# L 2 -Approximation rate of forward -backward SDEs using random walk

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# $L_2$-Approximation rate of forward - backward SDEs using random walk

Christel Geiss[1], Céline Labart[2], Antti Luoto[3]

**Abstract**

Let $(Y, Z)$ denote the solution to a forward-backward SDE. If one constructs a random walk $B^n$ from the underlying Brownian motion $B$ by Skorohod embedding, one can show $L_2$ convergence of the corresponding solutions $(Y^n, Z^n)$ to $(Y, Z)$. We estimate the rate of convergence in dependence of smoothness properties, especially for a terminal condition function in $C^{2,\alpha}$.
The proof relies on an approximative representation of $Z^n$ and uses the concept of discretized Malliavin calculus. Moreover, we use growth and smoothness properties of the PDE associated to the FBSDE as well as of the finite difference equations associated to the approximating stochastic equations. We derive these properties by stochastic methods.

**Keywords** : Backward stochastic differential equations, approximation scheme, finite difference equation, convergence rate, random walk approximation

**MSC codes** : 60H10, 60H35, 60G50, 60H30,

## 1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space carrying the standard Brownian motion $B = (B_t)_{t\geq 0}$ and assume $(\mathcal{F}_t)_{t\geq 0}$ is the augmented natural filtration. Let $(Y, Z)$ be the solution of the forward-backward SDE (FBSDE)

$$\begin{aligned} X_s &= x + \int_0^s b(r, X_r)dr + \int_0^s \sigma(r, X_r)dB_r, \\ Y_s &= g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r)dr - \int_s^T Z_r dB_r, \qquad 0 \leq s \leq T. \end{aligned} \tag{1}$$

Let $(Y^n, Z^n)$ be the solution of the FBSDE if the Brownian motion $B$ is replaced by a scaled random walk $B^n$ given by

$$B_t^n = \sqrt{h} \sum_{i=1}^{[t/h]} \varepsilon_i, \qquad 0 \leq t \leq T, \tag{2}$$

where $h = \frac{T}{n}$ and $(\varepsilon_i)_{i=1,2,\ldots}$ is a sequence of i.i.d. Rademacher random variables. Then $(Y^n, Z^n)$ solves the discretized FBSDE

$$X_s^n = x + \int_{(0,s]} b(r, X_{r^-}^n)d[B^n]_r + \int_{(0,s]} \sigma(r, X_{r^-}^n)dB_r^n,$$

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$$Y_s^n = g(X_T^n) + \int_{(s,T]} f(r, X_{r-}^n Y_{r-}^n, Z_{r-}^n) d[B^n]_r - \int_{(s,T]} Z_{r-}^n dB_r^n, \quad 0 \le s \le T. \tag{3}$$

In this paper, we study the rate of the $L_2$-approximation of $(Y_t^n, Z_t^n)$ to $(Y_t, Z_t)$. This extends the results of [5] where this question was considered for the special case $X = B$.

The approximation of BSDEs using random walk has been investigated by many authors, also numerically (see, for example, [2], [10], [12], [13], [14]). In 2001, Briand et al. [2] have shown weak convergence of $(Y^n, Z^n)$ to $(Y, Z)$ for a Lipschitz continuous generator $f$ and a terminal condition in $L_2$.

In [5], under the assumption that the forward process $X$ is the Brownian motion itself, a convergence rate in $L_2$ could be obtained for a locally Hölder continuous terminal function $g$ and Lipschitz continuous generators. However, if $X$ is a solution of the SDE in (1), we need rather strong conditions on the smoothness and boundedness on $f$ and $g$ and also on $b$ and $\sigma$. In Theorem 3.2, the main result of the paper, we show that the convergence rate for $(Y_t^n, Z_t^n)$ to $(Y_t, Z_t)$ in $L_2$ is of order $h^{\frac{1}{4}\wedge\frac{\alpha}{2}}$ provided that $g''$ is locally $\alpha$-Hölder continuous.

One reason behind the strong smoothness requirements on the coefficients is that the discretized Malliavin derivative, which describes the relation between $Y^n$ and $Z^n$, is not compatible with the variational equations related to $Y^n$ and $Z^n$. This problem becomes visible in Subsection 2.3 where we introduce a discretized Malliavin weight to obtain a representation $\hat{Z}^n$ for $Z^n$. While the continuous-time representation of $Z$ is exact, $\hat{Z}^n$ does not coincide with $Z^n$, but the difference converges to 0 in $L_2$ as $n \to \infty$. To prove our main result we also need strong smoothness properties on the solution $u^n$ of the difference equation associated to the discretized FBSDE (3). We sketch the proof by applying methods known for Lévy driven BSDEs.

The paper is organized as follows: Section 2 contains the setting, main assumptions and the approximative representation of $Z^n$. Our main results about the approximation rate for the case of no generator (i.e. $f = 0$) and for the general case are in Section 3. One can see that in contrast to what is known for time discretization schemes, for random walk schemes the Lipschitz generator seems to cause more difficulties than the terminal condition: while in the case $f = 0$ we need that $g'$ is locally $\alpha$-Hölder continuous, in the case of a Lipschitz continuous generator this property is required for $g''$. In Section 4 we recall some needed facts about Malliavin weights, about the regularity of solutions to BSDEs and properties of the associated PDEs. Finally, we sketch a proof for properties of solutions to the finite difference equation associated to the discretized FBSDE. Section 5 contains technical results which mainly arise from the fact that the construction of the random walk by Skorohod embedding forces us to compare our processes on different 'time lines', one coming from the stopping times of the Skorohod embedding and the other ruled by the equidistant deterministic times due to the quadratic variation process $[B^n]$.

## 2 Preliminaries

### 2.1 The SDE and its numerical scheme

We introduce

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \qquad 0 \le t \le T$$

and its discretized counterpart

$$X_{t_k}^n = x + h \sum_{j=1}^k b(t_j, X_{t_{j-1}}^n) + \sqrt{h} \sum_{j=1}^k \sigma(t_j, X_{t_{j-1}}^n) \varepsilon_j, \qquad t_j := j\tfrac{T}{n},\ j = 0, ..., n, \tag{4}$$

where $(\varepsilon_i)_{i=1,2,\dots}$ is a sequence of i.i.d. Rademacher random variables. Letting $\mathcal{G}_k := \sigma(\varepsilon_i : 1 \le i \le k)$ with $\mathcal{G}_0 := \{\emptyset, \Omega\}$, it follows that the associated discrete-time random walk $(B^n_{t_k})_{k=0}^n$ is $(\mathcal{G}_k)_{k=0}^n$-adapted. Recall (2) and $h = \frac{T}{n}$. If we extend the sequence $(X^n_{t_k})_{k\ge 0}$ to a process in continuous time by defining $X^n_t := X^n_{t_k}$ for $t \in [t_k, t_{k+1})$, it is the solution of the forward SDE (3).

**Assumption 2.1.**

*(i)* $b, \sigma \in C_b^{0,2}([0,T]\times\mathbb{R})$, *in the sense that the derivatives of order* $k = 0,1,2$ *w.r.t. the space variable are continuous and bounded on* $[0,T]\times\mathbb{R}$,

*(ii) the first and second derivatives of* $b$ *and* $\sigma$ *w.r.t. the space variable are assumed to be* $\gamma$*-Hölder continuous (for some* $\gamma \in (0,1]$, *w.r.t. the parabolic metric* $d((x,t),(x',t')) = (|x-x'|^2 + |t-t'|)^{\frac{1}{2}}$*) on all compact subsets of* $[0,T]\times\mathbb{R}$.

*(iii)* $b, \sigma$ *are* $\frac{1}{2}$*-Hölder continuous in time, uniformly in space,*

*(iv)* $\sigma(t,x) \ge \delta > 0$ *for all* $(t,x)$.

**Assumption 2.2.**

*(i)* $g$ *is locally Hölder continuous with order* $\alpha \in (0,1]$ *and polynomially bounded (*$p_0 \ge 0, C_g > 0$*) in the following sense*

$$\forall (x,x') \in \mathbb{R}^2, \quad |g(x) - g(x')| \le C_g(1 + |x|^{p_0} + |x'|^{p_0})|x - x'|^\alpha. \tag{5}$$

*(ii) The function* $[0,T]\times\mathbb{R}^3 : (t,x,y,z) \mapsto f(t,x,y,z)$ *satisfies*

$$|f(t,x,y,z) - f(t',x',y',z')| \le L_f(\sqrt{t-t'} + |x-x'| + |y-y'| + |z-z'|). \tag{6}$$

Notice that (5) implies

$$|g(x)| \le K(1 + |x|^{p_0+1}) =: \Psi(x), \quad x \in \mathbb{R}, \tag{7}$$

for some $K > 0$. From the continuity of $f$ we conclude that

$$K_f := \sup_{0\le t\le T} |f(t,0,0,0)| < \infty.$$

**Notation:**

- $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{P})}$ for $p \ge 1$ and for $p = 2$ simply $\|\cdot\|$.
- If $a$ is a function, $C(a)$ represents a generic constant which depends on $a$ and possibly also on its derivatives.

## 2.2 The FBSDE and its numerical scheme

Recall the FBSDE (1) and its approximation (3). The backward equation in (3) can equivalently be written in the form

$$Y^n_{t_k} = g(X^n_T) + h\sum_{m=k}^{n-1} f(t_{m+1}, X^n_{t_m}, Y^n_{t_m}, Z^n_{t_m}) - \sqrt{h}\sum_{m=k}^{n-1} Z^n_{t_m}\varepsilon_{m+1}, \quad 0 \le k \le n, \tag{8}$$

if one puts $X^n_r := X^n_{t_m}$, $Y^n_r := Y^n_{t_m}$ and $Z^n_r := Z^n_{t_m}$ for $r \in [t_m, t_{m+1})$.

For $n$ large enough, (3) has a unique solution $(Y^n, Z^n)$, and $(Y^n_{t_m}, Z^n_{t_m})_{m=0}^{n-1}$ is adapted to the filtration $(\mathcal{G}_m)_{m=0}^{n-1}$ (see [15, Proposition 1.2]).

### 2.3 Representations for $Z$ and $Z^n$

We will use the representation (see Ma and Zhang [11, Theorem 4.2])

$$Z_t = \mathbb{E}_t\left(g'(X_T)\nabla X_T + \int_t^T f(s, X_s, Y_s, Z_s)N_s^t ds\right)\sigma(t, X_t), \quad 0 \le t \le T \tag{9}$$

where $\mathbb{E}_t := \mathbb{E}(\cdot|\mathcal{F}_t)$, and for all $s \in (t, T]$, we have (cf. Lemma 4.1)

$$N_s^t = \frac{1}{s-t}\int_t^s \frac{\nabla X_r}{\sigma(r, X_r)\nabla X_t} dB_r, \tag{10}$$

where $\nabla X = (\nabla X_s)_{s\in[0,T]}$ is the variational process i.e. it solves

$$\nabla X_s = 1 + \int_0^s b_x(r, X_r)\nabla X_r dr + \int_0^s \sigma_x(r, X_r)\nabla X_r dB_r,$$

with $(X_s)_{s\in[0,T]}$ given in (1).

#### 2.3.1 Approximation for $Z^n$

A counterpart to (9) for $Z^n$ does in general only exist approximatively. In particular for $f \neq 0$ stronger smoothness assumptions are required:

**Assumption 2.3.** *Assumptions 2.1 and 2.2 hold. Additionally, we assume that all first and second derivatives w.r.t. the variables $x, y, z$ of $b(t, x), \sigma(t, x)$ and $f(t, x, y, z)$ exist and are bounded Lipschitz functions w.r.t. these variables, uniformly in time. Moreover, $g''$ satisfies (5).*

We shortly introduce the discretized Malliavin derivative and refer the reader to [1] for more information on this topic. We first define for any function $F : \{-1, 1\}^n \to \mathbb{R}$ the mappings $T_{m,+}$ and $T_{m,-}$ by

$$T_{m,\pm}F(\varepsilon_1, \dots, \varepsilon_n) := F(\varepsilon_1, \dots, \varepsilon_{m-1}, \pm 1, \varepsilon_{m+1}, \dots, \varepsilon_n), \qquad 1 \le m \le n,$$

and for any $\xi = F(\varepsilon_1, \dots, \varepsilon_n)$ the discretized Malliavin derivative

$$\mathcal{D}_m^n \xi := \frac{\mathbb{E}[\xi\varepsilon_m|\sigma((\varepsilon_l)_{l\in\{1,\dots,n\}\setminus\{m\}})]}{\sqrt{h}} = \frac{T_{m,+}\xi - T_{m,-}\xi}{2\sqrt{h}}, \quad 1 \le m \le n. \tag{11}$$

In contrast to the continuous time case where $\nabla X_t(\nabla X_s)^{-1} = D_s X_t(\sigma(s, X_s))^{-1}$ $(s \le t)$, we can not expect equality for the corresponding expressions if we use the discretized processes

$$\begin{aligned}
\nabla X_{t_m}^{n,t_k,x} &= 1 + h\sum_{l=k+1}^m b_x(t_l, X_{t_{l-1}}^{n,t_k,x})\nabla X_{t_{l-1}}^{n,t_k,x} + \sqrt{h}\sum_{l=k+1}^m \sigma_x(t_l, X_{t_{l-1}}^{n,t_k,x})\nabla X_{t_{l-1}}^{n,t_k,x}\varepsilon_l, \ 0 \le k \le m \le n,\\
\mathcal{D}_k^n X_{t_m}^n &= \sigma(t_k, X_{t_{k-1}}^n) + h\sum_{l=k+1}^m b_x^{(k,l)}\mathcal{D}_k^n X_{t_{l-1}}^n + \sqrt{h}\sum_{l=k+1}^m \sigma_x^{(k,l)}(\mathcal{D}_k^n X_{t_{l-1}}^n)\varepsilon_l, \quad 0 < k \le m \le n,
\end{aligned} \tag{12}$$

where for the latter we use for $\phi = b$ and $\phi = \sigma$ the notation (if $\mathcal{D}_k^n X_{t_{\ell-1}}^n \neq 0$ the second $'$ $:=$ holds as an identity)

$$\phi_x^{(k,l)} := \frac{\mathcal{D}_k^n \phi(t_l, X_{t_{l-1}}^n)}{\mathcal{D}_k^n X_{t_{l-1}}^n} := \int_0^1 \phi_x(t_l, \vartheta T_{k,+}\, X_{t_{l-1}}^n + (1-\vartheta)T_{k,-}\, X_{t_{l-1}}^n)d\vartheta. \tag{13}$$

However, we can show convergence of $\nabla X_{t_m}^{n,t_k,X_{t_k}^n} - \frac{\mathcal{D}_{k+1}^n X_{t_m}^n}{\sigma(t_{k+1}, X_{t_k}^n)}$ in $L_p$.

**Lemma 2.4.** *Under Assumption 2.1, and for $p \geq 2$, we have*

*(i)* $\mathbb{E}|X^n_{t_l} - T_{m,\pm}X^n_{t_l}|^p \leq C(b,\sigma,T,p)h^{\frac{p}{2}}, \quad 1 \leq l,m \leq n,$

*(ii)* $\mathbb{E}\left|\nabla X^{n,t_k,X^n_{t_k}}_{t_m} - \dfrac{\mathcal{D}^n_{k+1}X^n_{t_m}}{\sigma(t_{k+1},X^n_{t_k})}\right|^p \leq C(b,\sigma,T,p)h^{\frac{p}{2}}, \quad 0 \leq k < m \leq n.$

*(iii)* $\mathbb{E}|\mathcal{D}^n_k X^n_{t_m}|^p \leq C(b,\sigma,T,p), \quad 0 \leq k \leq m \leq n.$

*Proof.* (i) We only prove the '+'-case since the proof of the '−'-case is similar. By definition, $T_{m,+}X^n_{t_l} = X^n_{t_l}$ for $l \leq m-1$, and for $l \geq m$ we have

$$\begin{aligned} T_{m,+}X^n_{t_l} &= X^n_{t_{m-1}} + b(t_m, X^n_{t_{m-1}})h + \sigma(t_m, X^n_{t_{m-1}})\sqrt{h} \\ &\quad + h\sum_{j=m+1}^{l} b(t_j, T_{m,+}X^n_{t_{j-1}}) + \sqrt{h}\sum_{j=m+1}^{l}\sigma(t_j, T_{m,+}X^n_{t_{j-1}})\varepsilon_j. \end{aligned}$$

By the properties of $b$ and $\sigma$ and thanks to the inequality of Burkholder-Davis-Gundy and Hölder's inequality we see that

$$\begin{aligned} &\mathbb{E}|X^n_{t_l} - T_{m,+}X^n_{t_l}|^p \\ &\leq C(p)\Big(\mathbb{E}|\sigma(t_m, X^n_{t_{m-1}})\sqrt{h}(\varepsilon_m - 1)|^p + h^p\mathbb{E}\Big|\sum_{j=m+1}^{l}\big(b(t_j, X^n_{t_{j-1}}) - b(t_j, T_{m,+}X^n_{t_{j-1}})\big)\Big|^p \\ &\quad + h^{\frac{p}{2}}\mathbb{E}\Big|\sum_{j=m+1}^{l}\big(\sigma(t_j, X^n_{t_{j-1}}) - \sigma(t_j, T_{m,+}X^n_{t_{j-1}})\big)^2\Big|^{\frac{p}{2}}\Big) \\ &\leq C(p)\Big(\|\sigma\|^p_\infty h^{\frac{p}{2}} + h(\|b_x\|^p_\infty t^{p-1}_{l-m} + \|\sigma_x\|^p_\infty t^{\frac{p}{2}-1}_{l-m})\sum_{j=m+1}^{l}\mathbb{E}|X^n_{t_{j-1}} - T_{m,+}X^n_{t_{j-1}}|^p\Big). \end{aligned}$$

It remains to apply Gronwall's lemma.
(ii) By the inequality of Burkholder-Davis-Gundy (BDG) and Hölder's inequality,

$$\begin{aligned} \mathbb{E}\left|\nabla X^{n,t_k,X^n_{t_k}}_{t_m} - \frac{\mathcal{D}^n_{k+1}X^n_{t_m}}{\sigma(t_{k+1},X^n_{t_k})}\right|^p &\leq C(p,T)\Big(|b_x(t_{k+1},X^n_{t_k})h + \sigma_x(t_{k+1},X^n_{t_k})\sqrt{h}\varepsilon_{k+1}|^p \\ &\quad + h\sum_{l=k+2}^{m}\mathbb{E}\left|b_x(t_l,X^n_{t_{l-1}})\nabla X^{n,t_k,X^n_{t_k}}_{t_{l-1}} - b^{(k+1,l)}_x\frac{\mathcal{D}^n_{k+1}X^n_{t_{l-1}}}{\sigma(t_{k+1},X^n_{t_k})}\right|^p \\ &\quad + h\sum_{l=k+2}^{m}\mathbb{E}\left|\sigma_x(t_l,X^n_{t_{l-1}})\nabla X^{n,t_k,X^n_{t_k}}_{t_{l-1}} - \sigma^{(k+1,l)}_x\frac{\mathcal{D}^n_{k+1}X^n_{t_{l-1}}}{\sigma(t_{k+1},X^n_{t_k})}\right|^p\Big). \end{aligned}$$

Since by Lemma 2.4 (i) we conclude that

$$\mathbb{E}|b^{(k+1,l)}_x - b_x(t_l,X^n_{t_{l-1}})|^{2p} + \mathbb{E}|\sigma^{(k+1,l)}_x - \sigma_x(t_l,X^n_{t_{l-1}})|^{2p} \leq C(b,\sigma,T,p)h^p,$$

and Lemma 5.2 implies that

$$\mathbb{E}\sup_{k+1\leq l\leq m}\left|\nabla X^{n,t_k,X^n_{t_k}}_{t_{l-1}}\right|^{2p} \leq C(b,\sigma,T,p),$$

the assertion follows by Gronwall's lemma.

(iii) This is an immediate consequence of (i).

□

We introduce a discrete counterpart to the Malliavin weight given in (10) letting

$$N_{t_\ell}^{n,t_k} := \sqrt{h} \sum_{m=k+1}^{\ell} \frac{\nabla X_{t_{m-1}}^{n,t_k,X_{t_k}^n}}{\sigma(t_m, X_{t_{m-1}}^n)} \frac{\varepsilon_m}{t_\ell - t_k}, \quad k < \ell \leq n. \tag{14}$$

Notice that there is some constant $\widehat{\kappa}_2 > 0$ depending on $b, \sigma, T, \delta$ such that

$$\left(\mathbb{E}_k |N_{t_\ell}^{n,t_k}|^2\right)^{\frac{1}{2}} \leq \frac{\widehat{\kappa}_2}{(t_\ell - t_k)^{\frac{1}{2}}}, \quad 0 \leq k < \ell \leq n, \tag{15}$$

where $\mathbb{E}_k := \mathbb{E}(\cdot|\mathcal{G}_k)$. We define a process $\hat{Z}^n = (\hat{Z}_{t_k}^n)_{k=0}^{n-1}$ by

$$\hat{Z}_{t_k}^n := \mathbb{E}_k \left(\mathcal{D}_{k+1}^n g(X_T^n)\right) + \mathbb{E}_k \left( h \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) N_{t_m}^{n,t_k} \sigma(t_{k+1}, X_{t_k}^n) \right), \tag{16}$$

and compare it with $Z^n = (Z_{t_k}^n)_{k=0}^{n-1}$ given by

$$Z_{t_k}^n = \mathbb{E}_k \left(\mathcal{D}_{k+1}^n g(X_T^n)\right) + \mathbb{E}_k \left( \sqrt{h} \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \varepsilon_{k+1} \right). \tag{17}$$

The latter equation follows if one multiplies (8) by $\varepsilon_{k+1}$ and takes the conditional expectation w.r.t. $\mathcal{G}_k$. In (16) we could have used also the approximate expression $\mathbb{E}_k(g(X_T^n) N_{t_n}^{n,t_k} \sigma(t_{k+1}, X_{t_k}^n))$, but since we will assume that $g''$ exists, we work with the correct term.

**Proposition 2.5.** *If Assumption 2.3 holds, then*

$$\mathbb{E}_{0,x} |Z_{t_k}^n - \hat{Z}_{t_k}^n|^2 \leq C_{2.5} \Psi^2(x) h^\alpha,$$

*where* $\mathbb{E}_{0,x} := \mathbb{E}(\cdot|X_0 = x)$ *and* $C_{2.5}$ *depends on* $b, \sigma, T, p_0$ *and* $\delta$.

*Proof.* According to [2, Proposition 5.1] one has the representations

$$Y_{t_m}^n = u^n(t_m, X_{t_m}^n), \quad \text{and} \quad Z_{t_m}^n = \mathcal{D}_{m+1}^n u^n(t_{m+1}, X_{t_{m+1}}^n),$$

where $u^n$ is the solution of the 'discretised' PDE (32) with terminal condition $u^n(t_n, x) = g(x)$. Notice that by the definition of $\mathcal{D}_{m+1}^n$ in (11) the expression $\mathcal{D}_{m+1}^n u^n(t_{m+1}, X_{t_{m+1}}^n)$ depends in fact on $X_{t_m}^n$. Hence we can put

$$\begin{aligned} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) &= f(t_{m+1}, X_{t_m}^n, u^n(t_m, X_{t_m}^n), \mathcal{D}_{m+1}^n u^n(t_{m+1}, X_{t_{m+1}}^n)) \\ &=: F^n(t_{m+1}, X_{t_m}^n). \end{aligned}$$

By Proposition 4.5 we conclude that $u_x^n(t_m, x)$ and $\partial_x \mathcal{D}_{m+1}^n u^n(t_{m+1}, X_{t_{m+1}}^{n,t_m,x})$ (as functions of $x$) both satisfy (5), and by Assumption 2.3 on $f$ we derive this property also for $x \mapsto F_x^n(t_{m+1}, x)$. It holds that (we use $\mathbb{E} := \mathbb{E}_{0,x}$)

$$\begin{aligned} &\|Z_{t_k}^n - \hat{Z}_{t_k}^n\| \\ &= \left\| \mathbb{E}_k \left( \sqrt{h} \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \varepsilon_{k+1} \right) \right. \end{aligned}$$

$$-\mathbb{E}_k\left(h\sum_{m=k+1}^{n-1} f(t_{m+1},X^n_{t_m},Y^n_{t_m},Z^n_{t_m})N^{n,t_k}_{t_m}\sigma(t_{k+1},X^n_{t_k})\right)\Big\|$$

$$\le \sum_{m=k+1}^{n-1}\frac{h}{m-k}\sum_{\ell=k+1}^{m}\left\|\mathbb{E}_k\left[\mathcal{D}^n_{k+1}F^n(t_{m+1},X^n_{t_m})-\mathcal{D}^n_\ell F^n(t_{m+1},X^n_{t_m})\frac{\sigma(t_{k+1},X^n_{t_k})\nabla X^{n,t_k,X^n_{t_k}}_{t_{\ell-1}}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\right]\right\|.$$

With the notation introduced in (13),

$$\begin{aligned}
&\left\|\mathcal{D}^n_{k+1}F^n(t_{m+1},X^n_{t_m})-\mathcal{D}^n_\ell F^n(t_{m+1},X^n_{t_m})\frac{\sigma(t_{k+1},X^n_{t_k})\nabla X^{n,t_k,X^n_{t_k}}_{t_{\ell-1}}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\right\|\\
&\quad\le\quad \|(\mathcal{D}^n_{k+1}X^n_{t_m})(F_x^{n,(k+1,m+1)}-F_x^{n,(\ell,m+1)})\|\\
&\qquad+\left\|F_x^{n,(\ell,m+1)}\left((\mathcal{D}^n_{k+1}X^n_{t_m})-(\mathcal{D}^n_\ell X^n_{t_m})\frac{\sigma(t_{k+1},X^n_{t_k})\nabla X^{n,t_k,X^n_{t_k}}_{t_{\ell-1}}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\right)\right\|\\
&\quad=:\quad A_1+A_2.
\end{aligned}$$

For $A_1$ we use (13) again and exploit the fact that $x\mapsto F^n_x(t,x)$ satisfies (5). By Hölder's inequality and Lemma 2.4 (i) and (iii),

$$\begin{aligned}
A_1\le\|\mathcal{D}^n_{k+1}X^n_{t_m}\|_4\int_0^1\|F^n_x(t_{m+1},\vartheta T_{k+1,+}X^n_{t_m}+(1-\vartheta)T_{k+1,-}X^n_{t_m})\\
-F^n_x(t_{m+1},\vartheta T_{\ell,+}X^n_{t_m}+(1-\vartheta)T_{\ell,-}X^n_{t_m})\|_4 d\vartheta\le C(b,\sigma,T,p_0)\Psi(x)h^{\frac{\alpha}{2}}.
\end{aligned}$$

For the estimate of $A_2$ we notice that by our assumptions the $L_4$-norm of $F_x^{n,(\ell,m+1)}$ is bounded by $C\Psi(x)$, so that it suffices to estimate

$$\begin{aligned}
&\left\|(\mathcal{D}^n_{k+1}X^n_{t_m})-(\mathcal{D}^n_\ell X^n_{t_m})\frac{\sigma(t_{k+1},X^n_{t_k})\nabla X^{n,t_k,X^n_{t_k}}_{t_{\ell-1}}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\right\|_4\\
&\le\left\|(\mathcal{D}^n_{k+1}X^n_{t_m})-\frac{\sigma(t_{k+1},X^n_{t_k})\,\mathcal{D}^n_\ell X^n_{t_m}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\frac{\mathcal{D}^n_{k+1}X^n_{t_{\ell-1}}}{\sigma(t_{k+1},X^n_{t_k})}\right\|_4\\
&\qquad+\left\|\frac{\sigma(t_{k+1},X^n_{t_k})\,\mathcal{D}^n_\ell X^n_{t_m}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\left(\nabla X^{n,t_k,X^n_{t_k}}_{t_{\ell-1}}-\frac{\mathcal{D}^n_{k+1}X^n_{t_{\ell-1}}}{\sigma(t_{k+1},X^n_{t_k})}\right)\right\|_4.
\end{aligned}\tag{18}$$

The second expression on the r.h.s. of (18) is bounded by $C(b,\sigma,T,\delta)h^{\frac12}$ as a consequence of Lemma 2.4 (ii)-(iii). To analyze the first expression, we rewrite it by (12) and get

$$\begin{aligned}
&\left|\frac{\mathcal{D}^n_\ell X^n_{t_m}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}\mathcal{D}^n_{k+1}X^n_{t_{\ell-1}}-\mathcal{D}^n_{k+1}X^n_{t_m}\right|\\
&=\left|\left(1+\sum_{l=\ell+1}^{m}\frac{\mathcal{D}^n_\ell X^n_{t_{l-1}}}{\sigma(t_\ell,X^n_{t_{\ell-1}})}(b_x^{(\ell,l)}h+\sigma_x^{(\ell,l)}\sqrt{h}\varepsilon_l)\right)\right.\\
&\qquad\times\left(\sigma(t_{k+1},X^n_{t_k})+\sum_{l=k+2}^{\ell-1}\mathcal{D}^n_{k+1}X^n_{t_{l-1}}(b_x^{(k+1,l)}h+\sigma_x^{(k+1,l)}\sqrt{h}\varepsilon_l)\right)\\
&\qquad\left.-\left(\sigma(t_{k+1},X^n_{t_k})+\left(\sum_{l=k+2}^{\ell-1}+\sum_{l=\ell}^{m}\right)\mathcal{D}^n_{k+1}X^n_{t_{l-1}}(b_x^{(k+1,l)}h+\sigma_x^{(k+1,l)}\sqrt{h}\varepsilon_l)\right)\right|
\end{aligned}$$

$$\leq |\mathcal{D}^n_{k+1} X^n_{t_{\ell-1}} (b^{(k+1,\ell)}_x h + \sigma^{(k+1,\ell)}_x \sqrt{h}\varepsilon_\ell)|$$
$$+ \left| \sum_{l=\ell+1}^{m} \left[ \frac{\mathcal{D}^n_\ell X^n_{t_{l-1}}}{\sigma(t_\ell, X^n_{t_{\ell-1}})} \mathcal{D}^n_{k+1} X^n_{t_{\ell-1}} - \mathcal{D}^n_{k+1} X^n_{t_{l-1}} \right] \left( b^{(\ell,l)}_x h + \sigma^{(\ell,l)}_x \sqrt{h}\varepsilon_l \right) \right|$$
$$+ \left| \sum_{l=\ell+1}^{m} \mathcal{D}^n_{k+1} X^n_{t_{l-1}} \left[ b^{(\ell,l)}_x h + \sigma^{(\ell,l)}_x \sqrt{h}\varepsilon_l - \left( b^{(k+1,l)}_x h + \sigma^{(k+1,l)}_x \sqrt{h}\varepsilon_l \right) \right] \right|. \tag{19}$$

We take the $L_4$-norm of (19) and apply the BDG inequality and Hölder's inequality. The second term on the r.h.s. of (19) will be used for Gronwall's lemma, while the first and the last one can be bounded by $C(b, \sigma, T)h^{\frac{1}{2}}$, by using Lemma 2.4-(iii). For the last term we also use the Lipschitz continuity of $b_x$ and $\sigma_x$ in space and Lemma 2.4-(i). □

# 3 Main results

The following approximations will rely on the fact that the random walk $B^n$ can be constructed from the Brownian motion $B$ by Skorohod embedding. Let $\tau_0 := 0$ and define

$$\tau_k := \inf\{t > \tau_{k-1} : |B_t - B_{\tau_{k-1}}| = \sqrt{h}\}, \quad k \geq 1.$$

Then $(B_{\tau_k} - B_{\tau_{k-1}})_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with

$$\mathbb{P}(B_{\tau_k} - B_{\tau_{k-1}} = \pm\sqrt{h}) = \tfrac{1}{2},$$

which means that $\sqrt{h}\varepsilon_k \overset{d}{=} B_{\tau_k} - B_{\tau_{k-1}}$. In this case we also use the notation $\mathcal{X}_{\tau_k} := X^n_{t_k}$ for all $k = 0, \ldots, n$, so that (4) turns into

$$\mathcal{X}_{\tau_k} = x + \sum_{j=1}^{k} b(t_j, \mathcal{X}_{\tau_{j-1}})h + \sum_{j=1}^{k} \sigma(t_j, \mathcal{X}_{\tau_{j-1}})(B_{\tau_j} - B_{\tau_{j-1}}), \quad 0 \leq k \leq n,$$

and (3) holds for $B^n$ given by

$$B^n_t = \sum_{k=1}^{[t/h]} (B_{\tau_k} - B_{\tau_{k-1}}), \qquad 0 \leq t \leq T. \tag{20}$$

We will denote by $\mathbb{E}_{\tau_k}$ the conditional expectation w.r.t. $\mathcal{F}_{\tau_k}$.

## 3.1 Approximation rates for the zero generator case

We express conditional expectations with the help of an independent copy of $B$ denoted by $\tilde{B}$, for example $\mathbb{E}_{\tau_k} g(X^n_T) = \tilde{\mathbb{E}} g(\tilde{\mathcal{X}}^{\tau_k, \mathcal{X}_{\tau_k}}_{\tau_n})$ for $0 \leq k \leq n$, where

$$\tilde{\mathcal{X}}^{\tau_k, \mathcal{X}_{\tau_k}}_{\tau_n} = \mathcal{X}_{\tau_k} + \sum_{j=k+1}^{n} b(t_j, \tilde{\mathcal{X}}^{\tau_k, \mathcal{X}_{\tau_k}}_{\tau_{j-1}})h + \sum_{j=k+1}^{n} \sigma(t_j, \tilde{\mathcal{X}}^{\tau_k, \mathcal{X}_{\tau_k}}_{\tau_{j-1}})(\tilde{B}_{\tilde{\tau}_{j-k}} - \tilde{B}_{\tilde{\tau}_{j-k-1}}),$$

(we define $\tilde{\tau}_k := 0$ and $\tilde{\tau}_j := \inf\{t > \tilde{\tau}_{j-1} : |\tilde{B}_t - \tilde{B}_{\tilde{\tau}_{j-1}}| = \sqrt{h}\}$ for $j \geq 1$ and $\tau_n := \tau_k + \tilde{\tau}_{n-k}$ for $n \geq k$). In fact, to represent the conditional expectations $\mathbb{E}_{t_k}$ and $\mathbb{E}_{\tau_k}$ we work here with $\tilde{\mathbb{E}}$ and the Brownian motions $B'$ and $B''$, respectively, given by

$$B'_t = B_{t \wedge t_k} + \tilde{B}_{(t-t_k)^+} \quad \text{and} \quad B''_t = B_{t \wedge \tau_k} + \tilde{B}_{(t-\tau_k)^+}, \quad t \geq 0.$$

**Proposition 3.1.** *Let Assumption 2.1 and (20) hold. If $f = 0$ and $g \in C^1$ is such that $g'$ is a locally $\alpha$-Hölder continuous function in the sense of (5), then for all $0 \le v < T$, we have (for sufficiently large $n$) that*

$$\mathbb{E}_{0,x}|Y_v - Y_v^n|^2 \le C_{3.1}^y \Psi(x)^2 h^{\frac{1}{2}}, \quad \text{and} \quad \mathbb{E}_{0,x}|Z_v - Z_v^n|^2 \le C_{3.1}^z \Psi(x)^2 h^{\frac{\alpha}{2}},$$

*where $C_{3.1}^y = C(T, p_0, C_g, C_{4.2}^y, \sigma, b)$ and $C_{3.1}^z = C(T, p_0, C_{g'}, \sigma, b, \delta)$.*

*Proof.* To shorten the notation, we use $\mathbb{E} := \mathbb{E}_{0,x}$. Let us first deal with the error of $Y$. If $v$ belongs to $[t_k, t_{k+1})$ we have $Y_v^n = Y_{t_k}^n$. Then

$$\mathbb{E}|Y_v - Y_v^n|^2 \le 2(\mathbb{E}|Y_v - Y_{t_k}|^2 + \mathbb{E}|Y_{t_k} - Y_{t_k}^n|^2).$$

Using Theorem 4.2 we bound $\|Y_v - Y_{t_k}\|$ by $C_{4.2}^y \Psi(x)(v - t_k)^{\frac{1}{2}}$ (since $\alpha = 1$ can be chosen when $g$ is locally Lipschitz continuous). Then it remains to bound

$$\mathbb{E}|Y_{t_k} - Y_{t_k}^n|^2 \quad = \quad \mathbb{E}|\mathbb{E}_{t_k} g(X_T) - \mathbb{E}_{\tau_k} g(X_T^n)|^2 = \mathbb{E}|\tilde{\mathbb{E}} g(\tilde{X}_{t_n}^{t_k, X_{t_k}}) - \tilde{\mathbb{E}} g(\tilde{\mathcal{X}}_{\tau_n}^{\tau_k, \mathcal{X}_{\tau_k}})|^2.$$

By (5) and the Cauchy-Schwarz inequality ($\Psi_1 := C_g(1 + |\tilde{X}_{t_n}^{t_k, X_{t_k}}|^{p_0} + |\tilde{\mathcal{X}}_{\tau_n}^{\tau_k, \mathcal{X}_{\tau_k}}|^{p_0})$),

$$|\tilde{\mathbb{E}} g(\tilde{X}_{t_n}^{t_k, X_{t_k}}) - \tilde{\mathbb{E}} g(\tilde{\mathcal{X}}_{\tau_n}^{\tau_k, \mathcal{X}_{\tau_k}})|^2 \quad \le \quad (\tilde{\mathbb{E}}(\Psi_1 |\tilde{X}_{t_n}^{t_k, X_{t_k}} - \tilde{\mathcal{X}}_{\tau_n}^{\tau_k, \mathcal{X}_{\tau_k}}|))^2 \le \tilde{\mathbb{E}}(\Psi_1^2) \tilde{\mathbb{E}} |\tilde{X}_{t_n}^{t_k, X_{t_k}} - \tilde{\mathcal{X}}_{\tau_n}^{\tau_k, \mathcal{X}_{\tau_k}}|^2.$$

Finally, we get by Lemma 5.2-(v) that

$$\mathbb{E}|Y_{t_k} - Y_{t_k}^n|^2 \quad \le \quad \left(\mathbb{E}\tilde{\mathbb{E}}(\Psi_1^4)\right)^{\frac{1}{2}} \left(\mathbb{E}\tilde{\mathbb{E}} |\tilde{X}_{t_n}^{t_k, X_{t_k}} - \tilde{\mathcal{X}}_{\tau_n}^{\tau_k, \mathcal{X}_{\tau_k}}|^4\right)^{\frac{1}{2}} \le C(C_g, b, \sigma, T, p_0) \Psi(x)^2 h^{\frac{1}{2}}.$$

Let us now deal with the error of $Z$. We use $\|Z_v - Z_v^n\| \le \|Z_v - Z_{t_k}\| + \|Z_{t_k} - Z_{t_k}^n\|$ and the representation

$$Z_t = \sigma(t, X_t) \tilde{\mathbb{E}}(g'(\tilde{X}_T^{t, X_t}) \nabla \tilde{X}_T^{t, X_t})$$

(see Theorem 4.3), where

$$\nabla \tilde{X}_s^{t,x} = 1 + \int_t^s b_x(r, \tilde{X}_r^{t,x}) \nabla \tilde{X}_r^{t,x} dr + \int_t^s \sigma_x(r, \tilde{X}_r^{t,x}) \nabla \tilde{X}_r^{t,x} d\tilde{B}_{r-t}, \quad 0 \le t \le s \le T.$$

For the first term we get by the assumption on $g$ and Lemma 5.2-(i) and (iii)

$$\begin{aligned}
\|Z_v - Z_{t_k}\| \quad &= \quad \|\sigma(v, X_v) \tilde{\mathbb{E}}(g'(\tilde{X}_T^{v, X_v}) \nabla \tilde{X}_T^{v, X_v}) - \sigma(t_k, X_{t_k}) \tilde{\mathbb{E}}(g'(\tilde{X}_T^{t_k, X_{t_k}}) \nabla \tilde{X}_T^{t_k, X_{t_k}})\| \\
&\le \quad \|\sigma(v, X_v) - \sigma(t_k, X_{t_k})\|_4 \|\tilde{\mathbb{E}}(g'(\tilde{X}_T^{v, X_v}) \nabla \tilde{X}_T^{v, X_v})\|_4 \\
&\quad + \|\sigma\|_\infty \|\tilde{\mathbb{E}}(g'(\tilde{X}_T^{v, X_v}) \nabla \tilde{X}_T^{v, X_v}) - \tilde{\mathbb{E}}(g'(\tilde{X}_T^{t_k, X_{t_k}}) \nabla \tilde{X}_T^{v, X_v})\| \\
&\quad + \|\sigma\|_\infty \|\tilde{\mathbb{E}}(g'(\tilde{X}_T^{t_k, X_{t_k}}) \nabla \tilde{X}_T^{v, X_v}) - \tilde{\mathbb{E}}(g'(\tilde{X}_T^{t_k, X_{t_k}}) \nabla \tilde{X}_T^{t_k, X_{t_k}})\| \\
&\le \quad C(C_{g'}, b, \sigma, T, p_0) \Psi(x) \Big[ h^{\frac{1}{2}} + \|X_v - X_{t_k}\|_4 + \left(\mathbb{E}\tilde{\mathbb{E}} |\tilde{X}_T^{v, X_v} - \tilde{X}_T^{t_k, X_{t_k}}|^{4\alpha}\right)^{\frac{1}{4}} \\
&\quad + \left(\mathbb{E}\tilde{\mathbb{E}} |\nabla \tilde{X}_T^{v, X_v} - \nabla \tilde{X}_T^{t_k, X_{t_k}}|^4\right)^{\frac{1}{4}} \Big] \\
&\le \quad C(C_{g'}, b, \sigma, T, p_0) \Psi(x) h^{\frac{\alpha}{2}}.
\end{aligned}$$

We compute the second term using $Z^n_{t_k}$ as given in (17). Hence, with the notation from (13),

$$\begin{aligned}\|Z_{t_k}-Z^n_{t_k}\|^2 &= \mathbb{E}\big|\sigma(t_k,X_{t_k})\tilde{\mathbb{E}}g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}-\tilde{\mathbb{E}}\mathcal{D}^n_{k+1}g(\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n})\big|^2\\ &\le \|\sigma\|^2_\infty\,\mathbb{E}\left|\tilde{\mathbb{E}}(g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n})-\frac{\tilde{\mathbb{E}}\mathcal{D}^n_{k+1}g(\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n})}{\sigma(t_k,X_{t_k})}\right|^2\\ &= \|\sigma\|^2_\infty\,\mathbb{E}\Big|\tilde{\mathbb{E}}(g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n})-\tilde{\mathbb{E}}\Big(g_x^{(k+1,n+1)}\frac{\mathcal{D}^n_{k+1}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}}{\sigma(t_k,X_{t_k})}\Big)\Big|^2.\end{aligned}$$

We insert $\pm\tilde{\mathbb{E}}(\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}g_x^{(k+1,n+1)})$ and get by the Cauchy-Schwarz inequality that

$$\begin{aligned}&\Big|\tilde{\mathbb{E}}(g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n})-\tilde{\mathbb{E}}\Big(g_x^{(k+1,n+1)}\frac{\mathcal{D}^n_{k+1}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}}{\sigma(t_k,X_{t_k})}\Big)\Big|^2\\ &\quad\le 2\tilde{\mathbb{E}}|g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})-g_x^{(k+1,n+1)}|^2\tilde{\mathbb{E}}|\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}|^2+2\tilde{\mathbb{E}}|g_x^{(k+1,n+1)}|^2\tilde{\mathbb{E}}\Big|\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}-\frac{\mathcal{D}^n_{k+1}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}}{\sigma(t_k,X_{t_k})}\Big|^2.\end{aligned}\tag{21}$$

For the estimate of $\tilde{\mathbb{E}}|\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}|^2$ we use Lemma 5.2. Since $g'$ satisfies (5) we proceed with

$$\begin{aligned}&\tilde{\mathbb{E}}|g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})-g_x^{(k+1,n+1)}|^2\\ &\le \int_0^1\tilde{\mathbb{E}}\Big|g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})-g'(\vartheta T_{k+1,+}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}+(1-\vartheta)T_{k+1,-}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n})\Big|^2d\vartheta\\ &\le \int_0^1(\tilde{\mathbb{E}}\Psi_1^4)^{\frac12}\Big[\tilde{\mathbb{E}}\Big|\tilde{X}^{t_k,X_{t_k}}_{t_n}-\vartheta T_{k+1,+}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}-(1-\vartheta)T_{k+1,-}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}\Big|^{4\alpha}\Big]^{\frac12}d\vartheta,\end{aligned}$$

where $\Psi_1:=C_{g'}(1+|\tilde{X}^{t_k,X_{t_k}}_{t_n}|^{p_0}+|\vartheta T_{k+1,+}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}+(1-\vartheta)T_{k+1,-}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}|^{p_0})$. For $\tilde{\mathbb{E}}\Psi_1^4$ and

$$\begin{aligned}&\tilde{\mathbb{E}}\Big|\tilde{X}^{t_k,X_{t_k}}_{t_n}-(\vartheta T_{k+1,+}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}+(1-\vartheta)T_{k+1,-}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n})\Big|^{4\alpha}\\ &\quad\le 8\left(\vartheta^{2\alpha}\tilde{\mathbb{E}}\Big|\tilde{X}^{t_k,X_{t_k}}_{t_n}-T_{k+1,+}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}\Big|^{4\alpha}+(1-\vartheta)^{2\alpha}\tilde{\mathbb{E}}\Big|\tilde{X}^{t_k,X_{t_k}}_{t_n}-T_{k+1,-}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}\Big|^{4\alpha}\right)\\ &\quad\le C(b,\sigma,T)h^{2\alpha}+C(b,\sigma,T)(|X_{t_k}-\mathcal{X}_{\tau_k}|^{4\alpha}+h^\alpha),\end{aligned}$$

we use Lemma 2.4 and Lemma 5.2-(v). For the last term in (21) we notice that

$$\mathbb{E}\tilde{\mathbb{E}}|g_x^{(k+1,n+1)}|^4\le C(b,\sigma,T,p_0,C_{g'})\Psi^4(x).$$

By Lemma 5.2 we have $\mathbb{E}\tilde{\mathbb{E}}|\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}-\nabla\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}|^p\le C(b,\sigma,T,p)h^{\frac p4}$, and by Lemma 2.4,

$$\begin{aligned}&\mathbb{E}\tilde{\mathbb{E}}\left|\nabla\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}-\frac{\mathcal{D}^n_{k+1}\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_n}}{\sigma(t_k,X_{t_k})}\right|^p\\ &\le C(p)\mathbb{E}\left|\nabla X^{n,t_k,X^n_{t_k}}_{t_n}-\frac{\mathcal{D}^n_{k+1}X^n_{t_n}}{\sigma(t_{k+1},X^n_{t_k})}\right|^p+C(p)\mathbb{E}\left|\frac{\mathcal{D}^n_{k+1}X^n_{t_n}}{\sigma(t_{k+1},X^n_{t_k})}-\frac{\mathcal{D}^n_{k+1}X^n_{t_n}}{\sigma(t_k,X_{t_k})}\right|^p\le C(b,\sigma,T,p,\delta)h^{\frac p4}.\end{aligned}$$

Consequently, $\|Z_{t_k}-Z^n_{t_k}\|^2\le C(b,\sigma,T,p_0,C_{g'},\delta)\Psi^2(x)h^{\frac\alpha2}$. □

### 3.2 Approximation rates for the general case

**Theorem 3.2.** *Let Assumptions 2.3 be satisfied and $B^n$ be given by (20). Then for all $v \in [0,T)$ and large enough $n$, we have*

$$\mathbb{E}_{0,x}|Y_v - Y_v^n|^2 + \mathbb{E}_{0,x}|Z_v - Z_v^n|^2 \leq C_{3.2}\Psi(x)^2 h^{\frac{1}{2}\wedge\alpha}$$

*with $C_{3.2} = C(b,\sigma,f,g,T,p_0,\delta,\kappa_2,c_{4.3}^{2,3},C_{4.2}^y,C_{4.4})$.*

*Proof.* Let $u : [0,T)\times\mathbb{R} \to \mathbb{R}$ be the solution of the PDE (30) associated to (1). We use $Y_s = u(s,X_s)$ and $Z_s = \sigma(s,X_s)u_x(s,X_s)$ from Theorem 4.3 and define

$$F(s,x) := f(s,x,u(s,x),\sigma(s,x)u_x(s,x)). \tag{22}$$

We have

$$\begin{aligned}\|Y_{t_k} - Y_{t_k}^n\| &\leq \|\mathbb{E}_{t_k}g(X_T) - \mathbb{E}_{\tau_k}g(X_T^n)\| \\ &\quad + \left\|\mathbb{E}_{t_k}\int_{t_k}^T f(s,X_s,Y_s,Z_s)ds - h\mathbb{E}_{\tau_k}\sum_{m=k}^{n-1} f(t_{m+1},X_{t_m}^n,Y_{t_m}^n,Z_{t_m}^n)\right\|,\end{aligned}$$

where Proposition 3.1 provides the estimate for the terminal condition. We decompose the generator term as follows:

$$\begin{aligned}&\mathbb{E}_{t_k}f(s,X_s,Y_s,Z_s) - \mathbb{E}_{\tau_k}f(t_{m+1},X_{t_m}^n,Y_{t_m}^n,Z_{t_m}^n) \\ &= [\mathbb{E}_{t_k}f(s,X_s,Y_s,Z_s) - \mathbb{E}_{t_k}f(t_m,X_{t_m},Y_{t_m},Z_{t_m})] + [\mathbb{E}_{t_k}F(t_m,X_{t_m}) - \mathbb{E}_{\tau_k}F(t_m,X_{t_m}^n)] \\ &\quad +[\mathbb{E}_{\tau_k}F(t_m,X_{t_m}^n) - \mathbb{E}_{\tau_k}F(t_m,X_{t_m})] + [\mathbb{E}_{\tau_k}f(t_m,X_{t_m},Y_{t_m},Z_{t_m}) - \mathbb{E}_{\tau_k}f(t_{m+1},X_{t_m}^n,Y_{t_m}^n,Z_{t_m}^n)] \\ &=: d_1(s,m) + d_2(m) + d_3(m) + d_4(m).\end{aligned}$$

We use

$$\begin{aligned}&\left\|\mathbb{E}_{t_k}\int_{t_k}^T f(s,X_s,Y_s,Z_s)ds - h\mathbb{E}_{\tau_k}\sum_{m=k}^{n-1} f(t_{m+1},X_{t_m}^n,Y_{t_m}^n,Z_{t_m}^n)\right\| \\ &\qquad \leq \sum_{m=k}^{n-1}\left(\left\|\int_{t_m}^{t_{m+1}} d_1(s,m)ds\right\| + h\sum_{i=2}^4 \|d_i(m)\|\right)\end{aligned}$$

and estimate the expressions on the right hand side. For the function $F$ defined in (22) we use Assumption 2.3 (which implies that (5) holds for $\alpha = 1$) to derive by Theorem 4.3 and the mean value theorem that for $x_1, x_2 \in \mathbb{R}$ there exist $\xi_1, \xi_2 \in [\min\{x_1,x_2\}, \max\{x_1,x_2\}]$ such that

$$\begin{aligned}|F(t,x_1) - F(t,x_2)| &= |f(t,x_1,u(t,x_1),\sigma(t,x_1)u_x(t,x_1)) - f(t,x_2,u(t,x_2),\sigma(t,x_2)u_x(t,x_2))| \\ &\leq C(L_f,\sigma)\left(1 + c_{4.3}^2\Psi(\xi_1) + \frac{c_{4.3}^3\Psi(\xi_2)}{(T-t)^{\frac{1}{2}}}\right)|x_1 - x_2| \\ &\leq C(T,L_f,\sigma,c_{4.3}^{2,3})(1 + |x_1|^{p_0+1} + |x_2|^{p_0+1})\frac{|x_1 - x_2|}{(T-t)^{\frac{1}{2}}}.\end{aligned} \tag{23}$$

By (6), Theorem 4.2 and Proposition 4.4 we immediately get

$$\|d_1(s,m)\| \leq C(L_f,b,\sigma,T,C_{4.2}^y,C_{4.4})\Psi(x)\, h^{\frac{1}{2}}.$$

For the estimate of $d_2$ one exploits

$$\mathbb{E}_{t_k}F(t_m,X_{t_m})-\mathbb{E}_{\tau_k}F(t_m,X^n_{t_m})=\tilde{\mathbb{E}}F(t_m,\tilde{X}^{t_k,X_{t_k}}_{t_m})-\tilde{\mathbb{E}}F(t_m,\tilde{X}^{n,t_k,X^n_{t_k}}_{t_m})$$

and then uses (23) and Lemma 5.2-(v). This gives

$$\|d_2(m)\| \;\leq\; C(L_f,c^{2,3}_{4.3},b,\sigma,T,p_0)\Psi(x)\frac{1}{(T-t_m)^{\frac{1}{2}}}h^{\frac{1}{4}}.$$

For $d_3$ we start with Jensen's inequality and continue then similarly as above to get

$$\|d_3(m)\|\leq\|F(t_m,X^n_{t_m})-F(t_m,X_{t_m})\|\leq C(L_f,c^{2,3}_{4.3},b,\sigma,T,p_0)\Psi(x)\frac{1}{(T-t_m)^{\frac{1}{2}}}h^{\frac{1}{4}},$$

and for the last term we get

$$\|d_4(m)\| \;\leq\; L_f(h^{\frac{1}{2}}+\|X_{t_m}-X^n_{t_m}\|+\|Y_{t_m}-Y^n_{t_m}\|+\|Z_{t_m}-Z^n_{t_m}\|).$$

This implies

$$\|Y_{t_k}-Y^n_{t_k}\|\leq C\Psi(x)h^{\frac{1}{4}}+hL_f\sum_{m=k}^{n-1}(\|Y_{t_m}-Y^n_{t_m}\|+\|Z_{t_m}-Z^n_{t_m}\|), \tag{24}$$

where $C=C(L_f,C^y_{3.1},c^{2,3}_{4.3},C^y_{4.2},C_{4.4},b,\sigma,T,p_0)$.

For $\|Z_{t_k}-Z^n_{t_k}\|$ we use the representations (9), (17) and the approximation (16) as well as Proposition 2.5. Instead of $N^{n,t_k}_{t_n}$ we will use here the notation $N^{n,\tau_k}_{\tau_n}$ to indicate the measurability w.r.t. the filtration $(\mathcal{F}_t)$. It holds that

$$\begin{aligned}
\|Z^n_{t_k}-Z_{t_k}\| \;&\leq\; \|Z^n_{t_k}-\hat{Z}^n_{t_k}\|+\|Z_{t_k}-\hat{Z}^n_{t_k}\|\\
&\leq\; C_{2.5}\Psi(x)h^{\frac{\alpha}{2}}+\|\sigma(t_k,X_{t_k})\tilde{\mathbb{E}}g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}-\tilde{\mathbb{E}}\mathcal{D}^n_{k+1}g(\tilde{X}^{n,t_k,X^n_{t_k}}_{t_n})\|\\
&\quad+\Big\|\mathbb{E}_{t_k}\int_{t_{k+1}}^{T}f(s,X_s,Y_s,Z_s)N^{t_k}_s ds\,\sigma(t_k,X_{t_k})\\
&\qquad-\mathbb{E}_{\tau_k}h\sum_{m=k+1}^{n-1}f(t_{m+1},X^n_{t_m},Y^n_{t_m},Z^n_{t_m})N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},X^n_{t_k})\Big\|\\
&\quad+\Big\|\mathbb{E}_{t_k}\int_{t_k}^{t_{k+1}}f(s,X_s,Y_s,Z_s)N^{t_k}_s ds\,\sigma(t_k,X_{t_k})\Big\|.
\end{aligned}\tag{25}$$

For the terminal condition Proposition 3.1 provides

$$\|\sigma(t_k,X_{t_k})\tilde{\mathbb{E}}g'(\tilde{X}^{t_k,X_{t_k}}_{t_n})\nabla\tilde{X}^{t_k,X_{t_k}}_{t_n}-\tilde{\mathbb{E}}\mathcal{D}^n_{k+1}g(\tilde{X}^{n,t_k,X^n_{t_k}}_{t_n})\|\leq(C^z_{3.1})^{\frac{1}{2}}\Psi(x)h^{\frac{1}{4}}. \tag{26}$$

We continue with the generator terms and use $F$ defined in (22) to decompose the difference

$$\begin{aligned}
&\mathbb{E}_{t_k}f(s,X_s,Y_s,Z_s)N^{t_k}_s\sigma(t_k,X_{t_k})-\mathbb{E}_{\tau_k}f(t_{m+1},X^n_{t_m},Y^n_{t_m},Z^n_{t_m})N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},X^n_{t_k})\\
&\quad=\;\mathbb{E}_{t_k}f(s,X_s,Y_s,Z_s)N^{t_k}_s\sigma(t_k,X_{t_k})-\mathbb{E}_{t_k}f(t_m,X_{t_m},Y_{t_m},Z_{t_m})N^{t_k}_{t_m}\sigma(t_k,X_{t_k})\\
&\qquad+\mathbb{E}_{t_k}F(t_m,X_{t_m})N^{t_k}_{t_m}\sigma(t_k,X_{t_k})-\mathbb{E}_{\tau_k}F(t_m,X^n_{t_m})N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},X^n_{t_k})\\
&\qquad+\mathbb{E}_{\tau_k}\Big[[F(t_m,X^n_{t_m})-F(t_m,X_{t_m})]N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},X^n_{t_k})\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}_{\tau_k}\left[[f(t_m,X_{t_m},Y_{t_m},Z_{t_m})-f(t_{m+1},X^n_{t_m},Y^n_{t_m},Z^n_{t_m})]N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},X^n_{t_k})\right]\\
=:\;& t_1(s,m)+t_2(m)+t_3(m)+t_4(m)
\end{aligned}
$$

where $s\in[t_m,t_{m+1})$. For $t_1$ we use that $\mathbb{E}_{t_k}f(t_m,X_{t_k},Y_{t_k},Z_{t_k})(N^{t_k}_s-N^{t_k}_{t_m})=0$, so that

$$
\begin{aligned}
\|t_1(s,m)\| \;\le\;& \|\mathbb{E}_{t_k}f(s,X_s,Y_s,Z_s)N^{t_k}_s\sigma(t_k,X_{t_k})-\mathbb{E}_{t_k}f(t_m,X_{t_m},Y_{t_m},Z_{t_m})N^{t_k}_s\sigma(t_k,X_{t_k})\|\\
&+\|\mathbb{E}_{t_k}(f(t_m,X_{t_m},Y_{t_m},Z_{t_m})-f(t_m,X_{t_k},Y_{t_k},Z_{t_k}))(N^{t_k}_s-N^{t_k}_{t_m})\sigma(t_k,X_{t_k})\|.
\end{aligned}
$$

We rewrite the conditional expectations as before with the help of the independent copy $\tilde{B}$. Then

$$
\begin{aligned}
&\mathbb{E}_{t_k}f(s,X_s,Y_s,Z_s)N^{t_k}_s-\mathbb{E}_{t_k}f(t_m,X_{t_m},Y_{t_m},Z_{t_m})N^{t_k}_s\\
&\quad=\;\tilde{\mathbb{E}}[(f(s,\tilde{X}^{t_k,X_{t_k}}_s,\tilde{Y}^{t_k,X_{t_k}}_s,\tilde{Z}^{t_k,X_{t_k}}_s)-f(t_m,\tilde{X}^{t_k,X_{t_k}}_{t_m},\tilde{Y}^{t_k,X_{t_k}}_{t_m},\tilde{Z}^{t_k,X_{t_k}}_{t_m}))\tilde{N}^{t_k}_s]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_{t_k}(f(t_m,X_{t_m},Y_{t_m},Z_{t_m})-f(t_m,X_{t_k},Y_{t_k},Z_{t_k}))(N^{t_k}_s-N^{t_k}_{t_m})\\
&\quad=\;\tilde{\mathbb{E}}[(f(t_m,\tilde{X}^{t_k,X_{t_k}}_{t_m},\tilde{Y}^{t_k,X_{t_k}}_{t_m},\tilde{Z}^{t_k,X_{t_k}}_{t_m})-f(t_m,X_{t_k},Y_{t_k},Z_{t_k}))(\tilde{N}^{t_k}_s-\tilde{N}^{t_k}_{t_m})].
\end{aligned}
$$

We apply the conditional Hölder inequality, and from the estimates (29) and $\tilde{\mathbb{E}}|\tilde{N}^{t_k}_s-\tilde{N}^{t_k}_{t_m}|^2\le C(b,\sigma,T,\delta)\frac{h}{(s-t_k)^2}$ we get

$$
\begin{aligned}
\|t_1(s,m)\| \;\le\;& \frac{\kappa_2\|\sigma\|_\infty}{(s-t_k)^{\frac12}}\|f(s,X_s,Y_s,Z_s)-f(t_m,X_{t_m},Y_{t_m},Z_{t_m})\|\\
&+C(b,\sigma,T,\delta)\frac{h^{\frac12}}{s-t_k}\,\|f(t_m,X_{t_m},Y_{t_m},Z_{t_m})-f(t_k,X_{t_k},Y_{t_k},Z_{t_k})\|\\
\le\;& C(L_f,C^y_{4.2},C_{4.4},\kappa_2,b,\sigma,T,\delta,p_0)\Psi(x)\frac{h^{\frac12}}{(s-t_k)^{\frac12}},
\end{aligned}
$$

since for $0\le t<s\le T$ we have by Theorem 4.2 and Proposition 4.4

$$
\|f(s,X_s,Y_s,Z_s)-f(t,X_t,Y_t,Z_t)\| \;\le\; C(L_f,C^y_{4.2},C_{4.4},b,\sigma,T,p_0)\Psi(x)(s-t)^{\frac12}. \tag{27}
$$

For $t_2$ Lemma 5.2, Lemma 5.3, (23) and (29) yield

$$
\begin{aligned}
\|t_2(m)\| \;=\;& \|\tilde{\mathbb{E}}F(t_m,\tilde{X}^{t_k,X_{t_k}}_{t_m})\tilde{N}^{t_k}_{t_m}\sigma(t_k,X_{t_k})-\tilde{\mathbb{E}}F(t_m,\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_m})\tilde{N}^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},\mathcal{X}_{\tau_k})\|\\
\le\;& \frac{C(\sigma,\kappa_2)}{(t_m-t_k)^{\frac12}}\left(\mathbb{E}\tilde{\mathbb{E}}(F(t_m,\tilde{X}^{t_k,X_{t_k}}_{t_m})-F(t_m,\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_m}))^2\right)^{\frac12}\\
&+(\mathbb{E}\tilde{\mathbb{E}}|F(t_m,\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_m})-F(t_m,\mathcal{X}_{\tau_k})|^2\tilde{\mathbb{E}}|\tilde{N}^{t_k}_{t_m}\sigma(t_k,X_{t_k})-\tilde{N}^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},\mathcal{X}_{\tau_k})|^2)^{\frac12}\\
\le\;& C(L_f,c^{2,3}_{4.3},b,\sigma,T,p_0,\delta,\kappa_2)\frac{\Psi(x)}{(T-t_m)^{\frac12}}\frac{h^{\frac14}}{(t_m-t_k)^{\frac12}}.
\end{aligned}
$$

For $t_3$ we use the conditional Hölder inequality, (23), (15) and Lemma 5.2:

$$
\begin{aligned}
\|t_3(m)\| \;=\;& \|\mathbb{E}_{\tau_k}\left[[F(t_m,X^n_{t_m})-F(t_m,X_{t_m})]N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1},\mathcal{X}_{\tau_k})\right]\|\\
\le\;& \frac{C(\sigma,\widehat{\kappa}_2)}{(t_m-t_k)^{\frac12}}\,\|F(t_m,X^n_{t_m})-F(t_m,X_{t_m})\|
\end{aligned}
$$

$$\leq \quad C(L_f, c^{2,3}_{4.3}, b, \sigma, T, p_0, \delta)\frac{\Psi(x)}{(T-t_m)^{\frac{1}{2}}}\frac{h^{\frac{1}{4}}}{(t_m-t_k)^{\frac{1}{2}}}.$$

The term $t_4$ can be estimated as follows:

$$\begin{aligned}\|t_4(m)\| &= \|\mathbb{E}_{\tau_k}\left[[f(t_m, X_{t_m}, Y_{t_m}, Z_{t_m}) - f(t_{m+1}, X^n_{t_m}, Y^n_{t_m}, Z^n_{t_m})]N^{n,\tau_k}_{\tau_m}\sigma(t_{k+1}, \mathcal{X}_{\tau_k})\right]\| \\ &\leq \frac{C(L_f, b, \sigma, T, \delta)}{(t_m-t_k)^{\frac{1}{2}}}(h^{\frac{1}{2}} + \|X_{t_m} - X^n_{t_m}\| + \|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|).\end{aligned}$$

Finally, for the remaining term of the estimate of $\|Z_{t_k} - Z^n_{t_k}\|$, we use (27) and (29) to get

$$\begin{aligned}\left\|\mathbb{E}_{t_k} f(s, X_s, Y_s, Z_s)N^{t_k}_s\,\sigma(t_k, X_{t_k})\right\| &= \|\mathbb{E}_{t_k}[(f(s, X_s, Y_s, Z_s) - f(s, X_{t_k}, Y_{t_k}, Z_{t_k}))N^{t_k}_s]\,\sigma(t_k, X_{t_k})\| \\ &\leq C(L_f, C^y_{4.2}, C_{4.4}, b, \sigma, T, p_0, \kappa_2)\Psi(x).\end{aligned}$$

Consequently, from (25), (26), the estimates for the remaining term and for $t_1, ..., t_4$ it follows that

$$\begin{aligned}\|Z_{t_k} - Z^n_{t_k}\| \leq\ & C_{2.5}\Psi(x)h^{\frac{\alpha}{2}} + (C^z_{3.1})^{\frac{1}{2}}\Psi(x)h^{\frac{1}{4}} + C(L_f, C^y_{4.2}, C_{4.4}, b, \sigma, T, p_0, \kappa_2)\Psi(x)h \\ &+C(L_f, C^y_{4.2}, C_{4.4}, \kappa_2, b, \sigma, T, \delta, p_0)\Psi(x)h^{\frac{1}{2}}\int_{t_k}^T \frac{ds}{(s-t_k)^{\frac{1}{2}}} \\ &+C(L_f, c^{2,3}_{4.3}, b, \sigma, T, p_0, \delta, \kappa_2)h\sum_{m=k+1}^{n-1}\frac{\Psi(x)}{(T-t_m)^{\frac{1}{2}}}\frac{h^{\frac{1}{4}}}{(t_m-t_k)^{\frac{1}{2}}} \\ &+C(L_f, b, \sigma, T, \delta)h\sum_{m=k+1}^{n-1}(\|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|)\frac{1}{(t_m-t_k)^{\frac{1}{2}}} \\ \leq\ & C(C_{2.5}, C^z_{3.1})\Psi(x)h^{\frac{\alpha}{2}\wedge\frac{1}{4}} + C(L_f, c^{2,3}_{4.3}, C^y_{4.2}, C_{4.4}, \kappa_2, b, \sigma, T, p_0, \delta)\Psi(x)h^{\frac{1}{4}} \\ &+C(L_f, b, \sigma, T, \delta)\sum_{m=k+1}^{n-1}(\|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|)\frac{1}{(t_m-t_k)^{\frac{1}{2}}}h.\end{aligned}$$

Then we use (24) and the above estimate to get

$$\begin{aligned}&\|Y_{t_k} - Y^n_{t_k}\| + \|Z_{t_k} - Z^n_{t_k}\| \\ &\leq\ C(C_{2.5}, C^z_{3.1})\Psi(x)h^{\frac{\alpha}{2}\wedge\frac{1}{4}} + C(L_f, C^y_{3.1}, c^{2,3}_{4.3}, C^y_{4.2}, C_{4.4}, b, \sigma, T, p_0, \kappa_2, \delta)\Psi(x)h^{\frac{1}{4}} \\ &\quad +C(L_f, b, \sigma, T, \delta)\sum_{m=k+1}^{n-1}(\|Y_{t_m} - Y^n_{t_m}\| + \|Z_{t_m} - Z^n_{t_m}\|)\frac{1}{(t_m-t_k)^{\frac{1}{2}}}h.\end{aligned}$$

Consequently,

$$\|Y_{t_k} - Y^n_{t_k}\| + \|Z_{t_k} - Z^n_{t_k}\| \quad \leq \quad C_{3.2}\Psi(x)h^{\frac{\alpha}{2}\wedge\frac{1}{4}}.$$

By Theorem 4.2 it follows that

$$\|Y_v - Y^n_v\| \leq \|Y_v - Y_{t_k}\| + \|Y_{t_k} - Y^n_{t_k}\| \leq C(C_{3.2}, C^y_{4.2})\Psi(x)h^{\frac{\alpha}{2}\wedge\frac{1}{4}},$$

while Proposition 4.4 implies that

$$\|Z_v - Z_{t_k}\| \leq C_{4.4}\Psi(x)h^{\frac{1}{2}}.$$

□

# 4 Some properties of solutions to PDEs and BSDEs

## 4.1 Malliavin weights

We use the SDE from (1) started in $(t,x)$,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r, \quad 0 \le t \le s \le T \tag{28}$$

and recall the Malliavin weights and their properties from [4, Subsection 1.1 and Remark 3].

**Lemma 4.1.** *Let $H : \mathbb{R} \to \mathbb{R}$ be a polynomially bounded Borel function. If Assumption 2.1 holds and $X^{t,x}$ is given by (28) then setting*

$$G(t,x) := \mathbb{E}H(X_T^{t,x})$$

*implies that $G \in C^{1,2}([0,T) \times \mathbb{R})$. Especially it holds for $0 \le t \le r < T$ that*

$$\partial_x G(r, X_r^{t,x}) = \mathbb{E}[H(X_T^{t,x}) N_T^{r,(t,x)} | \mathcal{F}_r^t],$$

*where $(\mathcal{F}_r^t)_{r\in[t,T]}$ is the augmented natural filtration of $(B_r^{t,0})_{r\in[t,T]}$ and*

$$N_T^{r,(t,x)} = \frac{1}{T-r} \int_r^T \frac{\nabla X_s^{t,x}}{\sigma(s, X_s^{t,x}) \nabla X_r^{t,x}} dB_s.$$

*Moreover, for $q \in (0,\infty)$ it holds a.s.*

$$(\mathbb{E}[|N_T^{r,(t,x)}|^q | \mathcal{F}_r^t])^{\frac{1}{q}} \le \frac{\kappa_q}{(T-r)^{\frac{1}{2}}} \quad \text{and} \quad \mathbb{E}[N_T^{r,(t,x)} | \mathcal{F}_r^t] = 0 \text{ a.s.} \tag{29}$$

*and we have*

$$\|\partial_x G(r, X_r^{t,x})\|_{L_p(\mathbb{P})} \le \kappa_q \frac{\|H(X_T^{t,x}) - \mathbb{E}[H(X_T^{t,x})|\mathcal{F}_r^t]\|_p}{\sqrt{T-r}}$$

*for $1 < q, p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

## 4.2 Regularity of solutions to BSDEs

The following result originates from [4, Theorem 1] where also path dependent cases were included. We formulate it only for our Markovian setting but use $\mathbb{P}_{t,x}$ since we are interested in an estimate for all $(t,x) \in [0,T) \times \mathbb{R}$. A sketch of a proof of this formulation can be found in [5].

**Theorem 4.2.** *Let Assumption 2.1 and 2.2 hold. Then for any $p \in [2,\infty)$ the following assertions are true.*

*(i) There exists a constant $C_{4.2}^y > 0$ such that for $0 \le t < s \le T$ and $x \in \mathbb{R}$,*

$$\|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \le C_{4.2}^y \Psi(x) \left( \int_t^s (T-r)^{\alpha-1} dr \right)^{\frac{1}{2}},$$

*(ii) there exists a constant $C_{4.2}^z > 0$ such that for $0 \le t < s < T$ and $x \in \mathbb{R}$,*

$$\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})} \le C_{4.2}^z \Psi(x) \left( \int_t^s (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}}.$$

*The constants $C_{4.2}^y$ and $C_{4.2}^z$ depend on $K_f, L_f, C_g, c_{4.3}^{1,2}, T, p_0, b, \sigma, \kappa_q$ and $p$.*

### 4.3 Properties of the associated PDE

**Theorem 4.3** ([5], Theorem 5.6)**.** *Consider the FBSDE (1) and let Assumptions 2.1 and 2.2 hold. Then for the solution $u$ of the associated PDE*

$$\begin{cases} u_t(t,x) + \frac{\sigma^2(t,x)}{2} u_{xx}(t,x) + b(t,x)u_x(t,x) + f(t,x,u(t,x),\sigma(t,x)u_x(t,x)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T), x \in \mathbb{R}, \\ u(T,x) = g(x), \quad x \in \mathbb{R} \end{cases} \tag{30}$$

*we have*

*(i)* $Y_t = u(t,X_t)$ *where* $u(t,x) = \mathbb{E}_{t,x}\left(g(X_T) + \int_t^T f(r,X_r,Y_r,Z_r)dr\right)$ *and* $|u(t,x)| \leq c^1_{4.3}\Psi(x)$ *with* $\Psi$ *given in (7).*

*(ii)* *(a)* $\partial_x u$ *exists and is continuous in* $[0,T)\times\mathbb{R}$,
*(b)* $Z_s^{t,x} = u_x(s,X_s^{t,x})\sigma(s,X_s^{t,x})$,
*(c)* $|u_x(t,x)| \leq \frac{c^2_{4.3}\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}$.

*(iii)* *(a)* $\partial_x^2 u$ *exists and is continuous in* $[0,T)\times\mathbb{R}$,
*(b)* $|\partial_x^2 u(t,x)| \leq \frac{c^3_{4.3}\Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}$.

Using Assumption 2.3 we are now in the position to improve the bound on $\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})}$ given in Theorem 4.2.

**Proposition 4.4.** *If Assumption 2.3 holds, then there exists a constant $C_{4.4} > 0$ such that for $0 \leq t < s \leq T$ and $x \in \mathbb{R}$,*

$$\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})} \leq C_{4.4}\Psi(x)(s-t)^{\frac{1}{2}},$$

*where $C_{4.4}$ depends on $b,\sigma,T,p_0,g,f,p,c^{2,3}_{4.3}$.*

*Proof.* From $Z_s^{t,x} = u_x(s,X_s^{t,x})\sigma(s,X_s^{t,x})$ and $\nabla Y_s^{t,x} = \partial_x u(s,X_s^{t,x}) = u_x(s,X_s^{t,x})\nabla X_s^{t,x}$ we conclude

$$Z_s^{t,x} = \frac{\nabla Y_s^{t,x}}{\nabla X_s^{t,x}}\sigma(s,X_s^{t,x}), \quad 0 \leq t \leq s \leq T.$$

It is well-known (see e.g. [3]) that the solution $\nabla Y$ of the linear BSDE

$$\nabla Y_s = g'(X_T)\nabla X_T + \int_s^T f_x(\Theta_r)\nabla X_r + f_y(\Theta_r)\nabla Y_r + f_z(\Theta_r)\nabla Z_r dr - \int_s^T \nabla Z_r dB_r, \quad 0 \leq s \leq T, \tag{31}$$

can be represented as

$$\begin{aligned} \frac{\nabla Y_s}{\nabla X_s} &= \mathbb{E}_s\Big[g'(X_T)\nabla X_T\Gamma_T^s + \int_s^T f_x(\Theta_r)\nabla X_r\Gamma_r^s dr\Big]\frac{1}{\nabla X_s} \\ &= \tilde{\mathbb{E}}\Big[g'(\tilde{X}_T^{s,X_s})\nabla\tilde{X}_T^{s,X_s}\tilde{\Gamma}_T^{s,X_s} + \int_s^T f_x(\tilde{\Theta}_r^{s,X_s})\nabla\tilde{X}_r^{s,X_s}\tilde{\Gamma}_r^{s,X_s}dr\Big], \quad 0 \leq t \leq s \leq T, \end{aligned}$$

where $\Theta_r := (r,X_r,Y_r,Z_r)$ and $\Gamma^s$ denotes the adjoint process given by

$$\Gamma_r^s = 1 + \int_s^r f_y(\Theta_u)\Gamma_u^s du + \int_s^r f_z(\Theta_u)\Gamma_u^s dB_u, \quad s \leq r \leq T,$$

and

$$\tilde{\Gamma}_s^{t,x} = 1 + \int_t^s f_y(\tilde{\Theta}_r^{t,x})\tilde{\Gamma}_r^{t,x} dr + \int_t^s f_z(\tilde{\Theta}_r^{t,x})\tilde{\Gamma}_r^{t,x} d\tilde{B}_r, \quad t \le s \le T,\ x \in \mathbb{R}.$$

Especially, we have

$$\frac{\nabla Y_t^{t,x}}{\nabla X_t^{t,x}} = \nabla Y_t^{t,x} = \tilde{\mathbb{E}}\Big[g'(\tilde{X}_T^{t,x})\nabla \tilde{X}_T^{t,x}\tilde{\Gamma}_T^{t,x} + \int_t^T f_x(\tilde{\Theta}_r^{t,x})\nabla \tilde{X}_r^{t,x}\tilde{\Gamma}_r^{t,x} dr\Big].$$

Then,

$$\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})} \le C(\sigma)\Big[\Big\|\frac{\nabla Y_s}{\nabla X_s} - \frac{\nabla Y_t}{\nabla X_t}\Big\|_{L_p(\mathbb{P}_{t,x})} + \|\nabla Y_t\|_{L_{2p}(\mathbb{P}_{t,x})}[(s-t)^{\frac{1}{2}} + \|X_s^{t,x} - x\|_{L_{2p}(\mathbb{P}_{t,x})}]\Big].$$

Since $(\nabla Y_s, \nabla Z_s)$ is the solution to the linear BSDE (31) with bounded $f_x, f_y, f_z$, we have that $\|\nabla Y_t\|_{L_{2p}(\mathbb{P}_{t,x})} \le C(b,\sigma,T,p,f,g)$. Obviously, $\|X_s^{t,x} - x\|_{L_{2p}(\mathbb{P}_{t,x})} \le C(b,\sigma,T,p)(s-t)^{\frac{1}{2}}$. So it remains to show that

$$\Big\|\frac{\nabla Y_s}{\nabla X_s} - \frac{\nabla Y_t}{\nabla X_t}\Big\|_{L_p(\mathbb{P}_{t,x})} \le C\Psi(x)(s-t)^{\frac{1}{2}}.$$

From the above relations we get for $(X_s := X_s^{t,x})$

$$\begin{aligned}
\Big\|\frac{\nabla Y_s}{\nabla X_s} - \frac{\nabla Y_t}{\nabla X_t}\Big\|_{L_p(\mathbb{P}_{t,x})} &\le \Big\|\tilde{\mathbb{E}}\Big[g'(\tilde{X}_T^{s,X_s})\nabla \tilde{X}_T^{s,X_s}\tilde{\Gamma}_T^{s,X_s} - g'(\tilde{X}_T^{t,x})\nabla \tilde{X}_T^{t,x}\tilde{\Gamma}_T^{t,x}\Big]\Big\|_p \\
&\quad + \int_t^s \Big\|\tilde{\mathbb{E}}\Big[f_x(\tilde{\Theta}_r^{t,x})\nabla \tilde{X}_r^{t,x}\tilde{\Gamma}_r^{t,x}\Big]\Big\|_p dr \\
&\quad + \Big\|\int_s^T \tilde{\mathbb{E}}\Big[f_x(\tilde{\Theta}_r^{s,X_s})\nabla \tilde{X}_r^{s,X_s}\tilde{\Gamma}_r^{s,X_s} - f_x(\tilde{\Theta}_r^{t,x})\nabla \tilde{X}_r^{t,x}\tilde{\Gamma}_r^{t,x}\Big] dr\Big\|_p \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Since $g'$ is Lipschitz continuous and of polynomial growth, the estimate $J_1 \le C(b,\sigma,g,T,p)\Psi(x)(s-t)^{\frac{1}{2}}$ follows by Hölder's inequality and the $L_q$ -boundedness for any $q > 0$ of all the factors, as well as from the estimates for $\tilde{X}_T^{s,X_s} - \tilde{X}_T^{t,x}$ and $\nabla \tilde{X}_T^{s,X_s} - \nabla \tilde{X}_T^{t,x}$ like in Lemma 5.2. For the $\Gamma$ differences we first apply the inequalities of Hölder and BDG:

$$\begin{aligned}
\tilde{\mathbb{E}}|\tilde{\Gamma}_T^{s,X_s} - \tilde{\Gamma}_T^{t,x}|^q \le C(T,q) \quad & (s-t)^{q-1}\tilde{\mathbb{E}}\int_t^s |f_y(\tilde{\Theta}_r^{s,X_s})\tilde{\Gamma}_r^{s,X_s}|^q dr + \tilde{\mathbb{E}}\Big(\int_t^s |f_z(\tilde{\Theta}_r^{s,X_s})\tilde{\Gamma}_r^{s,X_s}|^2 dr\Big)^{\frac{q}{2}} \\
&+\tilde{\mathbb{E}}\int_s^T |f_y(\tilde{\Theta}_r^{s,X_s})\tilde{\Gamma}_r^{s,X_s} - f_y(\tilde{\Theta}_r^{t,x})\tilde{\Gamma}_r^{t,x}|^q dr \\
&+\tilde{\mathbb{E}}\Big(\int_s^T |f_z(\tilde{\Theta}_r^{s,X_s})\tilde{\Gamma}_r^{s,X_s} - f_z(\tilde{\Theta}_r^{t,x})\tilde{\Gamma}_r^{t,x}|^2 dr\Big)^{\frac{q}{2}}
\end{aligned}$$

Since $f_y$ and $f_z$ are bounded we have $\tilde{\mathbb{E}}|\tilde{\Gamma}_r^{s,X_s}|^q + \tilde{\mathbb{E}}|\tilde{\Gamma}_r^{t,x}|^q \le C(T,f,q)$. Similar to (23), since $f_x, f_y, f_z$ are Lipschitz continuous w.r.t. the space variables,

$$\begin{aligned}
|f_x(\tilde{\Theta}_r^{s,X_s}) - f_x(\tilde{\Theta}_r^{t,x})| &= |f_x(r,\tilde{X}_r^{s,X_s}, u(r,\tilde{X}_r^{s,X_s}), \sigma(r,\tilde{X}_r^{s,X_s})u_x(r,\tilde{X}_r^{s,X_s})) \\
&\quad - f_x(r,\tilde{X}_r^{t,x}, u(r,\tilde{X}_r^{t,x}), \sigma(r,\tilde{X}_r^{t,x})u_x(r,\tilde{X}_r^{t,x}))| \\
&\le C(T,f,\sigma,c_{4.3}^{2,3})(1 + |\tilde{X}_r^{s,X_s}|^{p_0+1} + |\tilde{X}_r^{t,x}|^{p_0+1})\frac{|\tilde{X}_r^{s,X_s} - \tilde{X}_r^{t,x}|}{(T-r)^{\frac{1}{2}}},
\end{aligned}$$

so that Lemma 5.2 yields

$$\tilde{\mathbb{E}}|f_x(\tilde{\Theta}_r^{s,X_s}) - f_x(\tilde{\Theta}_r^{t,x})|^q \leq C(b,\sigma,T,p_0,f,c_{4.3}^{2,3},q)(1+|X_s|^{p_0+1}+|x|^{p_0+1})^q \frac{|X_s-x|^q+|s-t|^{\frac{q}{2}}}{(T-r)^{\frac{1}{2}}}.$$

The same holds for $|f_y(\tilde{\Theta}_r^{s,X_s}) - f_y(\tilde{\Theta}_r^{t,x})|$ and $|f_z(\tilde{\Theta}_r^{s,X_s}) - f_z(\tilde{\Theta}_r^{t,x})|$. Applying these inequalities and Gronwall's Lemma, we arrive at

$$\|\tilde{\mathbb{E}}[\tilde{\Gamma}_T^{s,X_s} - \tilde{\Gamma}_T^{t,x}]\|_p \quad \leq \quad C(b,\sigma,T,p_0,f,g,c_{4.3}^{2,3},p)\Psi(x)|s-t|^{\frac{1}{2}}$$

for $p>0$.

For $J_2 \leq C(t-s)$ it is enough to realise that the integrand is bounded. The estimate for $J_3$ follows similarly to that of $J_1$. □

### 4.4 Properties of the solution to the finite difference equation

**Proposition 4.5.** *Let Assumption 2.3 hold and assume that $u^n$ is a solution of*

$$\begin{aligned} &u^n(t_m,x) - hf(t_{m+1},x,u^n(t_m,x),\mathcal{D}_{m+1}^n u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x})) \\ &\quad = \frac{1}{2}[T_{m+1,+}u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x}) + T_{m+1,-}u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x})], \quad m=0,\dots,n-1, \end{aligned} \tag{32}$$

*with terminal condition $u^n(t_n,x)=g(x)$. Then, for sufficiently small $h$, the map $x \mapsto u^n(t_m,x)$ is $C^2$, and $u_x^n(t_m,x)$ and $u_{xx}^n(t_m,x)$ satisfy (5) uniformly in $m=0,\dots,n-1$. The constants $C_{u^n}$, $C_{u_x^n}$ and $C_{u_{xx}^n}$ depend on the bounds of $f,g,b,\sigma$ and their derivatives and on $T$.*

*Proof.* **Step 1.** From (32), since $g$ is $C^2$ and $f_y$ is bounded, for sufficiently small $h$ we conclude by induction (backwards in time) that $u_x^n(t_m,x)$ exists for $m=0,...,n-1$, and that it holds

$$\begin{aligned} u_x^n(t_m,x) \quad = \quad & hf_x(t_{m+1},x,u^n(t_m,x),\mathcal{D}_{m+1}^n u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x})) \\ & + hf_y(t_{m+1},x,u^n(t_m,x),\mathcal{D}_{m+1}^n u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x}))u_x^n(t_m,x) \\ & + hf_z(t_{m+1},x,u^n(t_m,x),\mathcal{D}_{m+1}^n u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x}))\partial_x\mathcal{D}_{m+1}^n u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x}) \\ & + \tfrac{1}{2}(\partial_x T_{m+1,+}u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x}) + \partial_x T_{m+1,-}u^n(t_{m+1},X_{t_{m+1}}^{n,t_m,x})). \end{aligned}$$

Similarly one can show that $u_{xx}^n(t_m,x)$ exists and solves the derivative of the previous equation.

**Step 2.** We will use that $u^n(t_m,x) = Y_{t_m}^{n,t_m,x}$ and exploit the BSDE

$$Y_{t_m}^{n,t_m,x} = g(X_T^{n,t_m,x}) + \int_{(t_m,T]} f(s,X_{s^-}^{n,t_m,x},Y_{s^-}^{n,t_m,x},Z_{s^-}^{n,t_m,x})d[B^n]_s - \int_{(t_m,T]} Z_{s^-}^{n,t_m,x}dB_s^n, \tag{33}$$

where we will drop the superscript $t_m,x$ from now on. For $u_x^n(t_m,x)$ we will consider

$$\partial_x Y_{t_m}^n = g'(X_T^n)\partial_x X_T^n + \int_{(t_m,T]} f_x\partial_x X_{s^-}^n + f_y\partial_x Y_{s^-}^n + f_z\partial_x Z_{s^-}^n d[B^n]_s - \int_{(t_m,T]} \partial_x Z_{s^-}^n dB_s^n. \tag{34}$$

Similar as in the proof of [11, Theorem 3.1] the BSDE (34) can be derived as limit of difference quotients w.r.t. $x$ of (33). Notice that the generator of (34) is random but has the same Lipschitz constant and linear growth bound as $f$. In order to get estimates simultaneously for (33) and (34) we show the following lemma.

**Lemma 4.6.** *We fix $n$ and assume a BSDE*

$$\mathsf{Y}_{t_m} = \xi^n + \int_{(t_m,T]} f(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})d[B^n]_s - \int_{(t_m,T]} Z_{s^-}dB^n_s, \tag{35}$$

*with $\xi^n = g(X_T^{n,t_m,x})$ or $\xi^n = g'(X_T^{n,t_m,x})\partial_x X_T^{n,t_m,x}$ and $\mathsf{X}_s := X_s^{n,t_m,x}$ or $\mathsf{X}_s := \partial_x X_s^{n,t_m,x}$ such that $\mathsf{f}:\Omega\times[0,T]\times\mathbb{R}^3\to\mathbb{R}$ is measurable and satisfies*

$$\begin{aligned} |\mathsf{f}(\omega,t,x,y,z)-\mathsf{f}(\omega,t,x',y',z')| &\le L_f(|x-x'|+|y-y'|+|z-z'|),\\ |\mathsf{f}(\omega,t,x,y,z)| &\le (K_f+L_f)(1+|x|+|y|+|z|). \end{aligned} \tag{36}$$

*Then for any $p\ge 2$,*

*(i)* $\mathbb{E}|\mathsf{Y}_{t_m}|^p + \frac{\gamma_p}{4}\mathbb{E}\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s \le C(1+|x|^{(p_0+1)p}), \quad m=0,..,n$

*(ii)* $\mathbb{E}\sup_{t_m<s\le T}|\mathsf{Y}_{s^-}|^p \le C(1+|x|^{(p_0+1)p})$,

*(iii)* $\mathbb{E}\Big(\int_{(t_m,T]}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\Big)^{\frac{p}{2}} \le C(1+|x|^{(p_0+1)p})$ *for some constant $C=C(T,f,g,p,p_0,b,\sigma)$.*

*Proof.* (i) By Itô's formula (see [6, Theorem 4.57]) we get for $p\ge 2$

$$\begin{aligned} |\mathsf{Y}_{t_m}|^p &= |\xi^n|^p - p\int_{(t_m,T]}\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{Z}_{s^-}dB^n_s + p\int_{(t_m,T]}\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{f}(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})d[B^n]_s\\ &\quad - \sum_{s\in(t_m,T]}[|\mathsf{Y}_s|^p - |\mathsf{Y}_{s^-}|^p - p\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}(\mathsf{Y}_s-\mathsf{Y}_{s^-})]. \end{aligned} \tag{37}$$

Following the proof of [8, Proposition 2] (which is carried out there in the Lévy process setting but can be done also for martingales with jumps) we can use estimate

$$-\sum_{s\in(t_m,T]}[|\mathsf{Y}_s|^p - |\mathsf{Y}_{s^-}|^p - p\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}(\mathsf{Y}_s-\mathsf{Y}_{s^-})] \le -\gamma_p\sum_{s\in(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}(\mathsf{Y}_s-\mathsf{Y}_{s^-})^2$$

where $\gamma_p>0$ is computed in [17, Lemma A4]. Using our setting we have

$$\begin{aligned} &-\sum_{s\in(t_m,T]}[|\mathsf{Y}_s|^p - |\mathsf{Y}_{s^-}|^p - p\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}(\mathsf{Y}_s-\mathsf{Y}_{s^-})]\\ &\quad\le -\gamma_p\sum_{k=m}^{n-1}|\mathsf{Y}_{t_k}|^{p-2}(\mathsf{f}(t_{k+1},\mathsf{X}_{t_k},\mathsf{Y}_{t_k},\mathsf{Z}_{t_k})h - \mathsf{Z}_{t_k}\sqrt{h}\varepsilon_{k+1})^2\\ &\quad= -\gamma_p h\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{f}^2(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})d[B^n]_s - \gamma_p\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\\ &\qquad+2\gamma_p\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{f}(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})\mathsf{Z}_{s^-}(B^n_s - B^n_{s^-})d[B^n]_s. \end{aligned}$$

Hence we get from (37)

$$\begin{aligned} |\mathsf{Y}_{t_m}|^p &\le |\xi^n|^p - p\int_{(t_m,T]}\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{Z}_{s^-}dB^n_s + p\int_{(t_m,T]}\mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{f}(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})d[B^n]_s\\ &\quad-\gamma_p\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\\ &\quad+2\gamma_p\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{f}(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})\mathsf{Z}_{s^-}(B^n_s - B^n_{s^-})d[B^n]_s. \end{aligned}$$

From Young's inequality and (36) we conclude that there is a $c' = c'(p, K_f, L_f, \gamma_p) > 0$ such that

$$p|\mathsf{Y}_{s^-}|^{p-1}\,|\mathsf{f}(s, \mathsf{X}_{s^-}, \mathsf{Y}_{s^-}, \mathsf{Z}_{s^-})| \leq \tfrac{\gamma_p}{4}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 + c'(1 + |\mathsf{X}_{s^-}|^p + |\mathsf{Y}_{s^-}|^p)$$

and for $\sqrt{h} < \frac{1}{8(L_f+K_f)}$ we find a $c'' = c''(p, L_f, K_f, \gamma_p) > 0$ such that

$$2\gamma_p\sqrt{h}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{f}(s, \mathsf{X}_{s^-}, \mathsf{Y}_{s^-}, \mathsf{Z}_{s^-})||\mathsf{Z}_{s^-}| \leq \tfrac{\gamma_p}{4}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 + c''\,(1 + |\mathsf{X}_{s^-}|^p + |\mathsf{Y}_{s^-}|^p).$$

Then for $c = c' + c''$ we have

$$\begin{aligned} |\mathsf{Y}_{t_m}|^p \;\leq\; & |\xi^n|^p - p\int_{(t_m,T]} \mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{Z}_{s^-}dB^n_s + c\int_{(t_m,T]} 1 + |\mathsf{X}_{s^-}|^p + |\mathsf{Y}_{s^-}|^p d[B^n]_s \\ & -\tfrac{\gamma_p}{2}\int_{(t_m,T]} |\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s. \end{aligned} \tag{38}$$

By standard methods, approximating the terminal condition and the generator by bounded functions, it follows that for any $a > 0$

$$\mathbb{E}\sup_{t_m\leq s\leq T}|\mathsf{Y}_s|^a < \infty \quad \text{and} \quad \mathbb{E}\left(\int_{(t_m,T]}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\right)^{\frac{a}{2}} < \infty.$$

Hence $\int_{(t_m,T]} \mathsf{Y}_{s^-}|\mathsf{Y}_{s^-}|^{p-2}\mathsf{Z}_{s^-}dB^n_s$ has expectation zero. Taking the expectation in (38) yields

$$\mathbb{E}|\mathsf{Y}_{t_m}|^p + \tfrac{\gamma_p}{2}\mathbb{E}\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s \leq \mathbb{E}|\xi^n|^p + c\mathbb{E}\int_{(t_m,T]} 1 + |\mathsf{X}_{s^-}|^p + |\mathsf{Y}_{s^-}|^p d[B^n]_s. \tag{39}$$

By Gronwall's lemma and the polynomial growth of $x \mapsto \mathbb{E}|\xi^n|^p$, and $x \mapsto \mathbb{E}\int_{(t_m,T]} 1 + |\mathsf{X}_{s^-}|^p d[B^n]_s$,

$$\|\mathsf{Y}_{t_m}\|_p \leq C(T, f, g, p, p_0, b, \sigma)(1 + |x|^{p_0+1}), \quad m = 0, ..., n,$$

and inserting this into (39) yields

$$\left(\mathbb{E}\int_{(t_m,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\right)^{\frac{1}{p}} \leq c(T, f, g, p, p_0, b, \sigma)(1 + |x|^{p_0+1}), \quad m = 0, ..., n-1.$$

(ii) By the inequality of BDG and Young's inequality we derive from (38) that

$$\begin{aligned} & \mathbb{E}\sup_{t_k<s\leq T}|\mathsf{Y}_{s^-}|^p \\ \leq\; & \mathbb{E}|\xi^n|^p + C(p)\mathbb{E}\left(\int_{(t_k,T]}|\mathsf{Y}_{s^-}|^{2p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\right)^{\frac{1}{2}} + c\mathbb{E}\int_{(t_k,T]} 1 + |\mathsf{X}_{s^-}|^p + |\mathsf{Y}_{s^-}|^p d[B^n]_s \\ \leq\; & \mathbb{E}|\xi^n|^p + c\mathbb{E}\int_{(t_k,T]} 1 + |\mathsf{X}_{s^-}|^p d[B^n]_s + C(p)\mathbb{E}\left[\sup_{t_k<s\leq T}|\mathsf{Y}_{s^-}|^{\frac{p}{2}}\left(\int_{(t_k,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\right)^{\frac{1}{2}}\right] \\ & + c\mathbb{E}\int_{(t_k,T]}|\mathsf{Y}_{s^-}|^p d[B^n]_s \\ \leq\; & \mathbb{E}|\xi^n|^p + c\mathbb{E}\int_{(t_k,T]} 1 + |\mathsf{X}_{s^-}|^p d[B^n]_s + C(p)\mathbb{E}\int_{(t_k,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s \\ & + \mathbb{E}\sup_{t_k<s\leq T}|\mathsf{Y}_{s^-}|^p(\tfrac{1}{4} + c(T - t_k)). \end{aligned}$$

We assume that $h$ is sufficiently small so that we find a $t_k$ with $c(T-t_k)<\frac{1}{4}$. We rearrange the inequality to have $\mathbb{E}\sup_{t_k<s\le T}|\mathsf{Y}_{s^-}|^p$ on the l.h.s., and from (i) we conclude that

$$\begin{aligned}\mathbb{E}\sup_{t_k<s\le T}|\mathsf{Y}_{s^-}|^p &\le 2\mathbb{E}|\xi^n|^p+2c\mathbb{E}\int_{(t_k,T]}1+|\mathsf{X}_{s^-}|^p d[B^n]_s+2C(p)\mathbb{E}\int_{(t_k,T]}|\mathsf{Y}_{s^-}|^{p-2}|\mathsf{Z}_{s^-}|^2 d[B^n]_s\\ &\le C(T,f,g,p,p_0,b,\sigma)(1+|x|^{(p_0+1)p}).\end{aligned}$$

Now we may repeat the above step for $\mathbb{E}\sup_{t_\ell<s\le t_k}|\mathsf{Y}_{s^-}|^p$ with $c(t_k-t_\ell)<\frac{1}{4}$ and $\xi^n=\mathsf{Y}_T$ replaced by $\mathsf{Y}_{t_k}$, and continue doing so until we eventually get assertion (ii).
(iii) We proceed from (35),

$$\sup_{k\le\ell\le n}\Big|\int_{(t_\ell,T]}\mathsf{Z}_{s^-}dB^n_s\Big|^p\le C(p)\Big(|\xi^n|^p+\sup_{k\le\ell\le n}|\mathsf{Y}_{t_\ell}|^p+\Big|\int_{(t_k,T]}f(s,\mathsf{X}_{s^-},\mathsf{Y}_{s^-},\mathsf{Z}_{s^-})d[B^n]_s\Big|^p\Big),$$

so that by (36) and the inequalities of BDG and Hölder we have that

$$\begin{aligned}&\mathbb{E}\Big(\int_{(t_k,T]}|\mathsf{Z}_{s^-}|^2d[B^n]_s\Big)^{\frac{p}{2}}\\ &\le C(p)\Big(\mathbb{E}|\xi^n|^p+\mathbb{E}\sup_{k\le\ell\le n}|\mathsf{Y}_{t_\ell}|^p\Big)+C(p,L_f,K_f)\mathbb{E}\left(\int_{(t_k,T]}1+|\mathsf{X}_{s^-}|+|\mathsf{Y}_{s^-}|d[B^n]_s\right)^p\\ &\quad+C(p,L_f,K_f)(T-t_k)^{\frac{p}{2}}\mathbb{E}\left(\int_{(t_k,T]}|\mathsf{Z}_{s^-}|^2d[B^n]_s\right)^{\frac{p}{2}}.\end{aligned}$$

Hence for $C(p,L_f,K_f)(T-t_k)^{\frac{p}{2}}<\frac{1}{2}$ we derive from assertion (ii) that

$$\mathbb{E}\Big(\int_{(t_k,T]}|\mathsf{Z}_{s^-}|^2d[B^n]_s\Big)^{\frac{p}{2}}\le C(T,f,g,p,p_0,b,\sigma)(1+|x|^{(p_0+1)p}).$$

□

**Step 3** Applying Lemma 4.6 to (33) and (34) we see that for all $m=0,...,n$ we have

$$|u^n(t_m,x)|=|Y^{n,t_m,x}_{t_m}|\le(\mathbb{E}(Y^{n,t_m,x}_{t_m})^2)^{\frac{1}{2}}\le c(T,f,g,p_0,b,\sigma)(1+|x|^{p_0+1})$$

and

$$|u^n_x(t_m,x)|\le(\mathbb{E}(\partial_xY^{n,t_m,x}_{t_m})^2)^{\frac{1}{2}}\le c(T,f,g,p_0,b,\sigma)(1+|x|^{p_0+1}).$$

Our next aim is to show that $u^n_{xx}(t_m,x)$ satisfies (5). We first show that $u^n_{xx}(t_m,x)$ has polynomial growth. We introduce the BSDE which describes $u^n_{xx}(t_m,x)$ and denote for simplicity

$$f(t,x_1,x_2,x_3):=f(t,x,y,z)\quad\text{and}\quad D^a:=\partial^{i_1}_{x_1}\partial^{i_2}_{x_2}\partial^{i_3}_{x_3}\quad\text{with}\quad a:=(i_1,i_2,i_3)$$

and consider

$$\begin{aligned}\partial^2_xY^n_{t_m}&=g''(X^n_T)(\partial_xX^n_T)^2+g'(X^n_T)\partial^2_xX^n_T\\ &\quad+\int_{(t_m,T]}\sum_{\substack{a\in\{0,1,2\}^3\\ i_1+i_2+i_3=2}}(D^af)(s,X^n_{s^-},Y^n_{s^-},Z^n_{s^-})(\partial_xX^n_{s^-})^{i_1}(\partial_xY^n_{s^-})^{i_2}(\partial_xZ^n_{s^-})^{i_3}d[B^n]_s\\ &\quad+\int_{(t_m,T]}\sum_{\substack{a\in\{0,1\}^3\\ i_1+i_2+i_3=1}}(D^af)(s,X^n_{s^-},Y^n_{s^-},Z^n_{s^-})(\partial^2_xX^n_{s^-})^{i_1}(\partial^2_xY^n_{s^-})^{i_2}(\partial^2_xZ^n_{s^-})^{i_3}d[B^n]_s\end{aligned}$$

$$-\int_{(t_m,T]} \partial_x^2 Z^n_{s^-}\, dB^n_s. \tag{40}$$

We denote the generator of this BSDE by $\hat{f}$ and notice that it is of the structure

$$\hat{f}(\omega,t,x,y,z) = f_0(\omega,t) + f_1(\omega,t)x + f_2(\omega,t)y + f_3(\omega,t)z.$$

with $f_1, f_2, f_3$ bounded and $\mathbb{E}(\int_{(t_m,T]} f_0(s^-)d[B^n]s)^p < \infty$. We put

$$\hat{\xi}^n := g''(X^n_T)(\partial_x X^n_T)^2 + g'(X^n_T)\partial_x^2 X^n_T.$$

Denoting the solution by $(\hat{\mathsf{Y}}, \hat{\mathsf{Z}})$ we get for $C(f_3)(T-t_m) \le \frac{1}{2}$ that

$$\begin{aligned}
&\mathbb{E}|\hat{\mathsf{Y}}_{t_m}|^2 + \frac{1}{2}\mathbb{E}\int_{(t_m,T]} |\hat{\mathsf{Z}}_{s^-}|^2 d[B^n]_s \\
&\quad\le\quad C\Big[\mathbb{E}|\hat{\xi}^n|^2 + \mathbb{E}\Big(\int_{(t_m,T]} |f_0(s^-)|d[B^n]_s\Big)^2 + \mathbb{E}\int_{(t_m,T]} |\hat{\mathsf{X}}_{s^-}|^2 + |\hat{\mathsf{Y}}_{s^-}|^2 d[B^n]_s\Big].
\end{aligned} \tag{41}$$

Now we derive the polynomial growth of $\mathbb{E}|\hat{\xi}^n|^2$ from the properties of $g'$ and $g''$, and that of $\mathbb{E}(\int_{(t_m,T]} |f_0(\omega,s)|d[B^n,B^n]_s)^2$ from the previous steps. Indeed, $\mathbb{E}\sup_{t_m<s\le T} |\partial_x X^n_{s^-}|^p$ having polynomial growth is well-known under our conditions, for $\mathbb{E}\sup_{t_m<s\le T} |\partial_x Y^n_{s^-}|^p$ we have shown this in Lemma 4.6-(ii), and for $\mathbb{E}\Big(\int_{(t_m,T]} |\partial_x Z^n_{s^-}|^2 d[B^n]_s\Big)^2$ we use Lemma 4.6-(iii). Then Gronwall's Lemma implies polynomial growth of $\hat{\mathsf{Y}}^{t_m,x}_{t_m} = u_{xx}(t_m,x)$.

Finally, to show that $u^n_{xx}(t_m,x)$ satisfies (5), one uses (40) and derives an inequality as in (41) but now for the difference $\partial_x^2 Y^{n,t_m,x}_{t_m} - \partial_x^2 Y^{n,t_m,\bar{x}}_{t_m}$. Here one needs to exploit that moments of $X^n, Y^n, Z^n, \partial_x X^n, \partial_x Y^n$ and $\partial_x^2 X^n$ satisfy (5) (with $\alpha = 1$) (which easily follows by the mean value theorem if one uses the representation of the processes $Y^n$ and $Z^n$ by the help of $u^n$) and one also needs that

$$\mathbb{E}\Big(\int_{(t_m,T]} |\partial_x Z^{n,t_m,x}_{s^-} - \partial_x Z^{n,t_m,\bar{x}}_{s^-}|^2 d[B^n]_s\Big)^{\frac{p}{2}} \le C(1+|x|^{p\,p_0} + |\bar{x}|^{p\,p_0})|x-\bar{x}|^p, \quad p \ge 2.$$

The latter estimate one gets similarly as that in Lemma 4.6-(iii) if one considers the BSDE for the difference $\partial_x Y^{n,t_m,x}_{t_m} - \partial_x Y^{n,t_m,\bar{x}}_{t_m}$ instead of (34) itself. The result follows then from the estimate of $\mathbb{E}|\hat{\xi}^{n,t_m,x} - \hat{\xi}^{n,t_m,\bar{x}}|^2$ and Gronwall's lemma. □

# 5 Technical results and estimates

In this section we collect some facts which are needed for the proofs of our results. We start with properties of the stopping times used to construct a random walk.

**Lemma 5.1** (Lemma A.1 [5]). *For all $0 \le k \le m \le n$ and $p > 0$, it holds for $h = \frac{T}{n}$ that*

*(i)* $\mathbb{E}\tau_k = kh$,

*(ii)* $\mathbb{E}|\tau_1|^p \le C(p)h^p$,

*(iii)* $\mathbb{E}|B_{\tau_k} - B_{t_k}|^{2p} \le C(p)\mathbb{E}|\tau_k - t_k|^p \le C(p)(t_k h)^{\frac{p}{2}}$.

The next lemma lists some estimates concerning the diffusion $X$ and its discretisation.

**Lemma 5.2.** *Under Assumption 2.1 on $b$ and $\sigma$ it holds for $p \geq 2$ that there exists a constant $C = C(b, \sigma, T, p) > 0$ such that*

*(i)* $\mathbb{E}|X_T^{s,y} - X_T^{t,x}|^p \leq C(|y-x|^p + |s-t|^{\frac{p}{2}}), \quad x, y \in \mathbb{R},\ s, t \in [0,T],$

*(ii)* $\tilde{\mathbb{E}} \sup_{\tilde{\tau}_l \wedge t_m \leq r \leq \tilde{\tau}_{l+1} \wedge t_m} |\tilde{X}_{t_k+r}^{t_k,x} - \tilde{X}_{t_k+\tilde{\tau}_l \wedge t_m}^{t_k,x}|^p \leq C h^{\frac{p}{4}},\ 0 \leq k \leq n,\ 0 \leq l \leq n-k-1,\ 0 \leq m \leq n-k,$

*(iii)* $\mathbb{E}|\nabla X_T^{s,y} - \nabla X_T^{t,x}|^p \leq C(|y-x|^p + |s-t|^{\frac{p}{2}}), \quad x, y \in \mathbb{R},\ s, t \in [0,T],$

*(iv)* $\mathbb{E} \sup_{0 \leq l \leq m} |\nabla X_{t_k+t_l}^{n,t_k,x}|^p \leq C, \quad 0 \leq k \leq n,\ 0 \leq m \leq n-k,$

*(v)* $\tilde{\mathbb{E}}|\tilde{X}_{t_k+t_m}^{t_k,x} - \tilde{\mathcal{X}}_{\tau_k+\tilde{\tau}_m}^{\tau_k,y}|^p \leq C(|x-y|^p + h^{\frac{p}{4}}), \quad 0 \leq k \leq n,\ 0 \leq m \leq n-k,$

*(vi)* $\tilde{\mathbb{E}}|\nabla \tilde{X}_{t_k+t_m}^{t_k,x} - \nabla \tilde{\mathcal{X}}_{\tau_k+\tilde{\tau}_m}^{\tau_k,y}|^p \leq C(|x-y|^p + h^{\frac{p}{4}}), \quad 0 \leq k \leq n,\ 0 \leq m \leq n-k.$

*Proof.* (i): This estimate is well-known.
(ii): For the stochastic integral we use the inequality of BDG and then, since $b$ and $\sigma$ are bounded, we get by Lemma 5.1 (ii) that

$$\begin{aligned}\tilde{\mathbb{E}} \sup_{\tilde{\tau}_l \wedge t_m \leq r \leq \tilde{\tau}_{l+1} \wedge t_m} & |\tilde{X}_{t_k+r}^{t_k,x} - \tilde{X}_{t_k+\tilde{\tau}_l \wedge t_m}^{t_k,x}|^p \\ &\leq \quad C(p)(\|b\|_\infty^p \tilde{\mathbb{E}}|\tilde{\tau}_{l+1} - \tilde{\tau}_l|^p + \|\sigma\|_\infty^p \mathbb{E}|\tilde{\tau}_{l+1} - \tilde{\tau}_l|^{\frac{p}{2}}) \leq C(b, \sigma, T, p)\, h^{\frac{p}{2}}.\end{aligned}$$

(iii): This can be easily seen because the process $(\nabla X_r^{s,y})_{r \in [s,T]}$ solves a linear SDE with bounded coefficients.
(iv): The process solves (44). The estimate follows from the inequality of BDG and Gronwall's Lemma.
(v): Recall that

$$\tilde{\mathcal{X}}_{\tau_k+\tilde{\tau}_m}^{\tau_k,y} = \tilde{X}_{t_k+t_m}^{n,t_k,y} = y + \int_{(0,t_m]} b(t_k + r, \tilde{X}_{t_k+r^-}^{n,t_k,y}) d[\tilde{B}^n, \tilde{B}^n]_r + \int_{(0,t_m]} \sigma(t_k + r, \tilde{X}_{t_k+r^-}^{n,t_k,y}) d\tilde{B}_r^n$$

and

$$\tilde{X}_{t_k+t_m}^{t_k,x} = x + \int_0^{t_m} b(t_k + r, \tilde{X}_{t_k+r}^{t_k,x}) dr + \int_0^{t_m} \sigma(t_k + r, \tilde{X}_{t_k+r}^{t_k,y}) d\tilde{B}_r.$$

To compare the stochastic integrals of the previous two equations we use the relation

$$\int_{(0,t_m]} \sigma(t_k + r, \tilde{X}_{t_k+r^-}^{n,t_k,y}) d\tilde{B}_r^n = \int_0^\infty \sum_{l=0}^{m-1} \sigma(t_{k+l+1}, \tilde{X}_{t_{k+l}}^{n,t_k,y}) \mathbf{1}_{(\tilde{\tau}_l, \tilde{\tau}_{l+1}]}(r) d\tilde{B}_r.$$

We define an 'increasing' map $i(r) := t_{l+1}$ for $(t_l, t_{l+1}]$ and a 'decreasing' map $d(r) := t_l$ for $(t_l, t_{l+1}]$ and split the differences as follows using Assumption 2.1-(iii) for the coefficient $b$

$$\begin{aligned}&\tilde{\mathbb{E}}|\tilde{X}_{t_k+t_m}^{t_k,x} - \tilde{X}_{t_k+t_m}^{n,t_k,y}|^p \\ &\quad \leq \quad C(b,p) \left( |x-y|^p + \tilde{\mathbb{E}} \int_0^{t_m} |r - i(r)|^{\frac{p}{2}} + |\tilde{X}_{t_k+r}^{t_k,x} - \tilde{X}_{t_k+d(r)}^{t_k,x}|^p + |\tilde{X}_{t_k+d(r)}^{t_k,x} - \tilde{X}_{t_k+d(r)}^{n,t_k,y}|^p dr \right) \\ &\qquad + C(p)\tilde{\mathbb{E}}| \int_{t_m \wedge \tilde{\tau}_m}^{t_m} \sigma(t_k + r, \tilde{X}_{t_k+r}^{t_k,x}) d\tilde{B}_r|^p \\ &\qquad + C(p)\tilde{\mathbb{E}}| \int_{t_m \wedge \tilde{\tau}_m}^{\tilde{\tau}_m} \sum_{l=0}^{m-1} \sigma(t_{k+l+1}, \tilde{X}_{t_{k+l}}^{n,t_k,y}) \mathbf{1}_{(\tilde{\tau}_l, \tilde{\tau}_{l+1}]}(r) d\tilde{B}_r|^p\end{aligned}$$

$$+C(p)\tilde{\mathbb{E}}|\int_0^{t_m\wedge\tilde{\tau}_m}\sigma(t_k+r,\tilde{X}^{t_k,x}_{t_k+r})-\sum_{l=0}^{m-1}\sigma(t_{k+l+1},\tilde{X}^{n,t_k,y}_{t_{k+l}})\mathbf{1}_{(\tilde{\tau}_l,\tilde{\tau}_{l+1}]}(r)d\tilde{B}_r|^p. \tag{42}$$

We estimate the terms on the r.h.s as follows: by standard estimates for SDEs with bounded coefficients one has

$$\tilde{\mathbb{E}}\int_0^{t_m}|r-i(r)|^{\frac{p}{2}}+|\tilde{X}^{t_k,x}_{t_k+r}-\tilde{X}^{t_k,x}_{t_k+d(r)}|^p dr\le C(b,\sigma,T,p)h^{\frac{p}{2}}.$$

By the BDG inequality, the fact that $\sigma$ is bounded and Lemma 5.1 we conclude that

$$\begin{aligned}&\tilde{\mathbb{E}}\Big|\int_{t_m\wedge\tilde{\tau}_m}^{t_m}\sigma(t_k+r,\tilde{X}^{t_k,x}_{t_k+r})d\tilde{B}_r\Big|^p+\tilde{\mathbb{E}}\Big|\int_{t_m\wedge\tilde{\tau}_m}^{\tilde{\tau}_m}\sum_{l=0}^{m-1}\sigma(t_{k+l+1},\tilde{X}^{n,t_k,y}_{t_{k+l}})\mathbf{1}_{(\tilde{\tau}_l,\tilde{\tau}_{l+1}]}(r)d\tilde{B}_r\Big|^p\\ &\qquad\le C(\sigma,p)\|\sigma\|_\infty^p\tilde{\mathbb{E}}|\tilde{\tau}_m-t_m|^{\frac{p}{2}}\le C(\sigma,p)(t_mh)^{\frac{p}{4}}.\end{aligned}$$

Finally, by the BDG inequality

$$\begin{aligned}&\tilde{\mathbb{E}}\Big|\int_0^{t_m\wedge\tilde{\tau}_m}\sigma(t_k+r,\tilde{X}^{t_k,x}_{t_k+r})-\sum_{l=0}^{m-1}\sigma(t_{k+l+1},\tilde{X}^{n,t_k,y}_{t_{k+l}})\mathbf{1}_{(\tilde{\tau}_l,\tilde{\tau}_{l+1}]}(r)d\tilde{B}_r\Big|^p\\ &\quad\le\ C(p)\tilde{\mathbb{E}}\Big(\int_0^{t_m}\sum_{l=0}^{m-1}|\sigma(t_k+r,\tilde{X}^{t_k,x}_{t_k+r})-\sigma(t_{k+l+1},\tilde{X}^{n,t_k,y}_{t_{k+l}})|^2\mathbf{1}_{(\tilde{\tau}_l,\tilde{\tau}_{l+1}]}(r)dr\Big)^{\frac{p}{2}}\\ &\quad\le\ C(\sigma,p)\tilde{\mathbb{E}}\Big(\sum_{l=0}^{m-1}\int_{\tilde{\tau}_l\wedge t_m}^{\tilde{\tau}_{l+1}\wedge t_m}|\tilde{\tau}_{l+1}-t_{l+1}|^{\frac{p}{2}}+|\tilde{\tau}_l-t_{l+1}|^{\frac{p}{2}}+|\tilde{X}^{t_k,x}_{t_k+r}-\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}|^p\\ &\qquad\qquad+|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{n,t_k,y}_{t_{k+l}}|^p dr\Big)\\ &\quad\le\ C(\sigma,p,T)\Big(h^{\frac{p}{2}}+\max_{1\le l<m}(\tilde{\mathbb{E}}|\tilde{\tau}_l-t_l|^p)^{\frac{1}{2}}+\max_{0\le l<m}(\tilde{\mathbb{E}}\sup_{\tilde{\tau}_l\wedge t_m\le r\le\tilde{\tau}_{l+1}\wedge t_m}|\tilde{X}^{t_k,x}_{t_k+r}-\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}|^{2p})^{\frac{1}{2}}\\ &\qquad+\tilde{\mathbb{E}}\sum_{l=0}^{m-1}|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{n,t_k,y}_{t_{k+l}}|^p(\tilde{\tau}_{l+1}-\tilde{\tau}_l)\Big).\end{aligned}$$

Moreover, since $\tilde{\tau}_{l+1}-\tilde{\tau}_l$ is independent from $|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{n,t_k,y}_{t_k+t_l}|^p$ we get by Lemma 5.1-(i)

$$\begin{aligned}&\tilde{\mathbb{E}}\sum_{l=0}^{m-1}|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{n,t_k,y}_{t_{k+l}}|^p(\tilde{\tau}_{l+1}-\tilde{\tau}_l)\\ &\quad=\ \tilde{\mathbb{E}}\sum_{l=0}^{m-1}|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{n,t_k,y}_{t_{k+l}}|^p(t_{l+1}-t_l)\\ &\quad\le\ C(T,p)\Big(\tilde{\mathbb{E}}\int_0^{t_m}|\tilde{X}^{t_k,x}_{t_k+d(r)}-\tilde{X}^{n,t_k,y}_{t_k+d(r)}|^p dr+\max_{0\le l<m}\tilde{\mathbb{E}}|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{t_k,x}_{t_k+t_l}|^p\Big).\end{aligned}$$

Using Lemma 5.1 one concludes that $\tilde{\mathbb{E}}|\tilde{X}^{t_k,x}_{t_k+\tilde{\tau}_l\wedge t_m}-\tilde{X}^{t_k,x}_{t_k+t_l}|^p\le C(b,\sigma,T,p)h^{\frac{p}{4}}$. Then (42) combined with the above estimates implies that

$$\tilde{\mathbb{E}}|\tilde{X}^{t_k,x}_{t_k+t_m}-\tilde{X}^{n,t_k,y}_{t_k+t_m}|^p\le C(b,\sigma,T,p)\Big(|x-y|^p+h^{\frac{p}{4}}+\tilde{\mathbb{E}}\int_0^{t_m}|\tilde{X}^{t_k,x}_{t_k+d(r)}-\tilde{X}^{n,t_k,y}_{t_k+d(r)}|^p dr\Big). \tag{43}$$

Then Gronwall's Lemma yields

$$\tilde{\mathbb{E}}|\tilde{X}^{t_k,x}_{t_k+t_m}-\tilde{X}^{n,t_k,y}_{t_k+t_m}|^p\le C(b,\sigma,T,p)(|x-y|^p+h^{\frac{p}{4}}).$$

Notice that if we use the relation (43) for $k=0$ we get

$$\mathbb{E}|X_{t_m}-X^n_{t_m}|^p=\tilde{\mathbb{E}}|\tilde{X}^{0,x}_{t_m}-\tilde{X}^{n,0,x}_{t_m}|^p\leq C(b,\sigma,T,p)h^{\frac{p}{4}}.$$

(vi): We have

$$\begin{aligned}\nabla\tilde{X}^{n,t_k,y}_{t_k+t_m} &= 1+\int_{(0,t_m]}b_x(t_k+r,X^{n,t_k,y}_{t_k+r^-})\nabla\tilde{X}^{n,t_k,y}_{t_k+r^-}d[\tilde{B}^n,\tilde{B}^n]_r\\ &\quad+\int_{(0,t_m]}\sigma_x(t_k+r,\tilde{X}^{n,t_k,y}_{t_k+r^-})\nabla\tilde{X}^{n,t_k,y}_{t_k+r^-}d\tilde{B}^n_r \qquad (44)\end{aligned}$$

and

$$\nabla\tilde{X}^{t_k,x}_{t_k+t_m}=1+\int_0^{t_m}b_x(t_k+r,\tilde{X}^{t_k,x}_{t_k+r})\nabla\tilde{X}^{t_k,x}_{t_k+r}dr+\int_0^{t_m}\sigma_x(t_k+r,\tilde{X}^{t_k,x}_{t_k+r})\nabla\tilde{X}^{t_k,x}_{t_k+r}d\tilde{B}_r. \qquad (45)$$

We may proceed similarly as in (v) but this time the coefficients are not bounded but have linear growth. Here one uses that the integrands are bounded in any $L_p(\mathbb{P})$. □

Finally, we estimate the difference between the continuous-time Malliavin weight and its discrete-time counterpart.

**Lemma 5.3.** *Under Assumption 2.1 it holds that*

$$\tilde{\mathbb{E}}|\tilde{N}^{t_k}_{t_m}\sigma(t_k,X_{t_k})-\tilde{N}^{n,\tau_k}_{\tilde{\tau}_m}\sigma(t_{k+1},\mathcal{X}_{\tau_k})|^2\leq C(b,\sigma,\delta,T)\frac{|X_{t_k}-\mathcal{X}_{\tau_k}|^2+h^{\frac{1}{2}}}{(t_m-t_k)^{\frac{3}{2}}},\quad m=k+1,...,n.$$

*Proof.* For $N^{n,\tau_k}_{\tilde{\tau}_m}$ and $N^{t_k}_{t_m}$ given by (10) and (14), respectively, we introduce the notation

$$\tilde{N}^{t_k}_{t_m}\sigma(t_k,X_{t_k})=:\frac{1}{t_{m-k}}\int_0^{t_{m-k}}a_{t_k+s}d\tilde{B}_s\quad\text{and}\quad\tilde{N}^{n,\tau_k}_{\tilde{\tau}_m}\sigma(t_{k+1},\mathcal{X}_{\tau_k})=:\frac{1}{t_{m-k}}\int_0^{\tilde{\tau}_{m-k}}a^n_{\tau_k+s}d\tilde{B}_s$$

with

$$a_{t_k+s}:=\nabla\tilde{X}^{t_k,X_{t_k}}_{t_k+s}\frac{\sigma(t_k,X_{t_k})}{\sigma(t_k+s,\tilde{X}^{t_k,X_{t_k}}_{t_k+s})}\quad\text{and}\quad a^n_{\tau_k+s}:=\sum_{\ell=1}^{m-k}\nabla\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde{\tau}_{\ell-1}}\frac{\sigma(t_{k+1},\mathcal{X}_{\tau_k})}{\sigma(t_{k+\ell},\tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde{\tau}_{\ell-1}})}\mathbf{1}_{s\in(\tilde{\tau}_{\ell-1},\tilde{\tau}_\ell]}.$$

By the inequality of BDG,

$$\begin{aligned}&(t_m-t_k)^2\tilde{\mathbb{E}}|\tilde{N}^{t_k}_{t_m}\sigma(t_k,X_{t_k})-\tilde{N}^{n,\tau_k}_{\tilde{\tau}_m}\sigma(t_{k+1},\mathcal{X}_{\tau_k})|^2\\ &\quad\leq C\tilde{\mathbb{E}}\Big|\int_0^{t_{m-k}}a_{t_k+s}d\tilde{B}_s-\int_0^{\tilde{\tau}_{m-k}}a^n_{\tau_k+s}d\tilde{B}_s\Big|^2\\ &\quad\leq C\tilde{\mathbb{E}}\int_0^{t_{m-k}\wedge\tilde{\tau}_{m-k}}(a_{t_k+s}-a^n_{\tau_k+s})^2ds+\tilde{\mathbb{E}}\int_0^\infty a^2_{t_k+s}\mathbf{1}_{(\tilde{\tau}_{m-k},t_{m-k}]}(s)ds\\ &\qquad+\tilde{\mathbb{E}}\int_0^\infty(a^n_{\tau_k+s})^2\mathbf{1}_{(t_{m-k},\tilde{\tau}_{m-k}]}(s)ds\\ &\quad\leq C\sum_{\ell=1}^{m-k}\left(\tilde{\mathbb{E}}\sup_{s\in[0,t_{m-k}]\cap(\tilde{\tau}_{\ell-1},\tilde{\tau}_\ell]}|a_{t_k+s}-a^n_{\tau_k+\tilde{\tau}_\ell}|^4\right)^{\frac{1}{2}}(\tilde{\mathbb{E}}|\tilde{\tau}_\ell-\tilde{\tau}_{\ell-1}|^2)^{\frac{1}{2}}\\ &\qquad+\left(\tilde{\mathbb{E}}\sup_{s\in[0,t_{m-k}]}|a_{t_k+s}|^4+\tilde{\mathbb{E}}\max_{1\leq\ell\leq m-k}|a^n_{\tau_k+\tilde{\tau}_\ell}|^4\right)^{\frac{1}{2}}(\tilde{\mathbb{E}}|t_{m-k}-\tilde{\tau}_{m-k}|^2)^{\frac{1}{2}}.\end{aligned}$$

The assertion follows then from Lemma 5.1 and from the estimates

$$\tilde{\mathbb{E}} \sup_{s\in[0,t_{m-k}]\cap[\tilde\tau_{\ell-1},\tilde\tau_\ell]} |a_{t_k+s} - a^n_{\tau_k+\tilde\tau_\ell}|^4 \le C(b,\sigma,T,\delta)(|X_{t_k} - X^n_{t_k}|^4 + h) \tag{46}$$

$$\tilde{\mathbb{E}} \sup_{s\in[0,t_{m-k}]} |a_{t_k+s}|^4 + \tilde{\mathbb{E}} \max_{1\le\ell\le m-k} |a^n_{\tau_k+\tilde\tau_\ell}|^4 \le 2\|\sigma\|^4_\infty \delta^{-4}. \tag{47}$$

So it remains to show these inequalities. We put

$$\tilde K^{t_k}_{t_k+s} := \frac{\sigma(t_k, X_{t_k})}{\sigma(t_k+s, \tilde X^{t_k,X_{t_k}}_{t_k+s})} \quad \text{and} \quad \tilde K^{n,\tau_k}_{\tau_k+\tilde\tau_{\ell-1}} := \frac{\sigma(t_{k+1}, \mathcal{X}_{\tau_k})}{\sigma(t_{k+\ell}, \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}})}$$

and notice that by Assumption 2.1 both expressions are bounded by $\|\sigma\|_\infty\delta^{-1}$. To show (46) let us split $a_{t_k+s} - a^n_{\tau_k+\tilde\tau_\ell}$ in the following way:

$$\begin{aligned} a_{t_k+s} - a^n_{\tau_k+\tilde\tau_\ell} =& \tilde K^{t_k}_{t_k+s}(\nabla \tilde X^{t_k,X_{t_k}}_{t_k+s} - \nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}) + \nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}(\tilde K^{t_k}_{t_k+s} - \tilde K^{t_k}_{t_k+t_{\ell-1}}) \\ &+ \tilde K^{t_k}_{t_k+t_{\ell-1}}(\nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}} - \nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}}) + \nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}}(\tilde K^{t_k}_{t_k+t_{\ell-1}} - \tilde K^{n,\tau_k}_{\tau_k+\tilde\tau_{\ell-1}}). \end{aligned}$$

Then

$$\begin{aligned} &\tilde{\mathbb{E}} \sup_{s\in[\tilde\tau_{\ell-1}\wedge t_{m-k}, \tilde\tau_\ell\wedge t_{m-k}]} |\tilde K^{t_k}_{t_k+s}(\nabla \tilde X^{t_k,X_{t_k}}_{t_k+s} - \nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}})|^4 \\ &\quad\le \|\sigma\|^4_\infty \delta^{-4} \tilde{\mathbb{E}} \sup_{s\in[\tilde\tau_{\ell-1}\wedge t_{m-k}, \tilde\tau_\ell\wedge t_{m-k}]} |\nabla \tilde X^{t_k,X_{t_k}}_{t_k+s} - \nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}|^4 \le C(b,\sigma,T,\delta)h \end{aligned}$$

since one can show similarly to Lemma 5.2-(ii) that

$$\tilde{\mathbb{E}} \sup_{s\in[\tilde\tau_{\ell-1}\wedge t_{m-k}, \tilde\tau_\ell\wedge t_{m-k}]} |\nabla \tilde X^{t_k,X_{t_k}}_{t_k+s} - \nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}|^4 \le C(b,\sigma,T,\delta)h.$$

Notice that $\nabla \tilde X^{t_k,X_{t_k}}_t$ and $\nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_m}$ solve the linear SDEs (45) and (44), respectively. Therefore,

$$\tilde{\mathbb{E}} \sup_{s\in[0,t_{m-k}]} |\nabla \tilde X^{t_k,X_{t_k}}_{t_k+s}|^p \le C(b,\sigma,T,p) \quad \text{and} \quad \tilde{\mathbb{E}} \max_{0\le\ell\le m-k} |\nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tilde\tau_\ell+\tau_k}|^p \le C(b,\sigma,T,p). \tag{48}$$

We get

$$\begin{aligned} &\tilde{\mathbb{E}} \sup_{s\in[\tilde\tau_{\ell-1}\wedge t_{m-k}, \tilde\tau_\ell\wedge t_{m-k}]} |\nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}(\tilde K^{t_k}_{t_k+s} - \tilde K^{t_k}_{t_k+t_{\ell-1}})|^4 \\ &\quad\le C(\sigma,\delta)(\tilde{\mathbb{E}}|\nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}|^8)^{\frac12}(\tilde{\mathbb{E}} \sup_{s\in[\tilde\tau_{\ell-1}\wedge t_{m-k}, \tilde\tau_\ell\wedge t_{m-k}]} (|t_\ell - s|^4 + |\tilde X^{t_k,X_{t_k}}_{t_k+s} - \tilde X^{t_k,X_{t_k}}_{t_k+t_\ell}|^8)^{\frac12} \\ &\quad\le C(b,\sigma,\delta,T)h. \end{aligned}$$

For the third term Lemma 5.2-(vi) implies that

$$\tilde{\mathbb{E}}|\tilde K^{t_k}_{t_k+t_{\ell-1}}(\nabla \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}} - \nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}})|^4 \le C(b,\sigma,T)\|\sigma\|^4_\infty\delta^{-4}(|X_{t_k} - \mathcal{X}_{\tau_k}|^4 + h).$$

The last term we estimate similarly to the second one,

$$\begin{aligned} &\tilde{\mathbb{E}}|\nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}}(\tilde K^{t_k}_{t_k+t_{\ell-1}} - \tilde K^{n,\tau_k}_{\tau_k+\tilde\tau_{\ell-1}})|^4 \\ &\quad\le C(\sigma,\delta)(\tilde{\mathbb{E}}|\nabla \tilde{\mathcal{X}}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}}|^8)^{\frac12}(|X_{t_k} - \mathcal{X}_{\tau_k}|^8 + \tilde{\mathbb{E}}|\mathcal{X}^{\tau_k,\mathcal{X}_{\tau_k}}_{\tau_k+\tilde\tau_{\ell-1}} - \tilde X^{t_k,X_{t_k}}_{t_k+t_{\ell-1}}|^8)^{\frac12} \\ &\quad\le C(b,\sigma,T,\delta)(|X_{t_k} - \mathcal{X}_{\tau_k}|^4 + h). \end{aligned}$$

To see (47) use the estimates (48). □

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