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Abstract This article describes an elementary construction of a dual basis for non-uniform B-splines that is local, L^∞-stable and reproduces polynomials of any prescribed degree. This allows to define local projection operators with near-optimal approximation properties in any L^q, $1 \leq q \leq \infty$, and high order moment preserving properties. As the dual basis functions share the same piecewise polynomial structure as the underlying splines, simple quadrature formulas can be used to compute the projected spline coefficients.

Keywords Splines · Stable dual bases · Projection operators · Preservation of moments

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1 Introduction

Since the early developments of splines spaces and their description as linear combinations of locally supported B-splines $\{\varphi_i : 0 \leq i < n\}$, many local dual bases $\{\psi_i : 0 \leq i < n\}$ have been proposed, see e.g. [4,5,9,6,19,22,10]. Thanks to well-established stability properties these dual bases provide local projection operators $P : f \mapsto \sum_i \langle \psi_i, f \rangle \varphi_i$ that are stable in L^1 or L^∞, and are also high order accurate thanks to the polynomial reproduction properties of the splines. In particular, they may be used as efficient alternatives to the non-local L^2 projection or nodal interpolation operators.

In some cases, it is important that the dual basis itself has high order polynomial reproduction properties: on an abstract level this guarantees that the dual projection $Q : f \mapsto \sum_i \langle \varphi_i, f \rangle \psi_i$ also satisfies some high order approximation properties, and for the primal spline projection P it corresponds to a moment preserving property. In mortar methods for instance [3], the use of local dual bases spanning

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high order polynomials allows to localize the matching conditions at the interfaces without loosing the optimality of the method, see [23,20,17]. Another framework where such a tool is useful is the extension of classical commuting diagrams [18,1, 2,11] to non-conforming discretization spaces, see [13,12,14] and the presentation below.

One specific feature of B-splines compared to standard Finite Element spaces is their high order smoothness. This reduces their localization and makes it difficult to apply the abstract construction tools proposed in [20]. On the other hand, local dual bases with polynomial reproduction properties have been studied in the framework of spline wavelet multiresolutions [15,16] but their structure is made complicated by the refinability properties required by the multiresolution analysis, and their extension to non-uniform spline spaces is far from straightforward.

In this article we thus propose a direct and elementary construction of local dual bases for B-splines that reproduce polynomials of any desired degree. These dual bases are discontinuous but stable in L^∞, and they share the same (possibly non-uniform) piecewise polynomial structure as the splines. As a consequence, the resulting local projection operators are moment preserving, stable and near optimal in L^q, $1 \leq q \leq \infty$, and they can be implemented with simple quadrature formulas for the spline coefficients.

The outline is as follows. In the remainder of Section 1 we review a few classical tools that we will need, and specify some notation. Section 2 then specifies the role played by moment preserving projection operators in the design of non-conforming commuting diagrams, which was an important motivation for this work. A couple of classical and not-so-classical local dual bases are then presented in Section 3, and our new dual basis is presented in Section 4, together with its stability analysis. Section 5 then provides a couple of numerical tests to verify the practical validity of the proposed construction.

1.1 Univariate B-splines

We begin by collecting some well-known facts about splines which can be found in [8, Ch. IX] or [21, Ch. 4], and set some notation. Given two integers p, n, and a knot vector

$$\boldsymbol{\xi} = \{0 = \xi_0 \leq \cdots \leq \xi_{n+p} = 1\}, \tag{1}$$

we remind that B-splines $B_i^p = B^p[\xi_i, \ldots, \xi_{i+p+1}]$ can be recursively defined on the interval $\Omega = [0,1]$ by the Cox-de Boor formula, namely $B_i^0 = \mathbb{1}_{[\xi_i, \xi_{i+1})}$ for $0 \leq i < n+p$, and for $q = 1, \ldots p$,

$$B_i^q(x) = \frac{x - \xi_i}{\xi_{i+q} - \xi_i} B_i^{q-1}(x) + \frac{\xi_{i+q+1} - x}{\xi_{i+q+1} - \xi_{i+1}} B_{i+1}^{q-1}(x), \qquad 0 \leq i < n+p-q.$$

In this article we restrict ourselves to the case where the spline space

$$\mathbb{S}_{p,\boldsymbol{\xi}} := \mathrm{Span}\{B_i^p\}_{i=0}^{n-1} \tag{2}$$

contains every p-degree polynomial and is of maximal regularity, which corresponds to the following assumption.

Assumption 1 *The extremal knots have multiplicities $p+1$ (the knot vector is said* open*), and the inner knots have multiplicities 1, i.e.*

$$0 = \xi_0 = \cdots = \xi_p < \xi_{p+1} < \cdots < \xi_{n-1} < \xi_n = \cdots = \xi_{n+p} = 1.$$

Under Assumption 1 we may denote by

$$I_k := [\zeta_k, \zeta_{k+1}) \quad \text{with} \quad \zeta_k := \xi_{k+p}, \quad 0 \le k < N := n-p, \tag{3}$$

the non-empty intervals corresponding to the knot vector (1). The spline space then corresponds to

$$\mathbb{S}_{p,\boldsymbol{\xi}} = \{v \in \mathcal{C}^{p-1}(\Omega) : v|_{I_k} \in \mathbb{P}_p(I_k),\ 0 \le k < N\}$$

where $\mathbb{P}_p$ is the space of polynomials of degree less or equal to p, see [21, Ex. 4.3], and the B-splines

$$\varphi_i := B_i^p, \qquad 0 \le i < n, \tag{4}$$

form a basis of $\mathbb{S}_{p,\boldsymbol{\xi}}$ with local supports $\operatorname{supp}(\varphi_i) = [\xi_i, \xi_{i+p+1}]$, see [21, Th. 4.9].

1.2 Local projection operators with moment preservation properties

A convenient way of building local projection operators on $\mathbb{S}_{p,\boldsymbol{\xi}}$ that preserve polynomial moments up to some given degree m is to design functions

$$\psi_i \in L^\infty(\Omega), \qquad 0 \le i < n, \tag{5}$$

with the following properties:

- they are dual to the B-splines (4),

$$\langle \psi_i, \varphi_j \rangle = \delta_{i,j}, \qquad 0 \le i,j < n, \tag{P1}$$

- they are local, in the sense that there exists $K = K(p,m)$ such that

$$\operatorname{supp} \psi_i \subset I_i^+ \qquad \text{with} \qquad I_i^+ = I_i^+(K) := \bigcup_{|k-i| \le K} I_k \tag{P2}$$

- they are stable, i.e. there exists a constant $C = C(p,m)$ such that

$$\|\psi_i\|_{L^\infty} \le C h_i^{-1} \qquad \text{where} \qquad h_i := |\xi_{i+p+1} - \xi_i|, \tag{P3}$$

- they span every polynomial of degree m,

$$\mathbb{P}_m(\Omega) \subset \operatorname{Span}\{\psi_i\}_{i=0}^{n-1}. \tag{P4}$$

Thus, our problem is to design a dual basis with the properties listed above. To propose a solution we shall make the additional assumption that the grid is locally quasi-uniform.

Assumption 2 *There exists a constant c_* (possibly depending on p and m) such that*

$$\frac{1}{c_*} \le \frac{I_k}{I_{k-1}} \le c_*, \qquad k = 1, \ldots, N-1. \tag{6}$$

Using the above properties it is indeed standard to prove the following result, following for instance the arguments of Th. 6.60 in [21].

Theorem 3 *Let $p, n, m \in \mathbb{N}$ and let $\boldsymbol{\xi}$ be a knot vector satisfying Assumption 1 and 2. If* (P1)-(P3) *hold for given functions ψ_i, $0 \le i < n$, then the operator*

$$P : L^1(\Omega) \to \mathbb{S}_{p,\boldsymbol{\xi}}, \quad f \mapsto \sum_{i=0}^{n-1} \langle \psi_i, f \rangle \varphi_i \tag{7}$$

is a projection on $\mathbb{S}_{p,\boldsymbol{\xi}}$ which is locally near-optimal in the sense that for $1 \le q \le \infty$,

$$\|f - Pf\|_{L^q(I_k)} \le C \inf_{v \in \mathbb{S}_{p,\boldsymbol{\xi}}} \|f - v\|_{L^q(I_k^*)}, \qquad 0 \le k < N \tag{8}$$

holds with $I_k^* = \cup_{k \le i \le k+p} I_i^+ \subset I_k^+(K+p)$, and a constant C that only depends on p and m. Moreover if (P4) *holds, then P preserves the polynomial moments of degree m,*

$$\langle g, Pf \rangle = \langle g, f \rangle, \qquad \forall g \in \mathbb{P}_m, \quad f \in L^1(\Omega). \tag{9}$$

Proof The projection property is straightforward to verify using (P1), and the moment preserving property simply follows by noticing that any $g \in \mathbb{P}_m$ writes $g = \sum_{0 \le i < n} \gamma_i \psi_i$ according to (P4), therefore

$$\langle g, Pf \rangle = \sum_{0 \le i < n} \langle g, \varphi_i \rangle \langle \psi_i, f \rangle = \sum_{0 \le i < n} \gamma_i \langle \psi_i, f \rangle = \langle g, f \rangle, \qquad f \in L^1(\Omega).$$

For (8) we first establish a local stability estimate. Given k, we observe that φ_i vanishes on I_k for $i \notin \{k, \dots, k+p\}$, and $0 \le \varphi_i \le 1$ holds by definition. Hence we have $\|\varphi_i\|_{L^q(I_k)} \le |I_k|^{1/q} \le h_i^{1/q}$, and

$$\|Pf\|_{L^q(I_k)} \le \sum_{i=k}^{k+p} |\langle \psi_i, f \rangle| \|\varphi_i\|_{L^q(I_k)} \le \sum_{i=k}^{k+p} |\langle \psi_i, f \rangle| h_i^{\frac{1}{q}}.$$

The products are then bound using a Hölder inequality, (P2) and (P3): it yields

$$|\langle \psi_i, f \rangle| \le \|\psi_i\|_{L^{q'}(I_i^+)} \|f\|_{L^q(I_i^+)} \le \|\psi_i\|_{L^\infty} |I_i^+|^{\frac{1}{q'}} \|f\|_{L^q(I_i^+)} \le C h_i^{-1+\frac{1}{q'}} \|f\|_{L^q(I_i^+)}$$

where we have used the local quasi-uniformity property (6) and (P2) to estimate $|I_i^+| \le C h_i$ with a constant independent of i and N. As $-1 + \frac{1}{q'} = -\frac{1}{q}$, the above bounds show that

$$\|Pf\|_{L^q(I_k)} \le C \|f\|_{L^q(I_k^*)} \tag{10}$$

holds with a constant C depending only on p. The local error estimate is then easily obtained by observing that

$$\|f - Pf\|_{L^q(I_k)} \le \|f - v\|_{L^q(I_k)} + \|P(f-v)\|_{L^q(I_k)} \le (1 + C) \|f - v\|_{L^q(I_k^*)}$$

holds for all $v \in \mathbb{S}_{p,\boldsymbol{\xi}}$. □

2 Non-conforming commuting diagrams involving weak derivatives

An important motivation for moment preserving projection operators comes from the observation that they play a central role in commuting diagrams for non-conforming discretizations, which themselves are a key tool in the stable discretization of many systems of PDEs, see e.g. [18,1,2,11,13].

To describe this setting with univariate spline discretizations, we denote by

$$V_h = \mathbb{S}_{p,\boldsymbol{\xi}} \quad \text{and} \quad W_h = \mathbb{S}_{p-1,\boldsymbol{\xi}'}$$

two spline spaces of degree p and $p-1$ with open knot vectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ corresponding to the same grid $\zeta_0, \dots, \zeta_N$, under Assumption 1. This allows us to consider the standard derivative as a discrete operator mapping V_h to W_h, denoted

$$d_h : V_h \to W_h, \quad v \mapsto \partial_x v$$

and its discrete adjoint $d_h^* : W_h \to V_h$ defined by the relations

$$\langle d_h^* w, v\rangle = \langle w, d_h v\rangle \qquad v \in V_h,\ w \in W_h.$$

Formally the operator $-d_h^*$ can be seen as a weak approximation of ∂_x on $H_0^1(\Omega)$, and it is easy to verify that if one denotes by P_{V_h} the L^2 projection on V_h, namely

$$\langle P_{V_h} f, v\rangle = \langle f, v\rangle, \qquad f \in L^2(\Omega),\ \ v \in V_h, \tag{11}$$

and similarly for P_{W_h}, then the following diagram commutes

$$\begin{array}{ccc} H_0^1(\Omega) & \xrightarrow{\ \partial_x\ } & L^2(\Omega) \\ \Big\downarrow{\scriptstyle P_{W_h}} & & \Big\downarrow{\scriptstyle P_{V_h}} \\ W_h & \xrightarrow{\ -d_h^*\ } & V_h \end{array} \tag{12}$$

in the sense that $-d_h^* P_{W_h} u = P_{V_h} \partial_x u$ holds for all $u \in H_0^1(\Omega)$.

As described in [13,12], commuting diagrams can be extended to non-conforming discretizations in order to gain some locality and flexibility in the design of the Finite Element spaces. Here a natural choice for the non-conforming space consists of broken splines. Given a division of the grid into subdomains containing each $\tilde{N}$ cells (assuming $N = \tilde{N}S$ for some $S \in \mathbb{N}$) of the form

$$\tilde{I}_s = [\tilde{\zeta}_s, \tilde{\zeta}_{s+1}] \qquad \text{with} \qquad \tilde{\zeta}_s := \zeta_{s\tilde{N}}, \qquad 0 \le s < S,$$

and writing $\tilde{\boldsymbol{\xi}}^s = \{\tilde{\xi}_i^s : 0 \le i \le \tilde{n} + p\}$ with $\tilde{n} = \tilde{N} + p$ the open knot vector corresponding to p degree splines on the nodes ζ_k, $s\tilde{N} \le k \le (s+1)\tilde{N}$, we let

$$\tilde{V}_h := \{v \in L^2(\Omega) : v|_{\tilde{I}_s} \in \mathbb{S}_{p,\tilde{\boldsymbol{\xi}}^s},\ 0 \le s < S\} \tag{13}$$

denote the resulting broken spline space, which satisfies $V_h = \tilde{V}_h \cap \mathcal{C}^{p-1}(\Omega)$. A projection P of the form (7) can then be used as a local smoothing operator mapping $\tilde{V}_h$ on V_h, and this allows to define a non-conforming differential operator

$$\tilde{d}_h : \tilde{V}_h \to W_h, \quad \tilde{v} \mapsto \partial_x P \tilde{v}$$

and its discrete adjoint $\tilde{d}_h^* : W_h \to \tilde{V}_h$ characterized by

$$\langle \tilde{d}_h^* w, \tilde{v} \rangle = \langle w, \partial_x P \tilde{v} \rangle, \qquad \tilde{v} \in \tilde{V}_h.$$

A natural question is whether replacing d_h^*, V_h and P_{V_h} by their non-conforming counterparts $\tilde{d}_h^*$, $\tilde{V}_h$ and $P_{\tilde{V}_h}$ (the L^2 projection on $\tilde{V}_h$) in (12) results in a commuting diagram. The answer is negative, and instead one must consider the operator

$$P^* : L^2(\Omega) \to \tilde{V}_h, \qquad \langle P^* f, \tilde{v} \rangle = \langle f, P\tilde{v} \rangle, \qquad \tilde{v} \in \tilde{V}_h. \tag{14}$$

Indeed for all $u \in H_0^1(\Omega)$ and $\tilde{v} \in \tilde{V}_h$, we have

$$-\langle \tilde{d}_h^* P_{W_h} u, \tilde{v} \rangle = -\langle P_{W_h} u, \partial_x P \tilde{v} \rangle = -\langle u, \partial_x P \tilde{v} \rangle = \langle \partial_x u, P\tilde{v} \rangle = \langle P^* \partial_x u, \tilde{v} \rangle$$

which means that the following diagram commutes:

$$\begin{array}{ccc} H_0^1(\Omega) & \xrightarrow{\partial_x} & L^2(\Omega) \\ \Big\downarrow{\scriptstyle P_{W_h}} & & \Big\downarrow{\scriptstyle P^*} \\ W_h & \xrightarrow{-\tilde{d}_h^*} & \tilde{V}_h \end{array} \tag{15}$$

We note that applying P^* amounts to applying P on the broken B-splines and to inverting mass matrices in the subdomains. As the latter have a bounded number of cells this is a local computation, just as applying the L^2 projection on $\tilde{V}_h$,

$$\langle P_{\tilde{V}_h} f, \tilde{v} \rangle = \langle f, \tilde{v} \rangle, \qquad \tilde{v} \in \tilde{V}_h, \tag{16}$$

amounts to solving local problems on the subdomains. We also observe that if $Q : f \mapsto \sum_{i=0}^{n-1} \langle \varphi_i, f \rangle \psi_i$ denotes the dual projection operator mentioned in the introduction, we have $P^* = P_{\tilde{V}_h} Q$. Interestingly, P^* cannot be a projection if $\tilde{V}_h$ involves more than one subdomain: indeed that would imply

$$\langle \tilde{u}, \tilde{v} \rangle = \langle P^* \tilde{u}, \tilde{v} \rangle = \langle \tilde{u}, P\tilde{v} \rangle, \qquad \tilde{u}, \tilde{v} \in \tilde{V}_h$$

and taking $\tilde{u} = \tilde{v} = (I - P)\tilde{w}$ for an arbitrary $\tilde{w} \in \tilde{V}_h$ would give $\|(I - P)\tilde{w}\| = 0$, hence $\tilde{w} = P\tilde{w} \in V_h$, in contradiction with the fact that V_h is a proper subspace of $\tilde{V}_h$. However, one easily verifies that if P is a moment preserving projection then P^* is a quasi-interpolation.

Theorem 4 *Let $p, n, m \in \mathbb{N}$ with $m \leq p$, and let $\boldsymbol{\xi}$ be a knot vector satisfying Assumptions 1 and 2. Given $S, \tilde{N} \in \mathbb{N}$ such that $S\tilde{N} = N := n - p$, let then $\tilde{V}_h$ be the broken spline space defined by* (13)*. If* (P1)-(P4) *hold for given functions ψ_i, $0 \leq i < n$, then the operator $P^* : L^2 \to \tilde{V}_h$ defined by* (14) *is a quasi-interpolation,*

$$P^* g = g, \qquad g \in \mathbb{P}_m, \tag{17}$$

and it satisfies a local error estimate

$$\|f - P^* f\|_{L^q(\tilde{I}_s)} \leq C \inf_{g \in \mathbb{P}_m} \|f - g\|_{L^q(\tilde{I}_s^*)}, \qquad 0 \leq s < S,$$

with a constant C that depends on p and $\tilde{N}$ the number of elements in a subdomain, and $\tilde{I}_s^* \subset I_{s\tilde{N}}^+(\tilde{K})$, $\tilde{K} = K + \max(\tilde{N}, p)$, and interval which consists of a bounded number of cells close to the node $\zeta_{s\tilde{N}}$, see (P2)*.*

Proof To verify the quasi-interpolation property we use (14) and (9) to compute

$$\langle P^*g, \tilde{v}\rangle = \langle g, P\tilde{v}\rangle = \langle g, \tilde{v}\rangle, \qquad \tilde{v} \in \tilde{V}_h.$$

The equality $P^*g = g$ then follows from the fact that for $m \le p$, $\mathbb{P}_m$ is a subspace of $\tilde{V}_h$. Turning to the a priori estimate, we denote by $\{\tilde{\phi}_{s,a} \in L^2(\tilde{I}_s) : 0 \le a < \tilde{n}\}$ an orthonormal basis for the local spline space $\mathbb{S}_{p,\tilde{\boldsymbol{\xi}}^s}$, $0 \le s < S$, and set

$$\tilde{\varphi}_{s,a} = \tilde{h}_s^{\frac{1}{2}} \mathbb{1}_{\tilde{I}_s} \tilde{\phi}_{s,a} \quad \text{and} \quad \tilde{\psi}_{s,a} = \tilde{h}_s^{-1} \tilde{\varphi}_{s,a} \quad \text{with} \quad \tilde{h}_s = |\tilde{I}_s|.$$

These functions satisfy $\langle \tilde{\psi}_{s,a}, \tilde{\varphi}_{t,b}\rangle = \delta_{(s,a),(t,b)}$ for $0 \le s,t < S$, $0 \le a,b < \tilde{n}$, and standard arguments based on the piecewise polynomial structure of the space $\mathbb{S}_{p,\tilde{\boldsymbol{\xi}}^s}$ over its quasi-uniform grid (6) allow us to write

$$\|\tilde{\varphi}_{s,a}\|_{L^q} \le C\tilde{h}_s^{\frac{1}{q}} \quad \text{and} \quad \|\tilde{\psi}_{s,a}\|_{L^q} \le C\tilde{h}_s^{-1+\frac{1}{q}} \tag{18}$$

with constants depending only on p and $\tilde{N}$, the number of polynomial pieces in each subdomain. By definition of P^* we then have

$$P^*f = \sum_{s=0}^{S-1} \sum_{a=0}^{\tilde{n}-1} \langle P\tilde{\psi}_{s,a}, f\rangle\, \tilde{\varphi}_{s,a}.$$

A local stability estimate can then be obtained for P^* using similar arguments as above: On a subdomain $\tilde{I}_s$ we estimate

$$\|P^*f\|_{L^q(\tilde{I}_s)} \le \sum_{a=0}^{\tilde{n}-1} |\langle P\tilde{\psi}_{s,a}, f\rangle| \tilde{h}_s^{\frac{1}{q}}$$

and we infer from (P2) that the support of any $P\tilde{\psi}_{s,a} = \sum_{0\le i<n} \langle \psi_i, \tilde{\psi}_{s,a}\rangle \varphi_i$ with $0 \le a < \tilde{n}$ consists of a bounded number of intervals around $\tilde{\zeta}_s = \zeta_{s\tilde{N}}$, that is

$$\tilde{I}_s^* := \operatorname{supp}(P\tilde{\psi}_{s,a}) \subset \bigcup_{i\,:I^+(K)_i \cap \tilde{I}_s \neq \emptyset} \left(\bigcup_{i-p\le k<i} I_k \right) \subset I^+_{s\tilde{N}}(\tilde{K})$$

with $\tilde{K} = K + \max(\tilde{N}, p)$ as announced. It follows that we can bound

$$|\langle P\tilde{\psi}_{s,a}, f\rangle| \le \|P\tilde{\psi}_{s,a}\|_{L^{q'}} \|f\|_{L^q(\tilde{I}_s^*)} \le C\tilde{h}_s^{-1+\frac{1}{q'}} \|f\|_{L^q(\tilde{I}_s^*)}$$

where we have used the L^q stability of P, namely Eq. (10), and the estimates (18). Using again $-1 + \frac{1}{q'} = -\frac{1}{q}$, the above bounds give

$$\|P^*f\|_{L^q(\tilde{I}_s)} \le C\|f\|_{L^q(\tilde{I}_s^*)}$$

with a constant C depending only on p, q and $\tilde{N}$. The local error estimate is then easily derived from the quasi-projection property (17), indeed

$$\|f - P^*f\|_{L^q(\tilde{I}_s)} \le \|f - g\|_{L^q(\tilde{I}_s)} + \|P^*(f-g)\|_{L^q(\tilde{I}_s)} \le (1+C)\|f-g\|_{L^q(\tilde{I}_s^*)}$$

holds for all $g \in \mathbb{P}_m$. □

3 Local dual bases

Before describing a solution to our problem we recall a couple of local dual bases, i.e. functions ψ_i, $0 \le i < n$, that satisfy properties (P1)-(P3) but not (P4).

3.1 The dual basis of de Boor

A classical L^∞-stable local dual basis is described in [6], see also [21, Th. 4.41]. It is based on the perfect spline $B^p_* = \left(\frac{p+1}{2}\right) B^p[\xi^*_0, \dots, \xi^*_{p+1}]$ associated with the knots $\xi^*_i = \cos\left(\frac{(p+1-i)\pi}{p+1}\right)$, $i = 0, 1, \dots, p+1$ (called perfect because its p-th derivative is of constant absolute value on its support $[-1, 1]$). A scaled transition function is then defined as

$$G_i(x) = g\Big(\frac{2x - \xi_i - \xi_{i+p+1}}{\xi_{i+p+1} - \xi_i}\Big) \quad \text{with} \quad g(x) = \begin{cases} 0 & \text{if } x < -1, \\ \int_{-1}^{x} B^p_* & \text{if } -1 \le x < 1, \\ 1 & \text{if } 1 \le x \end{cases}$$

and the resulting dual basis functions read

$$\psi_i^{\mathsf{db}}(x) = \frac{\mathrm{d}^{p+1}}{\mathrm{d}x^{p+1}} \left(G_i(x) \frac{\prod_{j=i+1}^{i+p} (x - \xi_j)}{p!} \right), \qquad 0 \le i < n. \tag{19}$$

It is known (see, e.g. the proof of Theorem 4.41 in [21]) that these functions satisfy properties (P1) and (P3), and one easily verifies that $\mathrm{supp}(\psi_i^{\mathsf{db}}) \subset [\xi_i, \xi_{i+p+1}]$, so that (P2) holds with $K = p$.

3.2 Dual functions based on the piecewise polynomial structure of the B-splines

Another dual basis is easily derived from the piecewise polynomial structure of the B-splines. The resulting projection operator corresponds to the Bézier projection introduced in [22] and is studied as a particular case of local quasi-interpolation operators in [10].

An attractive feature of this basis is that it has the same polynomial structure as the B-splines, which simplifies the use of accurate quadrature formulas in numerical implementations. It can be defined as follows. For $0 \le k < N$, denote

$$\varphi_{i,k} := \varphi_i|_{I_k}, \qquad 0 \le i < n, \tag{20}$$

the restriction of the B-spline $\varphi_i = B^p_i$ to the cell $I_k = [\zeta_k, \zeta_{k+1})$. For $k \le i \le k+p$ these functions form a basis of $\mathbb{P}_p(I_k)$, see e.g. [21, Th. 4.21]: we may then let

$$\psi_{i,k} \in \mathcal{P}_p(I_k), \qquad k \le i \le k+p, \tag{21}$$

denote the dual polynomial basis on I_k, characterized by the relations

$$\langle \psi_{j,k}, \varphi_{i,k} \rangle_{I_k} = \delta_{i,j} \qquad \text{for} \quad k \le i, j \le k+p.$$

The resulting dual functions are then defined as

$$\psi_i^{\mathsf{ps}} := \sum_{k\in\mathcal{K}(i)} \alpha_{i,k} \mathbb{1}_{I_k} \psi_{i,k} \qquad \text{with} \qquad \alpha_{i,k} := \Big(\int \varphi_i\Big)^{-1} \int_{I_k} \varphi_i \tag{22}$$

where $\mathcal{K}(i) = \big\{k \in \{0,\dots,N-1\} : k \le i \le k+p\big\}$, $0 \le i < n$, denotes the cell indexes where the B-spline φ_i has a non zero contribution.

Lemma 5 *For $0 \le k < N$, the mapping $\Pi_{I_k} : f \mapsto \sum_{j=k}^{k+p} \langle \psi_{j,k}, f\rangle_{I_k} \varphi_{j,k}$ coincides with the L^2 (orthogonal) projection on $\mathbb{P}_p(I_k)$.*

Proof Writing $\psi_{j,k} = \sum_{l=k}^{k+p} c_{j,l}^k \varphi_{l,k}$ we see that

$$\delta_{i,j} = \langle \psi_{j,k}, \varphi_{i,k}\rangle_{I_k} = \sum_{l=k}^{k+p} c_{j,l}^k \langle \varphi_{l,k}, \varphi_{i,k}\rangle_{I_k},$$

which also shows that the matrix $(c_{j,l}^k)_{k\le j,l\le k+p}$ is symmetric. It follows that

$$\sum_{j=k}^{k+p} \langle \varphi_{i,k}, \varphi_{j,k}\rangle_{I_k} \langle \psi_{j,k}, f\rangle_{I_k} = \sum_{j,l=k}^{k+p} \langle \varphi_{i,k}, \varphi_{j,k}\rangle_{I_k} c_{l,j}^k \langle \varphi_{l,k}, f\rangle_{I_k} = \langle \varphi_{i,k}, f\rangle_{I_k}$$

which characterizes Π_{I_k} as the orthogonal projection on $\mathbb{P}_p(I_k)$. □

Remark 6 In particular, the linear forms $\lambda_\ell^{I_k}$, $\ell = 1,\dots,p+1$ from [10, Eq. (11)] and their analogs from [22, Eq. (58)] correspond to

$$\lambda_\ell^{I_k}(f) = \langle \psi_{i,k}, f\rangle_{I_k} \qquad \text{with} \quad i = k+\ell-1.$$

Theorem 7 *Let $p \in \mathbb{N}$. The functions ψ_i^{ps} defined by* (22) *satisfy the properties* (P1)-(P3)*, i.e. they form a local and stable dual basis to the B-splines. Moreover they satisfy* (P4) *with $m = 0$, i.e. they span the constant functions.*

Proof By definition of the set $\mathcal{K}(i)$, we compute

$$\sum_{k\in\mathcal{K}(i)} \alpha_{i,k} = \sum_{k\in\mathcal{K}(i)} \Big(\int \varphi_i\Big)^{-1} \int_{I_k} \varphi_i = 1, \tag{23}$$

hence

$$\langle \psi_i^{\mathsf{ps}}, \varphi_j\rangle = \sum_{k\in\mathcal{K}(i)} \alpha_{i,k} \langle \psi_{i,k}, \varphi_{j,k}\rangle_{I_k} = \sum_{k\in\mathcal{K}(i)} \alpha_{i,k}\delta_{i,j} = \delta_{i,j}$$

which proves the duality property (P1). The locality property (P2) is verified by construction. For the stability (P3) we can use Theorem 1 in [10] thanks to Remark 6 above. Under the local quasi-uniformity Assumption (2), this result states that

$$|\langle \psi_{i,k}, f\rangle_{I_k}| \le C|I_k|^{-1} \|f\|_{L^1(I_k)}$$

holds for $k \le i \le k+p$ with a constant depending only on p. This allows to write

$$\|\psi_{i,k}\|_{L^\infty} = \sup_{f\in L^1} \frac{\langle \psi_{i,k}, f\rangle_{I_k}}{\|f\|_{L^1(I_k)}} \le C|I_k|^{-1} \le Ch_i^{-1},$$

where the last inequality uses again the local quasi-uniformity (6). This clearly proves (P3) as the coefficients $\alpha_{i,k}$ are all bounded (by 1). To finally prove the conservation property we observe that $\big(\int \varphi_i\big)\alpha_{i,k} = \langle 1, \varphi_{i,k}\rangle$ is the i-th coefficient of the function 1 in the local basis (21). In particular, we have

$$\sum_{i=0}^{n-1} \Big(\int \varphi_i\Big)\psi_i^{\mathsf{ps}} = \sum_{i=0}^{n-1} \sum_{k\in\mathcal{K}(i)} \langle 1, \varphi_{i,k}\rangle \mathbb{1}_{I_k}\psi_{i,k} = \sum_{k=0}^{N-1} \mathbb{1}_{I_k} = \mathbb{1}_\Omega$$

which proves that (P4) indeed holds with $m = 0$. □

Remark 8 As studied in [22] and [10], any bounded combination of the form (22) with $\sum_{k\in\mathcal{K}(i)} \alpha_{i,k} = 1$ provides a stable and local dual basis for the B-splines. In [22] the values $\alpha_{i,k} = (\int \varphi_i)^{-1} \int_{I_k} \varphi_i$ are advocated as leading to more accurate projections, and they are the only ones for which the dual basis spans the constants.

4 Dual functions based on a macro-subdivision

We now present our solution to the problem described in Section 1.2.

4.1 Construction of the dual basis

To preserve an arbitrary number of moments when projecting a function in the spline space, we construct a dual basis that uses a subdivision of the domain Ω into macro-elements containing each $m + p + 1$ cells of the form $I_k = [\zeta_k, \zeta_{k+1})$, see (3). Assuming that such a subdivision is possible, i.e., that there exists a positive integer L such that

$$N = LM \quad \text{with} \quad M := m + p + 1$$

(under Assumption 1 we have $N = n - p$), we denote the elements of the macro-subdivision by

$$\hat{I}_\ell = [\hat{\zeta}_\ell, \hat{\zeta}_{\ell+1}) \quad \text{with} \quad \hat{\zeta}_\ell = \zeta_{\ell M}. \tag{24}$$

We next subdivide the B-splines φ_i, $0 \le i < n$, in two groups.

- B-splines φ_i that do not vanish on a macro-vertex $\hat{\zeta}_\ell = \zeta_{\ell M}$ will be said to be of **macro-vertex** type. Under Assumption 1 they are of the form

$$\varphi_i \quad \text{with} \quad i \in \mathcal{I}_\ell^{\mathsf{mv}} := \{\ell M, \cdots, \ell M + p - 1\} \quad \text{for some} \quad 0 \le \ell < L.$$

- B-splines φ_i that have their support contained in a single macro-element $\hat{I}_\ell$ will be said to be of **macro-element** type. They are of the form

$$\varphi_i \quad \text{with} \quad i \in \mathcal{I}_\ell^{\mathsf{me}} := \{\ell M + p, \cdots, (\ell + 1)M - 1\} \quad \text{for some} \quad 0 \le \ell < L.$$

We gather the previous index sets into

$$\mathcal{I}^{\mathsf{mv}} := \bigcup_{0\le\ell\le L} \mathcal{I}_\ell^{\mathsf{mv}} \qquad \text{and} \qquad \mathcal{I}^{\mathsf{me}} := \bigcup_{0\le\ell<L} \mathcal{I}_\ell^{\mathsf{me}}$$

and define the macro-subdivision dual basis functions ψ_i^{ms} as follows.

– If $i \in \mathcal{I}^{\mathsf{mv}}$ is the index of a macro-vertex spline, then we may take any stable dual basis function for the associated **macro-vertex dual function**. In order to have the same polynomial structure as the B-spline, we choose to set

$$\psi_i^{\mathsf{ms}} := \psi_i^{\mathsf{ps}} \quad \text{for} \quad i \in \mathcal{I}^{\mathsf{mv}}. \tag{25}$$

– If $i \in \mathcal{I}^{\mathsf{me}}$ is the index of a macro-element spline, we proceed in two steps. Given $0 \le \ell < L$, we let $\{\phi_j^\ell : j \in \mathcal{I}_\ell^{\mathsf{me}}\}$ be the polynomial basis of $\mathbb{P}_m(\hat{I}_\ell)$ that satisfies

$$\langle \phi_j^\ell, \varphi_l \rangle = \delta_{j,l} \qquad \text{for} \quad j, l \in \mathcal{I}_\ell^{\mathsf{me}}. \tag{26}$$

Here the important point is that such a basis is indeed well defined, and stable: it is guaranteed by Lemma 9 below. The **macro-element dual function** is then defined as

$$\psi_i^{\mathsf{ms}} := \phi_i^\ell - \sum_{j \in \mathcal{I}_\ell^{\mathsf{mv}} \cup \mathcal{I}_{\ell+1}^{\mathsf{mv}}} \langle \phi_i^\ell, \varphi_j \rangle \psi_j^{\mathsf{ms}} \qquad \text{for} \quad i \in \mathcal{I}_\ell^{\mathsf{me}}, \quad 0 \le \ell < L. \tag{27}$$

Lemma 9 *Let $0 \le \ell < L$. There exists a unique basis $\{\phi_i^\ell : i \in \mathcal{I}_\ell^{\mathsf{me}}\}$ of $\mathbb{P}_m(\hat{I}_\ell)$ that satisfies the relations*

$$\langle \phi_i^\ell, \varphi_j \rangle = \delta_{i,j} \qquad i, j \in \mathcal{I}_\ell^{\mathsf{me}}, \tag{28}$$

moreover the estimate

$$\|\phi_i^\ell\|_{L^\infty(\hat{I}_\ell)} \le C h_i^{-1}, \qquad i \in \mathcal{I}_\ell^{\mathsf{me}} \tag{29}$$

holds with a constant that only depends on p and m.

We postpone the proof of this lemma to Section 4.2 and state the desired result.

Theorem 10 *Let $p, m \in \mathbb{N}$. The functions ψ_i^{ms} defined by* (25) *and* (27) *satisfy the properties* (P1)-(P4)*, i.e. they form a local and stable dual basis to the B-splines that span the polynomials of degree $\le m$.*

Proof The duality property (P1) is straightforward to verify, given that the functions ψ_i^{ps} also satisfy this property. To verify the locality we first use the fact that the functions ψ_i^{ps} have the same local supports (ξ_i, ξ_{i+p+1}) as the B-splines, hence it is also the case for the dual functions of macro-vertex type. For the dual functions of macro-element type we observe that the functions ϕ_i^ℓ vanish by construction outside the macro-elements $\hat{I}_\ell$. It follows that

$$\operatorname{supp}(\psi_i^{\mathsf{ms}}) \subset \left(\hat{I}_\ell \cup \bigcup_{j \in \mathcal{I}_\ell^{\mathsf{mv}} \cup \mathcal{I}_{\ell+1}^{\mathsf{mv}}} \operatorname{supp}(\psi_j^{\mathsf{ps}}) \right) = (\xi_{\ell M}, \xi_{(\ell+1)M+2p})$$

which proves (P2) with $K = 2p + m$. The stability property (P3) is then easily derived from Theorem 7 and Lemma 9, using also the local quasi-uniformity (6) for the bound

$$|\langle \phi_i^\ell, \varphi_j \rangle| \le \|\phi_i^\ell\|_{L^\infty} \|\varphi_j\|_{L^1} \le C h_i^{-1} h_j \le C'$$

which holds with constants C, C' depending only on p and m. □

4.2 Stability analysis

We now prove Lemma 9. Given $0 \le \ell < L$ we denote

$$\hat{\varphi}_a = \varphi_{p+\ell M+a}, \qquad 0 \le a \le m, \tag{30}$$

the macro-element B-splines associated with $\hat{I} = \hat{I}_\ell = [\hat{\zeta}_0, \hat{\zeta}_M]$. We then let $\boldsymbol{M} \in \mathcal{M}_{m+1}(\mathbb{R})$ be the matrix of moments

$$\boldsymbol{M}_{a,b} := \frac{1}{a!(m-a)!} \int_{\hat{\zeta}_0}^{\hat{\zeta}_M} \frac{(\hat{\zeta}_M - x)^a}{(\hat{\zeta}_M - \hat{\zeta}_0)^a} \hat{\varphi}_b(x)\,\mathrm{d}x, \quad 0 \le a, b \le m.$$

Verifying (28) amounts to showing that $\boldsymbol{M}$ is invertible: the polynomials ϕ_i^ℓ are then given by

$$\phi^\ell_{p+\ell M+b} = \sum_{0\le a\le m} \frac{\eta_{b,a}}{a!(m-a)!} \frac{(\hat{\zeta}_M - x)^a}{(\hat{\zeta}_M - \hat{\zeta}_0)^a} \quad \text{with} \quad \eta_{b,a} := (\boldsymbol{M}^{-1})_{b,a}.$$

We will actually show that the inversion is stable, namely that

$$\|\boldsymbol{M}^{-1}\| \le C|\hat{I}|^{-1} \tag{31}$$

with a constant depending only on p and m. This in turn will allow us to write

$$\|\phi^\ell_{p+\ell M+b}\|_{L^\infty} \le \sum_{0\le a\le m} |\eta_{b,a}| C_m \le C|\hat{I}|^{-1}$$

which leads to the desired bound (29), thanks to the local quasi-uniformity (6). To prove (31) we consider an arbitrary vector $\gamma = (\gamma_a)_{0\le a\le m} \in \mathbb{R}^{m+1}$ and let

$$g := \sum_{0\le a\le m} \gamma_a \hat{\varphi}_a$$

be the associated spline. The vector $\mu := \boldsymbol{M}\gamma$ then corresponds to the moments of g: its coefficients read

$$\mu_a = \frac{1}{a!(m-a)!} \int_{\hat{\zeta}_0}^{\hat{\zeta}_M} \frac{(\hat{\zeta}_M - x)^a}{(\hat{\zeta}_M - \hat{\zeta}_0)^a} g(x)\,\mathrm{d}x, \qquad 0 \le a \le m. \tag{32}$$

We next let $G^0 := g$ and define

$$G^{a+1}(x) := \int_{\hat{\zeta}_0}^{x} G^a(y)\,\mathrm{d}y \qquad \text{for} \quad a = 0, \dots, m.$$

By construction, $G := G^{m+1}$ is a spline of degree M on $\hat{I} = [\hat{\zeta}_0, \hat{\zeta}_M]$, therefore it belongs to the space $\mathbb{S}_{M,\hat{\boldsymbol{\xi}}}$ associated with the open knot vector $\hat{\boldsymbol{\xi}} = \{\hat{\xi}_0, \dots, \hat{\xi}_{3M}\}$ obtained by repeating $M+1$ times the extremal nodes $\hat{\zeta}_0$ and $\hat{\zeta}_M$, namely

$$\hat{\xi}_0 = \cdots = \hat{\xi}_M < \cdots < \hat{\xi}_{2M} = \cdots = \hat{\xi}_{3M} \quad \text{and} \quad \hat{\xi}_{M+k} = \hat{\zeta}_k \quad \text{for } 0 \le k \le M. \tag{33}$$

It will be convenient to denote by $\hat{B}_i^q$ the B-splines of degree $q \leq M$ on these knots. The macro-element B-splines (30) then correspond to

$$\hat{\varphi}_a = \hat{B}^p_{M+a}, \qquad 0 \leq a \leq m, \tag{34}$$

and since G is a linear combination of M-degree B-splines on these knots, we may write $G = \sum_{i=0}^{2M-1} \Gamma_i \hat{B}_i^M$. One can then verify that the first M coefficients vanish in this sum: indeed by construction one has

$$G^{(r)} = \begin{cases} G^{m+1-r} & \text{for } 0 \leq r \leq m \\ g^{(r-m-1)} & \text{for } m+1 \leq r \leq m+p = M-1, \end{cases} \tag{35}$$

so that $G^{(r)}(\hat{\zeta}_0) = 0$ for $0 \leq b \leq M-1$. This guarantees that G only involves B-splines $\hat{B}_i^M$ with indexes $i \geq M$, which can also be seen as a consequence of the de Boor-Fix formula [9,7]

$$\Gamma_i = \sum_{r=0}^{M} (-1)^{M-r} \Psi_i^{(M-r)}(\tau_i) G^{(r)}(\tau_i) \quad \text{with} \quad \Psi_i(x) = \frac{(\hat{\xi}_{i+1} - x) \cdots (\hat{\xi}_{i+M} - x)}{M!}. \tag{36}$$

Here τ_i can be arbitrary in $(\hat{\xi}_i, \hat{\xi}_{i+M+1})$, and we further notice that any knot $\hat{\xi}_{i+k}$ with $1 \leq k \leq M$ is allowed (even if not in the open interval): indeed in the sum (36) the only discontinuous term corresponds to $r = M$, and on these knots the function $\Psi_i^{(M-r)} = \Psi_i$ vanishes, which makes the sum actually continuous. Thus, for $0 \leq i < M$ one may take $\tau_i = \hat{\xi}_M = \hat{\zeta}_0$ and the formula yields $\Gamma_i = 0$ indeed. For $M \leq i \leq 2M-1$ we then take $\tau_i = \hat{\xi}_{2M} = \hat{\zeta}_M$, which yields

$$|\Gamma_i| \leq \sum_{r=0}^{M-1} \frac{|\hat{I}|^r}{r!} |G^{(r)}(\hat{\zeta}_M)|.$$

Using again that g consists of macro-element B-splines, the relations (35) imply that $G^{(r)}(\hat{\zeta}_M) = 0$ if $m+1 \leq r$. For $0 \leq r \leq m$ we then compute

$$\begin{aligned} G^{(r)}(\hat{\zeta}_M) = G^{m+1-r}(\hat{\zeta}_M) &= \int_{\hat{\zeta}_0}^{\hat{\zeta}_M} G^{m-r}(x)\,\mathrm{d}x \\ &= \int_{\hat{\zeta}_0}^{\hat{\zeta}_M} (\hat{\zeta}_M - x) G^{m-r-1}(x)\,\mathrm{d}x \qquad (\text{if } r \leq m-1) \\ &\vdots \\ &= \int_{\hat{\zeta}_0}^{\hat{\zeta}_M} \frac{(\hat{\zeta}_M - x)^{m-r}}{(m-r)!} g(x)\,\mathrm{d}x = r!\,|\hat{I}|^{m-r} \mu_{m-r} \end{aligned}$$

where the last equality follows from (32). This yields

$$\|\Gamma\|_\infty \leq C_m |\hat{I}|^m \|\mu\|. \tag{37}$$

Turning to the function g, we use the recurrence relations (5.2) in [7] and the equality (34) to write

$$\sum_{a=0}^{m} \gamma_a \hat{B}^p_{M+a} = g = G^{(m+1)} = \Big(\sum_{i=M}^{2M-1} \Gamma_i \hat{B}_i^M \Big)^{(m+1)} = \sum_{i=M}^{2M-1} \Gamma_i^{(m+1)} \hat{B}_i^p$$

with differentiated coefficients defined by

$$\Gamma_i^{(r)} := \begin{cases} \Gamma_i & \text{if } r = 0 \\ \frac{M+1-r}{\hat{\xi}_{i+M+1-r} - \hat{\xi}_i} \left(\Gamma_i^{(r-1)} - \Gamma_{i-1}^{(r-1)} \right) & \text{if } r \geq 1. \end{cases} \tag{38}$$

This shows that $\gamma_a = \Gamma_{M+a}^{(m+1)}$, and in particular $\Gamma_{M+a}^{(m+1)} = 0$ for $a > m$. Moreover, arguing as above one verifies that $G^{(r)}$ only involves splines $\hat{B}_i^{M-r}$ with $i \geq M$, hence the recurrence relations [7, Eq. (5.2)] yield $\Gamma_i^{(r)} = 0$ for all $i < M$ and $0 \leq r \leq m+1$. According to (38) this allows us to only consider vectors of the form $(\Gamma_{M+a}^{(r)})_{a=0}^m$, (which we shall conveniently denote by $\Gamma^{(r)}$) when we evaluate the differentiated coefficients $\Gamma_{M+a}^{(m+1)}$. This restriction is helpful since for $0 \leq a \leq m$ and $1 \leq r \leq m+1$, the interval $(\hat{\xi}_{M+a}, \hat{\xi}_{2M+a+1-r})$ always contains at least one non-empty cell I_k of the original grid. The local quasi-uniformity property (6) then allows us to write

$$|\Gamma_a^{(r)}| \leq c|\hat{I}|^{-1} \|\Gamma^{(r-1)}\|$$

for $1 \leq r \leq m+1$, with a constant $c = c(p, m)$. Using (37) and (38) this leads to

$$\|\gamma\| = \|\Gamma^{(m+1)}\| \leq c^m |\hat{I}|^{-m-1} \|\Gamma\| \leq C|\hat{I}|^{-1} \|\mu\|$$

with another constant $C = C(p, m)$. This proves (31) and ends the proof of Lemma 9.

5 Numerical results

We end our study with a numerical assessment of the convergence properties of the various approximation operators described above.

5.1 Accuracy of the local projection operators P on the spline space

In Figure 1 we show the L^2 convergence curves of the local projection operators P on the smooth spline space $V_h = \mathbb{S}_{p,\boldsymbol{\xi}}$, as defined by (7). The different operators tested here are reminded in Table 1 and their accuracy is compared to that of the L^2 projection P_{V_h} which is optimal. The plots on the left correspond to the approximation of the H^1 function $f_{\mathsf{abs}}(x) = |x - 1/\pi|$, and the plots on the right correspond to the approximation of the smooth ($\mathcal{C}^\infty$) Runge function $f_{\mathsf{runge}}(x) = 1/(25(2x-1)^2 + 1)$. Both are computed with degrees $p \in \{1, 2, 3\}$, from top to bottom.

Overall the accuracies of the different projection operators are very similar, and close to that of the optimal L^2 projection, which confirms the result of Theorem 3. For f_{abs} (on the left) some differences are visible when cubic splines are employed: in particular the moment preserving P^{ms} projection has a reduced accuracy on the coarsest meshes, which is due to its oscillating behavior close to the point where the target function is not $\mathcal{C}^1$. For the smooth f_{runge} (on the right) the curves are hardly distinguishable

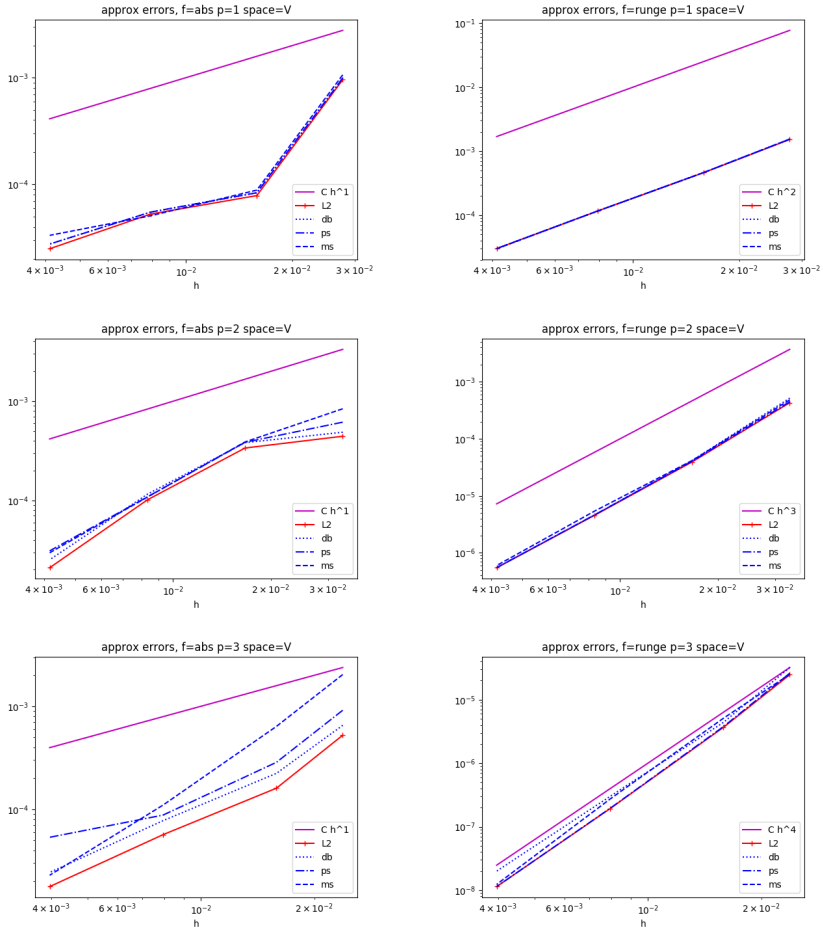

Fig. 1 Convergence curves corresponding to the projection operators $P : L^2 \to V_h$ defined on smooth spline spaces of specified degrees $1 \leq p \leq 3$, as indicated. Definitions of the different operators are reminded in Table 1.

5.2 Accuracy of the commuting approximations P^* on the broken splines

In Figure 2 we show the L^2 convergence curves of the operators P^* which provide a commuting diagram for the broken spline space $\tilde{V}_h$, as studied in Section 2. The definitions of the different operators are reminded in Table 2 and their accuracy is compared to that of the L^2 projection $P_{\tilde{V}_h}$ which is optimal. Again we note that the latter operator is local because of the broken nature of the space, but it does not yield a commuting diagram, as discussed in Section 2. The target functions and spline degrees are the same as in Figure 1.

In contrast to Figure 1, we find that the accuracies of the different operators are not comparable, even for the smooth target function f_{runge}. As announced by our a priori estimate from Theorem 4 the operator P^*_{ms} involving macro-subdivision

Table 1 Projection operators $P_{\mathsf{key}} : L^2 \to V_h$ used in Figure 1, for given values of key.

key	operator	reference
L2	non-local L^2 projection P_{V_h}	Eq. (11)
db	local projection of de Boor P_{db}	Eq. (7), (19)
ps	local projection P_{ps} based on the spline polynomial structure	Eq. (7), (22)
ms	local projection P_{ms} based on the macro-spline subdivision	Eq. (7), (25), (27)

dual functions ψ_i^{ms} seems to converge with an optimal rate, in accordance with their high order moment preserving properties. The operator P_{ps}^* involving ψ_i^{ps} dual functions has a lesser accuracy but still converges, in accordance with its first order moment preserving properties (see Theorem 7), and the operator P_{db}^* involving ψ_i^{db} dual functions hardly converges for degrees $p \geq 2$.

Table 2 Approximation operators $\tilde{P}_{\mathsf{key}} : L^2 \to \tilde{V}_h$ used in Figure 2, for given values of key.

key	operator	reference
L2	local L^2 projection $P_{\tilde{V}_h}$	Eq. (16)
db	local operator P_{db}^* using de Boor dual functions	Eq. (14), (19)
ps	local operator P_{ps}^* using ps dual functions	Eq. (14), (22)
ms	local operator P_{ms}^* using ms dual functions	Eq. (14), (25), (27)

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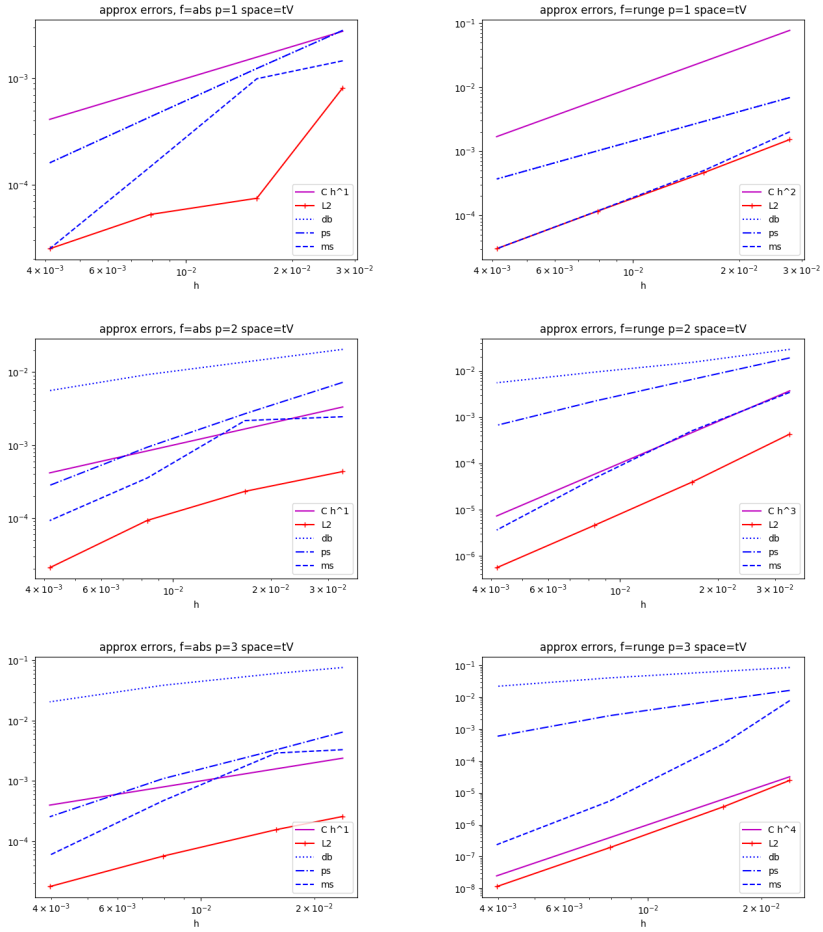

Fig. 2 Convergence curves corresponding to the projection operators $P^* : L^2 \to \tilde{V}_h$ defined on broken spline spaces with specified degrees.

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