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# 1 An application of parallel cut elimination in 2 unit-free multiplicative linear logic to the Taylor 3 expansion of proof nets

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## 11 — Abstract —

12 We examine some combinatorial properties of parallel cut elimination in multiplicative linear  
13 logic (MLL) proof nets. We show that, provided we impose some constraint on switching paths,  
14 we can bound the size of all the nets satisfying this constraint and reducing to a fixed resultant  
15 net. This result gives a sufficient condition for an infinite weighted sum of nets to reduce into  
16 another sum of nets, while keeping coefficients finite. We moreover show that our constraints are  
17 stable under reduction.

18 Our approach is motivated by the quantitative semantics of linear logic: many models have  
19 been proposed, whose structure reflect the Taylor expansion of multiplicative exponential linear  
20 logic (MELL) proof nets into infinite sums of differential nets. In order to simulate one cut  
21 elimination step in MELL, it is necessary to reduce an arbitrary number of cuts in the differential  
22 nets of its Taylor expansion. It turns out our results apply to differential nets, because their cut  
23 elimination is essentially multiplicative. We moreover show that the set of differential nets that  
24 occur in the Taylor expansion of an MELL net automatically satisfy our constraints.

25 In the present work, we stick to the unit-free and weakening-free fragment of linear logic, which  
26 is rich enough to showcase our techniques, while allowing for a very simple kind of constraint: a  
27 bound on the number of cuts that are crossed by any switching path.

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## 33 **1** Introduction

### 34 **1.1** Context: quantitative semantics and Taylor expansion

35 Linear logic takes its roots in the denotational semantics of  $\lambda$ -calculus: it is often presented,  
36 by Girard himself [15], as the result of a careful investigation of the model of coherence  
37 spaces. Since its early days, linear logic has thus generated a rich ecosystem of denotational  
38 models, among which we distinguish the family of *quantitative semantics*. Indeed, the first  
39 ideas behind linear logic were exposed even before coherence spaces, in the model of normal  
40 functors [16], in which Girard proposed to consider analyticity, instead of mere continuity, as



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41 the key property of the interpretation of  $\lambda$ -terms: in this setting, terms denote power series,  
42 representing analytic maps between modules.

43 This quantitative interpretation reflects precise operational properties of programs: the  
44 degree of a monomial in a power series is closely related to the number of times a function  
45 uses its argument. Following this framework, various models were considered — among which  
46 we shall include the multiset relational model as a degenerate, boolean-valued instance. These  
47 models allowed to represent and characterize quantitative properties such as the execution  
48 time [5], including best and worst case analysis for non-deterministic programs [18], or the  
49 probability of reaching a value [2]. It is notable that this whole approach gained momentum  
50 in the early 2000's, after the introduction by Ehrhard of models [7, 8] in which the notion  
51 of analytic maps interpreting  $\lambda$ -terms took its usual sense, while Girard's original model  
52 involved set-valued formal power series. Indeed, the keystone in the success of this line  
53 of work is an analogue of the Taylor expansion formula, that can be established both for  
54  $\lambda$ -terms and for linear logic proofs.

55 Mimicking this denotational structure, Ehrhard and Regnier introduced the differential  
56  $\lambda$ -calculus [12] and differential linear logic [13], which allow to formulate a syntactic version  
57 of Taylor expansion: to a  $\lambda$ -term (resp. to a linear logic proof), we associate an infinite linear  
58 combination of approximants [14, 11]. In particular, the dynamics (*i.e.*  $\beta$ -reduction or cut  
59 elimination) of those systems is dictated by the identities of quantitative semantics. In turn,  
60 Taylor expansion has become a useful device to design and study new models of linear logic,  
61 in which morphisms admit a matrix representation: the Taylor expansion formula allows to  
62 describe the interpretation of promotion — the operation by which a linear resource becomes  
63 freely duplicable — in an explicit, systematic manner. It is in fact possible to show that any  
64 model of differential linear logic without promotion gives rise to a model of full linear logic  
65 in this way [4]: in some sense, one can simulate cut elimination through Taylor expansion.

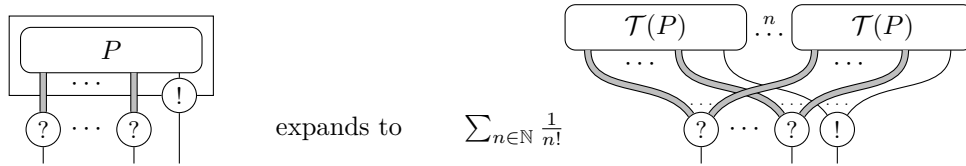
## 66 1.2 Motivation: reduction in Taylor expansion

67 There is a difficulty, however: Taylor expansion generates infinite sums and, *a priori*, there  
68 is no guarantee that the coefficients in these sums will remain finite under reduction. In  
69 previous works [4, 18], it was thus required for coefficients to be taken in a complete semiring:  
70 all sums should converge. In order to illustrate this requirement, let us first consider the  
71 case of  $\lambda$ -calculus.

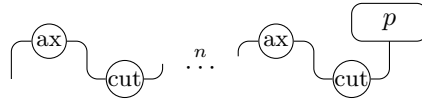
72 The linear fragment of differential  $\lambda$ -calculus, called *resource  $\lambda$ -calculus*, is the target  
73 of the syntactical Taylor expansion of  $\lambda$ -terms. In this calculus, the application of a  
74 term to another is replaced with a multilinear variant:  $\langle s \rangle [t_1, \dots, t_n]$  denotes the  $n$ -linear  
75 symmetric application of resource term  $s$  to the multiset of resource terms  $[t_1, \dots, t_n]$ .  
76 Then, if  $x_1, \dots, x_k$  denote the occurrences of  $x$  in  $s$ , the redex  $\langle \lambda x.s \rangle [t_1, \dots, t_n]$  reduces  
77 to the sum  $\sum_{f: \{1, \dots, k\} \xrightarrow{\sim} \{1, \dots, n\}} s[t_{f(1)}/x_1, \dots, t_{f(k)}/x_k]$ : here  $f$  ranges over all bijections  
78  $\{1, \dots, k\} \xrightarrow{\sim} \{1, \dots, n\}$  so this sum is zero if  $n \neq k$ . As sums are generated by reduction,  
79 it should be noted that all the syntactic constructs are linear, both in the sense that they  
80 commute to sums, and in the sense that, in the elimination of a redex, no subterm is copied  
81 nor erased. The key case of Taylor expansion is that of application:

$$82 \quad \mathcal{T}(MN) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \mathcal{T}(M) \rangle \mathcal{T}(N)^n \quad (1)$$

83 where  $\mathcal{T}(N)^n$  is the multiset made of  $n$  copies of  $\mathcal{T}(N)$  — by  $n$ -linearity,  $\mathcal{T}(N)^n$  is itself an  
84 infinite linear combination of multisets of resource terms appearing in  $\mathcal{T}(N)$ . Admitting that



■ **Figure 1** Taylor expansion of a promotion box (thick wires denote an arbitrary number of wires)



■ **Figure 2** Example of a family of nets, all reducing to a single net

85  $\langle M \rangle [N_1, \dots, N_n]$  represents the  $n$ -th derivative of  $M$ , computed at 0, and  $n$ -linearly applied  
 86 to  $N_1, \dots, N_n$ , one immediately recognizes the usual Taylor expansion formula.

87 From (1), it is immediately clear that, to simulate one reduction step occurring in  $N$ , it  
 88 is necessary to reduce in parallel in an unbounded number of subterms of each component of  
 89 the expansion. Unrestricted parallel reduction, however, is ill defined in this setting. Consider  
 90 the sum  $\sum_{n \in \mathbb{N}} \langle \lambda x x \rangle [\dots \langle \lambda x x \rangle [y] \dots]$  where each summand consists of  $n$  successive linear  
 91 applications of the identity to the variable  $y$ : then by simultaneous reduction of all redexes  
 92 in each component, each summand yields  $y$ , so the result should be  $\sum_{n \in \mathbb{N}} y$  which is not  
 93 defined unless the semiring of coefficients is complete in some sense.

94 Those considerations apply to linear logic as well as to  $\lambda$ -calculus. We will use proof nets  
 95 [15] as the syntax for proofs of multiplicative exponential linear logic (MELL). The target of  
 96 Taylor expansion is then in promotion-free differential nets [13], which we call *resource nets*  
 97 in the following, by analogy with resource  $\lambda$ -calculus: these form the multilinear fragment of  
 98 differential linear logic.

99 In linear logic, Taylor expansion consists in replacing duplicable subnets, embodied by  
 100 promotion boxes, with explicit copies, as in Fig. 1: if we take  $n$  copies of the box, the  
 101 main port of the box is replaced with an  $n$ -ary ! link, while the ? links at the border of  
 102 the box collect all copies of the corresponding auxiliary ports. Again, to follow a single  
 103 cut elimination step in  $P$ , it is necessary to reduce an arbitrary number of copies. And  
 104 unrestricted parallel cut elimination in an infinite sum of resource nets is broken, as one can  
 105 easily construct an infinite family of nets, all reducing to the same resource net  $p$  in a single  
 106 step of parallel cut elimination: see Fig. 2.

### 107 1.3 Our approach: taming the combinatorial explosion of antireduction

108 The problem of convergence of series of linear approximants under reduction was first tackled  
 109 by Ehrhard and Regnier, for the normalization of Taylor expansion of ordinary  $\lambda$ -terms [14].  
 110 Their argument relies on a uniformity property, specific to the pure  $\lambda$ -calculus: the support  
 111 of the Taylor expansion of a  $\lambda$ -term forms a clique in some fixed coherence space of resource  
 112 terms. This method cannot be adapted to proof nets: there is no coherence relation on  
 113 differential nets such that all supports of Taylor expansions are cliques [22, section V.4.1].

114 An alternative method to ensure convergence without any uniformity hypothesis was first  
 115 developed by Ehrhard for typed terms in a  $\lambda$ -calculus extended with linear combinations  
 116 of terms [9]: there, the presence of sums also forbade the existence of a suitable coherence

117 relation. This method can be generalized to strongly normalizable [20], or even weakly  
 118 normalizable [23] terms. One striking feature of this approach is that it concentrates on  
 119 the support (*i.e.* the set of terms having non-zero coefficients) of the Taylor expansion. In  
 120 each case, one shows that, given a normal resource term  $t$  and a  $\lambda$ -term  $M$ , there are finitely  
 121 many terms  $s$ , such that:

- 122 ■ the coefficient of  $s$  in  $\mathcal{T}(M)$  is non zero; and
- 123 ■ the coefficient of  $t$  in the normal form of  $s$  is non zero.

124 This allows to normalize the Taylor expansion: simply normalize in each component, then  
 125 compute the sum, which is component-wise finite.

126 The second author then remarked that the same could be done for  $\beta$ -reduction [23], even  
 127 without any uniformity, typing or normalizability requirement. Indeed, writing  $s \rightrightarrows t$  if  $s$   
 128 and  $t$  are resource terms such that  $t$  appears in the support of a parallel reduct of  $s$ , the size  
 129 of  $s$  is bounded by a function of the size of  $t$  and the height of  $s$ . So, given that if  $s$  appears  
 130 in  $\mathcal{T}(M)$  then its height is bounded by that of  $M$ , it follows that, for a fixed resource term  $t$   
 131 there are finitely many terms  $s$  in the support of  $\mathcal{T}(M)$  such that  $s \rightrightarrows t$ : in short, parallel  
 132 reduction is always well-defined on the Taylor expansion of a  $\lambda$ -term.

133 Our purpose in the present paper is to develop a similar technique for MELL proof nets:  
 134 we show that one can bound the size of a resource net  $p$  by a function of the size of any of its  
 135 parallel reducts, and of an additional quantity on  $p$ , yet to be defined. The main challenge is  
 136 indeed to circumvent the lack of inductive structure in proof nets: in such a graphical syntax,  
 137 there is no structural notion of height.

138 We claim that a side condition on switching paths, *i.e.* paths in the sense of Danos–  
 139 Regnier’s correctness criterion [3], is an appropriate replacement. Backing this claim, there  
 140 are first some intuitions:

- 141 ■ the culprits for the unbounded loss of size in reduction are the chains of consecutive cuts,  
 142 as in Fig. 2;
- 143 ■ we want the validity of our side condition to be stable under reduction so, rather than  
 144 chains of cuts, we should consider cuts in switching paths;
- 145 ■ indeed, if  $p$  reduces to  $q$  via cut elimination, then the switching paths of  $q$  are somehow  
 146 related with those of  $p$ ;
- 147 ■ and the switching paths of a resource net in  $\mathcal{T}(P)$  are somehow related with those of  $P$ .

148 In the following, we establish this claim up to some technical restrictions, which will allow us  
 149 to simplify the exposition:

- 150 ■ we use generalized  $n$ -ary exponential links rather than separate (co)dereliction and  
 151 (co)contraction, as this allows to reduce the dynamics of resource nets to that of multi-  
 152 plicative linear logic (MLL) proof nets;<sup>1</sup>
- 153 ■ we limit our study to a *strict* fragment of linear logic, *i.e.* we do not consider multiplicative  
 154 units, nor the 0-ary exponential links — weakening and coweakening — as dealing with  
 155 them would require us to introduce much more machinery.

## 156 1.4 Outline

157 In Section 2, we first introduce proof nets formally, in the term-based syntax of Ehrhard [10].  
 158 We define the parallel cut elimination relation  $\rightrightarrows$  in this setting, that we decompose into  
 159 multiplicative reduction  $\rightrightarrows_m$  and axiom-cut reduction  $\rightrightarrows_{ax}$ . We also present the notion of

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<sup>1</sup> In other words, we adhere to a version of linear logic proof nets and resource nets which is sometimes called *nouvelle syntaxe*, although it dates back to Regnier’s PhD thesis [21]. See also the discussion in our conclusion (Section 6).

160 switching path for this syntax, and introduce the quantity that will be our main object of  
 161 study in the following: the maximum number  $\mathbf{cc}(p)$  of cuts that are crossed by any switching  
 162 path in the net  $p$ . Let us mention that typing plays absolutely no role in our approach, so  
 163 we do not even consider formulas of linear logic: we will rely only on the acyclicity of nets.

164 Section 3 is dedicated to the proof that we can bound  $\mathbf{cc}(q)$  by a function of  $\mathbf{cc}(p)$ ,  
 165 whenever  $p \Rightarrow q$ : the main case is the multiplicative reduction, as this may create new  
 166 switching paths in  $q$  that we must relate with those in  $p$ . In this task, we concentrate on the  
 167 notion of *slipknot*: a pair of residuals of a cut of  $p$  occurring in a path of  $q$ . Slipknots are  
 168 essential in understanding how switching paths are structured after cut elimination.

169 We show in Section 4 that, if  $p \Rightarrow q$  then the size of  $p$  is bounded by a function of  $\mathbf{cc}(p)$   
 170 and the size of  $q$ . Although, as explained in our introduction, this result is motivated by the  
 171 study of quantitative semantics, it is essentially a theorem about MLL.

172 We establish the applicability of our approach to the Taylor expansion of MELL proof  
 173 nets in Section 5: we show that if  $p$  is a resource net of  $\mathcal{T}(P)$ , then the length of switching  
 174 paths in  $p$  is bounded by a function of the size of  $P$  — hence so is  $\mathbf{cc}(p)$ .

175 Finally, we discuss further work in the concluding Section 6.

## 176 2 Definitions

177 We provide here the minimal definitions necessary for us to work with MLL proof nets. We  
 178 use a term-based syntax, following Ehrhard [10].

179 As stated before, let us stress the fact that the choice of MLL is not decisive for the  
 180 development of Sections 2 to 4. The reader can check that we rely on two ingredients only:

- 181 ■ the definition of switching paths;
- 182 ■ the fact that multiplicative reduction amounts to plug bijectively the premises of a  $\otimes$   
 183 link with those of  $\wp$  link.

184 The results of those sections are thus directly applicable to resource nets, thanks to our  
 185 choice of generalized exponential links: this will be done in Section 6.

### 186 2.1 Structures

187 Our nets are finite families of trees and cuts; trees are inductively defined as MLL connectives  
 188 connecting trees, where the leaves are elements of a countable set of variables  $V$ . The duality  
 189 of two conclusions of an axiom is given by an involution  $x \mapsto \bar{x}$  over this set.

190 Formally, the set  $T$  of *raw trees* (denoted by  $s, t, \text{etc.}$ ) is generated as follows:

$$191 \quad t ::= x \mid \otimes(t_1, \dots, t_n) \mid \wp(t_1, \dots, t_n)$$

192 where  $x$  ranges over a fixed countable set of variables  $V$ , endowed with a fixpoint-free  
 193 involution  $x \mapsto \bar{x}$ .

194 We also define the subtrees of a given tree  $t$ , written  $\mathbf{T}(t)$ , in the natural way : if  $t \in V$ ,  
 195 then  $\mathbf{T}(t) = \{t\}$ . If  $t = \alpha(t_1, \dots, t_n)$ , then  $\mathbf{T}(t) = \{t\} \cup \bigcup_{i \in \{1, \dots, n\}} \mathbf{T}(t_i)$ , for  $\alpha \in \{\otimes, \wp\}$ . In  
 196 particular, we write  $\mathbf{V}(t)$  for  $\mathbf{T}(t) \cap V$ . A *tree* is a raw tree  $t$  such that if  $\alpha(t_1, \dots, t_n) \in \mathbf{T}(t)$   
 197 (with  $\alpha = \otimes$  or  $\wp$ ), then the sets  $\mathbf{V}(t_i)$  for  $1 \leq i \leq n$  are pairwise disjoint: in other words,  
 198 each variable  $x$  occurs at most once in  $t$ . A tree  $t$  is *strict* if  $\{\otimes(), \wp()\} \cap \mathbf{T}(t) = \emptyset$ .

199 From now on, we will consider strict trees only, *i.e.* we rule out the multiplicative units.  
 200 This restriction will play a crucial rôle in expressing and establishing the bounds of Sections 3

201 and 4. It is possible to generalize our results in presence of units: we postpone the discussion  
202 on this subject to Section 6.<sup>2</sup>

203 A *cut* is an unordered pair  $c = \langle t|s \rangle$  of trees such that  $\mathbf{V}(t) \cap \mathbf{V}(s) = \emptyset$ , and then we set  
204  $\mathbf{T}(c) = \mathbf{T}(t) \cup \mathbf{T}(s)$ . A *reducible cut* is a cut  $\langle t|s \rangle$  such that  $t$  is a variable and  $\bar{t} \notin \mathbf{V}(s)$ , or  
205 such that we can write  $t = \otimes(t_1, \dots, t_n)$  and  $s = \wp(s_1, \dots, s_n)$ , or *vice versa*. Note that, in  
206 the absence of typing, we do not require all cuts to be reducible, as this would not be stable  
207 under cut elimination.

208 Given a set  $A$ , we denote by  $\vec{a}$  any finite family of elements of  $A$ . In general, we  
209 abusively identify  $\vec{a}$  with any enumeration  $(a_1, \dots, a_n) \in A^n$  of its elements, and write  
210  $\vec{a}, \vec{b}$  for the union of disjoint families  $\vec{a}$  and  $\vec{b}$ . If  $\vec{\gamma}$  is a family of trees or cuts, we write  
211  $\mathbf{V}(\vec{\gamma}) = \bigcup_{\gamma \in \vec{\gamma}} \mathbf{V}(\gamma)$  and  $\mathbf{T}(\vec{\gamma}) = \bigcup_{\gamma \in \vec{\gamma}} \mathbf{T}(\gamma)$ . An MLL *proof net* is a pair  $p = (\vec{c}; \vec{t})$   
212 of a finite family  $\vec{c}$  of cuts and a finite family  $\vec{t}$  of trees, such that for all cuts or trees  
213  $\gamma, \gamma' \in \vec{c}, \vec{t}$ ,  $\mathbf{V}(\gamma) \cap \mathbf{V}(\gamma') = \emptyset$ , and such that for any  $x \in \mathbf{V}(p) = \mathbf{V}(\vec{c}) \cup \mathbf{V}(\vec{t})$ , we have  
214  $\bar{x} \in \mathbf{V}(p)$  too. We then write  $\mathbf{C}(p) = \vec{c}$ .

## 215 2.2 Cut elimination

216 The *substitution*  $\gamma[t/x]$  of a tree  $t$  for a variable  $x$  in a tree (or cut, or net)  $\gamma$  is defined in  
217 the usual way. By the definition of trees, we notice that this substitution is essentially linear,  
218 since each variable  $x$  appears at most once in a tree.

219 There are two basic cut elimination steps, one for each kind of reducible cut:

- 220 ■ the elimination of a connective cut yields a family of cuts: we write  $\langle \otimes(t_1, \dots, t_n) | \wp$   
221  $(s_1, \dots, s_n) \rangle \rightarrow_m ((t_i | s_i))_{i \in \{1, \dots, n\}}$  that we extend to nets by setting  $(c, \vec{c}; \vec{t}) \rightarrow_m$   
222  $(\vec{c}', \vec{c}; \vec{t})$  whenever  $c \rightarrow_m \vec{c}'$ ;
- 223 ■ the elimination of an axiom cut generates a substitution: we write  $(\langle x|t \rangle, \vec{c}; \vec{t}) \rightarrow_{ax}$   
224  $(\vec{c}; \vec{t})[t/\bar{x}]$  whenever  $\bar{x} \notin \mathbf{V}(t)$ .

225 We are in fact interested in the simultaneous elimination of any number of reducible cuts,  
226 that we describe as follows: we write  $p \rightrightarrows p'$  if  $p = (\langle x_1|t_1 \rangle, \dots, \langle x_n|t_n \rangle, c_1, \dots, c_k, \vec{c}; \vec{t})$  and  
227  $p' = (\vec{c}'_1, \dots, \vec{c}'_k, \vec{c}; \vec{t})[t_1/\bar{x}_1] \cdots [t_n/\bar{x}_n]$ , with  $c_i \rightarrow_m \vec{c}'_i$  for  $1 \leq i \leq k$ , and  $\bar{x}_i \notin \mathbf{V}(t_j)$   
228 for  $1 \leq i \leq j \leq n$ . We moreover write  $p \rightrightarrows_m p'$  (resp.  $p \rightrightarrows_{ax} p$ ) in case  $n = 0$  (resp.  $k = 0$ ).  
229 It is a simple exercise to check that if  $p \rightrightarrows p'$  then there exists  $q$  such that  $p \rightrightarrows_m q \rightrightarrows_{ax} p'$ :  
230 the converse does not hold, though, as the elimination of connective cuts may generate new  
231 axiom cuts.

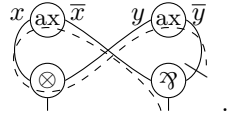
## 232 2.3 Paths

233 In order to control the effect of parallel reduction on the size of proof nets, we rely on a side  
234 condition involving the number of cuts crossed by switching paths, *i.e.* paths in the sense of  
235 Danos–Regnier’s correctness criterion [3].

236 In our setting, a *switching* of a net  $p$  is a partial map  $I : \mathbf{T}(p) \rightarrow \mathbf{T}(p)$  such that, for each  
237  $t = \wp(t_1, \dots, t_n) \in \mathbf{T}(p)$ ,  $I(t) \in \{t_1, \dots, t_n\}$ . Given a net  $p$  and a switching  $I$  of  $p$ , we define  
238 adjacency relations between the elements of  $\mathbf{T}(p)$ , written  $\sim_{t,s}$  for  $t, s \in \mathbf{T}(p)$  and  $\sim_c$  for  
239  $c \in \mathbf{C}(p)$ , as the least symmetric relations such that:

<sup>2</sup> An additional consequence is the fact that, given a (strict) tree  $t$ , any other tree  $u$  occurs at most once as a subtree of  $t$ : *e.g.*, in  $\wp^2(t_1, t_2)$ ,  $\mathbf{V}(t_1)$  and  $\mathbf{V}(t_2)$  are both non empty and disjoint, so that  $t_1 \neq t_2$ . In other words, we can identify  $\mathbf{T}(t)$  with the positions of subtrees in  $t$ , that play the rôle of vertices when considering  $t$  as a graphical structure. This will allow us to keep notations concise in our treatment of paths. This trick is of course inessential for our results.

- 240 ■ for any  $x \in \mathbf{V}(p)$ ,  $x \sim_{x,\bar{x}} \bar{x}$ ;
- 241 ■ for any  $t = \otimes(t_1, \dots, t_n) \in \mathbf{T}(p)$ ,  $t \sim_{t,t_i} t_i$  for each  $i \in \{1, \dots, n\}$ ;
- 242 ■ for any  $t = \mathfrak{A}(t_1, \dots, t_n) \in \mathbf{T}(p)$ ,  $t \sim_{t,I(t)} I(t)$ ;
- 243 ■ for any  $c = \langle t|s \rangle \in \mathbf{C}(p)$ ,  $t \sim_c s$ .
- 244 Whenever necessary, we may write, e.g.,  $\sim_{t,s}^p$  or  $\sim_{t,s}^{p,I}$  for  $\sim_{t,s}$  to make the underlying net and
- 245 switching explicit. Let  $l$  and  $m \in (\mathbf{T}(p) \times \mathbf{T}(p)) \cup \mathbf{C}(p)$  be two adjacency labels: we write
- 246  $l \equiv m$  if  $l = m$  or  $m = (x, \bar{x})$  and  $l = (\bar{x}, x)$  for some  $x \in V$ .
- 247 Given a switching  $I$  in  $p$ , an  $I$ -path is a sequence of trees  $t_0, \dots, t_n$  of  $\mathbf{T}(p)$  such that there
- 248 exists a sequence of pairwise  $\neq$  labels  $l_1, \dots, l_n$  with, for each  $i \in \{1, \dots, n\}$ ,  $t_{i-1} \sim_{l_i}^{p,I} t_i$ .<sup>3</sup>
- 249 For instance, if  $p = (; \otimes(x, y), \mathfrak{A}(\bar{y}, \bar{x}))$  and  $I(\mathfrak{A}(\bar{y}, \bar{x})) = \bar{x}$ , then the chain of adjacencies
- 250  $\mathfrak{A}(\bar{x}, \bar{y}) \sim_{\mathfrak{A}(\bar{x}, \bar{y}), \bar{x}} \bar{x} \sim_{x, \bar{x}} x \sim_{\otimes(x, y), x} \otimes(x, y) \sim_{\otimes(x, y), y} y \sim_{y, \bar{y}} \bar{y}$  defines an  $I$ -path in  $p$ , which
- 251 can be depicted as the dashed line in the following graphical representation of  $p$ :



252

253 We call *path* in  $p$  any  $I$ -path for  $I$  a switching of  $p$ , and we write  $\mathbf{P}(p)$  for the set of

254 all paths in  $p$ . We write  $t \rightsquigarrow s$  or  $t \rightsquigarrow_p s$  whenever there exists a path from  $t$  to  $s$  in  $p$ .

255 Given  $\chi = t_0, \dots, t_n \in \mathbf{P}(p)$ , we call *subpaths* of  $\chi$  the subsequences of  $\chi$ : a subpath is

256 either the empty sequence  $\epsilon$  or a path of  $p$ . We moreover write  $\bar{\chi}$  for the reverse path:

257  $\bar{\chi} = t_n, \dots, t_0 \in \mathbf{P}(p)$ . We say a net  $p$  is *acyclic* if for all  $\chi \in \mathbf{P}(p)$  and  $t \in \mathbf{T}(p)$ ,  $t$  occurs at

258 most once in  $\chi$ : in other words, there is no *cycle*  $t, \chi, t$ . From now on, we consider acyclic

259 nets only: it is well known that if  $p$  is acyclic and  $p \Rightarrow q$  then  $q$  is acyclic too.

260 If  $c = \langle t|s \rangle \in \mathbf{C}(p)$ , we may write  $\chi_1, c, \chi_2$  for either  $\chi_1, s, t, \chi_2$  or  $\chi_1, t, s, \chi_2$ : by acyclicity,

261 this notation is unambiguous, unless  $\chi_1 = \chi_2 = \epsilon$ .

262 For all  $\chi \in \mathbf{P}(p)$ , we write  $\mathbf{cc}_p(\chi)$ , or simply  $\mathbf{cc}(\chi)$ , for the number of cuts *crossed*

263 *by*  $\chi$ :  $\mathbf{cc}_p(\chi) = \#\{\langle t|s \rangle \in \mathbf{C}(p) \mid t \in \chi\}$  (recall that cuts are unordered). Observe that,

264 by acyclicity, a path  $\chi$  crosses each cut  $c = \langle t|s \rangle$  at most once: either  $\chi = \chi_1, c, \chi_2$ , or

265  $\chi = \chi_1, t, \chi_2$ , or  $\chi = \chi_1, s, \chi_2$ , with neither  $t$  nor  $s$  occurring in  $\chi_1, \chi_2$ . Finally, we write

266  $\mathbf{cc}(p) = \max\{\mathbf{cc}(\chi) \mid \chi \in \mathbf{P}(p)\}$ : in the following, we show that the maximal number of cuts

267 crossed by any switching path is a good parameter to limit the decrease in size induced by

268 parallel reduction.

### 269 3 Variations of $\mathbf{cc}(p)$ under reduction

270 Here we establish that the possible increase of  $\mathbf{cc}(p)$  under reduction is bounded. It should be

271 clear that if  $p \Rightarrow_{ax} q$  then  $\mathbf{cc}(q) \leq \mathbf{cc}(p)$ : intuitively, the only effect of  $\Rightarrow_{ax}$  is to straighten

272 some paths, thus decreasing the number of crossed cuts. In the case of connective cuts

273 however, cuts are duplicated and new paths are created.

274 Consider for instance a net  $r$ , as in Fig. 3, obtained from three nets  $p_1, p_2$  and  $q$ , by

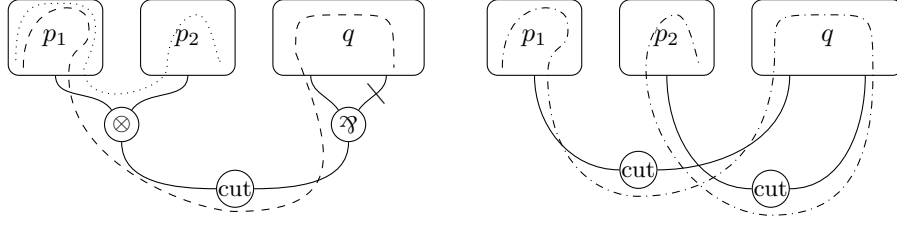
275 forming the cut  $\langle \otimes(t_1, t_2) | \mathfrak{A}(s_1, s_2) \rangle$  where  $t_1 \in \mathbf{T}(p_1)$ ,  $t_2 \in \mathbf{T}(p_2)$  and  $s_1, s_2 \in \mathbf{T}(q)$ . Observe

276 that, in the reduct  $r'$  obtained by forming two cuts  $\langle t_1 | s_1 \rangle$  and  $\langle t_2 | s_2 \rangle$ , we may very well

277 form a path that travels from  $p_1$  to  $q$  then  $p_2$ ; while in  $p$ , this is forbidden by any switching

<sup>3</sup> In standard terminology of graph theory, an  $I$ -path in  $p$  is a trail in the unoriented graph with vertices in  $\mathbf{T}(p)$  and edges given by the sum of adjacency relations defined by  $I$  (identifying  $\sim_{x,\bar{x}}$  with  $\sim_{\bar{x},x}$ ). The only purpose of our choice of labels for adjacency relations and the definition of  $\equiv$  is indeed to capture this notion of path in the unoriented graph of subtrees induced by a switching in a net.





■ **Figure 3** A cut, the resulting slipknot, and examples of paths before and after reduction

278 of  $\mathfrak{A}(s_1, s_2)$ . For instance, if we consider  $I(\mathfrak{A}(s_1, s_2)) = s_1$ , we may only form a path between  
 279  $p_1$  and  $p_2$  through  $\otimes(t_1, t_2)$ , or a path between  $q$  and one of the  $p_i$ 's, through  $s_1$  and the cut.

280 In the remainder of this section, we fix a reduction step  $p \Rightarrow_m q$ , and we show that the  
 281 previous example describes a general mechanism: if a new path is created in this step  $p \Rightarrow_m q$ ,  
 282 it must involve a path  $\xi$  between two premises of a  $\mathfrak{A}$  involved in a cut  $c$  of  $p$ , *unfolded* into  
 283 a path between the residuals of this cut. We call such an intermediate path  $\xi$  a *slipknot*.

### 284 3.1 Residual cuts and slipknots

285 Notice that  $\mathbf{T}(q) \subseteq \mathbf{T}(p)$ . Observe that, given a switching  $J$  of  $q$ , it is always possible to  
 286 extend  $J$  into a switching  $I$  of  $p$ , so that, for all  $t, s \in \mathbf{T}(q)$ :

- 287 ■ if  $t \sim_{t,s}^{q,J} s$  then  $t \sim_{t,s}^{p,I} s$ , and
- 288 ■ if  $c \in \mathbf{C}(p)$  and  $t \sim_{t,s}^{q,J} s$  then  $t \sim_{t,s}^{p,I} s$ .

289 To determine  $I$  uniquely, is remains only to select a premise for each  $\mathfrak{A}$  involved in an  
 290 eliminated cut. Consider  $c = \langle \otimes(t_1, \dots, t_n) | \mathfrak{A}(s_1, \dots, s_n) \rangle \in \mathbf{C}(p)$  and assume  $c$  is eliminated  
 291 in the reduction  $p \Rightarrow_m q$ . Then the *residuals* of  $c$  in  $q$  are the cuts  $\langle t_i | s_i \rangle \in \mathbf{C}(q)$  for  $1 \leq i \leq n$ .

292 If  $\xi \in \mathbf{P}(q)$ , a *slipknot* of  $\xi$  is any pair  $(d, d')$  of (necessarily distinct) residuals in  $q$  of a cut  
 293 in  $p$ , such that we can write  $\xi = \chi_1, d, \chi_2, d', \chi_3$ . We now show that a path in  $q$  is necessarily  
 294 obtained by alternating paths in  $p$  and paths between slipknots, that recursively consist  
 295 of such alternations. This will allow us to bound  $\mathbf{cc}(q)$  depending on  $\mathbf{cc}(p)$ , by reasoning  
 296 inductively on these paths. The main tool is the following lemma:

297 ► **Lemma 1.** *If  $\xi \in \mathbf{P}(q)$  then there exists a path  $\xi^- \in \mathbf{P}(p)$  with the same endpoints as  $\xi$ .*

298 **Proof.** Assuming  $\xi$  is a  $J$ -path of  $q$ , we construct an  $I$ -path  $\xi^-$  in  $p$  with the same endpoints  
 299 as  $\xi$  for an extension  $I$  of  $J$  as above. The definition is by induction on the number of  
 300 residuals occurring as subpaths of  $\xi$ . In the process, we must ensure that the constraints  
 301 we impose on  $I$  in each induction step can be satisfied globally: the trick is that we fix the  
 302 value of  $I(\mathfrak{A}(\vec{s}))$  only in case exactly one residual of the cut involving  $\mathfrak{A}(\vec{s})$  occurs in  $\xi$ .

303 First consider the case of  $\xi = \chi_1, d, \chi_2, d', \chi_3$ , for a slipknot  $(d, d')$ , where  $d$  and  $d'$  are  
 304 residuals of  $c \in \mathbf{C}(p)$ . We can assume, w.l.o.g. that: (i) no other residual of  $c$  occurs in  $\chi_1$ ,  
 305 nor in  $\chi_3$ ; (ii) no residual of a cut  $c' \neq c$  occurs in both  $\chi_1$  and  $\chi_3$ . By the definition of  
 306 residuals, we can write  $c = \langle \otimes(\vec{t}) | \mathfrak{A}(\vec{s}) \rangle \in \mathbf{C}(p)$ ,  $d = \langle t | s \rangle$  and  $d' = \langle t' | s' \rangle$  with  $t, t' \in \vec{t}$   
 307 and  $s, s' \in \vec{s}$ . It is then sufficient to prove that  $\xi = \chi_1, t, s, \chi_2, s', t', \chi_3$ , in which case we can  
 308 set  $\xi^- = \chi_1^-, t, \otimes(\vec{t}), t', \chi_3^-$ , where  $\chi_1^-$  and  $\chi_3^-$  are obtained from the induction hypothesis  
 309 (or by setting  $\epsilon^- = \epsilon$  for empty subpaths): by condition (ii), the constraints we impose on  $I$   
 310 by forming  $\chi_1^-$  and  $\chi_3^-$  are independent.

311 Let us rule out the other three orderings of  $d$  and  $d'$ : (a)  $\xi = \chi_1, s, t, \chi_2, t', s', \chi_3$ , (b)  
 312  $\xi = \chi_1, s, t, \chi_2, s', t', \chi_3$  or (c)  $\xi = \chi_1, t, s, \chi_2, t', s', \chi_3$ . First observe that  $\chi_2$  is not empty.

313 Indeed, if  $t \sim_l^q t'$  (or  $t \sim_l^q s'$ , or  $s \sim_l^q t'$ ) then:  $l$  cannot be a cut of  $q$  because  $\langle t|s \rangle$  and  
 314  $\langle t'|s' \rangle \in \mathbf{C}(q)$ ;  $l$  cannot be of the form  $(\alpha(t_1, \dots, t_n), t_n)$  because the trees  $t, t', s, s'$  are  
 315 pairwise disjoint; so  $l$  must be an axiom and we obtain a cycle in  $q$ .

316 Let  $u$  and  $v$  be the endpoints of  $\chi_2$ , and consider  $\chi_2^- \in \mathbf{P}(p)$  with the same endpoints,  
 317 obtained by induction hypothesis. Necessarily, we have  $t \sim_l^{q,J} u$  in cases (a) and (b),  $s \sim_l^{q,J} u$   
 318 in case (c),  $t' \sim_m^{q,J} v$  in cases (a) and (c), and  $s' \sim_m^{q,J} v$  in case (b), where  $l \neq m$ , and nor  $l$  nor  
 319  $m$  is a cut: it follows that the same adjacencies hold in  $p$  for any extension  $I$  of  $J$ . Observe  
 320 that  $\otimes(\vec{t}) \notin \chi_2^-$ : otherwise, we would obtain a path  $t \rightsquigarrow_p \otimes(\vec{t})$  (or  $\otimes(\vec{t}) \rightsquigarrow_p t'$ ) that we  
 321 could extend into a cycle. Then in case (a), we obtain a cycle in  $p$  directly:  $t, \chi_2^-, t', \otimes(\vec{t}), t$ .  
 322 In cases (b) and (c), we deduce that  $\mathfrak{A}(\vec{s}) \notin \chi_2^-$ , and we obtain a cycle, e.g. in case (b):  
 323  $t, \chi_2^-, s', \mathfrak{A}(\vec{s}), \otimes(\vec{t}), t'$ , for any  $I$  such that  $I(\mathfrak{A}(\vec{s})) = s'$ .

324 We can now assume that each cut of  $p$  has at most one residual occurring as a subpath of  
 325  $\xi$ . If no residual occurs in  $\xi$ , then we can set  $\xi^- = \xi$ . Now fix  $c = \langle \otimes(\vec{t}) | \mathfrak{A}(\vec{s}) \rangle \in \mathbf{C}(p)$  and  
 326 assume, w.l.o.g (otherwise, consider  $\bar{\xi}$ ), that  $\xi = \chi_1, t, s, \chi_2$  with  $t \in \vec{t}$  and  $s \in \vec{s}$ . Then we  
 327 set  $I(\mathfrak{A}(\vec{s})) = s$  and  $\xi^- = \chi_1^-, t, c, s, \chi_2^- \in \mathbf{P}(p)$ : this is the only case in which we impose a  
 328 value for  $I$  to construct  $\xi^-$ , so this choice, and the choices we make to form  $\chi_1^-$  and  $\chi_2^-$  are  
 329 all independent.  $\blacktriangleleft$

330 **► Lemma 2.** *If  $\xi \in \mathbf{P}(q)$  and  $c = \langle \otimes(\vec{t}) | \mathfrak{A}(\vec{s}) \rangle \in \mathbf{C}(p)$ , then at most two residuals of*  
 331  *$c$  occur as subpaths of  $\xi$ , and then we can write  $\xi = \chi_1, t, s, \chi_2, s', t', \chi_3$  with  $t, t' \in \vec{t}$  and*  
 332  *$s, s' \in \vec{s}$ .*

333 **Proof.** Assume  $\xi = \chi_1, d, \chi_2, d', \chi_3$  and  $d = \langle t|s \rangle$  and  $d' = \langle t'|s' \rangle$  with  $t, t' \in \vec{t}$  and  $s, s' \in \vec{s}$ .  
 334 Using Lemma 1, we establish that  $\xi = \chi_1, t, s, \chi_2, s', t', \chi_3$ : we can exclude the other cases  
 335 exactly as in the proof of Lemma 1. Then, as soon as three residuals of  $c$  occur in  $\xi$ , a  
 336 contradiction follows.  $\blacktriangleleft$

337 **► Lemma 3.** *Slipknots are well-bracketed in the following sense: there is no path  $\xi =$*   
 338  *$d_1, \chi_1, d_2, \chi_2, d'_1, \chi_3, d'_2 \in \mathbf{P}(q)$  such that both  $(d_1, d'_1)$  and  $(d_2, d'_2)$  are slipknots.*

339 **Proof.** Assume  $c_1 = \langle \otimes(\vec{t}_1) | \mathfrak{A}(\vec{s}_1) \rangle$ ,  $c_2 = \langle \otimes(\vec{t}_2) | \mathfrak{A}(\vec{s}_2) \rangle$ , and, for  $1 \leq i \leq 2$ ,  $d_i = (t_i, s_i)$   
 340 and  $d'_i = (t'_i, s'_i)$ , with  $t_i, t'_i \in \vec{t}_i$  and  $s_i, s'_i \in \vec{s}_i$ . By the previous lemma, we must have  
 341  $\xi = t_1, s_1, \chi_1, t_2, s_2, \chi_2, s'_1, t'_1, \chi_3, s'_2, t'_2$ . Observe that nor  $\chi_1^-$  nor  $\chi_3^-$  can cross  $c_1$  or  $c_2$ :  
 342 otherwise, we obtain a cycle in  $p$ . Then  $s_1, \chi_1^-, t_2, c_1, s'_2, \chi_3^-, t'_1, c_2, s_1$  is a cycle in  $p$ .  $\blacktriangleleft$

343 **► Corollary 4.** *Any path of  $q$  is of the form  $\zeta_1, c_1, \chi_1, c'_1, \zeta_2, \dots, \zeta_n, c_n, \chi_n, c'_n, \zeta_{n+1}$  where each*  
 344 *subpath  $\zeta_i$  is without slipknot, and each  $(c_i, c'_i)$  is a slipknot.*

345 The previous result describes precisely how paths in  $q$  are related with those in  $p$ : it will  
 346 be crucial in the following.

### 347 3.2 Bounding the growth of $\mathbf{cc}$

348 Now we show that we can bound  $\mathbf{cc}(q)$  depending only on  $\mathbf{cc}(p)$ . For each  $\xi \in \mathbf{P}(q)$ , we  
 349 define the *width*  $w_p(\xi)$  (or just  $w(\xi)$ ):  $w_p(\xi) = \max\{\mathbf{cc}_p(\chi^-) | \chi \text{ subpath of } \xi\}$ . We have:

350 **► Lemma 5.** *For any path  $\zeta \in \mathbf{P}(q)$ ,  $\mathbf{cc}_p(\zeta^-) \leq w_p(\zeta) \leq \mathbf{cc}(p)$  and  $w_p(\zeta) \leq \mathbf{cc}_q(\zeta)$ . If*  
 351 *moreover  $\zeta$  has no slipknot, then  $w_p(\zeta) = \mathbf{cc}_q(\zeta) = \mathbf{cc}_p(\zeta^-)$ .*

352 Defining  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  by  $\varphi(0) = 0$  and  $\varphi(n+1) = 2(n+1) + (n+1)(\varphi(n))$ , we obtain:

353 **► Lemma 6.** *If  $\xi \in \mathbf{P}(q)$  then  $\mathbf{cc}(\xi) \leq \varphi(w_p(\xi))$ .*

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354 **Proof.** The proof is by induction on  $w(\xi)$ . If  $w(\xi) = 0$ , then we can easily check that  $\mathbf{cc}(\xi) = 0$ .  
 355 Otherwise assume  $w(\xi) = n + 1$ . Then we set  $\xi = \zeta_1, c_1, \chi_1, c'_1, \zeta_2, \dots, \zeta_k, c_k, \chi_k, c'_k, \zeta_{k+1}$  as in  
 356 Corollary 4.

357 First observe that for all  $i \in \{1, \dots, k\}$ ,  $w(\chi_i) \leq w(\xi) - 1$ . Indeed,  $c_i, \chi_i$  is a subpath  
 358 of  $\xi$  and  $w(c_i, \chi_i) = w(\chi_i) + 1$  by the definition of width. So, by induction hypothesis,  
 359  $\mathbf{cc}(\chi_i) \leq \varphi(n)$ . We also have that  $\sum_{i=1}^{k+1} \mathbf{cc}(\zeta_i) \leq w(\xi) - k$ . Observe indeed that  $\mathbf{cc}(\xi^-) =$   
 360  $\sum_{i=1}^{k+1} \mathbf{cc}(\zeta_i) + k$ , because of Lemma 5 applied to  $\zeta_i$ , and because of the construction of  $\xi^-$   
 361 that contracts the slipknots  $c_i, \chi_i, c'_i$ ; also recall that  $\mathbf{cc}(\xi^-) \leq w(\xi)$ .

362 We obtain:

$$363 \quad \mathbf{cc}(\xi) = \sum_{1 \leq i \leq k} \mathbf{cc}(\chi_i) + \sum_{1 \leq j \leq k+1} \mathbf{cc}(\zeta_j) + 2k \leq k\varphi(n) + w(\xi) - k + 2k$$

364 and, since  $k \leq \mathbf{cc}(\xi^-) \leq w(\xi) = n+1$ , we obtain  $\mathbf{cc}(\xi) \leq (n+1)\varphi(n) + 2(n+1) = \varphi(n+1)$ . ◀

365 Using Lemma 5 again, we obtain:

366 ▶ **Corollary 7.** *Let  $p \rightrightarrows_m q$ . Then,  $\mathbf{cc}(q) \leq \varphi(\mathbf{cc}(p))$ .*

367 ▶ **Remark.** It is in fact possible to show that  $\mathbf{cc}(q) \leq 2n!\mathbf{cc}(p)$ , which is a better bound and  
 368 closer to the graphical intuition, but the proof is much longer, and we are only interested in  
 369 the existence of a bound.

### 370 4 Bounding the size of antireducts

371 For any tree, cut or net  $\gamma$ , we define the *size* of  $\gamma$  as  $\#\gamma = \mathbf{card}(\mathbf{T}(\gamma))$ : graphically,  $\#p$  is  
 372 nothing but the number of wires in  $p$ . In this section, we show that the loss of size during  
 373 parallel reduction is directly controlled by  $\mathbf{cc}(p)$  and  $\#q$ : more precisely, we show that the  
 374 ratio  $\frac{\#p}{\#q}$  is bounded by a function of  $\mathbf{cc}(p)$ .

375 First observe that the elimination of multiplicative cuts cannot decrease the size by more  
 376 than a half:

377 ▶ **Lemma 8.** *If  $p \rightrightarrows_m q$  then  $\#p \leq 2\#q$ .*

378 **Proof.** It is sufficient to observe that if  $c \rightarrow_m \vec{c}$  then  $\#c = 2 + \#\vec{c} \leq 2\#\vec{c}$ .<sup>4</sup> ◀

#### 379 4.1 Elimination of axiom cuts

380 Observe that:

- 381 ■ if  $x \in \mathbf{V}(\gamma)$  then  $\#\gamma[t/x] = \#\gamma + \#t - 1$ ;
- 382 ■ if  $x \notin \mathbf{V}(\gamma)$  then  $\#\gamma[t/x] = \#\gamma$ .

383 It follows that, in the elimination of a single axiom cut  $p \rightarrow_{ax} q$ , we have  $\#p = \#q + 1$ . But  
 384 we cannot reproduce the proof of Lemma 8 for  $\rightrightarrows_{ax}$ : as stated in our introduction, chains of  
 385 axiom cuts reducing into a single wire are the source of the collapse of size. We can bound  
 386 the length of those chains by  $\mathbf{cc}(p)$ , however, and this allows us to bound the loss of size  
 387 during reduction.

388 ▶ **Lemma 9.** *If  $p \rightrightarrows_{ax} q$  then  $\#p \leq (2\mathbf{cc}(p) + 1)\#q$ .*

<sup>4</sup> This is due to the fact that all the trees are strict, so  $\vec{c}$  is not empty and  $\#\vec{c} \geq 1$ . Without the strictness condition, we would have to deal with annihilating reductions  $\langle \otimes() | \wp() \rangle \rightarrow_m \epsilon$ : this will be discussed in the conclusion.

389 **Proof.** Assume  $p = (\langle x_1|t_1 \rangle, \dots, \langle x_n|t_n \rangle, \vec{c}; \vec{s})$  and  $q = (\vec{c}; \vec{s})[t_1/\bar{x}_1] \cdots [t_n/\bar{x}_n]$  with  $\bar{x}_i \notin$   
 390  $\mathbf{V}(t_j)$  for  $1 \leq i \leq j \leq n$ . In case  $\mathbf{cc}(p) = 0$ , we have  $n = 0$  and  $p = q$  so the result is  
 391 obvious. We thus assume  $\mathbf{cc}(p) > 0$ : to establish the result in this case, we make the chains  
 392 of eliminated axiom cuts explicit.

393 Due to the condition on free variables, there exists a (necessarily unique) permutation of  
 394  $\langle x_1|t_1 \rangle, \dots, \langle x_n|t_n \rangle$  yielding a family of the form  $\vec{c}_1, \dots, \vec{c}_k$  such that:

- 395 ■ for  $1 \leq i \leq k$ , we can write  $\vec{c}_i = \langle x_0^i|\bar{x}_1^i \rangle, \dots, \langle x_{n_i-1}^i|\bar{x}_{n_i}^i \rangle, \langle x_{n_i}^i|t^i \rangle$ ;
- 396 ■ each  $\vec{c}_i$  is maximal with this shape, *i.e.*  $\bar{x}_0^i \notin \{x_1, \dots, x_n, t_1, \dots, t_n\}$  and, in case  $t^i$  is a  
 397 variable,  $\bar{t}^i \notin \{x_1, \dots, x_n, t_1, \dots, t_n\}$ ;
- 398 ■ if  $i < j$ , then the cut  $\langle x_{n_i}^i|t^i \rangle$  occurs before  $\langle x_{n_j}^j|t^j \rangle$  in  $\langle x_1|t_1 \rangle, \dots, \langle x_n|t_n \rangle$ .

399 It follows that if  $\bar{x}_0^i \in \mathbf{V}(t_j)$  then  $j < i$ , and then  $q = (\vec{c}; \vec{s})[t^1/\bar{x}_0^1] \cdots [t^k/\bar{x}_0^k]$ , by applying  
 400 the same permutation to the substitutions as we did to cuts: we can do so because, by a  
 401 standard argument, if  $x \neq y$ ,  $x \notin \mathbf{V}(u)$  and  $y \notin \mathbf{V}(u)$  then  $\gamma[u/x][v/y] = \gamma[v/y][u/x]$ .

402 For  $1 \leq i \leq k$ , since  $\vec{c}_i$  is a chain of  $n_i + 1$  cuts, it follows that  $n_i \leq \mathbf{cc}(p) - 1$ . So  
 403  $\#p = \#\vec{c} + \#\vec{s} + \sum_{i=1}^k (\#t^i + 2n_i + 1) \leq \#\vec{c} + \#\vec{s} + \sum_{i=1}^k \#t^i + k(2\mathbf{cc}(p) - 1)$ . Moreover  
 404  $\#q = \#\vec{c} + \#\vec{s} + \sum_{i=1}^k \#t^i - k$ . It follows that  $\#p \leq \#q + 2k\mathbf{cc}(p)$  and, to conclude, it  
 405 will be sufficient to prove that  $\#q \geq k$ .

406 For  $1 \leq i \leq k$ , let  $A_i = \{j > i \mid \bar{x}_0^j \in \mathbf{V}(t^i)\}$ , and then let  $A_0 = \{i \mid \bar{x}_0^i \in \mathbf{V}(\vec{c}, \vec{s})\}$ . It fol-  
 407 lows from the construction that  $\{A_0, \dots, A_{k-1}\}$  is a partition (possibly including empty sets)  
 408 of  $\{1, \dots, k\}$ . By construction,  $\#t^i > \mathbf{card}(A_i)$ . Now consider  $q_i = (\vec{c}; \vec{s})[t^1/\bar{x}_0^1] \cdots [t^i/\bar{x}_0^i]$   
 409 for  $0 \leq i \leq k$  so that  $q = q_k$ . For  $1 \leq i \leq k$ , we obtain  $\#q_i = \#q_{i-1} + \#t^i - 1 \geq$   
 410  $\#q_{i-1} + \mathbf{card}(A_i)$ . Also observe that  $\#q_0 = \#(\vec{c}; \vec{s}) \geq \mathbf{card}(A_i)$ . We can then conclude:  
 411  $\#q = \#q_k \geq \sum_{i=0}^k \mathbf{card}(A_i) = k$ . ◀

## 4.2 General case

413 Recall that any parallel cut elimination step  $p \Rightarrow q$  can be decomposed into a multiplicative-  
 414 then-axiom pair of reductions:  $p \Rightarrow_m q' \Rightarrow_{ax} q$ . This allows us to bound the loss of size in  
 415 the reduction  $p \Rightarrow q$ , using the previous results:

416 ▶ **Theorem 10.** *If  $p \Rightarrow q$  then  $\#p \leq 4(\varphi(\mathbf{cc}(p)) + 1)\#q$ .*

417 **Proof.** Consider first  $q'$  such that  $p \Rightarrow_m q'$  and  $q' \Rightarrow_{ax} q$ . By Lemma 8,  $\#p \leq 2\#q'$ . Lemma  
 418 9 states that  $\#q' \leq (2\mathbf{cc}(q') + 1)\#q$ . Finally, Corollary 7, entails that  $\mathbf{cc}(q') \leq \varphi(\mathbf{cc}(p))$ , and  
 419 we can conclude:  $\#p \leq 2(\varphi(\mathbf{cc}(p)) + 1)\#q \leq 4(\varphi(\mathbf{cc}(p)) + 1)\#q$ . ◀

420 ▶ **Corollary 11.** *If  $q$  is an MLL net and  $n \in \mathbb{N}$ , then  $\{p \mid p \Rightarrow q \text{ and } \mathbf{cc}(p) \leq n\}$  is finite.*

421 To be precise, due to our term syntax, the previous corollary holds only up to renaming  
 422 variables in axioms: we keep this precision implicit in the following.

423 It follows that, given an infinite linear combination of  $\sum_{i \in I} a_i.p_i$ , such that  $\{\mathbf{cc}(p_i) \mid i \in I\}$   
 424 is finite, we can always consider an arbitrary family of reductions  $p_i \Rightarrow q_i$  for  $i \in I$  and form  
 425 the sum  $\sum_{i \in I} a_i.q_i$ : this is always well defined.

## 5 Taylor expansion

427 We now show how the previous results apply to Taylor expansion. For that purpose, we must  
 428 extend our syntax to MELL proof nets. Our presentation departs from Ehrhard's [11] in our  
 429 treatment of promotion boxes: instead of introducing boxes as tree constructors labelled by  
 430 nets, with auxiliary ports as inputs, we consider box ports as 0-ary trees, that are related

431 with each other in a *box context*, associating each box with its contents. This is in accordance  
 432 with the usual presentation of promotion as a black box, and has two motivations:

- 433 ■ In Ehrhard's syntax, the promotion is not a net but an open tree, for which the trees  
 434 associated with auxiliary ports must be mentioned explicitly: this would complicate the  
 435 expression of Taylor expansion.
- 436 ■ The *nouvelle syntaxe* imposes constraints on auxiliary ports, that are easier to express  
 437 when these ports are directly represented in the syntax.

438 Then we show that if  $p$  is a resource net in the support of the Taylor expansion of an MELL  
 439 proof net  $P$ , then  $\mathbf{cc}(p)$  (and in fact the length of any path in  $p$ ) is bounded by a function of  
 440  $P$ .

441 Observe that we need only consider the support of Taylor expansion, so we do not  
 442 formalize the expansion of MELL nets into infinite linear combinations of resource nets:  
 443 rather, we introduce  $\mathcal{T}(P)$  as a set of approximants. Also, as we limit our study to *strict*  
 444 nets, we will restrict  $\mathcal{T}(P)$  to those approximants that take at least one copy of each box of  
 445  $P$ : this is enough to cover the case of weakening-free MELL.

## 446 5.1 MELL nets

447 In addition to the set of variables, we fix a denumerable set  $\mathcal{A}$  of *box ports*: we assume given  
 448 an enumeration  $\mathcal{A} = \{a_i^b \mid i, b \in \mathbb{N}\}$ . We call *principal ports* the ports  $a_0^b$  and *auxiliary ports*  
 449 the other ports. In the so-called *nouvelle syntaxe* of MELL, contractions and derelictions are  
 450 merged together in a generalized contraction cell, and auxiliary ports must be premises of  
 451 such generalized contractions.

452 We introduce the corresponding term syntax, as follows. Raw pre-trees ( $S^\circ$ ,  $T^\circ$ , *etc.*)  
 453 and raw trees ( $S$ ,  $T$ , *etc.*) are defined by mutual induction as follows:

$$454 \quad T ::= x \mid a_0^b \mid \otimes(T_1, \dots, T_n) \mid \wp(T_1, \dots, T_n) \mid ?(T_1^\circ, \dots, T_n^\circ) \quad \text{and} \quad T^\circ ::= T \mid a_{i+1}^b$$

455 requiring that each  $\otimes$ ,  $\wp$  and  $?$  is of arity at least 1. We write  $\mathbf{V}(S)$  (resp.  $\mathbf{B}(S)$ ) for the set  
 456 of variables (resp. of principal and auxiliary ports) occurring in  $S$ . A *tree* (resp. a *pre-tree*)  
 457 is a raw tree (resp. raw pre-tree) in which each variable and port occurs at most once. A *cut*  
 458 is an unordered pair of trees  $C = \langle T \mid S \rangle$  with disjoint sets of variables and ports.

459 We now define *box contexts* and *pre-nets* by mutual induction as follows. A box context  
 460  $\Theta$  is the data of a finite set  $\mathcal{B}_\Theta \subset \mathbb{N}$ , and, for each  $b \in \mathcal{B}_\Theta$ , a closed pre-net  $\Theta(b)$ , of the form  
 461  $(\Theta_b; \vec{C}_b; T_b, \vec{S}_b^\circ)$ . Then we write  $\vec{S}_b^\circ = S_{b,1}^\circ, \dots, S_{b,n_b}^\circ$ . A pre-net is a triple  $P^\circ = (\Theta; \vec{C}; \vec{S}^\circ)$   
 462 where  $\Theta$  is a box context, each variable and port occurs at most once in  $\vec{C}$ ,  $\vec{S}^\circ$ , and moreover,  
 463 if  $a_i^b \in \mathbf{B}(\vec{C}; \vec{S}^\circ)$  then  $b \in \mathcal{B}_\Theta$  and  $i \leq n_b$ . A closed pre-net is a pre-net  $P^\circ = (\Theta; \vec{C}; \vec{S}^\circ)$   
 464 such that  $x$  occurs iff  $\bar{x}$  occurs, and moreover, if  $b \in \mathcal{B}_\Theta$  then each  $a_i^b$  with  $0 \leq i \leq n_b$  occurs.  
 465 Then a *net* is a closed pre-net of the form  $P = (\Theta; \vec{C}; \vec{S})$ .

466 We write  $\mathbf{T}(\gamma)$  for the set of sub-pre-trees of a pre-tree, or cut, or pre-net  $\gamma$ : the definition  
 467 extends that for subtrees in MLL nets, moreover setting  $\mathbf{T}(a) = \{a\}$  for any  $a \in \mathcal{A}$  (so we  
 468 do not look into the content of boxes). As for MLL, we set  $\#\gamma = \mathbf{card}(\mathbf{T}(\gamma))$ . We write  
 469  $\mathbf{depth}(P^\circ)$  for the maximum level of nesting of boxes in  $P^\circ$ , *i.e.* the inductive depth in the  
 470 previous definition. Also, the size of MELL pre-nets includes that of their boxes: we set  
 471  $\mathbf{size}(P^\circ) = \#P^\circ + \sum_{b \in \mathcal{B}_\Theta} \mathbf{size}(\Theta(b))$ .

472 We extend the switching functions of MLL to  $?$  links: for each  $T = ?(T_1, \dots, T_n)$ ,  
 473  $I(T) \in \{T_1, \dots, T_n\}$ , which induces a new adjacency relation  $T \sim_{T, I(T)} I(T)$ . We also  
 474 consider adjacency relations  $\sim_b$  for  $b \in \mathcal{B}_\Theta$ , setting  $a_i^b \sim_b a_j^b$  whenever  $0 \leq i < j \leq n_b$ : w.r.t.  
 475 paths, a box behaves like an  $(n_b + 1)$ -ary axiom link and the contents is not considered.

476 We write  $\mathbf{P}(P^\circ)$  for the set of paths in  $P^\circ$ . We say a pre-net  $P^\circ$  is *acyclic* if there is no cycle  
 477 in  $\mathbf{P}(P^\circ)$  and, inductively, each  $\Theta(b)$  is acyclic. From now on, we consider acyclic pre-nets  
 478 only.

## 479 5.2 Resource nets and Taylor expansion

480 The Taylor expansion of a net  $P$  will be a set of *resource nets*: these are the same as the  
 481 multiplicative nets introduced before, except we have two new connectives  $!$  and  $?$ . Raw trees  
 482 are given as follows:

$$483 \quad t ::= x \mid \otimes(t_1, \dots, t_n) \mid \wp(t_1, \dots, t_n) \mid !(t_1, \dots, t_n) \mid ?(t_1, \dots, t_n).$$

484 Again, we will consider strict trees only: each  $\otimes$ ,  $\wp$ ,  $!$  and  $?$  is of arity at least 1. In resource  
 485 nets, we extend switchings to  $?$  links as in MELL nets, and for each  $t = ?(t_1, \dots, t_n)$ , we set  
 486  $t \sim_{t, I(t)} I(t)$ . Moreover, for each  $t = !(t_1, \dots, t_n)$ , we set  $t \sim_{t, t_i} t_i$  for  $1 \leq i \leq n$ .

487 We are now ready to introduce the expansion of MELL nets. During the construction, we  
 488 need to track the conclusions of copies of boxes, in order to collect copies of auxiliary ports  
 489 in the external  $?$  links: this is the rôle of the intermediate notion of pre-Taylor expansion.

490 **► Definition 12.** Taylor expansion is defined by induction on depth as follows. Given a  
 491 closed pre-net  $P^\circ = (\Theta; \vec{C}; \vec{S}^\circ)$ , a *pre-Taylor expansion* of  $P^\circ$  is any pair  $(p, f)$  of a resource  
 492 net  $p = (\vec{c}; \vec{t})$ , together with a function  $f : \vec{t} \rightarrow \vec{S}^\circ$  such that  $f^{-1}(T)$  is a singleton  
 493 whenever  $T \in \vec{S}^\circ$  is a tree, obtained as follows:

- 494 ■ for each  $b \in \mathcal{B}_\Theta$ , fix a number  $k_b > 0$  of copies;
- 495 ■ for  $1 \leq j \leq k_b$ , fix a pre-Taylor expansion  $(p_j^b, f_j^b)$  of  $\Theta(b)$ , and write  $p_j^b = (\vec{c}_j^b; \vec{t}_j^b, \vec{s}_j^b)$  so  
 496 that  $f_j^b(\vec{t}_j^b) = T_b$ ;
- 497 ■ up to renaming the variables of the  $p_j^b$ 's, ensure that the sets  $\mathbf{V}(p_j^b)$  are pairwise disjoint,  
 498 and also disjoint from  $\mathbf{V}(\vec{C}) \cup \mathbf{V}(\vec{S}^\circ)$ ;
- 499 ■  $(\vec{c}; \vec{t})$  is obtained from  $(\vec{C}; \vec{S}^\circ)$  by replacing each  $a_0^b$  with  $!(t_1^b, \dots, t_{k_b}^b)$  and each  $a_{i+1}^b$   
 500 with an enumeration of  $\bigcup_{j=1}^{k_b} (f_j^b)^{-1}(S_{b, i+1}^\circ)$  — thus increasing the arity of the  $?$ -connective  
 501 having  $a_{i+1}^b$  as a premise, or increasing the number of trees in  $\vec{t}$  if  $a_{i+1}^b \in \vec{S}^\circ$  — and  
 502 then concatenating  $\vec{c}_j^b$  for  $b \in \mathcal{B}_\Theta$  and  $1 \leq j \leq k_b$ ;
- 503 ■ for  $t \in \vec{t}$ , set  $f(t) = a_{i+1}^b$  if  $f_j^b(t) = S_{b, i+1}^\circ$  for some  $j$ , otherwise let  $f(t)$  be the only  
 504 pre-tree of  $\vec{S}^\circ$  such that  $t$  is obtained from  $f(t)$  by the previous substitution.

505 The *Taylor expansion*<sup>5</sup> of a net  $P$  is then  $\mathcal{T}(P) = \{p \mid (p, f) \text{ is a pre-Taylor expansion of } P\}$ .

## 506 5.3 Paths in Taylor expansion

507 In the following, we fix a pre-Taylor expansion  $(p, f)$  of  $P^\circ = (\Theta; \vec{C}; \vec{S}^\circ)$ , and we describe  
 508 the structure of paths in  $p$ . Observe that if  $t \in \mathbf{T}(p)$  then:

- 509 ■ either  $t$  is at top level, *i.e.*  $t$  is obtained from some  $T \in \mathbf{T}(P^\circ) \setminus \mathcal{A}$  by substituting box  
 510 ports with trees from resource nets, and then we say  $t$  is *outer* and write  $t^* = T$ ;
- 511 ■ or  $t$  is in a copy of a box, *i.e.*  $t \in \mathbf{T}(p_j^b)$  for some  $b \in \mathcal{B}_\Theta$  and  $1 \leq j \leq k_b$ , and then we  
 512 say  $t$  is *inner* and write  $\beta(t) = b$  and  $\iota(t) = (b, j)$ ;

<sup>5</sup> More extensive presentations of Taylor expansion of MELL nets exist in the literature, in various styles [19, 17, 6]. Our only purpose here is to introduce sufficient notations to present our analysis of the length of paths in  $\mathcal{T}(P)$  by a function of the size of  $P$ .



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513 ■ or  $t$  is a *cocontraction*, i.e.  $t = !(t_1^b, \dots, t_{k_b}^b)$  for some  $b \in \mathcal{B}_\Theta$ , and then we write  $\beta(t) = b$   
 514 and  $t = !_b$ .

515 We moreover distinguish the *boundaries*, i.e. the cocontractions of  $p$ , together with all the  
 516 elements of the families  $\vec{s}_j^b$  of Definition 12: we write  $[\!_b] = a_0^b$  and  $[s] = f(s)$  if  $s \in \vec{s}_j^b$ .

517 We say a subpath  $\xi = t_1, \dots, t_n$  of  $\chi \in \mathbf{P}(p)$  is an *inner subpath* (resp. an *outer subpath*)  
 518 if each  $t_i$  is inner (resp. outer), and  $\xi$  is a *box subpath* if each  $t_i$  is inner or a cocontraction.

519 ► **Lemma 13.** *If  $\xi = t_0, \dots, t_n$  is an inner path of  $p$  then  $\iota(t_i) = \iota(t_j)$  for all  $i$  and  $j$ . We*  
 520 *then write  $\beta(\xi) = b$  and  $\iota(\xi) = (b, j)$ .*

521 **Proof.** If  $t \sim s$  and  $t$  and  $s$  are both inner then  $\iota(t) = \iota(s)$ . ◀

522 ► **Lemma 14.** *If  $\xi$  is a box path of  $p$  then  $\xi$  is an inner path or there is  $b \in \mathcal{B}_\Theta$  such that*  
 523  *$\xi = \chi_1, !_b, \chi_2$  with  $\chi_1$  and  $\chi_2$  inner subpaths. In the latter case: if  $\chi_1 \neq \epsilon$  then  $\beta(\chi_1) = b$ ; if*  
 524  *$\chi_2 \neq \epsilon$  then  $\beta(\chi_2) = b$ ; and  $\iota(\chi_1) \neq \iota(\chi_2)$  in case both subpaths are non empty.*

525 **Proof.** If  $t \sim s$  and  $t$  and  $s$  are both inner then  $\iota(t) = \iota(s)$ ; if  $t \sim !_b$  and  $t$  is inner then  
 526  $\beta(t) = b$ ; and no other adjacency relation can hold between the elements of a box path. ◀

527 ► **Lemma 15.** *If  $\xi = t_0, \dots, t_n$  is outer then  $\xi^* = t_0^*, \dots, t_n^* \in \mathbf{P}(P^\circ)$ .*

528 **Proof.** If  $t$  and  $s$  are outer, then  $t \sim_I^p s$  iff  $t^* \sim_{I^*}^{P^\circ} s^*$ , where  $I^*$  is obtained by restricting  
 529  $I$  to outer trees and then composing with  $-^*$ . Moreover,  $-^*$  is injective. ◀

530 ► **Lemma 16.** *Assume  $\xi = \xi_0, \chi_1, \xi_1, \dots, \chi_n, \xi_n \in \mathbf{P}(p)$  where each  $\chi_i$  is a box path and each*  
 531  *$\xi_i$  is outer. Then we can write  $\chi_i = u_i, \chi'_i, v_i$  where  $u_i$  and  $v_i$  are boundaries. Moreover,*  
 532  *$\beta(\chi_i) \neq \beta(\chi_j)$  when  $i \neq j$ , and we obtain  $\xi^* = \xi_0^*, [u_1], [v_1], \xi_1^*, \dots, [u_n], [v_n], \xi_n^* \in \mathbf{P}(P^\circ)$ .*

533 **Proof.** The proof is by induction on  $n$ . If  $n = 0$ , i.e.  $\xi$  is outer, then we conclude by the  
 534 previous lemma. We can thus assume  $n > 0$ .

535 The endpoints of  $\chi_i$  are boundaries, because  $\chi_i$  is a box path and the endpoints of  $\xi_{i-1}$   
 536 and  $\xi_i$  are outer. Since each boundary is adjacent to at most one outer tree, of which it is an  
 537 immediate subtree or against which it is cut,  $\chi_i$  is not reduced to a single boundary. For  
 538  $1 \leq i \leq n$ , write  $\chi_i = (u_i, \chi'_i, v_i)$ .

539 Write  $b_i = \beta(\chi_i)$ . Observe that, up to  $-^*$ , the only new adjacency relations in  $\xi^*$  are the  
 540  $[u_i] \sim_{b_i} [v_i]$  for  $1 \leq i \leq n$ . Hence, to conclude that  $\xi^*$  is indeed a path, it will be sufficient  
 541 to prove that  $b_i \neq b_j$  when  $i \neq j$ . If  $i < j$  then, by applying the induction hypothesis, we  
 542 obtain  $\zeta = \xi_i^*, \dots, [u_{j-1}], [v_{j-1}], \xi_{j-1}^* \in \mathbf{P}(P^\circ)$ . Then, if we had  $b_i = b_j$ , we would obtain a  
 543 cycle  $[v_i], \zeta, [u_j], [v_j]$  in  $P^\circ$ , which is a contradiction. ◀

544 From Lemma 16, we can derive that  $p$  is acyclic as soon as  $P^\circ$  is. Indeed, if  $\xi$  is a cycle  
 545 in  $p$ :

- 546 ■ either there is a tree at top level in  $\xi$  and we can apply Lemma 16 to obtain a cycle in  $P^\circ$ ;
- 547 ■ or  $\xi$  is an inner path, and we proceed inductively in  $\Theta(\beta(\xi))$ .

548 Our final result is a quantitative version of this corollary: not only there is no cycle in  
 549  $p$  but the length of paths in  $p$  is bounded by a function of  $P^\circ$ . If  $\xi = t_1, \dots, t_n$ , we write  
 550  $|\xi| = n$  for the *length* of  $\xi$ .

551 ► **Theorem 17.** *If  $p \in \mathcal{T}(P^\circ)$  and  $\xi \in \mathbf{P}(p)$  then  $|\xi| \leq 2^{\text{depth}(P^\circ)} \text{size}(P^\circ)$ .*

552 **Proof.** Write  $\xi = \xi_0, \chi_1, \xi_1, \dots, \chi_n, \xi_n \in \mathbf{P}(p)$  where each  $\chi_i$  is a box path and each  $\xi_i$  is an  
 553 outer path.

554 Write  $b_i = \beta(\chi_i)$ . By Lemma 14,  $\chi_i$  is either an inner path or of the form  $\zeta_i, !_{b_i}, \zeta'_i$  with  
 555  $\zeta_i$  and  $\zeta'_i$  inner subpaths in  $b_i$ . By induction hypothesis applied to those inner subpaths, we  
 556 obtain  $|\chi_i| \leq 1 + 2 \times 2^{\mathbf{depth}(\Theta(b_i))} \mathbf{size}(\Theta(b_i))$ .

557 Let  $\xi^*$  be as in Lemma 16: we have  $|\xi^*| = 2n + \sum_{i=0}^n |\xi_i^*| \leq \#(P^\circ)$ . It follows that  
 558  $\sum_{i=0}^n |\xi_i| \leq \#(P^\circ) - 2n$ .

559 We obtain:  $|\xi| = \sum_{i=0}^n |\xi_i| + \sum_{i=1}^n |\chi_i| \leq \#(P^\circ) - 2n + \sum_{i=1}^n (1 + 2^{\mathbf{depth}(\Theta(b_i)+1)} \mathbf{size}(\Theta(b_i)))$   
 560 hence  $|\xi| \leq \#(P^\circ) + \sum_{i=1}^n 2^{\mathbf{depth}(\Theta(b_i)+1)} \mathbf{size}(\Theta(b_i))$  and, since  $\mathbf{depth}(\Theta(b_i)) < \mathbf{depth}(P^\circ)$ ,  
 561  $|\xi| \leq 2^{\mathbf{depth}(P^\circ)} (\#(P^\circ) + \sum_{i=1}^n \mathbf{size}(\Theta(b_i)))$ . We conclude recalling that  $\mathbf{size}(P^\circ) = \#(P^\circ) +$   
 562  $\sum_{b \in \mathcal{B}_\Theta} \mathbf{size}(\Theta(b))$ . ◀

563 In particular, we obtain  $\mathbf{cc}(p) \leq 2^{\mathbf{depth}(P^\circ)} \mathbf{size}(P^\circ)$ .

## 564 5.4 Cut elimination in Taylor expansion

565 In resource nets, the elimination of the cut  $\langle ?(t_1, \dots, t_n) | (s_1, \dots, s_m) \rangle$  yields the finite sum  
 566  $\sum_{\sigma: \{1, \dots, n\} \xrightarrow{\sim} \{1, \dots, m\}} \langle t_1 | s_{\sigma(1)} \rangle, \dots, \langle t_n | s_{\sigma(n)} \rangle$ . It turns out that the results of Sections 3 and 4  
 567 apply directly to resource nets: setting  $\langle ?(t_1, \dots, t_n) | (s_1, \dots, s_n) \rangle \rightarrow \langle t_1 | s_{\sigma(1)} \rangle, \dots, \langle t_n | s_{\sigma(n)} \rangle$   
 568 for each permutation  $\sigma$ , we obtain an instance of multiplicative reduction, as the order of  
 569 premises is irrelevant from a combinatorial point of view — this is all the more obvious  
 570 because no typing constraint was involved in our argument. In other words, Corollary 11  
 571 also applies to the parallel reduction of resource nets. With Theorem 17, we obtain:

572 ▶ **Corollary 18.** *If  $q$  is a resource net and  $P$  is an MELL net,  $\{p \in \mathcal{T}(P); p \Rightarrow q\}$  is finite.*

## 573 6 Conclusion

574 Recall that our original motivation was the definition of a reduction relation on infinite linear  
 575 combinations of resource nets, simulating cut elimination in MELL through Taylor expansion.  
 576 We claim that a suitable notion is as follows:

577 ▶ **Definition 19.** Write  $\sum_{i \in I} a_i p_i \Rightarrow \sum_{i \in I} a_i q_i$  as soon as:

- 578 ■ for each  $i \in I$ , the resource net  $p_i$  reduces to  $q_i$  (which may be a finite sum);
- 579 ■ for any resource net  $q$ , there are finitely many  $i \in I$  such that  $q$  is a summand of  $q_i$ .

580 In particular, if  $\sum_{i \in I} a_i p_i$  is a Taylor expansion, then Theorem 18 ensures that the second  
 581 condition of the definition of  $\Rightarrow$  is automatically valid. The details of the simulation in a  
 582 quantitative setting remain to be worked out, but the main stumbling block is now over: the  
 583 necessary equations on coefficients are well established, as they have been extensively studied  
 584 in the various denotational models; it only remained to be able to form the associated sums  
 585 directly in the syntax.

586 Let us mention that another important incentive to publish our results is the *normalization-*  
 587 *by-evaluation* programme that we develop with Guerrieri, Pellissier and Tortora de Falco  
 588 [1] — which is limited to strict nets for independent reasons. Indeed, if  $P$  is cut-free, the  
 589 elements of the semantics of  $P$  are in one-to-one correspondence with  $\mathcal{T}(P)$ . Then, given  
 590 a sequence  $P_1, \dots, P_n$  of MELL nets such that  $P_i$  reduces to  $P_{i+1}$  by cut elimination and  
 591  $P_n$  is normal, from  $p_n \in \mathcal{T}(P_n)$  we can construct a sequence  $p_1, \dots, p_{n-1}$  of resource nets,  
 592 such that each  $p_i \in \mathcal{T}(P_i)$  and  $p_i \Rightarrow p_{i+1}$ . Then our results ensure that  $\#p_1$  is bounded by a  
 593 function of  $n$ ,  $\mathbf{size}(P_1)$  and  $\#p_n$ , which is a crucial step of our construction.



594 We finish the paper by reviewing the restrictions that we imposed on our framework.  
 595 Strictness is not an essential condition for the main results to hold. It is possible to deal with  
 596 units and weakenings (0-ary  $\mathcal{A}$ ,  $\otimes$  and  $?$  nodes), and then with complete Taylor expansion,  
 597 including 0-ary developments of boxes (generating weakenings and coweakenings). In this  
 598 case, we need to introduce additional structure — jumps from weakenings, that can be part  
 599 of switching paths — and some other constraint — a bound on the number of weakenings  
 600 that can jump to a given tree. The proof is naturally longer, and the bounds much greater,  
 601 but the finiteness property still holds. We leave a formal treatment of this extension for  
 602 further work.

603 The other notable constraint is the use of the *nouvelle syntaxe*, with generalized expo-  
 604 nential links. It is also possible to deal with a standard representation, including separate  
 605 derelictions and coderelictions, with a finer grained cut elimination procedure. This introduces  
 606 additional complexity in the formalism but, by contrast with lifting the strictness condition,  
 607 it essentially requires no new concept or technique: the difficulty in parallel reduction is to  
 608 control the chains of cuts to be simultaneously eliminated, and decomposing cut elimination  
 609 into finer reduction steps can only decrease the length of such chains.

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