MIX STAR-AUTONOMOUS QUANTALES AND THE CONTINUOUS WEAK ORDER

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Abstract. The set of permutations on a finite set can be given a lattice structure (known as the weak Bruhat order). The lattice structure is generalized to the set of words on a fixed alphabet $\Sigma = \{ x, y, z, \dots \}$, where each letter has a fixed number of occurrences (these lattices are known as multinomial lattices and, in dimension 2, as lattices of lattice paths). By interpreting the letters $x, y, z, \dots$ as axes, these words can be interpreted as discrete increasing paths on a grid of a d-dimensional cube, where $d = \mathrm{card}(\Sigma)$.

We show in this paper how to extend this order to images of continuous monotone paths from the unit interval to a d-dimensional cube. The key tool used to realize this construction is the quantale $\mathsf{L}_{\vee}(\mathbb{I})$ of join-continuous functions from the unit interval to itself; the construction relies on a few algebraic properties of this quantale: it is $\star$-autonomous and it satisfies the mix rule.

We begin developing a structural theory of these lattices by characterizing join-irreducible elements, and by proving these lattices are generated from their join-irreducible elements under infinite joins.

1. Introduction

Combinatorial objects (trees, permutations, discrete paths, ...) are pervasive in mathematics and computer science; often these combinatorial objects can be organised into some ordered collection in such a way that the underlying order is a lattice.

Building on our previous work on lattices of binary trees (known as Tamari lattices or associahedra) and lattices of permutations (known as weak Bruhat orders or permutohedra) as well as on related constructions [21, 22, 23, 24, 6, 25, 26], we have been led to ask whether these constructions can still be performed when the underlying combinatorial objects are replaced with geometric ones.

More precisely we have investigated the following problem. Multinomial lattices [3] generalize permutohedra in a natural way. Elements of a multinomial lattice are words on a finite totally ordered alphabet $\Sigma = \{ x, y, z \dots \}$ with a fixed number of occurrences of each letter. The order is obtained as the reflexive and transitive closure of the binary relation $\prec$ defined by $wabu \prec wbau$, whenever $a, b \in \Sigma$ and $a < b$ (if we consider words with exactly one occurrence of each letter, then we have a permutohedron). Now these words can be given a geometrical interpretation as discrete increasing paths in some Euclidean cube of dimension $d = \mathrm{card}(\Sigma)$, so the weak order can be thought of as a way of organising these paths into a lattice structure. When Σ contains only two letters, then these lattices are also known as lattices of (lattice) paths [9] and we did not hesitate in [21] to call the multinomial

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lattices "lattices of paths in higher dimensions". The question that we raised is therefore whether the weak order can be extended from discrete paths to continuous increasing paths.

We already presented at the conference TACL 2011 the following result, positively answering this question.

Proposition. Let $d \geq 2$. Images of increasing continuous paths from $\vec{0}$ to $\vec{1}$ in $\mathbb{R}^d$ can be given the structure of a lattice; moreover, all the permutohedra and all the multinomial lattices can be embedded into one of these lattices while respecting the dimension d.

We called this lattice the *continuous weak order*. The proof of this result was complicated by the many computations arising from the structure of the reals and from analysis. We recently discovered a cleaner proof of the above statement where all these computations are uniformly derived from a few algebraic properties. The algebra we need to consider is the one of the quantale $\mathsf{L}_\vee(\mathbb{I})$ of join-continuous functions from the unit interval to itself. This is a $\star$-autonomous quantale, see [2], and moreover it satisfies the mix rule, see [7]. The construction of the continuous weak order is actually an instance of a general construction of a lattice $\mathsf{L}_d(Q)$ from a $\star$-autonomous quantale Q satisfying the mix rule. When $Q = \mathbf{2}$ (the two-element Boolean algebra) this construction yields the usual weak Bruhat order; when $Q = \mathsf{L}_\vee(\mathbb{I})$, this construction yields the continuous weak order. Thus, the step we took is actually an instance of moving to a different set of (non-commutative, in our case) truth values, as notably suggested in [17]. What we found extremely surprising is that many deep geometric notions (continuous monotone path, maximal chains, ...) might be characterised via this simple move and using the algebra of quantales.

Let us state our first main result. Let $\langle Q, 1, \otimes, ^\star \rangle$ be a $\star$-autonomous quantale (or a residuated lattice), denote by 0 and $\oplus$ the dual monoidal operations. Q is not supposed to be commutative, but we assume that it is cyclic ($x^\star = x \multimap 0 = 0 \multimapinv x$, for each $x \in Q$) and that it satisfies the MIX rule ($x \otimes y \leq x \oplus y$, for each $x, y \in Q$). Let $d \geq 2$, $[d]_2 := \{ (i, j) \mid 1 \leq i < j \leq d \}$ and consider the product $Q^{[d]_2}$. Say that a tuple $f \in Q^{[d]_2}$ is *closed* if $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$, and that it is *open* if $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$; say that f is *clopen* if it is closed and open.

Theorem. The set of clopen tuples of $Q^{[d]_2}$, with the pointwise ordering, is a lattice.

The above lattice is the one we denoted $\mathsf{L}_d(Q)$. The second main result we aim to present relates the algebraic setting to the analytic one:

Theorem. Clopen tuples of $\mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ bijectively correspond to images of monotonically increasing continuous functions $\mathsf{p} : \mathbb{I} \to \mathbb{I}^d$ such that $\mathsf{p}(0) = \vec{0}$ and $\mathsf{p}(1) = \vec{1}$.

The results presented in this paper undoubtedly have a mathematical nature, yet our motivations for developing these ideas originate from various researches in computer science that we recall next.

Directed homotopy [13] was developed to understand behavioural equivalence of concurrent processes. Monotonically increasing paths might be seen as behaviours of distributed processes whose local state variable can only increase. The relationship between directed homotopies and lattice congruences (in lattices of lattice paths) was already pinpointed in [21]. In that paper we did not push further these ideas, mainly because the mathematical theory of a continuous weak order was not yet available.

In *discrete geometry* discrete paths (that is, words on the alphabet $\{ x, y, \ldots \}$) are used to approximate continuous lines. In dimension 2, Christoffel words [4] are well-established approximations of a straight segment from $(0,0)$ to some point (n,m). The lattice theoretic nature of this kind of approximation is apparent from the fact that Christoffel words can equivalently be defined as images of the identity/diagonal via the right/left adjoints to the canonical embedding of the binomial lattice $\mathsf{L}(n,m)$ into the lattice $\mathsf{L}_\vee(\mathbb{I})$. For higher dimensions, there are multiple proposals on how to approximate a straight segment, see for example [1, 10, 5, 19]. It is therefore tempting to give a lattice theoretic notion of approximation by replacing the binomial lattices with the multinomial lattices and the lattice $\mathsf{L}_\vee(\mathbb{I})$ with the lattice $\mathsf{L}(\mathbb{I}^d)$. The structural theory of the lattices $\mathsf{L}(\mathbb{I}^d)$ already identifies difficulties in defining such a notion of approximation. For $d \geq 3$, the lattice $\mathsf{L}(\mathbb{I}^d)$ is no longer the Dedekind-MacNeille completion of the sublattice of discrete paths whose steps are on rational points—this is the colimit of the canonical embeddings of the multinomial lattices into $\mathsf{L}(\mathbb{I}^d)$; defining approximations naively via right/left adjoints of these canonical embeddings is bound to be unsatisfying. This does not necessarily mean that we should discard lattice theory as an approach to discrete geometry; for example, we expect that notions of approximation that take into consideration the degree of generation of $\mathsf{L}(\mathbb{I}^d)$ from multinomial lattices will be more robust.

The paper is organized as follows. We recall in Section 2 some facts on join-continuous (or meet-continuous) functions and adjoints. Section 3 describes the construction of the lattice $\mathsf{L}_d(Q)$, for an integer $d \geq 2$ a lattice and a mix $\star$-autonomous quantale Q. In Section 4 we show that the quantale $\mathsf{L}_\vee(\mathbb{I})$ of continuous functions from the unit interval to itself is a mix $\star$-autonomous quantale, thus giving rise to a lattice $\mathsf{L}_d(\mathsf{L}_\vee(\mathbb{I}))$ (we shall denote this lattice $\mathsf{L}(\mathbb{I}^d)$, to ease reading). In the following sections we formally instantiate our geometrical intuitions. Section 5 introduces the crucial notion of path and discusses its equivalent characterizations. In Section 6 we shows that paths in dimension 2 are in bijection with elements of the quantale $\mathsf{L}_\vee(\mathbb{I})$. In Section 7 we argue that paths in higher dimensions bijectively correspond to clopen tuples of the lattice $\mathsf{L}_\vee(\mathbb{I})^{[d]_2}$. In Section 8 we discuss some structural properties of the lattices $\mathsf{L}_\vee(\mathbb{I})$. We add concluding remarks in the final section.

2. Elementary facts on join-continuous functions

Throughout this paper, $[d]$ shall denote the set $\{1,\ldots,d\}$ while we let $[d]_2 := \{(i,j) \mid 1 \leq i < j \leq d\}$.

Let P and Q be complete posets; a function $f : P \to Q$ is *join-continuous* (resp., *meet-continuous*) if

$$f(\bigvee X) = \bigvee_{x\in X} f(x)\,, \quad (\text{resp., } f(\bigwedge X) = \bigwedge_{x\in X} f(x))\,, \tag{1}$$

for every $X \subseteq P$ such that $\bigvee X$ (resp., $\bigwedge X$) exists. Recall that $\bot_P := \bigvee \emptyset$ (resp., $\top_P := \bigwedge \emptyset$) is the least (resp., greatest) element of P. Note that if f is join-continuous (resp., meet-continuous) then f is monotone and $f(\bot_P) = \bot_Q$ (resp., $f(\top_P) = \top_Q$). Let f be as above; a map $g : Q \to P$ is *left adjoint* to f if $g(q) \leq p$ holds if and only if $q \leq f(p)$ holds, for each $p \in P$ and $q \in Q$; it is *right adjoint* to f if $f(p) \leq q$ is equivalent to $p \leq g(q)$, for each $p \in P$ and $q \in Q$. Notice that there is at most one function g that is left adjoint (resp., right adjoint) to f; we write this relation by $g = f_\ell$ (resp., $g = f_\rho$). Clearly, when f has a right adjoint, then $f = (g_\rho)_\ell$, and a similar formula holds when f has a left adjoint. We shall often use the following fact:

Lemma 1. *If $f : P \to Q$ is monotone and P and Q are two complete posets, then the following are equivalent:*

(1) *f is join-continuous (resp., meet-continuous),*
(2) *f has a right adjoint (resp., left adjoint).*

If f is join-continuous (resp., meet-continuous), then we have

$$f_\rho(q) = \bigvee\{\, p \in P \mid f(p) \leq q \,\} \qquad (\,\text{resp., } f_\ell(q) = \bigwedge\{\, p \in P \mid q \leq f(p)\,\}\,)\,, \quad \text{for each } q \in Q.$$

Moreover, if f is surjective, then these formulas can be strengthened so to substitute inclusions with equalities:

$$f_\rho(q) = \bigvee\{\, p \in P \mid f(p) = q \,\} \qquad (\,\text{resp., } f_\ell(q) = \bigwedge\{\, p \in P \mid q = f(p)\,\}\,)\,, \quad \text{for each } q \in Q. \tag{2}$$

The set of monotone functions from P to Q can be ordered point-wise: $f \leq g$ if $f(p) \leq g(p)$, for each $p \in P$. Suppose now that f and g both have right adjoints; let us argue that $f \leq g$ implies $g_\rho \leq f_\rho$: for each $q \in Q$, the relation $g_\rho(q) \leq f_\rho(q)$ is obtained by transposing $f(g_\rho(q)) \leq g(g_\rho(q)) \leq q$, where the inclusion $g(g_\rho(q)) \leq q$ is the counit of the adjunction. Similarly, if f and g both have left adjoints, then $f \leq g$ implies $g_\ell \leq f_\ell$.

3. Lattices from mix ⋆-autonomous quantales

A *⋆-autonomous quantale* is a tuple $Q = \langle Q, 1, \otimes, 0, \oplus, (-)^\star \rangle$ where Q is a complete lattice, $\otimes$ is a monoid operation on Q that distributes over arbitrary joins, $(-)^\star : Q^{op} \to Q$ is an order reversing involution of Q, and $(0, \oplus)$ is a second monoid structure on Q which is dual to $(1, \otimes)$. This means that

$$0 = 1^\star \quad \text{and} \quad f \oplus g = (g^\star \otimes f^\star)^\star\,.$$

Last but not least, the following relation holds:

$$f \otimes g \leq h \quad \text{iff} \quad f \leq h \oplus g^\star \quad \text{iff} \quad g \leq f^\star \oplus h\,.$$

Let us mention that we could have also defined a ⋆-autonomous quantale as a residuated (bounded) lattice $\langle Q, \bot, \vee, \top, \wedge, 1, \otimes, \multimap, \multimapinv \rangle$ such that Q is complete and comes with a cyclic dualizing element 0. The latter condition means that, for each $x \in Q$, $x \multimap 0 = 0 \multimapinv x$ and, letting $x^\star := x \multimap 0$, $x^{\star\star} = x$. This sort of algebraic structure is also called *(pseudo) ⋆-autonomous lattice* or *involutive residuated lattice*, see e.g. [18, 28, 8].

Example 1. Boolean algebras are the ⋆-autonomous quantales such that $\wedge = \otimes$ and $\vee = \oplus$. For a further example consider the following structure on the ordered set $\{\, -1 < 0 < 1 \,\}$:

$\otimes$	-1	0	1
-1	-1	-1	-1
0	-1	0	1
1	-1	1	1

$\oplus$	-1	0	1
-1	-1	-1	1
0	-1	0	1
1	1	1	1

	$\star$
-1	1
0	0
1	-1

Together with the lattice structure on the chain, this structure yields a ⋆-autonomous quantale, known as the Sugihara monoid on the three-element chain, see e.g. [11].

We presented in [25] several ways on how to generalize the standard construction of the permutohedron (aka the weak Bruhat order). We give here a new one. Given a ⋆-autonomous quantale Q, we consider the product $Q^{[d]_2} := \prod_{1 \leq i < j \leq d} Q$. Observe that, as a product, $Q^{[d]_2}$ has itself the structure of a quantale, the structure being computed coordinate-wise. We shall say that a tuple $f = \langle f_{i,j} \mid 1 \leq i < j \leq d \rangle$ is *closed* (resp., *open*) if

$$f_{i,j} \otimes f_{j,k} \leq f_{i,k} \qquad (\mathit{resp.},\ f_{i,k} \leq f_{i,j} \oplus f_{j,k})\,.$$

Clearly, closed tuples are closed under arbitrary meets and open tuples are closed under arbitrary joins. Observe that f is closed if and only if $f^{\star} = \langle (f_{\sigma(j),\sigma(i)})^{\star} \mid 1 \leq i < j \leq d \rangle$ is open, where for $i \in [d]$, $\sigma(i) = d - i + 1$. Thus, the correspondence sending f to $f^{\star}$ is an anti-isomorphism of $Q^{[d]_2}$, sending closed tuples to open ones, and vice versa. We shall be interested in tuples $f \in Q^{[d]_2}$ that are *clopen*, that is, they are at the same time closed and open.

For $(i, j) \in [d]_2$, a subdivision of the interval $[i, j]$ is a sequence of the form $i = \ell_0 < \ell_1 < \ldots \ell_{k-1} < \ell_k = j$ with $\ell_i \in [d]$, for $i = 1, \ldots, k$. We shall use $S_{i,j}$ for the set of subdivisions of the interval $[i, j]$. As closed tuples are closed under arbitrary meets, for each $f \in Q^{[d]_2}$ there exists a least tuple $\overline{f}$ such that $f \leq \overline{f}$ and $\overline{f}$ is closed; this tuple is easily computed as follows:

$$\overline{f}_{i,j} = \bigvee_{i<\ell_1<\ldots\ell_{k-1}<j\in S_{i,j}} f_{i,\ell_1} \otimes f_{\ell_1,\ell_2} \otimes \ldots \otimes f_{\ell_{k-1},j}\,.$$

Similarly and dually, if we set

$$f^{\circ}_{i,j} := \bigwedge_{i<\ell_1<\ldots\ell_{k-1}<j\in S_{i,j}} f_{i,\ell_1} \oplus f_{\ell_1,\ell_2} \oplus \ldots \oplus f_{\ell_{k-1},j}\,.$$

then f° is the greatest open tuple f° below f.

Proposition 1. *Suppose that, for each $f \in Q^{[d]_2}$, $\overline{(\overline{f})^{\circ}} = (\overline{f})^{\circ}$. Then, for each $f \in Q^{[d]_2}$, $(\overline{f^{\circ}})^{\circ} = (\overline{f^{\circ}})$ as well. The set of clopen tuples is then a lattice.*

Proof. The first statement is a consequence of the duality sending $f \in Q^{[d]_2}$ to $f^{\star} \in Q^{[d]_2}$. Now the relation $\overline{(\overline{f})^{\circ}} = (\overline{f})^{\circ}$ amounts to saying that the interior of any closed f is again closed. The other relation amounts to saying that the closure of an open is open. For a family $\{ f_i \mid i \in I \}$, with each f_i clopen, define then

$$\bigvee_{\mathsf{L}_d(Q)} \{ f_i \mid i \in I \} := \overline{\bigvee_{Q^{[d]_2}} \{ f_i \mid i \in I \}}\,, \qquad \bigwedge_{\mathsf{L}_d(Q)} \{ f_i \mid i \in I \} := (\bigwedge_{Q^{[d]_2}} \{ f_i \mid i \in I \})^{\circ}\,,$$

and remark that, by our assumptions, the expressions on the right of the equalities denote clopen tuples. It is easily verified then these are the joins and meets, respectively, among clopen tuples. □ □

Lemma 2. *Consider the following inequalities:*

$$(\alpha \oplus \beta) \otimes (\gamma \oplus \delta) \leq \alpha \oplus (\beta \otimes \gamma) \oplus \delta \tag{3}$$
$$\alpha \otimes \beta \leq \alpha \oplus \beta\,. \tag{4}$$

Then (3) *is valid and* (4) *is equivalent to* $0 \leq 1$.

The inequation (4) is known as the *mix rule*. We say that a $\star$-autonomous quantale Q is a mix $\star$-autonomous quantale if the mix rule holds in Q.

Theorem 1. *If Q is a mix $\star$-autonomous quantale and $f \in Q^{[d]_2}$ is closed, then so is f°. Consequently, the set of clopen tuples of $Q^{[d]_2}$ is a lattice.*

Proof. Let $i, j, k \in [d]$ with $i < j < k$. We need to show that

$$f^{\circ}_{i,j} \otimes f^{\circ}_{j,k} \leq f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_{n-1},k}$$

whenever $i < \ell_1 < \ldots \ell_{n-1} < k \in S_{i,k}$. This is achieved as follows. Let $u \in \{ 0, 1, \ldots, n-1 \}$ such that $j \in [\ell_u, \ell_{u+1})$ and put

$$\alpha := f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_u}\,, \qquad \delta := f_{\ell_{u+1}} \oplus \ldots \oplus f_{\ell_{n-1},k}$$
$$\beta := f_{\ell_u,j} \qquad \gamma := f_{j,\ell_{u+1}}\,.$$

We let in the above definition $f_{\ell_u,j} := 0$ when $j = \ell_u$. Then

$$\begin{aligned} f^{\circ}_{i,j} \otimes f^{\circ}_{j,k} &\leq (\alpha \oplus \beta) \otimes (\gamma \oplus \delta)\,, && \text{by definition of } f^{\circ}_{i,j} \text{ and } f^{\circ}_{j,k},\\ &\leq \alpha \oplus (\beta \otimes \gamma) \oplus \delta\,, && \text{by the inequation (3)},\\ &\leq \alpha \oplus f_{\ell_u,\ell_{u+1}} \oplus \delta\,, && \text{since } f \text{ is closed,} \end{aligned}$$

(or, when $j = \ell_u$, by using $\beta = 0 \leq 1$ and $\gamma = f_{\ell_u,\ell_{u+1}}$)

$$= f_{i,\ell_1} \oplus \ldots \oplus f_{\ell_{n-1},k}\,.$$

The last statement of the theorem is an immediate consequence of Proposition 1. □ □

Definition 1. For Q a mix ⋆-autonomous quantale, $\mathsf{L}_d(Q)$ shall denote the lattice of clopen tuples of $Q^{[d]_2}$.

Example 2. Suppose $Q = \mathbf{2}$, the Boolean algebra with two elements $0, 1$. Identify a tuple $\chi \in \mathbf{2}^{[d]_2}$ with the characteristic map of a subset S_χ of $\{ (i, j) \mid 1 \leq i < j \leq d \}$. Think of this subset as a relation. Then χ is clopen if both S_χ and its complement in $\{ (i, j) \mid 1 \leq i < j \leq d \}$ are transitive relations. These subsets are in bijection with permutations of the set $[d]$, see [6]; the lattice $\mathsf{L}_d(\mathbf{2})$ is therefore isomorphic to the well-known permutohedron, aka the weak Bruhat order. On the other hand, if Q is the Sugihara monoid on the three-element chain described in Example 1, then the lattice of clopen tuples is isomorphic to the lattice of pseudo-permutations, see [16, 25].

Remark 1. For a fixed integer d the definition of the lattice $\mathsf{L}_d(Q)$ relies only on the algebraic structure of Q. This allows to say that the construction $\mathsf{L}_d(-)$ is functorial: if $f : Q_0 \to Q_1$ is a ⋆-autonomous quantale homomorphism, then we shall have a lattice homomorphism $\mathsf{L}_d(f) : \mathsf{L}_d(Q_0) \to \mathsf{L}_d(Q_1)$ (it might be also argued that if f is injective, then so is $\mathsf{L}_d(f)$). It also means that we can interpret the theories of the lattices $\mathsf{L}_d(Q)$ in the theory of the quantale Q. For example, if the equational theory of a quantale Q is decidable, then the equational theory of the lattice $\mathsf{L}_d(Q)$ is decidable as well.

4. The mix ⋆-autonomous quantale $\mathsf{L}_\vee(\mathbb{I})$

In this paper $\mathbb{I}$ shall denote the unit interval of the reals, that is $\mathbb{I} := [0, 1]$. We use $\mathsf{L}_\vee(\mathbb{I})$ for the set of join-continuous functions from $\mathbb{I}$ to itself. Notice that a monotone function $f : \mathbb{I} \to \mathbb{I}$ is join-continuous if and only if

$$f(x) = \bigvee_{y<x,\, y\in\mathbb{I}\cap\mathbb{Q}} f(y)\,, \tag{5}$$

see Proposition 2.1, Chapter II of [12]. As the category of complete lattices and join-continuous functions is a symmetric monoidal closed category, for every complete lattice L the set of join-continuous functions from Q to itself is a monoid object in that category, that is, a quantale, see [14, 20]. Thus, we have:

Lemma 3. *Composition induces a quantale structure on* $\mathsf{L}_\vee(\mathbb{I})$.

Let now $\mathsf{L}_\wedge(\mathbb{I})$ denote the collection of meet-continuous functions from $\mathbb{I}$ to itself. By duality, we obtain:

Lemma 4. *Composition induces a dual quantale structure on $\mathsf{L}_\wedge(\mathbb{I})$.*

With the next set of observations we shall see $\mathsf{L}_\vee(\mathbb{I})$ and $\mathsf{L}_\wedge(\mathbb{I})$ are order isomorphic. For a monotone function $f : \mathbb{I} \to \mathbb{I}$, define

$$f^\wedge(x) = \bigwedge_{x<x'} f(x')\,, \qquad\qquad f^\vee(x) = \bigvee_{x'<x} f(x')\,.$$

Lemma 5. *If $x < y$, then $f^\wedge(x) \leq f^\vee(y)$.*

Proof. Pick $z \in \mathbb{I}$ such that $x < z < y$ and observe then that $f^\wedge(x) \leq f(z) \leq f^\vee(y)$. □ □

Proposition 2. *$f^\wedge$ is the least meet-continuous function above f and $f^\vee$ is the greatest join-continuous function below f. The relations $f^{\vee\wedge} = f^\wedge$ and $f^{\wedge\vee} = f^\vee$ hold and, consequently, the operations $(\,\cdot\,)^\vee : \mathsf{L}_\wedge(\mathbb{I}) \to \mathsf{L}_\vee(\mathbb{I})$ and $(\,\cdot\,)^\wedge : \mathsf{L}_\vee(\mathbb{I}) \to \mathsf{L}_\wedge(\mathbb{I})$ are inverse order preserving bijections.*

Proof. We prove only one statement. Let us show that $f^\wedge$ is meet-continuous; to this goal, we use equation (5):

$$\bigwedge_{x<t} f^\wedge(t) = \bigwedge_{x<t}\bigwedge_{t<t'} f(t') = \bigwedge_{x<t} f(t') = f^\wedge(x)\,.$$

We observe next that $f \leq f^\wedge$, as if $x < t$, then $f(x) \leq f(t)$. This implies that if $g \in \mathsf{L}_\wedge(\mathbb{I})$ and $f^\wedge \leq g$, then $f \leq f^\wedge \leq g$. Conversely, if $g \in \mathsf{L}_\wedge(\mathbb{I})$ and $f \leq g$, then

$$f^\wedge(x) = \bigwedge_{x<t} f(t) \leq \bigwedge_{x<t} g(t) = g(x)\,.$$

Let us prove the last sentence. Clearly, both maps are order preserving. Let us show that $f^{\vee\wedge} = f^\wedge$ whenever f is order preserving. We have $f^{\vee\wedge} \leq f^\wedge$, since $f^\vee \leq f$ and $(-)^\wedge$ is order preserves the pointwise ordering. For the converse inclusion, recall from the previous lemma that if $x < y$, then $f^\wedge(x) \leq f^\vee(y)$, so

$$f^\wedge(x) \leq \bigwedge_{x<y} f^\vee(y) = f^{\vee\wedge}(x)\,,$$

for each $x \in \mathbb{I}$. Finally, to see that $(-)^\wedge$ and $(-)^\vee$ are inverse to each other, observe that of $f \in \mathsf{L}_\wedge(\mathbb{I})$, then $f^{\vee\wedge} = f^\wedge = f$. The equality $f^{\wedge\vee} = f$ for $f \in \mathsf{L}_\vee(\mathbb{I})$ is derived similarly.
□ □

Recall that if $f \in \mathsf{L}_\vee(\mathbb{I})$ (resp., $g \in \mathsf{L}_\wedge(\mathbb{I})$), then $f_\rho \in \mathsf{L}_\wedge(\mathbb{I})$ (resp., $f_\ell \in \mathsf{L}_\vee(\mathbb{I})$) denotes the right adjoint of f (resp., left adjoint of g). The following relation is the key observation to uncover the $\star$-autonomous quantale structure on $\mathsf{L}_\vee(\mathbb{I})$.

Lemma 6. *For each $f \in \mathsf{L}_\vee(\mathbb{I})$, the relation $(f_\rho)^\vee = (f^\wedge)_\ell$ holds.*

Proof. Let $f \in \mathsf{L}_\vee(\mathbb{I})$; we shall argue that $x \leq f^\wedge(y)$ if and only if $(f_\rho)^\vee(x) \leq y$, for each $x, y \in \mathbb{I}$.

We begin by proving that $x \leq f^\wedge(y)$ implies that $(f_\rho)^\vee(x) \leq y$. Suppose $x \leq f^\wedge(y)$ so, for each z with $y < z$, we have $x \leq f(z)$. Suppose that $(f_\rho)^\vee(x) \not\leq y$, thus there exists $w < x$ such that $f_\rho(w) \not\leq y$. Then $y < f_\rho(w)$, so $x \leq f(f_\rho(w)) \leq w$, contradicting $w < x$. Therefore, $(f_\rho)^\vee(x) \not\leq y$.

Dually, we can argue that if $g \in \mathsf{L}_\wedge(\mathbb{I})$, then $g^\vee(x) \leq y$ implies $x \leq (g_\ell)^\wedge(y)$. Letting $g := f_\rho$ in this statement we obtain the converse implication: $(f_\rho)^\vee(x) \leq y$ implies $x \leq ((f_\rho)_\ell)^\wedge(y) = f^\wedge(y)$. □ □

For $f, g \in \mathsf{L}_\vee(\mathbb{I})$, let us define

$$f \otimes g := g \circ f\,, \qquad f \oplus g := (g^\wedge \circ f^\wedge)^\vee\,, \qquad f^\star = (f_\rho)^\vee = (f^\wedge)_\ell\,.$$

Proposition 3. *The tuple $\langle \mathsf{L}_\vee(\mathbb{I}), id, \otimes, id, \oplus, (-)^\star \rangle$ is a mix $\star$-autonomous quantale.*

Proof. The correspondence $(\cdot)^\star$ is order reversing as it is the composition of an order reversing function with a monotone one; by Lemma 6, it is an involution:

$$f^{\star\star} = (((f_\rho)^\wedge)^\vee)_\ell = (f_\rho)_\ell = f\,.$$

To verify that

$$(f \otimes g)^\star = g^\star \oplus f^\star \tag{6}$$

holds, for any $f, g \in \mathsf{L}_\vee(\mathbb{I})$, we compute as follows:

$$\begin{aligned} g^\star \oplus f^\star &= ((f^\star)^\wedge \circ (g^\star)^\wedge)^\vee \\ &= (f_\rho^{\vee\wedge} \circ g_\rho^{\vee\wedge})^\vee = (f_\rho \circ g_\rho)^\vee = (g \circ f)_\rho^\vee = (g \circ f)^\star = (f \otimes g)^\star\,. \end{aligned}$$

We verify next that, for any $f, g, h \in \mathsf{L}_\vee(\mathbb{I})$,

$$f \otimes g \leq h \quad \text{iff} \quad f \leq h \oplus g^\star\,. \tag{7}$$

Notice that $h \oplus g^\star = ((g^\star)^\wedge \circ h^\wedge)^\vee = (g_\rho^{\vee\wedge} \circ h^\wedge)^\vee = (g_\rho \circ h^\wedge)^\vee$, so

$$\begin{aligned} f \leq h \oplus g^\star \quad &\text{iff} \quad f \leq (g_\rho \circ h^\wedge)^\vee\,, && \text{by the equality just established,} \\ &\text{iff} \quad f \leq g_\rho \circ h^\wedge\,, && \text{by Proposition 2,} \\ &\text{iff} \quad g \circ f \leq h^\wedge\,, && \text{since } g(x) \leq h \text{ iff } x \leq g_\rho(y), \\ &\text{iff} \quad f \otimes g = g \circ f \leq h^{\wedge\vee} = h\,, && \text{using again Proposition 2.} \end{aligned}$$

It is an immediate algebraic consequence of (6) and (7) that $f \otimes g \leq h$ is equivalent to $g \leq f^\star \oplus h$, for any $f, g, h \in \mathsf{L}_\vee(\mathbb{I})$. Namely, we have

$$\begin{aligned} f \otimes g \leq h \quad &\text{iff} \quad f \leq h \oplus g^\star \\ &\text{iff} \quad (h \oplus g^\star)^\star \leq f^\star \\ &\text{iff} \quad g \otimes h^\star = g^{\star\star} \otimes h^\star \leq f^\star \\ &\text{iff} \quad g \leq f^\star \oplus h^{\star\star} = f^\star \oplus h\,. \end{aligned}$$

Finally, recall that the identity *id* is both join-continuous and meet-continuous and therefore $id^\wedge = id$. Then it is easily seen that *id* is both a unit for $\otimes$ and for $\oplus$. As seen in Lemma 2, this implies that $\mathsf{L}_\vee(\mathbb{I})$ satisfies the mix rule. □ □

5. Paths

Let in the following $d \geq 2$ be a fixed integer; we shall use $\mathbb{I}^d$ to denote the d-fold product of $\mathbb{I}$ with itself. That is, $\mathbb{I}^d$ is the usual geometric cube in dimension d. Let us recall that $\mathbb{I}^d$, as a product of the poset $\mathbb{I}$, has itself the structure of a poset (the order being coordinate-wise) which, moreover, is complete.

Definition 2. A *path in* $\mathbb{I}^d$ is a chain $C \subseteq \mathbb{I}^d$ with the following properties:

(1) if $X \subseteq C$, then $\bigwedge X \in C$ and $\bigvee X \in C$,
(2) C is dense as an ordered set: if $x, y \in C$ and $x < y$, then $x < z < y$ for some $z \in C$.

We have given a working definition of the notion of path in $\mathbb{I}^d$, as a totally ordered dense sub-complete-lattice of $\mathbb{I}^d$. The next theorem state the equivalence among several properties, each of which could be taken as a definition of the notion of path.

Theorem 2. *Let $d \geq 2$ and let $C \subseteq \mathbb{I}^d$. The following conditions are then equivalent:*

(1) *C is a path as defined in Definition 2;*
(2) *C is a maximal chain of the poset $\mathbb{I}^d$;*
(3) *There exists a monotone (increasing) topologically continuous map $\mathsf{p} : \mathbb{I} \to \mathbb{I}^d$ such that $\mathsf{p}(0) = \vec{0}$, $\mathsf{p}(1) = \vec{1}$, whose image is C.*

6. Paths in dimension 2

We give next a further characterization of the notion of path, valid in dimension 2. The principal result of this Section, Theorem 3, states that paths in dimension 2 are (up to isomorphism) just elements of the quantale $\mathsf{L}_\vee(\mathbb{I})$.

For a monotone function $f : \mathbb{I} \to \mathbb{I}$ define $C_f \subseteq \mathbb{I}^2$ by the formula

$$C_f := \bigcup_{x \in \mathbb{I}} \{ x \} \times [f^\vee(x), f^\wedge(x)] \,. \tag{8}$$

Notice that, by Proposition 2, $C_f = C_{f^\vee} = C_{f^\wedge}$.

Proposition 4. *C_f is a path in $\mathbb{I}^2$.*

Proof. We prove first that C_f, with the product ordering induced from $\mathbb{I}^2$, is a linear order. To this goal, we shall argue that, for $(x, y), (x', y') \in C_f$, we have $(x, y) < (x', y')$ iff either $x < x'$ or $x = x'$ and $y < y'$. That is, C_f is a lexicographic product of linear orders, whence a linear order. Let us suppose that one of these two conditions holds: a) $x < x'$, b) $x = x'$ and $y < y'$. If a), then $f^\wedge(x) \leq f^\vee(x')$. Considering that $y \in [f^\vee(x), f^\wedge(x)]$ and $y' \in [f^\vee(x'), f^\wedge(x')]$ we deduce $y \leq y'$. This proves that $(x, y) < (x', y')$ in the product ordering. If b) then we also have $(x, y) < (x', y')$ in the product ordering. The converse implication, $(x, y) < (x', y')$ implies $x < x'$ or $x = x'$ and $y < y'$, trivially holds.

We argue next that C_f is closed under joins from $\mathbb{I}^2$. Let (x_i, y_i) be a collection of elements in C_f, we aim to show that $(\bigvee x_i, \bigvee y_i) \in C_f$, i.e. $\bigvee y_i \in [f^\vee(\bigvee x_i), f^\wedge(\bigvee x_i)]$. Clearly, as $y_i \leq f^\wedge(x_i)$, then $\bigvee y_i \leq \bigvee f^\wedge(x_i) \leq f^\wedge(\bigvee x_i)$. Next, $f^\vee(x_i) \leq y_i$, whence $f^\vee(\bigvee x_i) = \bigvee f^\vee(x_i) \leq \bigvee y_i$. By a dual argument, we have that $(\bigwedge x_i, \bigwedge y_i) \in C_f$.

Finally, we show that C_f is dense; to this goal let $(x, y), (x', y') \in C_f$ be such that $(x, y) < (x', y')$. If $x < x'$ then we can find a z with $x < z < x'$; of course, $(z, f(z)) \in C_f$ and, but the previous characterisation of the order, $(x, y) < (z, f(z)) < (x', y')$ holds. If $x = x'$ then $y < y'$ and we can find a w with $y < w < y'$; as $w \in [y, y'] \subseteq [f^\vee(x), f^\wedge(x)]$, then $(x, w) \in C_f$; clearly, we have then $(x, y) < (x, w) < (x, y') = (x', y')$. □ □

For C a path in $\mathbb{I}^2$, define

$$f_C^-(x) := \bigwedge \{ y \mid (x, y) \in C \} \,, \qquad f_C^+(x) := \bigvee \{ y \mid (x, y) \in C \} \,. \tag{9}$$

Recall that a path $C \subseteq \mathbb{I}^2$ comes with bi-continuous surjective projections $\pi_1, \pi_2 : C \to \mathbb{I}$. Observe that the following relations hold:

$$f_C^- = \pi_2 \circ (\pi_1)_\ell \,, \qquad f_C^+ = \pi_2 \circ (\pi_1)_\rho \,. \tag{10}$$

Indeed, we have

$$\pi_2((\pi_1)_\ell(x)) = \pi_2(\bigwedge\{\,(x',y) \in C \mid x = x'\,\})\,, \qquad \text{using equation (2)}$$
$$= \bigwedge \pi_2(\{\,(x',y) \in C \mid x = x'\,\}) = \bigwedge\{\,y \mid (x,y) \in C\,\}\,.$$

The other expression for f^+ is derived similarly. In particular, the expressions in (10) show that $f^- \in \mathsf{L}_\vee(\mathbb{I})$ and $f^+ \in \mathsf{L}_\wedge(\mathbb{I})$.

Lemma 7. *We have*

$$f_C^- = (f_C^+)^\vee\,, \quad f_C^+ = (f_C^-)^\wedge\,, \quad \textit{and} \quad C = C_{f_C^+} = C_{f_C^-}\,.$$

Proof. Let us firstly argue that $(x,y) \in C$ if and only if $f_C^-(x) \leq y \leq f_C^+(y)$. The direction from left to right is obvious. Conversely, it is easily verified that if $f_C^-(x) \leq y \leq f_C^+(y)$, then the pair (x,y) is comparable with all the elements of C; then, since C is a maximal chain, necessarily $(x,y) \in C$.

Therefore, let us argue that $f_C^+ = (f_C^-)^\wedge$; we do this by showing that f_C^+ is the least meet-continuous function above f_C^-. We have $f_C^-(x) \leq f_C^+(x)$ for each $x \in \mathbb{I}$ since the fiber sets $\pi_1^{-1}(x) = \{\,(x',y) \in C \mid x' = x\,\}$ are non empty. Suppose now that $f_C^- \leq g \in \mathsf{L}_\wedge(\mathbb{I})$. In order to prove that $f_C^+ \leq g$ it will be enough to prove that $f_C^+(x) \leq g(x')$ whenever $x < x'$. Observe that if $x < x'$ then $f_C^+(x) \leq f_C^-(x')$: this is because if $(x,y),(x',y') \in C$, then $x < x'$ and C a chain imply $y \leq y'$. We deduce therefore $f_C^+(x) \leq f_C^-(x') \leq g(x')$. The relation $f_C^- = (f_C^+)^\vee$ is proved similarly. □ □

Lemma 8. *Let $f : \mathbb{I} \to \mathbb{I}$ be monotone and consider the path C_f. Then $f^\vee = f_{C_f}^-$ and $f^\wedge = f_{C_f}^+$.*

Proof. For a monotone $f : \mathbb{I} \to \mathbb{I}$ define $f' : \mathbb{I} \to C_f$ by $f' := \langle id_{\mathbb{I}}, f\rangle$, so $f = \pi_2 \circ f'$. Recall that $f_{C_f}^- = \pi_2 \circ (\pi_1)_\ell$. Therefore, in order to prove the relation $f^\vee = f_{C_f}^- = \pi_2 \circ (\pi_1)_\ell$ it shall be enough to prove that $\langle id, f^\vee\rangle$ is left adjoint to the first projection (that is, we prove that $\langle id, f^\vee\rangle = (\pi_1)_\ell$, from which it follows that $f^\vee = \pi_1 \circ \langle id, f^\vee\rangle = \pi_2 \circ (\pi_1)_\ell$). This amounts to verify that, for $x \in \mathbb{I}$ and $(x',y) \in C_f$ we have $x \leq \pi_1(x',y)$ if and only if $(x, f^\vee(x)) \leq (x',y)$. To achieve this goal, the only non trivial observation is that if $x \leq x'$, then $f^\vee(x) \leq f^\vee(x') \leq y$. The relation $f^\wedge = \pi_2 \circ (\pi_1)_\rho$ is proved similarly. □ □

Theorem 3. *There is a bijective correspondence between the following data:*

(1) *paths in $\mathbb{I}^2$,*
(2) *join-continuous functions in $\mathsf{L}_\vee(\mathbb{I})$,*
(3) *meet-continuous functions in $\mathsf{L}_\wedge(\mathbb{I})$.*

Proof. According to Lemmas 7 and 8, the correspondence sending a path C to $f_C^- \in \mathsf{L}_\vee(\mathbb{I})$ has the mapping sending f to C_f as an inverse. Similarly, the correspondence $C \mapsto f_C^+ \in \mathsf{L}_\wedge(\mathbb{I})$ has $f \mapsto C_f$ as inverse. □ □

7. Paths in higher dimensions

We show in this Section that paths in dimension d, as defined in Section 5, are in bijective correspondence with clopen tuples of $\mathsf{L}_\vee(\mathbb{I})^{[d]_2}$, as defined in Section 3; therefore, as established in that Section, there is a lattice $\mathsf{L}_d(\mathsf{L}_\vee(\mathbb{I}))$ whose underlying set can be identified with the set of paths in dimension d.

Let $f \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$, so $f = \{\, f_{i,j} \mid 1 \leq i < j \leq d\,\}$. We define then, for $1 \leq i < j \leq d$,

$$f_{j,i} := (\,f_{i,j}\,)^\star = ((f_{i,j})_\rho)^\vee\,.$$

Moreover, for $i \in [d]$, we let $f_{i,i} := id$.

Definition 3. We say that a tuple $f \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ is *compatible* if $f_{j,k} \circ f_{i,j} \leq f_{i,k}$, for each triple of elements $i, j, k \in [d]$.

Lemma 9. *A tuple is compatible if and only if it is clopen.*

Proof. For $i < j < k$, compatibility yields $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$ (closedness) and $f_{k,j} \otimes f_{j,i} \leq f_{k,i}$ which in turn is equivalent to $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$ (openness).

Conversely, suppose that f is clopen. Say that the pattern (ijk) is satisfied by f if $f_{i,j} \otimes f_{i,j} \leq f_{i,k}$. If $\mathrm{card}(\{ i, j, k \}) \leq 2$, then f satisfies the pattern (ijk) if $i = j$ or $j = k$, since then $f_{i,j} = id$ or $f_{j,k} = id$. If $i = k$, then $f_{i,j} \otimes f_{j,i} \leq id$ is equivalent to $f_{i,j} \leq id \oplus f_{i,j}$. Suppose therefore that $\mathrm{card}(\{ i, j, k \}) = 3$.

By assumption, f satisfies (ijk) and (kji) whenever $i < j < k$. Then it is possible to argue that all the patterns on the set $\{ i, j, k \}$ are satisfied by observing that if (ijk) is satisfied, then (jki) is satisfied as well: from $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$, derive $f_{i,j} \leq f_{i,k} \oplus f_{k,j}$ and then $f_{j,k} \otimes f_{k,i} \leq f_{j,i}$. □ □

Remark 2. Let $f \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ and suppose that, for some $i, j, k \in [d]$, with $i < j < k$, $f_{i,k} = f_{i,k} \circ f_{i,j}$. That is, we have $f_{i,k} = f_{i,j} \otimes f_{j,k}$ and, using the mix rule, we derive $f_{i,k} \leq f_{i,j} \oplus f_{j,k}$. Dually, a relation of the form $f^{\wedge}_{i,k} = f^{\wedge}_{j,k} \circ f^{\wedge}_{i,j}$ is equivalent to $f_{i,k} = f_{i,j} \oplus f_{j,k}$ and implies $f_{i,j} \otimes f_{j,k} \leq f_{i,k}$.

Remark 3. Lemma 9 shows that a clopen tuple of $\mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ can be extended in a unique way to a *skew* enrichment of the set $[n]$ over $\mathsf{L}_\vee(\mathbb{I})$, see [17, 27]. Dually, a clopen tuple gives rise to a unique skew metric on the set $[n]$ with values in $\mathsf{L}_\vee(\mathbb{I})$. For a skew enrichment (or metric) we mean, here, that the law $f_{j,i} = f^{\star}_{i,j}$ holds; this law, which replaces the more usual requirement that a metric is symmetric, has been considered e.g. in [15].

If $C \subseteq \mathbb{I}^d$ is a path, then we shall use $\pi_i : C \to \mathbb{I}$ to denote the projection onto the i-th coordinate. Then $\pi_{i,j} := \langle \pi_i, \pi_j \rangle : C \to \mathbb{I} \times \mathbb{I}$.

Definition 4. For a path C in $\mathbb{I}^d$, let us define $v(C) \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ by the formula:

$$v(C)_{i,j} := \pi_j \circ (\pi_i)_\ell \,, \quad (i, j) \in [d]_2. \tag{11}$$

Remark 4. An explicit formula for $v(C)_{i,j}(x)$ is as follows:

$$v(C)_{i,j}(x) = \bigwedge \{ \pi_j(y) \in C \mid \pi_i(y) = x \} \,. \tag{12}$$

Let $C_{i,j}$ be the image of C via the projection $\pi_{i,j}$. Then $C_{i,j}$ is a path, since it is the image of a bi-continuous function from $\mathbb{I}$ to $\mathbb{I} \times \mathbb{I}$. Some simple diagram chasing (or the formula in (12)) shows that $v(C)_{i,j} = f^{-}_{C_{i,j}}$ as defined in (9).

Definition 5. For a compatible $f \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$, define

$$C_f := \{ (x_1, \ldots, x_d) \mid f_{i,j}(x_i) \leq x_j, \text{ for all } i, j \in [d] \} \,.$$

Remark 5. Notice that the condition $f_{i,j}(x) \leq y$ is equivalent (by definition of $f_{i,j}$ or $f_{j,i}$) to the condition $x \leq f^{\wedge}_{j,i}(y)$. Thus, there are in principle many different ways to define C_f; in particular, when $d = 2$ (so any tuple $\mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ is compatible), the definition given above is equivalent to the one given in (8).

Proposition 5. *C_f is a path.*

The proposition is an immediate consequence of the following Lemmas 10, 11 and 13.

Lemma 10. *C_f is a total order.*

Proof. Let $x, y \in C_f$ and suppose that $x \not\leq y$, so there exists $i \in [d]$ such that $x_i \not\leq y_i$. W.l.o.g. we can suppose that $i = 1$, so $y_1 < x_1$ and then, for $i > 1$, we have $f^{\wedge}_{1,i}(y_1) \leq f_{1,i}(x_1)$, whence $y_i \leq f^{\wedge}_{1,i}(y_1) \leq f_{1,i}(x_1) \leq x_1$. This shows that $y < x$. □ □

Lemma 11. *C_f is closed under arbitrary meets and joins.*

Proof. Let $\{ x^\ell \mid \ell \in I \}$ be a family of tuples in C_f. For all $i, j \in [d]$ and $\ell \in I$, we have $f_{i,j}(\bigwedge_{\ell \in I} x^\ell_i) \leq f_{i,j}(x^\ell_i) \leq x^\ell_j$, and therefore $f_{i,j}(\bigwedge_{\ell \in I} x^\ell_i) \leq \bigwedge_{\ell \in I} x^\ell_j$. Since meets in $\mathbb{I}^d$ are computed coordinate-wise, this shows that C_f is closed under arbitrary meets. Similarly, $f_{i,j}(x^\ell_i) \leq \bigvee_{\ell \in I} x^\ell_j$ and

$$f_{i,j}(\bigvee_{\ell \in I} x^\ell_i) = \bigvee_{\ell \in I} f_{i,j}(x^\ell_i) \leq \bigvee_{\ell \in I} x^\ell_j ,$$

so C_f is also closed under arbitrary joins. □ □

Lemma 12. *Let $f \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ be compatible. Let $i_0 \in [d]$ and $x_0 \in \mathbb{I}$; define $x \in \mathbb{I}^d$ by setting $x_i := f_{i_0,i}(x_0)$ for each $i \in [d]$. Then $x \in C_f$ and $x = \bigwedge \{ y \in C_f \mid \pi_{i_0}(y) = x_0 \}$.*

Proof. Since f is compatible, $f_{i,j} \circ f_{i_0,i} \leq f_{i_0,j}$, for each $i, j \in [d]$, so

$$f_{i,j}(x_i) = f_{i,j}(f_{i_0,i}(x_0)) \leq f_{i_0,j}(x_0) = x_j .$$

Therefore, $x \in C_f$. Observe that since $f_{i_0,i_0} = id$, we have $x_{i_0} = x_0$ and x so defined is such that $\pi_{i_0}(x) = x_0$. On the other hand, if $y \in C_f$ and $x_0 \leq \pi_{i_0}(y) = y_{i_0}$, then $x_i = f_{i,i_0}(x_0) \leq f_{i,i_0}(y_{i_0}) \leq y_i$, for all $i \in [d]$. Thus $x = \bigwedge \{ y \in C_f \mid \pi_{i_0}(y) = x_0 \}$. □ □

Lemma 13. *C_f is dense.*

Proof. Let $x, y \in C_f$ and suppose that $x < y$, so there exists $i_0 \in [d]$ such that $x_{i_0} < y_{i_0}$. Pick $z_0 \in \mathbb{I}$ such that $x_{i_0} < z_0 < y_{i_0}$ and define $z \in C_f$ as in Lemma 12, $z_i := f_{i_0,i}(z_0)$, for all $i \in [d]$. We claim that $x_i \leq z_i \leq y_i$, for each $i \in [d]$. From this and $x_{i_0} < z_{i_0} < y_0$ it follows that $x < z < y$. Indeed, we have $z_i = f_{i_0,i}(z_0) \leq f_{i_0,i}(y_{i_0}) \leq y_i$. Moreover, $x_{i_0} < z_0$ implies $f^{\wedge}_{i_0,i}(x_{i_0}) \leq f_{i_0,i}(z_0)$; by Remark 5, we have $x_i \leq f^{\wedge}_{i_0,i}(x_{i_0})$. Therefore, we also have $x_i \leq f^{\wedge}_{i_0,i}(x_{i_0}) \leq f_{i_0,i}(z_0) = z_i$. □ □

Lemma 14. *If $f \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ is compatible, then $\nu(C_f) = f$.*

Proof. By Lemma 12, the correspondence sending x to $(f_{i,1}(x), \ldots, f_{d,1}(x))$ is left adjoint to the projection $\pi_i : C_f \to \mathbb{I}$. In turn, this gives that $\nu(C_f)_{i,j}(x) = \pi_j((\pi_i)_\ell(x)) = f_{i,j}(x)$, for any $i, j \in [d]$. It follows that $\nu(C_f) = f$. □ □

Lemma 15. *For C a path in $\mathbb{I}^d$, we have $C_{\nu(C)} = C$.*

Proof. Let us show that $C \subseteq C_{\nu(C)}$. Let $c \in C$; notice that for each $i, j \in [d]$, we have

$$\nu(C)_{i,j}(c_i) = \pi_j((\pi_i)_\ell(c_i)) = \pi_j((\pi_i)_\ell(\pi_i(c)) \leq \pi_j(c) = c_j ,$$

so $c \in C_{\nu(C)}$. For the converse inclusion, notice that $C \subseteq C_{\nu(C)}$ implies $C = C_{\nu(C)}$, since every path is a maximal chain. □ □

Putting together Lemmas 14 and 15 we obtain:

Theorem 4. *The correspondences, sending a path C in $\mathbb{I}^d$ to the tuple $\nu(C)$, and a compatible tuple f to the path C_f, are inverse bijections.*

8. Structure of the lattices $\mathsf{L}(\mathbb{I}^d)$

As final remarks, we present and discuss some structural properties of the lattices $\mathsf{L}(\mathbb{I}^d)$.

Recall that an element p of a lattice L is *join-prime* if, for any *finite* family $\{ x_i \mid i \in I \}$, $p \leq \bigvee_{i \in I} x_i$ implies $p \leq x_i$, for some $i \in I$. A *completely join-prime* element is defined similarly, by considering arbitrary families in place of finite ones. An element p of a lattice L is *join-irreducible* if, for any *finite* family $\{ x_i \mid i \in I \}$, $p = \bigvee_{i \in I} x_i$ implies $p = x_i$, for some $i \in I$; *completely join-irreducible* elements are defined similarly, by considering arbitrary families. If p is join-prime, then it is also join-irreducible, and the two notions coincide on distributive lattices.

Join-prime elements of $\mathsf{L}_\vee(\mathbb{I})$. We begin by describing the join-prime elements of $\mathsf{L}_\vee(\mathbb{I})$; this lattice being distributive, join-prime and join-irreducible elements coincide. For $x, y \in \mathbb{I}$, let us put

$$e_{x,y}(t) := \begin{cases} 0\,, & 0 \leq t \leq x\,, \\ y\,, & x < t\,, \end{cases} \qquad\qquad E_{x,y}(t) := \begin{cases} 0\,, & 0 \leq t < x\,, \\ y\,, & x \leq t < 1\,, \\ 1\,, & t = 1\,. \end{cases}$$

so $e_{x,y} \in \mathsf{L}_\vee(\mathbb{I})$, $E_{x,y} \in \mathsf{L}_\wedge(\mathbb{I})$ and $E_{x,y} = e^{\wedge}_{x,y}$. We call a function of the form $e_{x,y}$ a *one step function*. Notice that if $x = 1$ or $y = 0$, then $e_{x,y}$ is the constant function that takes 0 as its unique value; said otherwise, $e_{x,y} = \bot$. We say that $e_{x,y}$ is a *prime one step function* if $x < 1$ and $0 < y$; we say that $e_{x,y}$ is *rational* if $x, y \in \mathbb{I} \cap \mathbb{Q}$.

Proposition 6. *Prime one step functions are exactly the join-prime elements of* $\mathsf{L}_\vee(\mathbb{I})$.

There are no completely join-prime elements in $\mathsf{L}_\vee(\mathbb{I})$. Yet we have:

Proposition 7. *Every element of* $\mathsf{L}_\vee(\mathbb{I})$ *is a join of rational one step functions.*

Meet-irreducible elements are easily characterized using duality; they belong to the join-semilattice generated by the join-prime elements. Using duality, the following proposition is derived.

Proposition 8. $\mathsf{L}_\vee(\mathbb{I})$ *is the Dedekind-MacNeille completion of the sublattice generated by the rational one step functions.*

Join-irreducible elements of $\mathsf{L}(\mathbb{I}^d)$. Let now $d \geq 3$ be fixed. The lattice $\mathsf{L}(\mathbb{I}^d)$ is no more distributive; we characterize therefore its join-irreducible elements. We associate to a vector $p \in \mathbb{I}^d$ the tuple $e_p \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ defined as follows:

$$e_p := \langle\, e_{p_i, p_j} \mid (i, j) \in [d]_2 \,\rangle\,.$$

Proposition 9. *The elements of the form* $e_p \in \mathsf{L}_\vee(\mathbb{I})^{[d]_2}$ *are clopen and they are exactly the join-irreducible elements of* $\mathsf{L}(\mathbb{I}^d)$ *(whenever* $e_p \neq \bot$*). Every element of* $\mathsf{L}(\mathbb{I}^d)$ *is the join of the join-irreducible elements below it.*

As before $\mathsf{L}(\mathbb{I}^d)$ is the Dedekind-MacNeille completion of its sublattice generated by the join-irreducible elements. Yet, it is no longer true that every element of $\mathsf{L}(\mathbb{I}^d)$ is a join of elements of the form e_p with all the p_i rational and therefore $\mathsf{L}(\mathbb{I}^d)$ is not the Dedekind-MacNeille completion of its sublattice generated by this kind of elements.

Let us explain the significance of the previous observations. For each vector $v \in \mathbb{N}^d$ there is an embedding ι_v of the multinomial lattice $\mathsf{L}(v)$ (see [3, 25]) into $\mathsf{L}(\mathbb{I}^d)$, as in the diagram on the right, where ℓ_v and ρ_v are, respectively, the left and right adjoint to ι_v. These embeddings form a directed diagram whose colomit can be identified with the sublattice of $\mathsf{L}(\mathbb{I}^d)$ generated by the elements e_p with all the p_i, $i \in [d]$, rational. The fact $\mathsf{L}(\mathbb{I}^d)$ is not the Dedekind-MacNeille completion of this sublattice means that, while we can still define approximations of elements of $\mathsf{L}(\mathbb{I}^d)$ in the multinomial lattices via adjoints, these approximations do not converge to what they are meant to approximate. For example, we could define $\mathtt{appr}_v(f) := \ell_v(f)$ and yet have $\bigvee_{v\in\mathbb{N}^d} \iota_v(\mathtt{appr}_v(f)) < f$. On the other hand, it is possible to prove that every meet-irreducible element is an infinite join of join-irreducible elements arising from a rational point. Therefore we can state:

$$\begin{tikzcd} \mathsf{L}(v) \arrow[r, tail, "\iota_v"] & \mathsf{L}(\mathbb{I}^d) \arrow[l, bend right, "\ell_v"'] \arrow[l, bend left, "\rho_v"] \end{tikzcd}$$

Proposition 10. *Every element of $\mathsf{L}(\mathbb{I}^d)$ is a meet of joins (and a join of meets) of elements in the sublattice of $\mathsf{L}(\mathbb{I}^d)$ generated by the e_p such that p_i is rational for each $i \in [d]$.*

Whether the last proposition is the key to use the lattices $\mathsf{L}(\mathbb{I}^d)$ as well as the multinomial lattices for higher dimensional approximations in discrete geometry is an open problem that we shall tackle in future research.

9. Conclusions

In this paper we have shown how to extend the lattice structure on a set of discrete paths (known as a multinomial lattice, or weak Bruhat order, if the words coding these paths are permutations) to a lattice structure on the set of (images of) continuous paths from $\mathbb{I}$, the unit interval of the reals, to the cube $\mathbb{I}^d$, for some $d \geq 2$.

By studying the structure of these lattices, called here $\mathsf{L}(\mathbb{I}^d)$, we have been able to identify an intrinsic difficulty in defining discrete approximations of lines in dimensions $d \geq 3$ (a problem that motivated us to develop this research). This stems from the fact that $\mathsf{L}(\mathbb{I}^d)$ is no longer (when $d \geq 3$) generated by its sublattice of discrete paths as a Dedekind-Macneille completion. Proposition 10 exactly describes how the lattice $\mathsf{L}(\mathbb{I}^d)$ is generated from discrete paths and might be the key to use the lattices $\mathsf{L}(\mathbb{I}^d)$ as well as the multinomial lattices for defining higher dimensional approximations of lines. We shall tackle this problem in future research.

As a byproduct, our paper also pinpoints that various generalizations of permutohedra crucially rely on the algebraic (but also logical) notion of mix $\star$-autonomous quantale. Every such quantale yields an infinite family of lattices indexed by positive integers. While the definition of these lattices becomes straightforward by means of the algebra, it turns out that the elements of these lattices are (as far as observed up to now) in bijective correspondence either with interesting combinatorial objects (permutations, pseudo-permutations) or with geometric ones (continuous paths, as seen in this paper). These intriguing correspondences suggest the existence of a deep connection between combinatorics/geometry and logic. Future research shall unravel these phenomena. A first step, already under way for the Sugihara monoids on a chain, shall systematically identify the combinatorial objects arising from a given mix $\star$-autonomous quantale Q.

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