proved in [3]. Let $\mathbb{B}(u) = \mathbb{B}^X(u) - \mathbb{B}^Y(u)$, $u \in (0, 1)$, that is the Gaussian process driving the limit distribution in Propositions 2 and 3 as well as in forthcoming results.

We are now ready to state our main results. Remind (11) and set

$$v_n = \frac{1}{\rho(1/\sqrt{n})} = \frac{n^{b/2}}{L(1/\sqrt{n})} \quad (19)$$

hence, in our first statement we have $K\sqrt{n} \leq v_n = o(n)$ for some $K > 0$. The constants π_- and π_+ come from (C4). Our first result concerns $F = G$.

Theorem 9 *Assume (FG), (C), $E = (0, 1)$ and (CFG_E) , in which case $1 \leq b^-, b^+ < 2$. Then*

$$v_n W_c(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{weak} \pi_- \int_0^1 1_{\{\mathbb{B}(u) < 0\}} |\mathbb{B}(u)|^{b^-} du + \pi_+ \int_0^1 1_{\{\mathbb{B}(u) > 0\}} |\mathbb{B}(u)|^{b^+} du$$

and the limiting r.v. is finite and, if $\mathbb{P}(X = Y) < 1$, positive.

Remark 10 *As shown in [7], and since \mathbb{B}^X is a centered Gaussian process,*

$$\mathbb{P}\left(\int_0^1 |\mathbb{B}^X(u)|^b du < +\infty\right) = 1 \text{ is equivalent to } \int_0^1 \left(\frac{\sqrt{u(1-u)}}{h_X(u)}\right)^b du < +\infty.$$

The latter bound being guaranteed by (CFG_E) and $(FG3)$, which imply (17), the finiteness of the limiting r.v. in Theorem 9 follows.

For light tails one can handle the limiting case $b = 2$ – here stated with $b = b^+ = b^- = 2$ and $L(x) = 1$ for $|x| < x_0$ for sake of simplicity.

Theorem 11 *Assume that $E = (0, 1)$, $(FG1)$, $(FG2)$ and*

$$\lim_{u \rightarrow 0} \frac{u}{h_X(u)} = \lim_{u \rightarrow 1} \frac{1-u}{h_X(u)} = 0, \quad \int_0^1 \frac{u(1-u)}{h_X^2(u)} du < +\infty. \quad (20)$$

Moreover assume (C0) with $\rho_c(x) = x^2$ for $|x| \leq x_0$, hence $b = b^+ = b^- = 2$. Then

$$n W_c(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{weak} \int_0^1 \mathbb{B}(u)^2 du.$$

Notice that Theorem 11 includes the case $W_c = W_2^2$ and shows that the cost function only matters at 0.

Example 12 *For light tails of Weibull type it holds, for some $w > 0$,*

$$h_X(u) = w(1-u) (\log(1/(1-u)))^{1-1/w}$$

and $(1-u)/h_X^2(u) = 1/w ((1-u) \log(1/(1-u)))^{2(1-1/w)}$. The first condition in (20) is then satisfied for $w > 1$ and the second for $w > 2$, so that $w > 2$ is required. This excludes Gaussian tails, as in Theorem 4.6 in [9].

Remark 13 *Theorem 11 requires no assumption on the cost $\rho(y)$ as $y \rightarrow +\infty$. In particular, (C3) may hold with any γ_+, γ_- . Since only sub-Gaussian tails are allowed by (20) the tail part of $W_c(\mathbb{F}_n, \mathbb{G}_n)$ indeed behaves the same as for compactly supported distributions. Namely, empirical extremes of both samples remain simultaneously stuck together very closely to their common deterministic counterpart F^{-1} that increases very slowly.*

Our second main statement is an extension of the main theorem of [3] which now allows F and G to have arbitrarily close tails.

Theorem 14 *Assume (FG), (C), $D = (0, 1)$ and (CFG_D) . Then*

$$\sqrt{n} (W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) \rightarrow_{weak} \mathcal{N}(0, \sigma^2)$$

where

$$\sigma^2 = \mathbb{E}\left(\left(\int_0^1 |\rho'_c(F^{-1}(u) - G^{-1}(u))| \mathbb{B}(u) du\right)^2\right) < +\infty.$$

Remark 15 *The finiteness and a closed form expression for $\sigma^2 = \sigma^2(c, H)$ have been proved in [3]. We also refer to the latter paper for explicit examples in the independent samples case.*

Our third result shows that if there exists a point, or equivalently an open interval, where $F \neq G$ then the rate is \sqrt{n} , whether $E \neq \emptyset$ or not.

Theorem 16 *Assume (FG) , (C) , $D \neq \emptyset$ and (CFG_{ED}) . If $1 < b < 2$ then*

$$\sqrt{n} (W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) \rightarrow_{weak} \mathcal{N}(0, \sigma_D^2)$$

where

$$\sigma_D^2 = \mathbb{E} \left(\left(\int_D |\rho'_c(F^{-1}(u) - G^{-1}(u))| \mathbb{B}(u) du \right)^2 \right) < +\infty.$$

If $b = 1$ then, for $L_{\pm}(0)$ from (10),

$$\begin{aligned} \sqrt{n} (W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) &\rightarrow_{weak} \int_D |\rho'_c(F^{-1}(u) - G^{-1}(u))| \mathbb{B}(u) du \\ &+ 1_{\{b_- = 1\}} L_-(0) \int_E 1_{\{\mathbb{B}(u) < 0\}} |\mathbb{B}(u)| du \\ &+ 1_{\{b_+ = 1\}} L_+(0) \int_E 1_{\{\mathbb{B}(u) > 0\}} |\mathbb{B}(u)| du. \end{aligned}$$

Remark 17 *In the second part of Theorem 16 the first term in the limiting r.v. has distribution $\mathcal{N}(0, \sigma_D^2)$ and is correlated in an explicit way to the other two terms. Theorem 16 also shows that whenever $1 < b < 2$ Theorem 14 remains true if F and G are not stochastically ordered but cross each other at a finite number of points, since this implies $\sigma_D^2 = \sigma^2$.*

The next corollary concerns the L_1 -distance $W_1(\mathbb{F}_n, \mathbb{G}_n) = \|\mathbb{F}_n - \mathbb{G}_n\|_{L_1}$. Remind that $c_{a,1}(z, z') = a_-(z' - z) 1_{z < z'} + a_+(z - z') 1_{z' < z}$.

Corollary 18 *Assume (FG) , (C) and (CFG_{ED}) . Then*

$$\begin{aligned} &\sqrt{n} (W_{c_{a,1}}(\mathbb{F}_n, \mathbb{G}_n) - W_{c_{a,1}}(F, G)) \\ &\rightarrow_{weak} \int_D (a_- 1_{\{F^{-1}(u) < G^{-1}(u)\}} + a_+ 1_{\{F^{-1}(u) > G^{-1}(u)\}}) \mathbb{B}(u) du \\ &+ \int_E (a_- 1_{\{\mathbb{B}(u) < 0\}} + a_+ 1_{\{\mathbb{B}(u) > 0\}}) |\mathbb{B}(u)| du \end{aligned}$$

and, in particular for $a_- = a_+ = 1$,

$$\sqrt{n} (W_1(\mathbb{F}_n, \mathbb{G}_n) - W_1(F, G)) \rightarrow_{weak} \int_D \mathbb{B}(u) du + \int_E |\mathbb{B}(u)| du.$$

It is easily seen that straightforward adaptations of the proof of Theorems 9 to 11 leads to analog results for $\sqrt{n} (W_c(\mathbb{F}_n, G) - W_c(F, G))$ and $v_n W_c(\mathbb{F}_n, F)$ by just replacing $\mathbb{B}(u) = \mathbb{B}^X(u) - \mathbb{B}^Y(u)$ with $\mathbb{B}^X(u)$. In particular we get the following corollary of Theorem 9.

Corollary 19 *Let $1 \leq p < 2$. Assume that F satisfies (FG) and has tails lighter than a Pareto tail with index strictly larger than $2(p+2)/(2-p)$. Then*

$$n^{p/2} W_p^p(\mathbb{F}_n, F) \rightarrow_{weak} \int_0^1 |\mathbb{B}^X(u)|^p du$$

and the limiting r.v. is positive and finite.

We conclude this section by stating the counterpart of Theorem 9 for compactly supported probability distributions. Other extensions to this case of the above results are likewise easy.

Corollary 20 Assume wlog that $F = G$ has support $[0, 1]$ and is twice differentiable with positive derivative f on $(0, 1)$. Assume moreover $(FG2)$, $(FG3)$ and (C) except $(C3)$ with $b^- < b'$ and $b^+ < b'$ where $b' > 1$ and

$$\int_0^1 \left(\frac{\sqrt{u(1-u)}}{h_X(u)} \right)^{b'} du < +\infty. \quad (21)$$

Then

$$v_n W_c(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{weak} \pi_- \int_0^1 1_{\{\mathbb{B}(u) < 0\}} |\mathbb{B}(u)|^{b^-} du + \pi_+ \int_0^1 1_{\{\mathbb{B}(u) > 0\}} |\mathbb{B}(u)|^{b^+} du$$

and the limiting r.v. is finite and, if $\mathbb{P}(X = Y) < 1$, positive.

This extends Theorem 19 of [3] to the case $F = G$ and reduces (CFG_E) to the integrability assumption with no restriction on b , since the influence of the cost is limited to its behaviour near 0.

Example 21 The Beta distribution with parameters $\alpha > 0$ and $\beta > 0$, has density $f(x) = \mathcal{B}(\alpha, \beta) x^{\alpha-1} (1-x)^{\beta-1}$ on $(0, 1)$. Clearly $(FG2)$ and $(FG3)$ are satisfied, and since (21) is true for any $b' > 1$, the previous result applies for any $b^-, b^+ \geq 1$.

This is not always the case. For instance, consider a c.d.f. F on $(0, 1)$ equal to $e^{-1/|\log x|^w}$, $w > 0$ near 0 – and symmetrically near 1. Then it satisfies $(FG2)$ and $(FG3)$ but only satisfies (21) for $b' \leq 2$. Hence the previous result applies for $1 \leq b^-, b^+ < 2$.

4 Applications

4.1 Comparison and goodness-of-fit tests

A consequence of Theorems 9 and 16 is the construction of a statistical test of the hypothesis $\mathcal{H}_0 : F = G$ against $\mathcal{H}_1 : F \neq G$, based on two samples that may arise from correlated experiments. Let us choose the b -Wasserstein distance with $1 < b < 2$. The distributions F and G are supposed to be C^2 on \mathbb{R} or \mathbb{R}^+ and satisfy (CFG_{ED}) and (FG) . By Theorem 9, under \mathcal{H}_0 the statistic $n^{b/2} W_c(\mathbb{F}_n, \mathbb{G}_n)$ converges to a positive finite random variable while by Theorem 16, under \mathcal{H}_1 it converges almost surely to $+\infty$ at the rate $n^{b/2} W_c(F, G)$. Mathematically this test is effectively valid when the set $D = \{F^{-1} \neq G^{-1}\}$ is a finite union of non empty intervals, but we think that its validity could be extended to the more general case where D is of positive Lebesgue measure in $(0, 1)$. The use of W_2^2 , with a rate n is more restrictive since it needs very light tails. Nevertheless if sub-Gaussian tails can be asserted, by Theorem 11 the previous test works with $b = 2$, which actually is a new test.

In each case the rather minimal (CFG) type conditions have to be checked. They are close to be necessary in the proofs to overcome the difficulty of controlling how close the empirical tails of \mathbb{F}_n and \mathbb{G}_n must be under \mathcal{H}_0 , and how far $|\mathbb{F}_n - \mathbb{G}_n|$ can deviate from $|F - G|$ in tails under \mathcal{H}_1 . Interestingly the choice of $\rho(x)$ may be with a locally polynomial shape as $x \rightarrow 0$ and a different shape as $x \rightarrow +\infty$ possibly linear, polynomial or exponential. This flexibility allows to test the tail or the mid-quantiles with more or less accuracy.

In the same vein, concerning the distribution functions, Corollary 18 yields

$$\begin{aligned} & \sqrt{n} \left(\int_{-\infty}^{+\infty} |\mathbb{F}_n(t) - \mathbb{G}_n(t)| dt - \int_{F^{-1}(D)} |F(t) - G(t)| dt \right) \\ & \rightarrow_{weak} \int_D \mathbb{B}(u) du + \int_E |\mathbb{B}(u)| du \end{aligned}$$

which seems not to have been already obtained. This provides weak limits for the power of the test under alternatives to $\mathcal{H}_0 : F = G$ of the kind $\mathcal{H}_1 : F = G_1$ where G_1^{-1} only differs from G^{-1} on an interval D , for instance with a slightly different right hand tail only. The test statistic $\sqrt{n} \int_{-\infty}^{+\infty} |\mathbb{F}_n(t) - \mathbb{G}_n(t)| dt$ has an almost sure first order rate of escape $\sqrt{n} \int_{G_1^{-1}(D)} |G_1(t) - G(t)| dt$.

As a by-product of the results of Section 3 one can similarly build goodness-of-fit tests $\mathcal{H}_0 : F = F_0$ against $\mathcal{H}_1 : F \neq F_0$ by using one sample under F or by using an additional sample distributed as F_0 . Notice that the test associated to $b = 2$ was a consequence of [9].

4.2 An application

The motivation of our initial work was intimately related to the field of computer experiments. Many computer codes produce as output values of a function computed on so many points that it can be considered as a functional output. The case we are interested in is when this function is the *c.d.f.* of a real *r.v.* It turns out that Wasserstein distances are now commonly used to analyze such outputs. In view of defining new features for random *c.d.f.* such as median or quantiles, more general Wasserstein costs may be used as contrasts to compute these features by solving an optimization problem – see [11]. Nevertheless computer codes only provide samples of the underlying distributions. Whence the importance of an efficient estimation of distances between *c.d.f.* and goodness-of-fit tests through random samples.

As an illustration, let us conclude with a notion of quantile for a *r.v.* taking values in the set of continuous *c.d.f.*'s. A useful new result of this article is the first part of Corollary 18 which is strongly related to the preprint [11]. Let $0 < \alpha < 1$. In [11] the α -quantile F_α of a random continuous *c.d.f.* \mathbb{F} is defined to be

$$F_\alpha = \underset{\theta \in \mathcal{F}}{\text{Argmin}} \mathbb{E} W_c(\mathbb{F}, \theta),$$

where $c(x, y) = (x - y)(\alpha - \mathbf{1}_{x-y < 0})$ is the non-symmetric contrast for the α -quantile of a real *r.v.* and \mathcal{F} is the set of continuous *c.d.f.* As previously mentioned, in practice a realization $\mathbb{F}(\omega)$ of \mathbb{F} is known through a n -sample of the distribution $\mathbb{F}(\omega)$. Hence we may assume that a N -sample $\mathbb{F}_n^1, \dots, \mathbb{F}_n^N$ is available, where each \mathbb{F}_n^i is a n -empirical *c.d.f.* of \mathbb{F}^i and $\mathbb{F}^1, \dots, \mathbb{F}^N$ are *i.i.d.* according to \mathbb{F} . Define

$$F_{N,n,\alpha} = \underset{\theta \in \mathcal{F}_n}{\text{Argmin}} \frac{1}{N} \sum_{i=1}^N W_c(\mathbb{F}_n^i, \theta),$$

where \mathcal{F}_n is the set of *c.d.f.* with at most n different values. Then one could use Corollary 18 to prove that $F_{N,n,\alpha}$ is a consistent estimator of F_α when N and n tend to $+\infty$, and determine the rate of convergence.

5 Proofs

In the forthcoming proofs the high order quantiles are shown to have a secondary order impact compared to the mid-order quantiles that impose the rate as well as the limiting distribution under our sufficient conditions ensuring that the tails are not too heavy. For sake of simplicity we only work on the right hand tail, with quantiles of order $u \in (\underline{u}, 1)$ for an arbitrary small $\underline{u} > 0$. The counterpart for the left hand tail is immediate by using the same arguments.

To help the reader the variable of frequently used deterministic functions defined on \mathbb{R}_+ like $\rho_\pm, \rho_\pm^{-1}, l_\pm, l_\pm^{-1}$ or L_\pm is denoted x when considered as $x \rightarrow 0$ and y when considered as $y \rightarrow +\infty$. In the subsequent proofs the constant $K > 0$ may change at each appearance.

In steps numbered 0 we remind active hypotheses while introducing local notation. The non standard Steps 1, 2 and 3 of the four proofs – including the one in [3] – are designed to address the non trivial problem of controlling the high order and extreme order quantiles under an explicit and almost minimal assumption on tails, namely (CFG_E) , (CFG_D) or (CFG_{ED}) . The secondary order terms in these conditions could be balanced slightly more sharply but at the price of adding technicalities to connect Steps 1 and 2. Finally we point out that the convergence at Steps 3 is weaker than in probability, due to the coupling approach.

5.1 The case $F = G$

We prove Theorem 9.

Step 0. In this section $F = G$ and hence $E = \mathbb{R}$. For short, the key functions common to X, Y are denoted F^{-1}, ψ, H and h . Let assume (FG) , (C) and (CFG_E) with $1 \leq b_\pm < 2$ in $(C2)$. Hence $\rho(x) = \max(\rho_+(x), \rho_-(x)) \geq \rho_c(x)$ and $\rho_\pm(x)$ are positive convex increasing functions defined on \mathbb{R}_+^* with $\rho_\pm(0) = 0$. For $0 \leq x \leq x_0$ we have $\rho_\pm(x) = x^{b_\pm} L_\pm(x)$ and, whenever $b_\pm = 1$ it is also assumed through (10) that $\lim_{x \rightarrow 0} L_\pm(x) = L_\pm(0) < +\infty$. Recall that $b = \min(b_+, b_-)$ and, for $0 \leq x \leq x_0$, $\rho(x) = \max(\rho_+(x), \rho_-(x)) = x^b L(x)$ where $L(x)$ is defined at (11) and is slowly varying as $x \rightarrow 0$. We then have

$$v_n = \frac{1}{\rho(1/\sqrt{n})}, \quad \lim_{n \rightarrow +\infty} \frac{\sqrt{n}}{v_n} = 1_{\{b=1\}} L(0).$$

Since $L \in RV(0, 0)$ we have, by the Karamata representation theorem,

$$L(x) = \exp \left(\eta(x) + \int_B^{1/x} \frac{s(y)}{y} dy \right), \quad 0 < x \leq x_0, \quad (22)$$

with $B > 0$, $\eta(x)$ and $s(y)$ are bounded measurable functions such that

$$\lim_{x \rightarrow 0} \eta(x) = \eta_\infty \in \mathbb{R}, \quad \lim_{y \rightarrow +\infty} s(y) = 0.$$

We can then define

$$\eta_0 = \sup_{0 < x \leq x_0} |\eta(x)| \in \mathbb{R}_+, \quad c_0 = e^{2\eta_0} \geq 1. \quad (23)$$

For y large it holds $\rho_\pm(y) = \exp(l_\pm(y))$ where the functions $l_\pm(y)$ are not asked to be in $RV(+\infty, \gamma_\pm)$ in this proof, but (7) does matter. However in practice if (C3) would not hold then (CFG_E) would be more difficult to translate in terms of admissible F . Hence, for some $y_0 > x_0$,

$$\rho(y) = \exp(l(y)), \quad l(y) = \max(l_+(y), l_-(y)), \quad y \geq y_0.$$

Since ρ_\pm and ρ are convex, by (7) there exists $d_\pm \geq 1$, $d = \min(d_-, d_+)$ and $d_{0,\pm}$, d_0 such that

$$l_\pm(y) \geq d_\pm \log y + d_{0,\pm}, \quad l(y) \geq d \log y + d_0, \quad y \geq y_0. \quad (24)$$

By (CFG_E), the joint influence of l , L and b on the allowed tails F^{-1} is expressed at (13) if $b > 1$ and (14) if $b = 1$.

We decompose the integral $W_c(\mathbb{F}_n, \mathbb{G}_n)$ as follows, with the three remainder terms implicitly treated in a similar way for left hand tails. We will specify later two positive sequences i_n and j_n such that $n > j_n > i_n \rightarrow +\infty$. The proof consists in four steps, each dealing with one of the four terms

$$W_c(\mathbb{F}_n, \mathbb{G}_n) = I_{\mathcal{I}_n} + I_{\mathcal{J}_n} + I_{\mathcal{K}_n} + I_{\mathcal{L}}, \quad I_A = \int_A \rho_c(\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u)) du, \quad (25)$$

where $\mathcal{I}_n = (1 - i_n/n, 1]$, $\mathcal{J}_n = (1 - j_n/n, 1 - i_n/n]$, $\mathcal{K}_n = (\bar{u}, 1 - j_n/n]$, $\mathcal{L} = [\underline{u}, \bar{u}]$ and $0 < \underline{u} < 1/2 < \bar{u} < 1$. In order to accurately choose i_n and j_n one has to take into account two difficulties. First, the rate $1/v_n$ is faster than $1/\sqrt{n}$ so that $\mathcal{I}_n \cup \mathcal{J}_n$ should be sufficiently small. Second, the empirical extreme quantile difference $\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u)$ may be either very large or very small as $u \rightarrow 1$, thus the cost function $\rho_c(\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u))$ is evaluated at 0 on some random subsets of $\mathcal{I}_n \cup \mathcal{J}_n$ and at $+\infty$ on some others. The later problem is the most difficult to address.

Step 1. Let K_n be a positive sequence such that $K_n \rightarrow +\infty$ and define

$$i_n = \frac{n}{v_n K_n \rho(\psi^{-1}(\log n + K_n))}. \quad (26)$$

Notice that (FG1) and (24) imply that $\rho(\psi^{-1}(\log n + K_n)) \rightarrow +\infty$ and $i_n = o(n^{1-b/2} L(1/\sqrt{n})/K_n)$ as $n \rightarrow +\infty$, so that $i_n/\sqrt{n} \rightarrow 0$ even when $b = 1$, thanks to (10). The following lemma ensures that $i_n/\log \log n \rightarrow +\infty$. Observe also that $\psi^{-1}(\log n + K_n) = F^{-1}(1 - 1/ne^{K_n})$ is an extreme quantile just beyond the expected order $F^{-1}(1 - 1/n)$ for $X_{(n)}$ and $Y_{(n)}$, which is the key to Lemma 22. Let $[y]$ denote the integer part of y . Consider the *r.v.*

$$I_{\mathcal{I}_n} \leq \int_{\mathcal{I}_n} \rho(\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u)) du = \frac{1}{n} \sum_{i=n-[i_n]}^n \rho(X_{(i)} - Y_{(i)}).$$

Lemma 22 Assume (FG1), (C) and (CFG_E). There exists K_n such that

$$K_n \rightarrow +\infty, \quad \lim_{n \rightarrow +\infty} \frac{K_n}{\log \log n} = 0, \quad \liminf_{n \rightarrow +\infty} \frac{\log i_n}{\log \log n} \geq \theta_2 > 0$$

and

$$\lim_{n \rightarrow +\infty} v_n I_{\mathcal{I}_n} = 0 \text{ in probability.}$$

Proof. (i) Let $K_n \rightarrow +\infty$, $K_n/\log \log n \rightarrow 0$ be as slow as needed later. By (FG1) we have $F^{-1}(1 - 1/ne^{K_n}) \rightarrow +\infty$ as $n \rightarrow +\infty$, yet arbitrarily slowly. Thus, by (13) and (26) we have, for any $\theta'' > 1 - b/2$, any $\theta' < \theta_2$ and all n large,

$$\begin{aligned} i_n &= \frac{n^{1-b/2} L(1/\sqrt{n})}{K_n \rho(\psi^{-1}(\log n + K_n))} \\ &\geq \frac{1}{K_n} \frac{L(1/\sqrt{n})}{L(1/\sqrt{ne^{K_n}})} \exp\left(-\left(1 - \frac{b}{2}\right) K_n + 2 \log \psi^{-1}(\log n + K_n) + \theta_2 \log(\log n + K_n)\right) \\ &\geq \frac{L(1/\sqrt{n})}{L(1/\sqrt{ne^{K_n}})} \frac{1}{e^{\theta'' K_n}} \left(F^{-1}\left(1 - \frac{1}{ne^{K_n}}\right)\right)^2 (\log n + K_n)^{\theta_2} \\ &\geq \frac{L(1/\sqrt{n})}{L(1/\sqrt{ne^{K_n}})} (\log n)^{\theta'}. \end{aligned}$$

Applying (22) and $K_n \rightarrow +\infty$ we get

$$\frac{L\left(\frac{1}{\sqrt{ne^{K_n}}}\right)}{L(1/\sqrt{n})} = \exp\left(\eta\left(\frac{1}{\sqrt{ne^{K_n}}}\right) - \eta\left(\frac{1}{\sqrt{n}}\right) + \int_{\sqrt{n}}^{\sqrt{ne^{K_n}}} \frac{s(y)}{y} dy\right).$$

Since $e^{K_n} < \log n$ we can furthermore choose K_n such that

$$K_n < \frac{1}{s_n}, \quad s_n = \sup_{\sqrt{n} \leq y \leq \sqrt{n \log n}} s(y),$$

where $s_n \rightarrow 0$ as $n \rightarrow +\infty$. The slower is L the faster is $1/s_n$ hence the resulting requirement is sometimes only the initial $K_n/\log \log n \rightarrow 0$. We readily obtain, by (23),

$$\limsup_{n \rightarrow +\infty} \frac{L\left(\frac{1}{\sqrt{ne^{K_n}}}\right)}{L(1/\sqrt{n})} \leq \limsup_{n \rightarrow +\infty} \exp\left(2\eta_0 + s_n \frac{K_n}{2}\right) < +\infty.$$

The claimed deterministic lim inf is proved by letting $\theta' \rightarrow \theta_2$. Notice that (CFG_E) was crucially required.

(ii) Concerning the stochastic integral $I_{\mathcal{I}_n}$ the choice of i_n in (26) is minimal to guaranty the rate v_n and (CFG_E) is not required. Recall that F has support \mathbb{R} . Fix $\varepsilon > 0$ and consider the events

$$A_n = \{v_n I_{\mathcal{I}_n} \geq 4\varepsilon\}, \quad B_n = \{X_{(n-[i_n])} > 0 \cap Y_{(n-[i_n])} > 0\}.$$

We have $\mathbb{P}(A_n) \leq \mathbb{P}(A_n \cap B_n) + \mathbb{P}(B_n^c)$ and $\mathbb{P}(B_n^c) \rightarrow 0$ as $n \rightarrow +\infty$. On B_n it holds

$$v_n I_{\mathcal{I}_n} \leq \frac{v_n}{n} \sum_{i=n-[i_n]}^n (\rho_+(X_{(i)}) + \rho_-(Y_{(i)})) \leq \frac{v_n}{n} (i_n + 1) (\rho_+(X_{(n)}) + \rho_-(Y_{(n)}))$$

hence $\mathbb{P}(A_n \cap B_n) \leq \mathbb{P}(C_{n,X}) + \mathbb{P}(C_{n,Y})$ where

$$C_{n,X} = \left\{ \rho_+(X_{(n)}) \geq \varepsilon \frac{n}{v_n i_n} \right\}, \quad C_{n,Y} = \left\{ \rho_-(Y_{(n)}) \geq \varepsilon \frac{n}{v_n i_n} \right\}.$$

In order to evaluate $\mathbb{P}(C_{n,X}) = 1 - (1 - \mathbb{P}(\rho_+(X) > \varepsilon n/v_n i_n))^n$ we combine $\rho_+^{-1}(x) = l_+^{-1}(\log x)$, $l^{-1} \leq l_+^{-1}$ and $\psi_X = \psi$ with (26) to obtain, for n large enough to have $K_n > 1/\varepsilon$,

$$\begin{aligned} & \mathbb{P}(\rho_+(X) > \varepsilon K_n \rho(\psi^{-1}(\log n + K_n))) \\ & \leq \exp(-\psi \circ l^{-1}(\log \varepsilon + \log K_n + l(\psi^{-1}(\log n + K_n)))) \\ & \leq \frac{1}{ne^{K_n}}. \end{aligned}$$

Therefore $\mathbb{P}(C_{n,X}) \leq 1 - \exp(-\exp(-K_n)) \sim \exp(-K_n) \rightarrow 0$ as $n \rightarrow +\infty$, and similarly $\mathbb{P}(C_{n,Y}) \rightarrow 0$. This implies that $v_n I_{\mathcal{I}_n} \rightarrow 0$ in probability. \square

Step 2. Write $\beta_n(u) = \beta_n^X(u) - \beta_n^Y(u)$ with

$$\beta_n^X(u) = \sqrt{n}(\mathbb{F}_n^{-1}(u) - F^{-1}(u)), \quad \beta_n^Y(u) = \sqrt{n}(\mathbb{G}_n^{-1}(u) - F^{-1}(u)), \quad (27)$$

thus $I_A = \int_A \rho_c(\beta_n(u)/\sqrt{n}) du$ in (25). Let $\Delta_n = \mathcal{J}_n \cup \mathcal{K}_n \cup \mathcal{L} = [\underline{u}, 1 - i_n/n]$. The next lemma shows that in the integral I_{Δ_n} the cost function ρ is evaluated near 0 provided that n is large.

Lemma 23 Assume (FG) and (CFG_E) . For any $0 < \xi < 1/2 - b/4$ it holds

$$\lim_{n \rightarrow +\infty} (\log n)^\xi \sup_{u \in \Delta_n} \frac{|\beta_n(u)|}{\sqrt{n}} = 0 \quad \text{a.s.}$$

Proof. (i) Assuming $(FG1)$, $(FG2)$ and since $i_n/\log \log n \rightarrow +\infty$ by Lemma 22 we can apply the classical hungarian results to $|\beta_n(u)| \leq |\beta_n^X(u)| + |\beta_n^Y(u)|$ exactly as for Lemma 23 in [3] to get

$$\limsup_{n \rightarrow +\infty} \sup_{u \in \Delta_n} \frac{h(u) |\beta_n(u)|}{\sqrt{(1-u) \log \log n}} \leq 8 \quad \text{a.s.} \quad (28)$$

Next observe that (FG3) implies, for some $0 < M < +\infty$ and $u \in \Delta_n$,

$$\begin{aligned} \frac{1}{M} \frac{\sqrt{1-u}}{h(u)} \sqrt{\frac{\log \log n}{n}} &\leq \frac{F^{-1}(u)}{\sqrt{1-u}} \sqrt{\frac{\log \log n}{n}} \\ &\leq \varepsilon_n = \frac{F^{-1}(1-i_n/n)}{\sqrt{i_n}} \sqrt{\log \log n}. \end{aligned} \quad (29)$$

(ii) Remind that $e^{-K_n} < 1 < i_n$ for all n large, and $F^{-1}(1-i_n/n) \rightarrow +\infty$ as $n \rightarrow +\infty$ with no obvious control on the rate. By (26) and the consequence (13) of (CFG_E) we have already seen in the proof of Lemma 22 that if $\theta'' < 1 - b/2$ and $\theta'' < \theta' < \theta_2$ then it holds, for all n large enough,

$$\begin{aligned} i_n &\geq \frac{1}{e^{\theta'' K_n}} \left(F^{-1} \left(1 - \frac{1}{ne^{K_n}} \right) \right)^2 (\log n + K_n)^{\theta_2} \\ &\geq \left(F^{-1} \left(1 - \frac{1}{n} \right) \right)^2 (\log n)^{\theta'} \\ &\geq \left(F^{-1} \left(1 - \frac{i_n}{n} \right) \right)^2 (\log \log n)(\log n)^{\theta''} \end{aligned}$$

hence for any $0 < \xi < \theta''/2$ it holds $\lim_{n \rightarrow +\infty} (\log n)^\xi \varepsilon_n = 0$. The conclusion follows, by (28) and (29). \square

Let $j_n = n^\beta$ with $1/2 < \beta < 1$, so that $i_n < \sqrt{n} < j_n$ for all n large. Remind ε_n from (29). Let introduce

$$\varepsilon_n(u) = 9 \frac{\sqrt{1-u}}{h(u)} \sqrt{\frac{\log \log n}{n}} \leq 9\varepsilon_n, \quad u \in \mathcal{J}_n. \quad (30)$$

Lemma 24 Assume (FG), (C) and (CFG_E). Then we have

$$\lim_{n \rightarrow +\infty} v_n I_{\mathcal{J}_n} = 0 \quad a.s.$$

Proof. (i) By Lemma 23, for all n large enough and any $u \in \mathcal{J}_n$ it holds

$$\frac{1}{n} \leq \sup_{u \in \mathcal{J}_n} \sqrt{\frac{1-u}{n}} \leq \varepsilon_n(u) \leq \frac{1}{(\log n)^\xi}.$$

Consider L defined in (11). Using (22) and (23) we get

$$L_n = \sup_{u \in \mathcal{J}_n} \frac{L(\varepsilon_n(u))}{L(1/\sqrt{n})} \leq \exp \left(2\eta_0 + \int_{(\log n)^\xi}^n \frac{|s(y)|}{y} dy \right) \quad (31)$$

hence

$$\lim_{n \rightarrow +\infty} \frac{\log L_n}{\log n} \leq \lim_{n \rightarrow +\infty} \frac{1}{\log n} \left(2\eta_0 + \log n \sup_{(\log n)^\xi \leq y \leq n} |s(y)| \right) = 0. \quad (32)$$

(ii) Remind that ρ_\pm are increasing. By Lemma 23 and (C2) we almost surely have, for all n large,

$$I_{\mathcal{J}_n} \leq \int_{\mathcal{J}_n \cap \{\beta_n \geq 0\}} \rho_+(\varepsilon_n(u)) du + \int_{\mathcal{J}_n \cap \{\beta_n < 0\}} \rho_-(\varepsilon_n(u)) du$$

where, by (28), (29) and (30), $\sup_{u \in \mathcal{J}_n} \varepsilon_n(u) \leq 9\varepsilon_n \rightarrow 0$. Hence, recalling (11) we are reduced to study the bounding deterministic integral

$$I_{\mathcal{J}_n} \leq \int_{\mathcal{J}_n} \rho(\varepsilon_n(u)) du = \int_{\mathcal{J}_n} (\varepsilon_n(u))^b L(\varepsilon_n(u)) du.$$

By (11), L_n from (31) and (FG3) we further have

$$v_n I_{\mathcal{J}_n} \leq L_n (\log \log n)^{b/2} \int_{\mathcal{J}_n} \left(\frac{F^{-1}(u)}{\sqrt{1-u}} \right)^b du. \quad (33)$$

We next show that $L_n (\log \log n)^{b/2}$ is a secondary order factor compared to the integral in (33), whatever the choice of $1/2 < \beta < 1$ defining j_n in \mathcal{J}_n .

(iii) The fact that $l(y) \geq \log y$ as $y \rightarrow +\infty$ combined to (CFG_E) shows that for all u large enough, we have

$$\begin{aligned} F^{-1}(u) &= \psi^{-1} \left(\log \frac{1}{1-u} \right) \\ &\leq \exp \left(l \circ \psi^{-1} \left(\log \frac{1}{1-u} \right) \right) \\ &\leq \exp \left(\left(1 - \frac{b}{2} \right) \log \frac{1}{1-u} + \log L(\sqrt{1-u}) - 2 \log F^{-1}(u) - \theta_2 \log \log \frac{1}{1-u} \right). \end{aligned}$$

Therefore we get

$$F^{-1}(u)^3 \leq \left(\frac{1}{1-u} \right)^{1-b/2} \frac{L(\sqrt{1-u})}{(\log(1/(1-u)))^{\theta_2}}$$

and

$$\begin{aligned} \int_{\mathcal{J}_n} \left(\frac{F^{-1}(u)}{\sqrt{1-u}} \right)^b du &\leq \int_{\mathcal{J}_n} \left(\frac{1}{1-u} \right)^{(1-b/2)b/3+b/2} \frac{L(\sqrt{1-u})^{b/3}}{(\log(1/(1-u)))^{\theta_2 b/3}} du \\ &\leq \int_{\mathcal{J}_n} \left(\frac{1}{1-u} \right)^{b(5-b)/6} \frac{L(\sqrt{1-u})^{b/3}}{(\log(1/(1-u)))^{\theta_2 b/3}} du. \end{aligned} \quad (34)$$

Since $1 \leq b < 2$ we can find γ such that $0 < b(5-b)/6 < \gamma < 1$. The second factor in the integral (34) is slowly varying in $1-u$ as $u \rightarrow 1$ thus the whole integral is ultimately bounded from above by

$$(1-\gamma) \int_{\mathcal{J}_n} \left(\frac{1}{1-u} \right)^\gamma du = [-(1-u)^{1-\gamma}]_{1-j_n/n}^{1-i_n/n} \leq \frac{1}{n^{(1-\gamma)(1-\beta)}}. \quad (35)$$

We deduce from (32), (33), (34) and (35) the convergence

$$\lim_{n \rightarrow +\infty} v_n I_{\mathcal{J}_n} \leq \lim_{n \rightarrow +\infty} \frac{L_n(\log \log n)^{b/2}}{n^{(1-\gamma)(1-\beta)}} = 0 \quad a.s.$$

at a power rate. \square

Step 3. Compared to \mathcal{J}_n the interval \mathcal{K}_n is so large that $v_n I_{\mathcal{K}_n}$ can no more converge to zero in probability. Instead it is made small with high probability by choosing \bar{u} and β properly, at Lemma 26. Moreover, in order to evaluate the integral of $\rho_c(\beta_n(u)/\sqrt{n})$ over \mathcal{K}_n accurately enough it is no more sufficient to bound the process, therefore we approximate it at Lemma 25 by a Gaussian process which helps revealing the underlying deterministic integral to compute. Lastly the fact that $\beta_n(u)$ itself may be very small or very large along \mathcal{K}_n makes a bit tedious the uniform control of the slowly varying part $L(x)$ of $\rho(x)$.

Define $\Delta'_n = (j_n/n, 1 - j_n/n)$. We first recall the strong approximation of the joint quantile processes

$$\mathcal{Q}_n(u) = (\beta_n^X(u), \beta_n^Y(u)), \quad u \in \Delta'_n,$$

by the joint Gaussian processes

$$\mathcal{G}_n(u) = (\mathbb{B}_n^X(u), \mathbb{B}_n^Y(u)), \quad \mathbb{B}_n^X(u) = \frac{B_n^X(u)}{h_X(u)}, \quad \mathbb{B}_n^Y(u) = \frac{B_n^Y(u)}{h_Y(u)}, \quad u \in \Delta'_n,$$

where $B_n^X(u) = \mathbb{H}_n(H_X(u))$, $B_n^Y(u) = \mathbb{H}_n(H^Y(u))$ and \mathbb{H}_n is a $\mathbb{P}^{X,Y}$ -Brownian bridge indexed by the halfplanes

$$H_X(u) = \{(x, y) : x \leq F^{-1}(u)\}, \quad H^Y(u) = \{(x, y) : y \leq F^{-1}(u)\}.$$

Therefore B_n^X and B_n^Y are two standard Brownian bridges with cross covariance given for $u, v \in (0, 1)$ by

$$\begin{aligned} cov(B_n^X(u), B_n^Y(v)) &= \mathbb{P}^{X,Y}(H_X(u) \cap H^Y(v)) - \mathbb{P}^{X,Y}(H_X(u))\mathbb{P}^{X,Y}(H^Y(v)) \\ &= \mathbb{P}(X \leq F^{-1}(u), Y \leq F^{-1}(v)) - uv \\ &= H(F^{-1}(u), F^{-1}(v)) - uv. \end{aligned}$$

Notice that $H(F^{-1}(u), F^{-1}(v))$ is the copula function of (X, Y) . From now and for the remainder of the proof we work on the probability space of the following Lemma 25. The weak convergence finally established on this space at Steps 4 and 5 remains valid on any probability space.

Lemma 25 Assume (FG) . Then we can build on the same probability space versions of $(X_n, Y_n)_{n \geq 1}$ and $(\mathbb{H}_n)_{n \geq 1}$ such that $\mathcal{Q}_n(u) = \mathcal{G}_n(u) + \mathcal{Z}_n(u)$ for all $n \geq 1$ and $u \in \Delta'_n$ where $\mathcal{Z}_n(u) = (Z_n^X(u)/h_X(u), Z_n^Y(u)/h_Y(u))$ satisfies, for some $v \in (0, 1/22)$,

$$\lim_{n \rightarrow +\infty} n^v \sup_{u \in \Delta'_n} |Z_n^X(u)| = \lim_{n \rightarrow +\infty} n^v \sup_{u \in \Delta'_n} |Z_n^Y(u)| = 0 \quad a.s.$$

Proof. This follows from Theorem 28 in [3] with $F = G$. \square

The joint strong approximation of Lemma 25 applied with $F = G$ and $h_X = h_Y = h$ combined to (CFG_E) provides a stochastic control of the deviations of $v_n I_{\mathcal{K}_n}$ that is weaker than in probability but sufficient for the targeted weak convergence. Since it concerns the probability distribution of $I_{\mathcal{K}_n}$ the following lemma remains true on any probability space.

Lemma 26 Assume (FG) , (C) and (CFG_E) . There exists $\beta \in (1/2, 1)$ such that for any choice of $\lambda > 0$ and $\varepsilon > 0$ one can find $\bar{u}_0 \in (1/2, 1)$ and $n_0 > 0$ such that, for all $\bar{u} \in [\bar{u}_0, 1)$ and $n > n_0$,

$$\mathbb{P}(v_n I_{\mathcal{K}_n} > \lambda) < \varepsilon.$$

Proof. Fix $\lambda > 0$ and $\varepsilon > 0$ then consider, with β_n as in (27) the event

$$\mathcal{C}_n^\lambda = \left\{ v_n \int_{\mathcal{K}_n} \rho_c \left(\frac{\beta_n(u)}{\sqrt{n}} \right) du > \lambda \right\}.$$

(i) For $0 < \tau < \min(1, \lambda/2)$ define the random sets

$$\mathcal{K}_n^{<\tau} = \{u \in \mathcal{K}_n : |\beta_n(u)| < \tau\}, \quad \mathcal{K}_n^{>\tau} = \mathcal{K}_n \setminus \mathcal{K}_n^{<\tau}.$$

Recalling that the cost ρ_\pm is convex, positive and such that $\rho_\pm(0) = 0$ we have $\rho_\pm(\tau x) \leq \tau \rho_\pm(x)$ for all $x \geq 0$. It follows that

$$\begin{aligned} v_n I_{\mathcal{K}_n^{<\tau}} &\leq v_n \int_{\mathcal{K}_n^{<\tau} \cap \{\beta_n < 0\}} \tau \rho_+ \left(\frac{1}{\sqrt{n}} \right) du + v_n \int_{\mathcal{K}_n^{<\tau} \cap \{\beta_n \geq 0\}} \tau \rho_- \left(\frac{1}{\sqrt{n}} \right) du \\ &\leq \frac{\max(\rho_-(1/\sqrt{n}), \rho_+(1/\sqrt{n}))}{\rho(1/\sqrt{n})} \tau \int_{\mathcal{K}_n^{<\tau}} du \leq \tau. \end{aligned}$$

As a consequence,

$$\mathbb{P}(\mathcal{C}_n^\lambda) = \mathbb{P}(v_n(I_{\mathcal{K}_n^{<\tau}} + I_{\mathcal{K}_n^{>\tau}}) > \lambda) \leq \mathbb{P}(v_n I_{\mathcal{K}_n^{>\tau}} \geq \lambda - \tau) \leq \mathbb{P}\left(v_n I_{\mathcal{K}_n^{>\tau}} \geq \frac{\lambda}{2}\right).$$

(ii) For all $n \geq n_0$ and $n_0 = n_0(\varepsilon, \xi)$ large enough we have $(\log n)^\xi < \sqrt{n}$ together with, by Lemma 23 and since $\mathcal{K}_n^{>\tau} \subset \mathcal{K}_n \subset \Delta_n$,

$$\mathbb{P}(\mathcal{D}_n) > 1 - \frac{\varepsilon}{2}, \quad \mathcal{D}_n = \left\{ \sup_{u \in \mathcal{K}_n^{>\tau}} \frac{|\beta_n(u)|}{\sqrt{n}} \leq \frac{1}{(\log n)^\xi} \right\}.$$

Assume now that $n \geq n_0$. On the event \mathcal{D}_n , for any $u \in \mathcal{K}_n^{>\tau}$ we have

$$\frac{\tau}{\sqrt{n}} \leq \min \left(\frac{1}{\sqrt{n}}, \frac{|\beta_n(u)|}{\sqrt{n}} \right) \leq \frac{1}{(\log n)^\xi}$$

which by (11), (22) and (23) yields

$$\begin{aligned} \frac{L(|\beta_n(u)|/\sqrt{n})}{L(1/\sqrt{n})} &\leq \exp \left(2\eta_0 + \int_{\min(\sqrt{n}, \sqrt{n}/|\beta_n(u)|)}^{\sqrt{n}/\tau} \frac{|s(y)|}{y} dy \right) \\ &\leq c_0 \exp \left(s_n \int_{\min(\sqrt{n}, \sqrt{n}/|\beta_n(u)|)}^{\sqrt{n}/\tau} \frac{1}{y} dy \right) \\ &= c_0 \exp(s_n (\max(0, \log(|\beta_n(u)|)) - \log \tau)) \\ &\leq c_0 \left| \frac{\beta_n(u)}{\tau} \right|^{q_n(u)} \end{aligned}$$

where the sequence s_n and the stochastic process $q_n(u)$ are defined by

$$s_n = \sup_{(\log n)^\varepsilon \leq y \leq \sqrt{n}/\tau} |s(y)|, \quad q_n(u) = s_n 1_{\{|\beta_n(u)| > 1\}}. \quad (36)$$

Since $s(y) \rightarrow 0$ as $y \rightarrow +\infty$ we have

$$\lim_{n \rightarrow +\infty} \sup_{u \in \mathcal{K}_n^{\geq \tau}} q_n(u) \leq \lim_{n \rightarrow +\infty} s_n = 0 \quad (37)$$

and this uniform convergence of q_n is certain, not almost sure. In other words, the uncertainty in the following inequality only comes from $\mathbb{P}(\mathcal{D}_n)$. We have shown that for all n large, on the event \mathcal{D}_n , it holds

$$v_n I_{\mathcal{K}_n^{\geq \tau}} \leq v_n \int_{\mathcal{K}_n^{\geq \tau}} \rho \left(\frac{|\beta_n(u)|}{\sqrt{n}} \right) du \leq \frac{c_0}{\tau^{s_n}} \int_{\mathcal{K}_n^{\geq \tau}} |\beta_n(u)|^{b+q_n(u)} du \quad (38)$$

where $\tau^{s_n} \rightarrow 1$ as $n \rightarrow +\infty$. We are ready to bound $\mathbb{P}(\mathcal{D}_n \cap \{v_n I_{\mathcal{K}_n^{\geq \tau}} \geq \lambda/2\})$.

(iii) On the probability space of Lemma 25 we have

$$|\beta_n(u)| \leq \frac{|B_n^X(u)|}{h(u)} + \frac{|B_n^Y(u)|}{h(u)} + \frac{|Z_n^X(u)|}{h(u)} + \frac{|Z_n^Y(u)|}{h(u)}.$$

If $\alpha \geq 1$ then $(x+y)^\alpha \leq 2^{\alpha-1}(x^\alpha + y^\alpha)$ for all $x, y \geq 0$. Combining this fact with $b + q_n(u) \geq b \geq 1$ and (37) thus implies that, for $K > 1$ fixed and all n large enough,

$$\frac{1}{K4^{b-1}} \int_{\mathcal{K}_n^{\geq \tau}} |\beta_n(u)|^{b+q_n(u)} du \leq R_n^X + R_n^Y + S_n^X + S_n^Y$$

where

$$R_n^X = \int_{\mathcal{K}_n^{\geq \tau}} \left| \frac{B_n^X(u)}{h(u)} \right|^{b+q_n(u)} du, \quad S_n^X = \int_{\mathcal{K}_n^{\geq \tau}} \left| \frac{Z_n^X(u)}{h(u)} \right|^{b+q_n(u)} du.$$

It remains to prove that for an appropriate choice of \bar{u} and β we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{P} \left(\mathcal{D}_n \cap \left\{ R_n^X \geq \frac{\lambda \tau^{s_n}}{8c_0} \right\} \right) &< \frac{\varepsilon}{8}, \\ \limsup_{n \rightarrow +\infty} \mathbb{P} \left(\mathcal{D}_n \cap \left\{ S_n^X \geq \frac{\lambda \tau^{s_n}}{8c_0} \right\} \right) &< \frac{\varepsilon}{8}, \end{aligned}$$

which ensures by (38) that $\mathbb{P}(\mathcal{D}_n \cap \{v_n I_{\mathcal{K}_n^{\geq \tau}} \geq \lambda/2\}) \leq \varepsilon/2$. For short, it is assumed below that $1/9 < \tau^{s_n}/8$.

(iv) The following integral T_n is crucial with respect to the integrability of the processes B_n^X and Z_n^X . Let $b' > b$ be so close to b that $0 < b'(5-b')/6 < \gamma < 1$. Consider the random function $q_n(u)$ from (36). For all n large enough we have $b \leq b + q_n(u) < b'$ hence (34) and (35) entail

$$\begin{aligned} T_n &= \int_{\mathcal{K}_n} \left| \frac{\sqrt{1-u}}{h(u)} \right|^{b+q_n(u)} du \leq \int_{\mathcal{K}_n} \left| \frac{F^{-1}(u)}{\sqrt{1-u}} \right|^{b+q_n(u)} du \leq \int_{\mathcal{K}_n} \left| \frac{F^{-1}(u)}{\sqrt{1-u}} \right|^{b'} du \\ &\leq \left[-\frac{(1-u)^{1-\gamma}}{1-\gamma} \right]_{\bar{u}}^{1-j_n/n} \leq \frac{(1-\bar{u})^{1-\gamma}}{1-\gamma}. \end{aligned}$$

(v) On the one hand we have, by Fubini-Tonelli and recalling that B_n^X is a standard Brownian bridge and the sequence s_n is defined at (37),

$$\begin{aligned} \mathbb{E}(R_n^X) &\leq T_n \sup_{u \in \mathcal{K}_n} \mathbb{E} \left(\left| \frac{B_n^X(u)}{\sqrt{u(1-u)}} \right|^{b+q_n(u)} \right) \\ &\leq T_n \sup_{0 \leq s \leq s_n} \mathbb{E} \left(|\mathcal{N}(0,1)|^{b+s} \right) = T_n \mathbb{E} \left(|\mathcal{N}(0,1)|^{b+s_n} \right). \end{aligned}$$

Assuming n so large that $s_n < 2-b$ we get $\mathbb{E}(R_n^X)/T_n < \mathbb{E}(|\mathcal{N}(0,1)|^2) = 1$ then choosing \bar{u}_0 such that $(1-\bar{u}_0)^{1-\gamma} < 8(1-\gamma)\lambda/9c_0\varepsilon$ yields, for all $\bar{u} \in [\bar{u}_0, 1)$,

$$\mathbb{P} \left(R_n^X \geq \frac{\lambda}{9c_0} \right) < \frac{9c_0}{\lambda} T_n < \frac{\varepsilon}{8}. \quad (39)$$

On the other hand we have $\mathcal{K}_n^{>\tau} \subset \mathcal{K}_n \subset \Delta'_n$ and

$$S_n^X \leq \sup_{u \in \mathcal{K}_n^{>\tau}} \left| \frac{Z_n^X(u)}{\sqrt{1-u}} \right|^{b+q_n(u)} T_n.$$

By Lemma 25 it almost surely holds, for $b' > b$ and all n large,

$$\sup_{u \in \mathcal{K}_n^{>\tau}} \left| \frac{Z_n^X(u)}{\sqrt{1-u}} \right| \leq \sup_{u \in \mathcal{K}_n} \frac{1}{n^v \sqrt{1-u}} = \frac{1}{n^v} \sqrt{\frac{n}{j_n}} \leq n^{(1-\beta)/2-v}$$

which vanishes provided $1 - 2v < \beta < 1$. Therefore, for this choice of β ,

$$\lim_{n \rightarrow +\infty} S_n^X = 0 \text{ a.s.}, \quad \lim_{n \rightarrow +\infty} \mathbb{P} \left(S_n^X \geq \frac{\lambda}{9c_0} \right) = 0. \quad (40)$$

(vi) Putting together the conclusions of (i)-(v), and especially (38), (39) and (40), implies

$$\mathbb{P}(\mathcal{C}_n^\lambda) \leq 1 - \mathbb{P}(\mathcal{D}_n) + \mathbb{P} \left(\mathcal{D}_n \cap \left\{ v_n I_{\mathcal{K}_n^{>\tau}} \geq \frac{\lambda}{2} \right\} \right) < \frac{\varepsilon}{2} + 4\frac{\varepsilon}{8} = \varepsilon.$$

Finally notice that the same β works whatever the choice of λ, ε . \square

Step 4. Now $\mathcal{L} = [\underline{u}, \bar{u}]$ is fixed. By Lemmas 23 and 25 there almost surely exists $n_0(\omega)$ such that, for all $n \geq n_0(\omega)$, $\varepsilon_n(u)$ from (30), $B_n(u) = B_n^X(u) - B_n^Y(u)$ and $Z_n(u) = Z_n^X(u) - Z_n^Y(u)$,

$$\left| \frac{\beta_n(u)}{\sqrt{n}} \right| \leq \varepsilon_n(u) \leq x_0, \quad \beta_n(u) = \frac{B_n(u) + Z_n(u)}{h(u)}, \quad u \in \mathcal{L}.$$

As a consequence, the cost ρ_c is evaluated at 0 all along this step. Let $\alpha > 0$ and consider $I_{\mathcal{L}} = I_{\mathcal{L}_{1,n}} + I_{\mathcal{L}_{2,n}} + I_{\mathcal{L}_{3,n}}$ where, for $n \geq n_0(\omega)$,

$$I_{\mathcal{L}_{k,n}} = \int_{\mathcal{L}_{k,n}} \rho_c \left(\frac{B_n(u) + Z_n(u)}{\sqrt{n}h(u)} \right) du, \quad k = 1, 2, 3, \quad (41)$$

and $\mathcal{L} = \mathcal{L}_{1,n} \cup \mathcal{L}_{2,n} \cup \mathcal{L}_{3,n}$ with $\mathcal{L}_{1,n} = \mathcal{L} \cap \{|B_n(u)| \leq \alpha\}$, $\mathcal{L}_{2,n} = \mathcal{L} \cap \{|B_n(u)| \geq 1/\alpha\}$ and $\mathcal{L}_{3,n} = \mathcal{L} \cap \{\alpha < |B_n(u)| < 1/\alpha\}$. Also define

$$0 < \underline{h} = \min_{u \in \mathcal{L}} h(u) \leq \bar{h} = \max_{u \in \mathcal{L}} h(u) < +\infty.$$

Step 4.1 Choose $\alpha \in (0, 1)$ arbitrarily small. In view of the almost sure rate $1/n^v$ from Lemma 25 and (11) we have, given \underline{u}, \bar{u} then \underline{h} ,

$$\begin{aligned} \lim_{n \rightarrow +\infty} v_n I_{\mathcal{L}_{1,n}} &\leq \lim_{n \rightarrow +\infty} \frac{1}{\rho(1/\sqrt{n})} \int_{\mathcal{L}_{1,n}} \rho \left(\frac{\alpha + 1/n^v}{\sqrt{n}\underline{h}} \right) du \\ &\leq \lim_{n \rightarrow +\infty} \frac{\rho(2\alpha/\sqrt{n}\underline{h})}{\rho(1/\sqrt{n})} = \frac{(2\alpha)^b}{\underline{h}^b} \text{ a.s.} \end{aligned} \quad (42)$$

The last equality holds by definition of $\rho \in RV(0, b)$.

Step 4.2 Write $\mathcal{L}_{2,n}^+ = \mathcal{L} \cap \{B_n(u) \geq 1/\alpha\}$ and $\mathcal{L}_{2,n}^- = \mathcal{L} \cap \{B_n(u) \leq -1/\alpha\}$. By Lemma 25 we have, for n large enough,

$$v_n I_{\mathcal{L}_{2,n}^+} = \frac{1}{\rho(1/\sqrt{n})} \int_{\mathcal{L}_{2,n}^+} \rho_+ \left(\frac{\beta_n(u)}{\sqrt{n}} \right) du \leq \frac{1}{\rho(1/\sqrt{n})} \int_{\mathcal{L}_{2,n}^+} \rho_+ \left(\frac{2B_n(u)}{h(u)\sqrt{n}} \right) du$$

then similar arguments as for (ii) in the proof of Lemma 26 yield

$$v_n I_{\mathcal{L}_{2,n}^+} \leq c_0 \frac{\rho_+(1/\sqrt{n})}{\rho(1/\sqrt{n})} \int_{\mathcal{L}_{2,n}^+} \left(\frac{2B_n(u)}{h(u)} \right)^{b_+ + s_n} du$$

where $s_n \rightarrow 0$ is defined at (36) with $\tau = 2/\alpha$. By replacing $\min(u, 1-u)$ with $u(1-u) \leq \min(u, 1-u)$ in (CFG3) it follows that

$$v_n I_{\mathcal{L}_{2,n}^+} \leq K \int_{\mathcal{L}} 1_{\{B_n(u) \geq 1/\alpha\}} \left| \frac{F^{-1}(u)}{\sqrt{u(1-u)}} \right|^{b_+ + s_n} \left(\frac{B_n(u)}{\sqrt{u(1-u)}} \right)^{b_+ + s_n} du$$

where $K > 0$. As a consequence of (CFG_E) we obtain exactly as for (34) and (35) that if $b' \in (b, 2)$ is chosen sufficiently close to b then

$$\int_{(0,1)} \left| \frac{F^{-1}(u)}{\sqrt{u(1-u)}} \right|^{b'} du = K' < +\infty. \quad (43)$$

Since $2u - 1 \leq H(u, u) \leq u$ for $u \in (0, 1)$ we have

$$-(1-u)^2 \leq \text{Cov}(B_n^X(u), B_n^Y(u)) = H(u, u) - u^2 \leq u(1-u)$$

hence

$$0 \leq \text{Var}(B_n(u)) \leq 2u(1-u) + 2(1-u)^2 = 2(1-u)$$

and the r.v. $B_n(u)/\sqrt{u(1-u)}$ is centered Gaussian with variance bounded above by $2/\underline{u}$. Let denote $\mathcal{N}(0, 1)$ the standard normal distribution. By Hölder inequality we have, for $u \in \mathcal{L}$ and n large,

$$\mathbb{E} \left(\mathbf{1}_{\{B_n(u) \geq 1/\alpha\}} \left| \frac{B_n(u)}{\sqrt{u(1-u)}} \right|^{b_+ + s_n} \right) \leq K'' \mathbb{P} \left(\sup_{\underline{u} \leq u \leq \bar{u}} |B_n(u)| \geq \frac{1}{\alpha} \right)^{1/2}$$

where $K'' = (3/\underline{u}) \sup_{b_+ \leq s \leq b} \left(\mathbb{E} |\mathcal{N}(0, 1)|^{2s} \right)^{1/2} < +\infty$ only depends on b . We conclude that it asymptotically holds

$$\mathbb{E} \left(v_n I_{\mathcal{L}_{2,n}^+} \right) \leq K K' K'' \mathbb{P} \left(\sup_{\underline{u} \leq u \leq \bar{u}} |B_n(u)| > \frac{1}{\alpha} \right)^{1/2} \leq C \exp \left(-\frac{1}{\alpha^2} \right) \quad (44)$$

where C depends on M, b, F and α was left arbitrary from the beginning. Clearly $\mathbb{E}(v_n I_{\mathcal{L}_{2,n}^-})$ also obeys (44) by the same arguments. Notice that for the left hand tail \underline{u} and $1 - \bar{u}$ play a symmetric role in the previous control of the variance of $B_n(u)$ by $u(1-u)$.

Step 4.3 Let introduce $\mathcal{L}_{3,n}^- = \mathcal{L} \cap \{-1/\alpha < B_n(u) < -\alpha\}$ and $\mathcal{L}_{3,n}^+ = \mathcal{L} \cap \{\alpha < B_n(u) < 1/\alpha\}$. By Lemma 25 we almost surely ultimately have

$$\text{sign}(B_n(u) + Z_n(u)) \mathbf{1}_{\mathcal{L}_{3,n}}(u) = \text{sign}(B_n(u)) \mathbf{1}_{\mathcal{L}_{3,n}}(u)$$

where $\text{sign}(x) = \mathbf{1}_{x>0} - \mathbf{1}_{x<0}$. Therefore, (C2) implies, for all n large enough,

$$\begin{aligned} & \mathbf{1}_{\mathcal{L}_{3,n}}(u) \rho_c \left(\frac{B_n(u) + Z_n(u)}{\sqrt{nh(u)}} \right) \\ &= \mathbf{1}_{\mathcal{L}_{3,n}^+}(u) \rho_+ \left(\frac{B_n(u) + Z_n(u)}{\sqrt{nh(u)}} \right) + \mathbf{1}_{\mathcal{L}_{3,n}^-}(u) \rho_- \left(\frac{|B_n(u) + Z_n(u)|}{\sqrt{nh(u)}} \right). \end{aligned}$$

Now assume that $\alpha < 2/\bar{h}$ and $\mathcal{L}_{3,n} \neq \emptyset$, so that

$$\begin{aligned} v_n I_{\mathcal{L}_{3,n}} &= \frac{1}{\rho(1/\sqrt{n})} \left(\int_{\mathcal{L}_{3,n}^+} \rho_+ \left(\frac{|B_n(u)|}{\sqrt{nh(u)}} \right) du + R_n^+ \right) \\ &\quad + \frac{1}{\rho(1/\sqrt{n})} \left(\int_{\mathcal{L}_{3,n}^-} \rho_- \left(\frac{|B_n(u)|}{\sqrt{nh(u)}} \right) du + R_n^- \right) \end{aligned}$$

where we have, by convexity and differentiability of ρ_{\pm} on $(0, +\infty)$,

$$\begin{aligned} R_n^{\pm} &= \int_{\mathcal{L}_{3,n}^{\pm}} \left(\rho_{\pm} \left(\frac{|B_n(u) + Z_n(u)|}{\sqrt{nh(u)}} \right) - \rho_{\pm} \left(\frac{|B_n(u)|}{\sqrt{nh(u)}} \right) \right) du \\ &\leq \sup_{u \in \mathcal{L}_{3,n}^{\pm}} \rho'_{\pm} \left(\frac{|B_n(u)| + |Z_n(u)|}{\sqrt{nh(u)}} \right) \frac{|Z_n(u)|}{\sqrt{nh(u)}} \end{aligned}$$

The regular variation (C2) further implies $x \rho'_{\pm}(x)/\rho_{\pm}(x) \rightarrow 1$ as $x \rightarrow 0$. As a consequence, with probability one, for all n large it holds

$$\begin{aligned} \frac{R_n^{\pm}}{\rho(1/\sqrt{n})} &\leq \frac{1}{\rho(1/\sqrt{n})} \rho_{\pm} \left(\frac{|B_n(u)| + |Z_n(u)|}{\sqrt{nh(u)}} \right) \sup_{u \in \mathcal{L}_{3,n}} \frac{|Z_n(u)|}{|B_n(u)| + |Z_n(u)|} \\ &\leq \frac{\rho_{\pm}(2/\sqrt{nh}\alpha)}{\rho(1/\sqrt{n})} \frac{2}{\alpha n^v} \leq \frac{\rho_{\pm}(2/\sqrt{nh}\alpha)}{\rho_{\pm}(1/\sqrt{n})} \frac{2}{\alpha n^v} \leq \left(\frac{2}{h\alpha} \right)^{b_{\pm}} \frac{3}{\alpha n^v} \end{aligned}$$

which vanishes as $n \rightarrow +\infty$. Here we have used that $\rho_{\pm}(\theta x)/\rho_{\pm}(x) \rightarrow \theta^{b_{\pm}}$ as $x \rightarrow 0$ for any fixed $\theta > 0$, and Lemma 25. Finally we see that

$$\frac{1}{\rho_{\pm}(1/\sqrt{n})} \int_{\mathcal{L}_{3,n}^{\pm}} \rho_{\pm} \left(\frac{|B_n(u)|}{\sqrt{nh}(u)} \right) du = \int_{\mathcal{L}_{3,n}^{\pm}} \left(\frac{|B_n(u)|}{h(u)} \right)^b du + R_{3,n}^{\pm}$$

with

$$R_{3,n}^{\pm} = \int_{\mathcal{L}_{3,n}^{\pm}} L_n^{\pm}(u) \left(\frac{|B_n(u)|}{h(u)} \right)^b du, \quad L_n^{\pm}(u) = \frac{L_{\pm}(|B_n(u)|/\sqrt{nh}(u))}{L_{\pm}(1/\sqrt{n})} - 1.$$

Clearly, it follows

$$|R_{3,n}^{\pm}| \leq \left(\frac{1}{\underline{h}\alpha} \right)^b \sup_{u \in \mathcal{L}_{3,n}^{\pm}} |L_n^{\pm}(u)| \leq \left(\frac{1}{\underline{h}\alpha} \right)^b \left(\sup_{\alpha/\sqrt{h}\sqrt{n} \leq x \leq 1/\alpha\sqrt{h}\sqrt{n}} \frac{L_{\pm}(x)}{L_{\pm}(1/\sqrt{n})} - 1 \right)$$

thus, by (22) and (23) we get $|R_{3,n}^{\pm}| \rightarrow 0$ as $n \rightarrow +\infty$. We conclude that

$$I_{\mathcal{L}_{3,n}^*}^* = \rho_+(1/\sqrt{n}) \int_{\mathcal{L}_{3,n}^+} \left(\frac{|B_n(u)|}{h(u)} \right)^{b_+} du + \rho_-(1/\sqrt{n}) \int_{\mathcal{L}_{3,n}^-} \left(\frac{|B_n(u)|}{h(u)} \right)^{b_-} du \quad (45)$$

almost surely satisfies $\lim_{n \rightarrow +\infty} v_n |I_{\mathcal{L}_{3,n}} - I_{\mathcal{L}_{3,n}^*}^*| = 0$.

Step 5. Consider $W_c(\mathbb{F}_n, \mathbb{G}_n) = \int_0^1 \rho_c(\mathbb{F}_n^{-1}(u), \mathbb{G}_n^{-1}(u)) du$. As we assumed that

$$\lim_{n \rightarrow +\infty} \frac{\rho_+(1/\sqrt{n})}{\rho(1/\sqrt{n})} = \pi_+, \quad \lim_{n \rightarrow +\infty} \frac{\rho_-(1/\sqrt{n})}{\rho(1/\sqrt{n})} = \pi_-$$

by (C4) and $E = \mathbb{R}$ we have established that $v_n W_c^E(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{weak} W$ with

$$W = \pi_+ \int_0^1 1_{\{B(u) > 0\}} \left(\frac{|B(u)|}{h(u)} \right)^{b_+} du + \pi_- \int_0^1 1_{\{B(u) < 0\}} \left(\frac{|B(u)|}{h(u)} \right)^{b_-} du$$

and B is a standard Brownian bridge. To see this write $W_c^E(\mathbb{F}_n, \mathbb{G}_n) = I_{\mathcal{I}_n} + I_{\mathcal{J}_n} + I_{\mathcal{K}_n} + I_{\mathcal{L}_{1,n}} + I_{\mathcal{L}_{2,n}} + I_{\mathcal{L}_{3,n}}$ where each of the first three integrals is indeed the sum of its left hand tail and right hand tail version, likewise for $I_{\mathcal{L}_{3,n}}^*$ defined at (45). We have shown that $v_n(I_{\mathcal{I}_n} + I_{\mathcal{J}_n}) \rightarrow 0$ in probability. Let Ψ be a real valued k -Lipschitz function on \mathbb{R} , bounded by m . Given arbitrarily small constants $\lambda > 0$, $\varepsilon > 0$ and $\alpha > 0$ then an appropriate choice of $0 < \underline{u}, \bar{u} < 1$ and thus \underline{h} it holds, for all n large enough, by Lemma 26 and Step 4,

$$\begin{aligned} & \mathbb{E} \left(\Psi \left(v_n (I_{\mathcal{K}_n} + I_{\mathcal{L}_{1,n}} + I_{\mathcal{L}_{2,n}} + I_{\mathcal{L}_{3,n}}) \right) - \Psi \left(v_n I_{\mathcal{L}_{3,n}}^* \right) \right) \\ & \leq 4m\mathbb{P}(v_n I_{\mathcal{K}_n} > \lambda) + 4m\mathbb{P} \left(v_n |I_{\mathcal{L}_{3,n}} - I_{\mathcal{L}_{3,n}}^*| > \lambda \right) \\ & + 4m\mathbb{P} \left(v_n I_{\mathcal{L}_{1,n}} > \frac{(5\alpha)^b}{\underline{h}^b} \right) + k\mathbb{E} \left(4\lambda + \frac{(5\alpha)^b}{\underline{h}^b} + v_n I_{\mathcal{L}_{2,n}} \right) \\ & \leq 12m\varepsilon + 4k\lambda + \frac{k(5\alpha)^b}{\underline{h}^b} + 2kC \exp \left(-\frac{1}{\alpha^2} \right) \end{aligned}$$

which is as small as desired. Finally it is easily seen that $v_n I_{\mathcal{L}_{3,n}}^* \rightarrow_{weak} W$ as $(\underline{u}, \bar{u}) \rightarrow (0, 1)$ and $\alpha \rightarrow 0$ so that $\mathbb{E}(\Psi(W))$ can replace $\mathbb{E}(\Psi(v_n I_{\mathcal{L}_{3,n}}^*))$ above with an asymptotically arbitrarily small error. \square

5.2 The case $F < G$

We establish Theorem 14.

Step 0. In this section $D = (0, 1)$. Without loss of generality, assume that $F^{-1} > G^{-1}$ everywhere. We again focus on arguments for the right hand tail, thus we write $\psi_X = \psi_X^+$ and $\psi_Y = \psi_Y^+$ on $(y_0, +\infty)$. Therefore $\psi_X^{-1} > \psi_Y^{-1}$ and $\psi_X^{-1} > 0$ on $(u_0, 1)$ where $u_0 = F^{-1}(y_0)$. We need this stochastic ordering only to simplify the control of extremes without imposing (CFG_E) . Let assume (FG) , (C) with $b \in [1, 2)$ and (CFG_D) . For y large it holds

$\rho_{\pm}(y) = \exp(l_{\pm}(y))$ with $l_{\pm} \in RV_2^+(\gamma_{\pm}, +\infty)$. By (15), for $y_0 > 0$ and $\theta_+, \theta_- > 1$ playing exactly the role of θ in (CFG) of [3] we have

$$(\psi_X \circ l_+^{-1})'(y) \geq 2 + \frac{2\theta_+}{y}, \quad (\psi_Y \circ l_-^{-1})'(y) \geq 2 + \frac{2\theta_-}{y}, \quad y > y_0. \quad (46)$$

In particular, this implies

$$l_+ \circ \psi_X^{-1}(y) \leq \frac{y}{2} - \theta_+ \log y + K, \quad y > y_0. \quad (47)$$

By (16), whenever $F^{-1}(u) - G^{-1}(u) > 0$ is not asymptotically away from 0 as $u \rightarrow 1$ we further ask that, for some $\theta_2 > 0$,

$$l_+ \circ \psi_X^{-1}(y) \leq \frac{y}{2} - 2 \log \psi_X^{-1}(y) - \theta_2 \log y, \quad y > y_0. \quad (48)$$

Notice that if F is logconvex then $\log \psi_X^{-1}(y) > \log y$ and (48) already implies (47) with $\theta_+ > 2$ whereas if F is logconcave then $\log \psi_X^{-1}(y) < \log y$ and (47) implies (48) with $\theta_2 > 1$. Since (CFG_D) implies (CFG) of [3] through (16) hence (46), we are allowed to use most results of the latter paper. In particular Theorem 14 is true when $F^{-1}(u) - G^{-1}(u) > \delta$ for some $\delta > 0$ and $b > 1$ to ensure (C3) in [3]. We thus focus on the case $F^{-1}(u) - G^{-1}(u) \rightarrow 0$ as $u \rightarrow 1$ which requires (48) whatever b , and we isolate out the case $b = 1$ only when necessary to extend the main result of [3], at Step 4. We often use $F^{-1}(u) = \psi_X^{-1}(\log(1/(1-u)))$. A consequence is that (48) also reads

$$\rho \circ F^{-1}(u) = \rho_+ \circ F^{-1}(u) \leq \frac{1}{F^{-1}(u)^2 \sqrt{1-u} |\log(1-u)|^{\theta_2}}, \quad u > u_0.$$

Let us study $W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G) = I_{\mathcal{I}_n} + I_{\mathcal{J}_n} + I_{\mathcal{K}_n} + I_{\mathcal{L}}$ with the notation

$$I_A = \int_A (\rho_c(\tau(u) + \tau_n(u)) - \rho_c(\tau(u))) du, \quad A \subset (0, 1), \quad (49)$$

$$\tau(u) = F^{-1}(u) - G^{-1}(u), \quad \tau_n(u) = \frac{\beta_n(u)}{\sqrt{n}} = \frac{\beta_n^X(u) - \beta_n^Y(u)}{\sqrt{n}},$$

and $\mathcal{I}_n = (1 - i_n/n, 1]$, $\mathcal{J}_n = (1 - j_n/n, 1 - i_n/n]$, $\mathcal{K}_n = (\bar{u}, 1 - j_n/n]$, $\mathcal{L} = [\underline{u}, \bar{u}]$ with $0 < \underline{u} < 1/2 < \bar{u} < 1$.

Step 1. Consider a non negative increasing sequence $K_n \rightarrow +\infty$ to be chosen later in such a way that $K_n/\log \log n \rightarrow 0$. Define

$$i_n = \frac{\sqrt{n}}{K_n \exp(l \circ \psi_X^{-1}(\log n + K_n))}. \quad (50)$$

We have $l \circ \psi_X^{-1}(y) = l_+ \circ \psi_X^{-1}(y) \rightarrow +\infty$ as $y \rightarrow +\infty$ thus $i_n = o(\sqrt{n}/K_n)$. When (48) is enforced then for any $\theta' \in (0, \theta_2)$ and all n large enough,

$$i_n \geq \frac{K}{K_n} \left(F^{-1} \left(1 - \frac{1}{ne^{K_n}} \right) \right)^2 \exp \left(-\frac{K_n}{2} + \theta_+ \log(\log n + K_n) \right)$$

$$> \left(F^{-1} \left(1 - \frac{1}{n} \right) \right)^2 (\log n)^{\theta'}. \quad (51)$$

Otherwise, when only (47) holds then for $\theta' \in (1, \theta_+)$,

$$i_n \geq \frac{K}{K_n} \exp \left(-\frac{K_n}{2} + \theta_+ \log(\log(n + K_n)) \right) > (\log n)^{\theta'}. \quad (52)$$

Hence in both case we have $i_n/\log \log n \rightarrow +\infty$ and $i_n/\sqrt{n} \rightarrow 0$. Let us define

$$I_{\mathcal{I}_n}^1 = \int_{\mathcal{I}_n} \rho_+(\tau(u)) du,$$

$$I_{\mathcal{I}_n}^2 = \int_{\mathcal{I}_n} \rho_c(\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u)) du = \frac{1}{n} \sum_{i=n-[i_n]}^n \rho_c(X_{(i)} - Y_{(i)}).$$

Lemma 27 Assume that (C), (FG) and (CFG_D) hold. Then $\sqrt{n}I_{\mathcal{I}_n}^1 \rightarrow 0$ and $\sqrt{n}I_{\mathcal{I}_n}^2 \rightarrow 0$ in probability.

Proof. This readily follows from Lemma 22 in [3]. For $\sqrt{n}I_{\mathcal{L}_n}^1$ the mentioned proof only needed $\theta > 0$ hence $\theta_+, \theta_- > 0$. For $\sqrt{n}I_{\mathcal{L}_n}^2$ the initial expansion

$$\sum_{i=n-[i_n]}^n \rho_c(X_{(i)} - Y_{(i)}) \leq \sum_{i=n-[i_n]}^n \rho_+(X_{(i)}) + \sum_{i=n-[i_n]}^n \rho_-(Y_{(i)})$$

almost surely holds for n large enough, when $\min(X_{(n-[i_n])}, Y_{(n-[i_n])}) > 0$. \square

Step 2. We now study $I_{\mathcal{J}_n}$ with $j_n = n^\beta$, $\beta \in (1/2, 1)$. Recall that $\Delta_n = \mathcal{J}_n \cup \mathcal{K}_n \cup \mathcal{L}$ and $\tau(u) = F^{-1}(u) - G^{-1}(u) > 0$ for all $u \in \Delta_n$.

(i) Define $\varepsilon_n = \sup_{u \in \Delta_n} \varepsilon_n(u)$ where $\varepsilon_n(u) = \varepsilon_n^X(u) + \varepsilon_n^Y(u)$ and

$$\varepsilon_n^X(u) = \frac{\sqrt{\log \log n}}{\sqrt{n}} \frac{\sqrt{1-u}}{h_X(u)}, \quad \varepsilon_n^Y(u) = \frac{\sqrt{\log \log n}}{\sqrt{n}} \frac{\sqrt{1-u}}{h_Y(u)}.$$

The current ε_n is bounded by the one of (29). By combining (28) and (29) with (51) as in Lemma 23 we get, for some $\zeta > 0$,

$$\lim_{n \rightarrow +\infty} (\log n)^\zeta \sup_{u \in \Delta_n} \tau_n(u) \leq 9 \lim_{n \rightarrow +\infty} (\log n)^\zeta \varepsilon_n = 0 \quad a.s.$$

Let $m_n \rightarrow +\infty$ be a non negative sequence so slow that $m_n \varepsilon_n \rightarrow 0$. Consider $\mathcal{J}_n = \mathcal{J}_n^< \cup \mathcal{J}_n^>$ where

$$\begin{aligned} \mathcal{J}_n^< &= \{u \in \mathcal{J}_n : 0 < \tau(u) \leq m_n \varepsilon_n(u)\}, \\ \mathcal{J}_n^> &= \{u \in \mathcal{J}_n : 0 < m_n \varepsilon_n(u) < \tau(u)\}. \end{aligned}$$

By (28) again we almost surely ultimately have

$$-9\varepsilon_n(u) < \tau_n(u) = \frac{\beta_n^X(u)}{\sqrt{n}} - \frac{\beta_n^Y(u)}{\sqrt{n}} < 9\varepsilon_n(u), \quad u \in \mathcal{J}_n.$$

Notice that if $u \in \mathcal{J}_n^>$ then

$$0 < (m_n - 9)\varepsilon_n(u) < \tau(u) + \tau_n(u) < \tau(u) + 9\varepsilon_n(u) < \tau(u) \left(1 + \frac{9}{m_n}\right) \quad (53)$$

whereas if $u \in \mathcal{J}_n^<$ then it is possible that $\tau(u) + \tau_n(u) < 0$ since

$$-9\varepsilon_n(u) < \tau_n(u) < \tau(u) + \tau_n(u) < (m_n + 9)\varepsilon_n(u). \quad (54)$$

Let us control $|I_{\mathcal{J}_n}| \leq |I_{\mathcal{J}_n^<}| + |I_{\mathcal{J}_n^>}|$, starting with the first term.

(ii) Recall that $\sup_{u \in \mathcal{J}_n^<} m_n \varepsilon_n(u) \rightarrow 0$ as $n \rightarrow +\infty$. By (54) we have, for $u \in I_{\mathcal{J}_n^<}$ and $m_n > 9$,

$$\begin{aligned} |\rho_c(\tau(u) + \tau_n(u)) - \rho_c(\tau(u))| &\leq \rho_c(\tau(u) + \tau_n(u)) + \rho_+(\tau(u)) \\ &\leq \rho_-(9\varepsilon_n(u)) + 2\rho_+(2m_n \varepsilon_n(u)). \end{aligned}$$

hence $\sqrt{n} |I_{\mathcal{J}_n^<}| \leq R_{1,n} + R_{2,n}$ for all n large enough, with

$$\begin{aligned} R_{1,n} &= K \sqrt{n} \int_{\mathcal{J}_n^<} \varepsilon_n(u)^{b_-} L_-(9\varepsilon_n(u)) du, \\ R_{2,n} &= K \sqrt{n} \int_{\mathcal{J}_n^<} (m_n \varepsilon_n(u))^{b_+} L_+(2m_n \varepsilon_n(u)) du. \end{aligned}$$

Lemma 28 Assume (C), (FG) and (CFG). We have $R_{1,n} \rightarrow 0$ and $R_{2,n} \rightarrow 0$.

Proof. If $F^{-1}(u) - G^{-1}(u) > \delta$ then the set $\mathcal{J}_n^<$ is ultimately empty. Otherwise (48) holds. We have $\sqrt{1-u}(1/h_X(u) + 1/h_Y(u)) \leq 2F^{-1}(u)/\sqrt{1-u}$ for $u \in \mathcal{J}_n$ in view of $F^{-1}(u) > G^{-1}(u)$ and (FG3). If $\min(b_+, b_-) - 1 > 0$ this extra power cancels the slowly varying functions and we asymptotically have

$$R_{1,n} + R_{2,n} \leq K \sqrt{n} \int_{\mathcal{J}_n^<} m_n \varepsilon_n(u) du \leq K m_n \sqrt{\log \log n} \int_{\mathcal{J}_n} \frac{F^{-1}(u)}{\sqrt{1-u}} du.$$

If $b_+ = 1$ then $L_+(x)$ is bounded on $[0, x_0]$ since $xL_+(x)$ is convex non negative and starts from 0. Hence $L_+(2m_n\varepsilon_n(u))$ is bounded on \mathcal{J}_n , and the above upper bound remains true. Likewise if $b_- = 1$ then $L_-(9\varepsilon_n(u))$ is bounded on \mathcal{J}_n . Observe that (48) and $l(y) > \log y$ imply

$$\psi_X^{-1}(y) \leq \exp(l \circ \psi_X^{-1}(y)) \leq \frac{1}{\psi_X^{-1}(y)^2} \exp\left(\frac{y}{2} - \theta \log y\right)$$

thus $\psi_X^{-1}(y)^6 \leq e^y$ and $F^{-1}(u) < 1/(1-u)^{1/6}$. Therefore

$$\int_{\mathcal{J}_n} \frac{F^{-1}(u)}{\sqrt{1-u}} du \leq K \left(\frac{j_n}{n}\right)^{1/3} = Kn^{(\beta-1)/3}$$

with $\beta < 1$ and the conclusion follows since $m_n \rightarrow +\infty$ is arbitrarily slow. \square

We have shown that $\sqrt{n}I_{\mathcal{J}_n^<} \rightarrow 0$ almost surely.

(iii) By (53) we ultimately have, for all $u \in \mathcal{J}_n$,

$$|\rho_c(\tau(u) + \tau_n(u)) - \rho_c(\tau(u))| = |\rho_+(\tau(u) + \tau_n(u)) - \rho_+(\tau(u))|.$$

Consider now $\mathcal{J}_n^> = \mathcal{J}_n^{<\delta} \cup \mathcal{J}_n^{>\delta}$ with

$$\mathcal{J}_n^{<\delta} = \{u \in \mathcal{J}_n : m_n\varepsilon_n(u) < \tau(u) < \delta\}, \quad \mathcal{J}_n^{>\delta} = \{u \in \mathcal{J}_n : \tau(u) > \delta\}.$$

Since $\tau(u) > \delta$ on $\mathcal{J}_n^{>\delta}$ and Proposition 31 and Lemma 25 of [3] are satisfied by ρ_+ – thanks to (7) and (8) – we readily deduce from Lemma 23 of [3] that

$$\lim_{n \rightarrow +\infty} \sqrt{n} \int_{\mathcal{J}_n^{>\delta}} |\rho_+(\tau(u) + \tau_n(u)) - \rho_+(\tau(u))| du = 0 \quad a.s.$$

Concerning $\mathcal{J}_n^{<\delta}$ observe that by (53) again $0 < \tau(u) + \tau_n(u) < 2\delta$ for all n large. Since ρ_+ is convex it ensues

$$\begin{aligned} |\rho_+(\tau(u) + \tau_n(u)) - \rho_+(\tau(u))| &\leq \max(\rho'_+(\tau(u) + \tau_n(u)), \rho'_+(\tau(u))) |\tau_n(u)| \\ &\leq K_\delta |\tau_n(u)| \end{aligned}$$

with $K_\delta = \rho'_+(2\delta)$. Therefore, with probability one, for all n large enough

$$\sup_{u \in \mathcal{J}_n^{<\delta}} |\rho_+(\tau(u) + \tau_n(u)) - \rho_+(\tau(u))| \leq K_\delta \sup_{u \in \mathcal{J}_n^{<\delta}} |\tau_n(u)| \leq K \sup_{u \in \mathcal{J}_n^{<\delta}} |\varepsilon_n(u)|.$$

As already seen, (16) implies $F^{-1}(u) < 1/(1-u)^{1/6}$ for all $u < 1$ large enough. As a consequence, with probability one it ultimately holds

$$\begin{aligned} \sqrt{n} \int_{\mathcal{J}_n^{<\delta}} |\rho_+(\tau(u) + \tau_n(u)) - \rho_+(\tau(u))| du &\leq K\sqrt{n} \int_{\mathcal{J}_n} \varepsilon_n(u) du \\ &\leq K\sqrt{\log \log n} \int_{\mathcal{J}_n} \frac{F^{-1}(u)}{\sqrt{1-u}} du \leq Kn^{(\beta-1)/3} \sqrt{\log \log n} \end{aligned}$$

which vanishes as $n \rightarrow +\infty$. We conclude that $\sqrt{n}I_{\mathcal{J}_n^>} \rightarrow 0$ almost surely.

Step 3. The convergence of $I_{\mathcal{K}_n}$ is weaker than in probability.

Lemma 29 Assume (FG), (C) and (CFG_D). There exists $\beta \in (1/2, 1)$ such that for any choice of $\lambda > 0$ and $\varepsilon > 0$ one can find $\bar{u}_0 \in (1/2, 1)$ and $n_0 > 0$ such that, for all $\bar{u} \in [\bar{u}_0, 1)$ and all $n > n_0$,

$$\mathbb{P}(\sqrt{n}I_{\mathcal{K}_n} > \lambda) < \varepsilon.$$

Proof. Fix $\delta > 0$ and consider

$$\mathcal{K}_n^{<\delta} = \{u \in \mathcal{K}_n : 0 < \tau(u) < \delta\}, \quad \mathcal{K}_n^{>\delta} = \{u \in \mathcal{K}_n : \tau(u) > \delta\}.$$

The claimed result holds for $I_{\mathcal{K}_n^{>\delta}}$ by applying Lemma 26 from [3] with $\delta = \tau_0$ and $\bar{u} = F(M)$. Let us apply Lemma 25 to get, for $K > \sup_{|x| < 2\delta} \rho'_c(x)$,

$$\begin{aligned} \sqrt{n}I_{\mathcal{K}_n^{<\delta}} &= \sqrt{n} \int_{\mathcal{K}_n^{<\delta}} |\rho_c(\tau(u) + \tau_n(u)) - \rho_c(\tau(u))| du \leq K \int_{\mathcal{K}_n^{<\delta}} |\beta_n(u)| du \\ &\leq K \int_{\mathcal{K}_n} \left(\frac{|B_n^X(u)|}{h_X(u)} + \frac{|B_n^Y(u)|}{h_Y(u)} \right) du + \int_{\mathcal{K}_n} \left(\frac{|Z_n^X(u)|}{h_X(u)} + \frac{|Z_n^Y(u)|}{h_Y(u)} \right) du. \end{aligned}$$

The first two terms satisfy

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{K}_n} \frac{|B_n^X(u)|}{h_X(u)} du \right) &\leq \int_{\mathcal{K}_n} \frac{\sqrt{1-u} \mathbb{E}(|B_n^X(u)|)}{h_X(u) \sqrt{u(1-u)}} du \\ &\leq \int_{\bar{u}}^1 \frac{F^{-1}(1-u)}{\sqrt{1-u}} du \leq 3(1-\bar{u})^{1/3} \end{aligned}$$

and the last two terms obey, with probability one as $n \rightarrow +\infty$,

$$\begin{aligned} \int_{\mathcal{K}_n} \frac{|Z_n^X(u)|}{h_X(u)} du &\leq \sup_{u \in \mathcal{K}_n} |Z_n^X(u)| \int_{\mathcal{K}_n} \frac{F^{-1}(1-u)}{1-u} du \\ &\leq \frac{1}{n^v} \int_{\bar{u}}^{1-j_n/n} \frac{F^{-1}(1-u)}{1-u} du \\ &\leq \frac{1}{n^v} \int_{\bar{u}}^{1-j_n/n} \frac{1}{(1-u)^{7/6}} du \leq \frac{6}{n^v} n^{(1-\beta)/6} \end{aligned}$$

which vanishes if $\beta > 1 - 6v$ is chosen close enough to 1. \square

Step 4. Here we recall that (C2) with $b_{\pm} > 1$ and (15) respectively imply (C3) and (CFG) in [3]. Clearly Steps 4 and 5 of [3] remain true in the current framework and lead to the same conclusion as the main theorem in the latter paper, whence Theorem 14. The new case to conclude with is $b = 1$. By Glivenko-Cantelli and (FG), we almost surely have

$$0 \leq |\tau_n(u)| = \frac{|\beta_n(u)|}{\sqrt{n}} < \tau = \min_{u \in \mathcal{L}} \tau(u)$$

for all n large enough, we only deal with ρ_+ . Assuming that $b_+ = 1$ and $\rho_+(x) = xL_+(x)$ we have, for some $\varepsilon > 0$ such that $\mathcal{L}_\varepsilon \subset (0, 1)$ is an ε -neighborhood of \mathcal{L} ,

$$\begin{aligned} &\left| \sqrt{n} \int_{\mathcal{L}} (\rho_+(\tau(u) + \tau_n(u)) - \rho_+(\tau(u))) du - \sqrt{n} \int_{\bar{u}} \rho'_+(\tau(u)) \tau_n(u) du \right| \\ &\leq \frac{\sqrt{n}}{2} \sup_{u \in \mathcal{L}_\varepsilon} |\rho''_+(\tau(u))| \int_{\bar{u}} \tau_n^2(u) du \leq \frac{K}{\sqrt{n}} \int_{\bar{u}} \beta_n^2(u) du \end{aligned}$$

which almost surely vanishes by the law of the iterated logarithm. Thus we can conclude as in [3] by combining this with the previous Steps 1, 2, 3. In particular, the limiting variance is finite as a consequence of (15).

In order to complete the proof of Theorem 14 note that whenever $F > G$ we similarly get

$$\begin{aligned} &\left| \sqrt{n} \int_{\mathcal{L}} (\rho_-(-\tau(u) - \tau_n(u)) - \rho_-(-\tau(u))) du - \sqrt{n} \int_{\bar{u}} \rho'_-(-\tau(u)) \tau_n(u) du \right| \\ &\leq \frac{K}{\sqrt{n}} \int_{\bar{u}} \beta_n^2(u) du \end{aligned}$$

which explains why the term $\rho'_-(-\tau(u)) = |\rho'_c(\tau(u))|$ shows up.

5.3 The general case

We now prove Theorem 16. Recall that (FG0) implies the existence of $0 = u_0 < u_1 < \dots < u_\kappa = 1$ such that $F^{-1}(u_k) = G^{-1}(u_k)$ and $A_k = (u_{k-1}, u_k) \subset E$ or $A_k \subset D$ for $k = 1, \dots, \kappa$. We now study the mixed case where at least one of these intervals is included in E and one in D , so that $\kappa \geq 2$. Consider, using notation (49),

$$\sqrt{n}(W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) = \sqrt{n} \sum_{k=1}^{\kappa} I_{A_k}.$$

Let $0 \leq \lambda < \min_{1 \leq k \leq \kappa} (u_k - u_{k-1})/2$. Define the intervals $A_{k,\lambda}^+ = (u_{k-1}, u_{k-1} + \lambda) \subset A_k$ for $2 \leq k \leq \kappa$ and $A_{k,\lambda}^- = (u_k - \lambda, u_k) \subset A_k$ for $1 \leq k \leq \kappa - 1$. If $A_k \subset D$ we have $F^{-1}(u) \neq G^{-1}(u)$ for $u \in A_{k,\lambda}^+ \cup A_{k,\lambda}^-$. If

$A_k \subset E$ the intervals $A_{k,\lambda}^+$ and $A_{k,\lambda}^-$ are assumed to be empty instead. Consider first the intervals $A_{k,\lambda}^+$ for $2 \leq k \leq \kappa$ and set $0 < u_- < u_1 < u_{\kappa-1} < u_+ < 1$. Since

$$\lim_{n \rightarrow +\infty} \sup_{u_- < u < u_+} \left| \frac{\beta_n(u)}{\sqrt{n}} \right| = 0 \quad a.s.$$

we have, by (C2), for $K = \sup_{u_- < u < u_+} (\rho'_-(2|\tau(u)|), \rho'_+(2|\tau(u)|)) < +\infty$,

$$\lim_{n \rightarrow +\infty} \sup_{u_- < u < u_+} \frac{\rho_c(\beta_n(u)/\sqrt{n})}{|\beta_n(u)/\sqrt{n}|} \leq K \quad a.s.$$

Therefore, in view of Step 4 in the previous proof for $F \neq G$ we get

$$\sqrt{n} |I_{A_{k,\lambda}^+}| \leq K \int_{A_{k,\lambda}^+} |\beta_n(u)| du \leq \frac{K}{\underline{h}} \left(\int_{A_{k,\lambda}^+} |B_n(u)| du + \int_{A_{k,\lambda}^+} |Z_n(u)| du \right)$$

where $\underline{h} = \min_{u_- < u < u_+} \min(h_X(u), h_Y(u)) > 0$. Lemma 25 further yields

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\sqrt{n} |I_{A_{k,\lambda}^+}| > \alpha \right) \leq \mathbb{P} \left(\int_{A_{k,\lambda}^+} |B(u)| du > \frac{2\alpha \underline{h}}{K} \right)$$

for any $\alpha > 0$ and all $2 \leq k \leq \kappa$, where B has the same law as $B_n = B_n^X - B_n^Y$. The latter upper bound vanishes as $\lambda \rightarrow 0$. A similar conclusion holds for $A_{k,\lambda}^-$ and $1 \leq k \leq \kappa - 1$. Write $A_{1,\lambda}^* = A_1 \setminus A_{1,\lambda}^-$, $A_{\kappa,\lambda}^* = A_\kappa \setminus A_{\kappa,\lambda}^+$ and $A_{k,\lambda}^* = A_k \setminus (A_{k,\lambda}^+ \cup A_{k,\lambda}^-)$ for $2 \leq k \leq \kappa - 1$.

(i) Consider the case $1 < b < 2$. Fix $\lambda > 0$ arbitrarily small and write

$$\sqrt{n}(W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) = \sqrt{n}I_E + \sqrt{n}I_{D,\lambda}^* + \sqrt{n}I_{D,\lambda}^\pm \quad (55)$$

where

$$I_E = \sum_{A_k \subset E} I_{A_{k,\lambda}^*} = \sum_{A_k \subset E} I_{A_k}, \quad I_{D,\lambda}^* = \sum_{A_k \subset D} I_{A_{k,\lambda}^*}, \quad I_{D,\lambda}^\pm = \sum_{A_k \subset D} I_{A_{k,\lambda}^+ \cup A_{k,\lambda}^-}.$$

We just proved that

$$\lim_{\lambda \rightarrow 0} \lim_{n \rightarrow +\infty} \mathbb{P} \left(\sqrt{n}I_{D,\lambda}^\pm > \alpha \right) = 0.$$

Since $b > 1$ we have $v_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow +\infty$. Therefore Steps 1 to 4 of Section 5.1 when $F = G$ show that

$$\lim_{n \rightarrow +\infty} \sqrt{n}I_E = \lim_{n \rightarrow +\infty} \frac{\sqrt{n}}{v_n} v_n I_E = 0 \quad \text{in probability.}$$

In the case $\kappa \geq 3$ then for all $2 \leq k \leq \kappa - 1$ with $A_k \subset D$ we have $\delta_k = \inf_{u \in A_k^*} |\tau(u)| > \delta > 0$ and $\tau(u)$ has constant sign on A_k . It follows from Steps 1 to 4 of Section 5.2 when $F \neq G$ that the weak limit of $\sqrt{n}I_{D,\lambda}^*$ is $\int_{D_\lambda} \rho'_c(\tau(u)) \mathbb{B}(u) du$ where $D_\lambda = \bigcup_{A_k \subset D} I_{A_{k,\lambda}^*}$ and $\mathbb{B}(u) = B^X(u)/h_X(u) - B^Y(u)/h_Y(u)$. By letting $\lambda \rightarrow 0$ we conclude that

$$\sqrt{n}(W_c(\mathbb{F}_n, \mathbb{G}_n) - W_c(F, G)) \rightarrow_{weak} \int_D \rho'_c(\tau(u)) \mathbb{B}(u) du$$

which is easily seen to have the normal distribution $\mathcal{N}(0, \sigma_D^2)$.

(ii) Assume that $b = 1$. Starting again from (55) we again obtain that

$$\sqrt{n}I_{D,\lambda}^* \rightarrow_{weak} \int_{D_\lambda} \rho'_c(\tau(u)) \mathbb{B}(u) du$$

while the Steps 1 to 4 of Section 5.2 now entails, for v_n from (19),

$$v_n I_E \rightarrow_{weak} \pi_+ \int_E 1_{\{\mathbb{B}(u) > 0\}} |\mathbb{B}(u)| du + \pi_- \int_E 1_{\{\mathbb{B}(u) < 0\}} |\mathbb{B}(u)| du.$$

The above approximation with the same \mathbb{B} proves that the weak convergence of the couple $(\sqrt{n}I_{D,\lambda}^*, v_n I_E)$ holds, thus the sum weakly converges.

Finally observe that (C4) implies $\sqrt{n}/v_n \rightarrow L_+(0)/\pi_+$ and $\sqrt{n}/v_n \rightarrow L_-(0)/\pi_-$ as $n \rightarrow +\infty$. As previously we conclude by letting $\lambda \rightarrow 0$.

5.4 A first special case : $F = G$ and $b = 2$

We establish Theorem 11.

Step 0. Assume (C0), $\rho_c(x) = x^2$ for $|x| < x_0$, $E = \mathbb{R}$, (FG1), (FG2) and

$$\lim_{u \rightarrow 0} \frac{u}{h(u)} = \lim_{u \rightarrow 1} \frac{1-u}{h(u)} = 0, \quad \int_0^1 \frac{u(1-u)}{h^2(u)} du < +\infty.$$

This proof partially follows the line of the proof of Lemma 2.4 of [9].

Step 1. We show that $\sup_{1/n \leq u \leq 1-1/n} |\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u)| \rightarrow 0$ in probability, so that the behaviour of ρ_c near 0 only matters. Write $h = h_X$. Define $U_i = F(X_i)$ and $V_i = F(Y_i)$, $i = 1, \dots, n$. Consider $nI_{\mathcal{I}_n}$ with $i_n = 1$ and

$$\begin{aligned} I_{\mathcal{I}_n} &= \int_{1-1/n}^1 (\mathbb{F}_n^{-1}(u) - \mathbb{G}_n^{-1}(u))^2 du \\ &= \int_{1-1/n}^1 (F^{-1}(U_{(n)}) - F^{-1}(V_{(n)}))^2 \\ &\leq \frac{2}{n} \left(F^{-1}(U_{(n)}) - F^{-1}\left(1 - \frac{1}{n}\right) \right)^2 du \\ &\quad + \frac{2}{n} \left(F^{-1}(V_{(n)}) - F^{-1}\left(1 - \frac{1}{n}\right) \right)^2. \end{aligned}$$

By the mean theorem, for some random $U_{(n)}^*$ between $U_{(n)}$ and $1 - 1/n$,

$$F^{-1}(U_{(n)}) - F^{-1}\left(1 - \frac{1}{n}\right) = \frac{U_{(n)} - 1 + 1/n}{h(U_{(n)}^*)} = \frac{U_{(n)} - 1 + 1/n}{h(U_{(n)})} \frac{h(U_{(n)})}{h(U_{(n)}^*)}.$$

By a classical argument – see [3] – we have, thanks to (FG2),

$$\max \left(\frac{h(U_{(n)})}{h(U_{(n)}^*)}, \frac{h(U_{(n)}^*)}{h(U_{(n)})} \right) \leq \max \left(\frac{1 - U_{(n)}}{1 - U_{(n)}^*}, \frac{1 - U_{(n)}^*}{1 - U_{(n)}} \right)^K.$$

Now recall that $U_{(n)} - 1 + 1/n = O_P(1/n)$ and $d_{(n)} = n(1 - U_{(n)}) \rightarrow_{weak} d_{(\infty)}$ where $d_{(\infty)}$ is a positive finite r.v. Hence

$$\begin{aligned} \left(F^{-1}(U_{(n)}) - F^{-1}\left(1 - \frac{1}{n}\right) \right)^2 &\leq \frac{(U_{(n)} - 1 + 1/n)^2}{h^2(U_{(n)})} \max \left(\frac{1}{d_{(n)}}, d_{(n)} \right)^{2K} \\ &= \frac{(1 - U_{(n)})^2}{h^2(U_{(n)})} \left(1 - \frac{1}{d_{(n)}} \right)^2 \max \left(\frac{1}{d_{(n)}}, d_{(n)} \right)^{2K} \end{aligned}$$

where $(1 - U_{(n)})^2/h^2(U_{(n)}) \rightarrow 0$ almost surely and $(1 - 1/d_{(n)})^2 \max(1/d_{(n)}, d_{(n)})^{2K} = O_P(1)$. Hence $nI_{\mathcal{I}_n} = o_P(1)$.

Step 2. Now consider, for $j_n = n^\beta$,

$$nI_{\mathcal{J}_n} = \int_{1-j_n/n}^{1-1/n} (\beta_n^X(u) - \beta_n^Y(u))^2 du.$$

Lemma 30 *There exists a sequence of processes \mathbb{B}_n^X having the same law as \mathbb{B}^X of (18) such that*

$$\Xi_n = \sup_{1/n \leq u \leq 1-1/n} |\beta_n^X(u) - \mathbb{B}_n^X(u)| \frac{h(u)}{\sqrt{1-u}} = O_P(1).$$

Proof. It is an immediate extension of Corollary 4.2.1, page 382 of [6] starting from (4.2.2) of Theorem 4.2.1 of [6]. \square

As a consequence,

$$\begin{aligned} \mathbb{P}(nI_{\mathcal{J}_n} > 3\alpha) &\leq \mathbb{P}\left(\int_{1-j_n/n}^{1-1/n} (\mathbb{B}_n^X(u) - \mathbb{B}_n^Y(u))^2 du > \alpha\right) \\ &\quad + 2\mathbb{P}\left(\Xi_n^2 \int_{1-j_n/n}^{1-1/n} \frac{1-u}{h^2(u)} du > \alpha\right) \end{aligned}$$

hence $nI_{\mathcal{J}_n} \rightarrow 0$ in probability. We conclude the proof by applying the Steps 3 to 5 in Section 5.1 with many simplifications since $L(x) = 1$ now.

5.5 A second special case : $F = G$ has compact support

The proof of Corollary 20 follows exactly the same path as the proof of Theorem 9 up to the following slight changes.

Step 0. We mainly require $(FG2)$, $(FG3)$ to apply the Hungarian construction but not $(C3)$ for the cost at $+\infty$ since the support is bounded.

Step 1. In Step 1 of Section 5.1 we only need $K_n \rightarrow +\infty$.

Step 2. It is much shortened thanks to the boundedness of F^{-1} by taking K_n such that $i_n/\log \log n \rightarrow +\infty$ and (29) is no more required since by (21) $(\sqrt{u(1-u)}/h(u))^{b'}$ is integrable.

Steps 3 and 4. Since F^{-1} is bounded we use (21) that implies the *a.s.* finiteness of $\int_0^1 |\mathbb{B}^X(u)|^{b'} du$ and $\int_0^1 |\mathbb{B}^Y(u)|^{b'} du$.

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