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# Exponential approximation and enriched FEM for rods and Timoshenko beams

**Rémi Cornaggia**, Université Rennes 1

**Éric Darrigrand**, Université Rennes 1

**Loïc Le Marrec**, Université Rennes 1

**Fabrice Mahé**, Université Rennes 1

**Mots-clés** : Enriched finite elements, rods, Timoshenko beams

This study proposes a new enriched finite element method to handle vibrations of rods (traction-compression) and beams (bending) with varying cross-sections. In time-harmonic domain, closed-form analytical solutions for Timoshenko beams exist only for exponential cross-sections [1], and we therefore focus on finite elements enriched with such solutions. More precisely, for a given beam or rod with varying section, we use the enrichment corresponding to an optimal approximation by an exponential-by-parts beam or rod. The best approximation is established using an energetic criterion. Thereafter, the proposed method is presented for a rod whose section is approximated by a unique exponential profile, and extensions are then briefly discussed.

**Approximation by an exponential rod.** Working with non-dimensional quantities, the longitudinal displacement  $u(x)$  in a rod made of an homogeneous material obeys the wave equation  $(\mathcal{A}u)' + \mathcal{A}\omega^2 u = f$ , where  $\mathcal{A}(x)$  is the cross-section,  $\omega$  is the considered frequency and  $f$  represents distributed forces.

Given a rod with a given cross-section  $\mathcal{A}$ , we define its optimal exponential approximation as the rod of section  $\mathcal{A}_\delta(x) = \mathcal{A}_0 e^{2\delta x}$  (where  $\mathcal{A}_0 > 0$  and  $\delta \in \mathbb{R}$ ) which has the closest total energy (kinetic plus elastic) under any kind of boundary excitations (i.e. for  $f = 0$ ).

This criterion can be written in terms of (i) the solutions  $u$  corresponding to  $\mathcal{A}$ , and (ii) the solutions  $u_\delta$  corresponding to the sought  $\mathcal{A}_\delta$ , which are of the form:

$$u_\delta(x) = e^{-\delta x} (c_+ e^{i\tilde{k}x} + c_- e^{-i\tilde{k}x}), \quad \text{with } \tilde{k} = \sqrt{\omega^2 - \delta^2}. \quad (1)$$

Preliminary results are obtained for linear and quadratic rods, for which there also exist analytical solutions [3]. The optimality condition can therefore be expressed explicitly, resulting in semi-analytical determination of optimal  $(\mathcal{A}_0, \delta)$ . For an arbitrary cross-section  $\mathcal{A}$ , an analysis of the error  $u - u_\delta$  must be conducted.

**Enriched finite elements.** To address the case  $f \neq 0$ , and foreseeing an extension to the time domain, a finite element discretization is studied. The variable  $\tilde{u}$  defined by  $\tilde{u}(x) = u(x)e^{\delta x}$  is used instead of  $u$ , where  $\delta$  is determined as described above. Following e.g. [2], the considered finite element basis (e.g.  $\mathbb{P}_1$  basis) is then enriched by the basis  $\{e^{\pm i\tilde{k}x}\}$  of solutions corresponding to the exponential approximation. For linear and quadratic rods, using a regular mesh of element length  $h$ , this enrichment brings significant improvements of the FE solution, reaching  $O(h^4)$  convergence rate for enriched  $\mathbb{P}_1$  basis (instead of the well-known  $O(h^2)$  rate for  $\mathbb{P}_1$  elements).

**Extensions.** To address rods with roughly varying cross-section, such rod is divided into subdomains, and the optimal approximation procedure is then applied to each subdomain, obtaining an exponential-by-parts rod. Finally, the whole method is transposed to beams modeled by the Timoshenko system.

## Références

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