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Subdivisions of oriented cycles in digraphs with large chromatic number

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Abstract

An oriented cycle is an orientation of a undirected cycle. We first show that for any oriented cycle $C$, there are digraphs containing no subdivision of $C$ (as a subdigraph) and arbitrarily large chromatic number. In contrast, we show that for any $C$ a cycle with two blocks, every strongly connected digraph with sufficiently large chromatic number contains a subdivision of $C$. We prove a similar result for the antidirected cycle on four vertices (in which two vertices have out-degree 2 and two vertices have in-degree 2).

1 Introduction

What can we say about the subgraphs of a graph $G$ with large chromatic number? Of course, one way for a graph to have large chromatic number is to contain a large complete subgraph. However, if we consider graphs with large chromatic number and small clique number, then we can ask what other subgraphs must occur. We can avoid any graph $H$ that contains a cycle because, as proved by Erdős [7], there are graphs with arbitrarily high girth and chromatic number. Reciprocally, one can easily show that every $n$-chromatic graph contains every tree of order $n$ as a subgraph.

The following more general question attracted lots of attention.

Problem 1. Which are the graph classes $\mathcal{G}$ such that every graph with sufficiently large chromatic number contains an element of $\mathcal{G}$?

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If such a class is finite, then it must contain a tree, by the above-mentioned result of Erdős. If it is infinite however, it does not necessarily contains a tree. For example, every graph with chromatic number at least 3 contains an odd cycle. This was strengthened by Erdős and Hajnal [8] who proved that every graph with chromatic number at least $k$ contains an odd cycle of length at least $k$. A counterpart of this theorem for even length was settled by Mihók and Schiermeyer [16]: every graph with chromatic number at least $k$ contains an even cycle of length at least $k$. Further results on graphs with prescribed lengths of cycles have been obtained [10, 16, 20, 15, 13].

In this paper, we consider the analogous problem for directed graphs, which is in fact a generalization of the undirected one. The chromatic number $\chi(D)$ of a digraph $D$ is the chromatic number of its underlying graph. The chromatic number of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest $k$ such that $\chi(D) \leq k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such $k$ exists. By convention, if $\mathcal{D} = \emptyset$, then $\chi(\mathcal{D}) = 0$. If $\chi(\mathcal{D}) \neq +\infty$, we say that $\mathcal{D}$ has bounded chromatic number.

We are interested in the following question: which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$? Let us denote by $\text{Forb}(H)$ (resp. $\text{Forb}(\mathcal{H})$) the class of digraphs that do not contain $H$ (resp. any element of $\mathcal{H}$) as a subdigraph. The above question can be restated as follows:

**Problem 2.** Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\text{Forb}(\mathcal{D})) < +\infty$?

This is a generalization of Problem 1. Indeed, let us denote by $\text{Dig}(G)$ the set of digraphs whose underlying digraph is in $G$; Clearly, $\chi(G) = \chi(\text{Dig}(G))$.

An oriented graph is an orientation of a (simple) graph; equivalently it is a digraph with no directed cycles of length 2. Similarly, an oriented path (resp. oriented cycle, oriented tree) is an orientation of a path (resp. cycle, tree). An oriented path (resp., an oriented cycle) is said directed if all nodes have in-degree and out-degree at most 1.

Observe that if $D$ is an orientation of a graph $G$ and $\text{Forb}(D)$ has bounded chromatic number, then $\text{Forb}(G)$ has also bounded chromatic number, so $G$ must be a tree. Burr [5] proved that every $(k - 1)^2$-chromatic digraph contains an oriented tree of order $k$. This was slightly improved by Addario-Berry et al. [2] who proved the following.

**Theorem 3** (Addario-Berry et al. [2]). Every $(k^2/2 - k/2 + 1)$-chromatic oriented graph contains every oriented tree of order $k$. In other words, for every oriented tree $T$ of order $k$, $\chi(\text{Forb}(T)) \leq k^2/2 - k/2$.

**Conjecture 4** (Burr [5]). Every $(2k - 2)$-chromatic digraph $D$ contains a copy of any oriented tree $T$ of order $k$.

For special oriented trees $T$, better bounds on the chromatic number of $\text{Forb}(T)$ are known. The most famous one, known as Gallai-Roy Theorem, deals with directed paths (a directed path is an oriented path in which all arcs are in the same direction) and can be restated as follows, denoting by $P^+(k)$ the directed path of length $k$.

**Theorem 5** (Gallai [9], Hasse [11], Roy [17], Vitaver [19]). $\chi(\text{Forb}(P^+(k))) = k$. 

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The chromatic number of the class of digraphs not containing a prescribed oriented path with two blocks (blocks are maximal directed subpaths) has been determined by Addario-Berry et al. [1].

**Theorem 6** (Addario-Berry et al. [1]). Let $P$ be an oriented path with two blocks on $k$ vertices.

- If $k = 3$, then $\chi(\text{Forb}(P)) = 3$.
- If $k \geq 4$, then $\chi(\text{Forb}(P)) = k - 1$.

In this paper, we are interested in the chromatic number of $\text{Forb}(H)$ when $H$ is an infinite family of oriented cycles. Let us denote by $\text{S-Forb}(D)$ (resp. $\text{S-Forb}(\mathcal{D})$) the class of digraphs that contain no subdivision of $D$ (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of $\text{S-Forb}(C)$, where $C$ is a family of oriented cycles.

Let us denote by $\vec{C}_k$ the directed cycle of length $k$. For all $k$, $\chi(\text{S-Forb}(\vec{C}_k)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k, \ell)$ the oriented cycle with two blocks, one of length $k$ and the other of length $\ell$. Observe that the oriented cycles with two blocks are the subdivisions of $C(1, 1)$. As pointed out by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k, \ell))) = +\infty$. In fact, the following construction, communicated to us by J. Nešetřil 1, generalises this result to any number of blocks.

**Theorem 7.** For any positive integers $b, c$, there exists an acyclic digraph $D$ with $\chi(D) \geq c$ in which all oriented cycles have more than $b$ blocks.

**Proof.** By [7], there exist graphs with chromatic number $c$ and girth greater than $cb$. Let $G$ be such a graph and consider a proper $c$-colouring $\phi$ of it. Let $D$ be the acyclic orientation of $G$ in which an edge $uv$ of $G$ is oriented from $u$ to $v$ if and only if $\phi(u) < \phi(v)$. By construction, the length of all directed paths in $D$ is less than $c$ and since each cycle of $D$ has length more than $cb$, they all have more than $b$ blocks. \qed

This directly implies the following theorem.

**Theorem 8.** For any finite family $C$ of oriented cycles,

$$\chi(\text{S-Forb}(C)) = +\infty.$$ 

In contrast, if $C$ is an infinite family of oriented cycles, $\text{S-Forb}(C)$ may have bounded chromatic number. By the above argument, such a family must contain a cycle with at least $b$ blocks for every positive integer $b$. A cycle $C$ is antidirected if any vertex of $C$ has either in-degree 2 or out-degree 2 in $C$. In other words, it is an oriented cycle in which all blocks have length 1. Let us denote by $\mathcal{A}_{\geq 2k}$ the family of antidirected cycles of length at least $2k$. In Theorem 13, we prove that $\chi(\text{Forb}(\mathcal{A}_{\geq 2k})) \leq 8k - 8$. Hence we are left with the following questions.

**Problem 9.** What are the infinite families of oriented cycles $C$ such that $\text{Forb}(C) < +\infty$? What are the infinite families of oriented cycles $C$ such that $\text{S-Forb}(C) < +\infty$?

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1An earlier version of this manuscript contained a much more complicated proof of this result.

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On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different results. An example is provided by the following celebrated result due to Bondy [3]: every strong digraph of chromatic number at least \( k \) contains a directed cycle of length at least \( k \). Denoting the class of strong digraphs by \( S \), this result can be rephrased as follows.

**Theorem 10** (Bondy [3]). \( \chi(S-Forb(\bar{C}_k) \cap S) = k - 1 \).

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

**Problem 11.** Let \( k \) and \( \ell \) be two positive integers. Does \( S-Forb(C(k, \ell) \cap S) \) have bounded chromatic number?

In Subsection 4.2, we answer this problem in the affirmative. In Theorem 21 we prove

\[
\chi(S-Forb(C(k, \ell) \cap S)) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1), \text{ for all } k \geq \ell \geq 2, k \geq 3. \tag{1}
\]

Note that since \( \chi(S-Forb(C(k', \ell') \cap S)) \leq \chi(S-Forb(C(k, \ell) \cap S)) \) if \( k' \leq k \) and \( \ell' \leq \ell \), this gives also an upper bound when \( k \) or \( \ell \) are small.

The bound given in Equation (1) is certainly not tight.\(^2\) In Subsection 4.3 and Section 5, we establish better upper bounds in some particular cases. In Corollary 26, we prove

\[
\chi(S-Forb(C(k, 1)) \cap S) \leq \max\{k + 1, 2k - 4\} \text{ for all } k.
\]

We also give in Subsection 4.2 the exact value of \( S-Forb(C(k, \ell) \cap S) \) for \( (k, \ell) \in \{(1, 2), (2, 2), (1, 3), (2, 3)\} \).

More generally, one may wonder what happens for other oriented cycles.

**Problem 12.** Let \( C \) be an oriented cycle with at least four blocks. Is \( \chi(S-Forb(C) \cap S) \) bounded?

In Section 6, we show that \( \chi(S-Forb(\hat{C}_4) \cap S) \leq 24 \) where \( \hat{C}_4 \) is the antidirected cycle of order 4.

## 2 Definitions

We follow [4] for basic notions and notations. Let \( D \) be a digraph. \( V(D) \) denotes its vertex-set and \( A(D) \) its arc-set.

If \( uv \in A(D) \) is an arc, we sometimes write \( u \to v \) or \( v \leftarrow u \).

For any \( v \in V(D) \), \( d^+(v) \) (resp. \( d^-(v) \)) denotes the out-degree (resp. in-degree) of \( v \). \( \delta^+(D) \) (resp. \( \delta^-(D) \)) denotes the minimum out-degree (resp. in-degree) of \( D \).

An **oriented path** is any orientation of a path. The **length** of a path is the number of its arcs. Let \( P = (v_1, \ldots, v_n) \) be an oriented path. If \( v_iv_{i+1} \in A(D) \), then \( v_iv_{i+1} \) is a **forward arc** and \( v_{i+1}v_i \) is a **backward arc**. \( P \) is a **directed path** if either all of its arcs are forward ones or all of its arcs are backward ones. For convenience, a directed path with forward arcs only is called a **dipath**.

\(^2\)While this paper was under review, Kim et al. [14] improved on (1) by showing \( \chi(S-Forb(C(k, \ell) \cap S)) \leq 2(2k - 3)(k + 2\ell - 1) \). This bound is however certainly not tight either.
A block of $P$ is a maximal directed subpath of $P$. A path is entirely determined by the sequence $(b_1, \ldots, b_p)$ of the lengths of its blocks and the sign $+$ or $-$ indicating if the first arc is forward or backward respectively. Therefore we denote by $P^+(b_1, \ldots, b_p)$ (resp. $P^-(b_1, \ldots, b_p)$) an oriented path whose first arc is forward (resp. backward) with $p$ blocks, such that the $i$th block along it has length $b_i$.

Let $P = (x_1, x_2, \ldots, x_n)$ be an oriented path. We say that $P$ is an $(x_1, x_n)$-path. For every $1 \leq i \leq j \leq n$, we note $P[x_i, x_j]$ (resp. $P[x_i, x_j]$, $P[x_i, x_j]$, $P[x_i, x_j]$) the oriented subpath $(x_i, \ldots, x_j)$ (resp. $(x_{i+1}, \ldots, x_{j-1})$, $(x_i, \ldots, x_{j-1})$, $(x_{i+1}, \ldots, x_j)$).

The vertex $x_1$ is the initial vertex of $P$ and $x_n$ its terminal vertex. Let $P_1$ be an $(x_1, x_2)$-dipath and $P_2$ an $(x_2, x_3)$-dipath which are disjoint except in $x_2$. Then $P_1 \circ P_2$ denotes the $(x_1, x_3)$-dipath obtained from the concatenation of these dipaths.

The above definitions and notations can also be used for oriented cycles. Since a cycle has no initial and terminal vertex, we have to choose one as well as a direction to run through the cycle. Therefore if $C = (x_1, x_2, \ldots, x_n, x_1)$ is an oriented cycle, we always assume that $x_1x_2$ is an arc, and if $C$ is not directed that $x_1x_n$ is also an arc.

A path or a cycle (not necessarily directed) is Hamiltonian in a digraph if it goes through all vertices of $D$.

The digraph $D$ is connected (resp. $k$-connected) if its underlying graph is connected (resp. $k$-connected). It is strongly connected, or strong, if for any two vertices $u, v$, there is a $(u, v)$-dipath in $D$. It is $k$-strongly connected or $k$-strong, if for any set $S$ of $k - 1$ vertices $D - S$ is strong. A strong component of a digraph is an inclusionwise maximal strong subdigraph. Similarly, a $k$-connected component of a digraph is an inclusionwise maximal $k$-connected subdigraph.

## 3 Antidirected cycles

The aim of this section is to prove the following theorem, that establish that $\chi(\text{Forb} (\mathcal{A}_{\geq 2k})) \leq 8k - 8$.

**Theorem 13.** Let $D$ be an oriented graph and $k$ an integer greater than 1. If $\chi(D) \geq 8k - 7$, then $D$ contains an antidirected cycle of length at least $2k$.

A graph $G$ is $k$-critical if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph $H$ of $G$. Every graph with chromatic number $k$ contains a $k$-critical graph. We denote by $\delta(G)$ the minimum degree of the graph $G$. The following easy result is well-known.

**Proposition 14.** If $G$ is a $k$-critical graph, then $\delta(G) \geq k - 1$.

Let $(A, B)$ be a bipartition of the vertex set of a digraph $D$. We denote by $E(A, B)$ the set of arcs with tail in $A$ and head in $B$ and by $\epsilon(A, B)$ its cardinality.

**Lemma 15** (Burr [6]). Every digraph $D$ contains a partition $(A, B)$ such that $\epsilon(A, B) \geq |E(D)|/4$.

**Lemma 16** (Burr [6]). Let $G$ be a bipartite graph and $p$ be an integer. If $|E(G)| \geq p|V(G)|$, then $G$ has a subgraph with minimum degree at least $p + 1$. 

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Lemma 17. Let $k \geq 1$ be an integer. Every bipartite graph with minimum degree $k$ contains a cycle of order at least $2k$.

Proof. Let $G$ be a bipartite graph with bipartition $(A, B)$. Consider a longest path $P$ in $G$. Without loss of generality, we may assume that one of its ends $a$ is in $A$. All neighbours of $a$ are in $P$ (otherwise $P$ can be lengthened). Let $b$ be the furthest neighbour of $a$ in $B$ along $P$. Then $C = P[a, b] \cup ab$ is a cycle containing at least $k$ vertices in $B$, namely the neighbours of $a$. Hence $C$ has length at least $2k$, since $G$ is bipartite. □

Proof of Theorem 13. It suffices to prove that every $(8k-7)$-critical oriented graph contains an antidirected cycle of length at least $2k$.

Let $D$ be a $(8k-7)$-critical oriented graph. By Proposition 14, it has minimum degree at least $8k-8$, so $|E(D)| \geq (4k-4)|V(D)|$. By Lemma 15, $D$ contains a partition such that $e(A, B) \geq |E(D)|/4 \geq (k-1)|V(D)|$. Consequently, by Lemma 16, there are two sets $A' \subseteq A$ and $B' \subseteq B$ such that every vertex in $A'$ (resp. $B'$) has at least $k$ out-neighbours in $B'$ (resp. $k$ in-neighbours in $A'$). Therefore, by Lemma 17, the bipartite oriented graph induced by $E(A', B')$ contains a cycle of length at least $2k$, which is necessarily antidirected. □

Problem 18. Let $\ell$ be an even integer. What is the minimum integer $a(\ell)$ such that every oriented graph with chromatic number at least $a(\ell)$ contains an antidirected cycle of length at least $\ell$?

4 Cycles with two blocks in strong digraphs

In this section we first prove that $S$-Forb($C(k, \ell)$) $\cap$ $S$ has bounded chromatic number for every $k, \ell$. We need some preliminaries.

4.1 Definitions and tools

4.1.1 Levelling

In a digraph $D$, the distance from a vertex $x$ to another $y$, denoted by dist$_D(x, y)$ or simply dist$(x, y)$ when $D$ is clear from the context, is the minimum length of an $(x, y)$-dipath or $+\infty$ if no such dipath exists. For a set $X \subseteq V(D)$ and vertex $y \in V(D)$, we define dist$(X, y) = \min\{\text{dist}(x, y) \mid x \in X\}$ and dist$(y, X) = \min\{\text{dist}(y, x) \mid x \in X\}$, and for two sets $X, Y \subseteq V(D)$, dist$(X, Y) = \min\{\text{dist}(x, y) \mid x \in X, y \in Y\}$.

An out-generator in a digraph $D$ is a vertex $u$ such that for any $x \in V(D)$, there is an $(u, x)$-dipath. Observe that in a strong digraph every vertex is an out-generator.

Let $u$ be an out-generator of $D$. For every nonnegative integer $i$, the $i$th level from $u$ in $D$ is $L_i^u = \{v \mid \text{dist}_D(u, v) = i\}$. Because $u$ is an out-generator, $\bigcup_i L_i^u = V(D)$. Let $v$ be a vertex of $D$, we set lvl$^u(v) = \text{dist}_D(u, v)$, hence $v \in L_{\text{lvl}(v)}^u$. In the following, the vertex $u$ is always clear from the context. Therefore, for sake of clarity, we drop the superscript $u$.

The definition immediately implies the following.

Proposition 19. Let $D$ be a digraph having an out-generator $u$. If $x$ and $y$ are two vertices of $D$ with lvl$(y) > lvl(x)$, then every $(x, y)$-dipath has length at least lvl$(y) - lvl(x)$. 
Let $D$ be a digraph and $u$ be an out-generator of $D$. A Breadth-First-Search Tree or BFS-tree $T$ with root $u$, is a subdigraph of $D$ spanning $V(D)$ such that $T$ is an oriented tree and, for any $v \in V(D)$, $\text{dist}_T(u,v) = \text{dist}_D(u,v)$. It is well-known that if $u$ is an out-generator of $D$, then there exist BFS-trees with root $u$.

Let $T$ be a BFS-tree with root $u$. For any vertex $x$ of $D$, there is an unique $(u,x)$-dipath in $T$. The ancestors of $x$ are the vertices on this dipath. For an ancestor $y$ of $x$, we note $y \geq_T x$. If $y$ is an ancestor of $x$, we denote by $T[x,y]$ the unique $(y,x)$-dipath in $T$. For any two vertices $v_1$ and $v_2$, the least common ancestor of $v_1$ and $v_2$ is the common ancestor $x$ of $v_1$ and $v_2$ for which $\text{lvl}(x)$ is maximal. (This is well-defined since $u$ is an ancestor of all vertices.)

### 4.1.2 Decomposing a digraph

The union of two digraphs $D_1$ and $D_2$ is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$. Note that $V(D_1)$ and $V(D_2)$ are not necessarily disjoint.

The following lemma is well-known.

**Lemma 20.** Let $D_1$ and $D_2$ be two digraphs. $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.

**Proof.** Let $D = D_1 \cup D_2$. For $i \in \{1, 2\}$, let $c_i$ be a proper colouring of $D_i$ with $\{1, \ldots, \chi(D_i)\}$. Extend $c_i$ to $(V(D), A(D_i))$ by assigning the colour 1 to all vertices in $V_3-i$. Now the function $c$ defined by $c(v) = (c_1(v), c_2(v))$ for all $v \in V(D)$ is a proper colouring of $D$ with colour set $\{1, \ldots, \chi(D_1)\} \times \{1, \ldots, \chi(D_2)\}$. \hfill $\square$

### 4.2 General upper bound

**Theorem 21.** Let $k$ and $\ell$ be two positive integers such that $k \geq \max\{\ell, 3\}$ and $\ell \geq 2$, and let $D$ be a digraph in $S$-$\text{Forb}(C(k, \ell)) \cap S$. Then, $\chi(D) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1)$.

**Proof.** Since $D$ is strongly connected, it has an out-generator $u$. Let $T$ be a BFS-tree with root $u$. We define the following sets of arcs.

\[
A_0 = \{xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\
A_1 = \{xy \in A(D) \mid 0 < |\text{lvl}(x) - \text{lvl}(y)| < k + \ell - 3\}; \\
A' = \{xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3\}.
\]

Since $k + \ell - 3 > 0$ and there is no arc $xy$ with $\text{lvl}(y) > \text{lvl}(x) + 1$, $(A_0, A_1, A')$ is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition $A'$ into two sets $A_2$ and $A_3$, where $A_2 = \{xy \in A' \mid y$ is an ancestor of $x$ in $T\}$ and $A_3 = A' \setminus A_2$. Then $(A_0, A_1, A_2, A_3)$ is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0, 1, 2, 3\}$.

**Claim 21.1.** $\chi(D_0) \leq k + \ell - 2$.

**Subproof.** Observe that $D_0$ is the disjoint union of the $D[L_i]$ where $L_i = \{v \mid \text{dist}_D(u,v) = i\}$. Therefore it suffices to prove that $\chi(D[L_i]) \leq k + \ell - 2$ for all non-negative integer $i$.

$L_0 = \{u\}$ so the result holds trivially for $i = 0$.  

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Assume now \( i \geq 1 \). Suppose for a contradiction \( \chi(D[L_i]) \geq k + \ell - 1 \). Since \( k \geq 3 \), by Theorem 6, \( D[L_i] \) contains a copy \( Q \) of \( P^+(k-1, \ell-1) \). Let \( v_1 \) and \( v_2 \) be the initial and terminal vertices of \( Q \), and let \( x \) be the least common ancestor of \( v_1 \) and \( v_2 \). By definition, for \( j \in \{1, 2\} \), there exists a \((x, v_j)\)-dipath \( P_j \) in \( T \). By definition of least common ancestor, \( V(P_1) \cap V(P_2) = \{x\} \), \( V(P_j) \cap L_i = \{v_j\} \), \( j = 1, 2 \), and both \( P_1 \) and \( P_2 \) have length at least 1. Consequently, \( P_1 \cup P_2 \cup Q \) is a subdivision of \( C(k, \ell) \), a contradiction. \( \diamond \)

**Claim 21.2.** \( \chi(D_1) \leq k + \ell - 3 \).

*Subproof.* Let \( \phi_1 \) be the colouring of \( D_1 \) defined by \( \phi_1(x) = \text{lvl}(x) \mod k + \ell - 3 \). By definition of \( D_1 \), this is clearly a proper colouring of \( D_1 \). \( \diamond \)

**Claim 21.3.** \( \chi(D_2) \leq 2\ell + 2 \).

*Subproof.* Suppose for a contradiction that \( \chi(D_2) \geq 2\ell + 3 \). By Theorem 6, \( D_2 \) contains a copy \( Q \) of \( P^-(-1, \ell + 1) \), which is the union of two disjoint dipaths which are disjoint except in there initial vertex \( y \), say \( Q_1 = (y_0, y_1, y_2, \ldots, y_{\ell+1}) \) and \( Q_2 = (z_0, z_1, z_2, \ldots, z_{\ell+1}) \) with \( y_0 = z_0 = y \). Since \( Q \) is in \( D_2 \), all vertices of \( Q \) belong to \( T[u, y] \). Without loss of generality, we can assume \( z_1 \geq_T y_1 \).

If \( z_{\ell+1} \geq_T y_{\ell+1} \), then let \( j \) be the smallest integer such that \( z_j \geq_T y_{\ell+1} \). Then the union of \( T[y_1, y] \circ Q_1[y, z_j] \circ T[z_j, y_{\ell+1}] \) and \( Q_1[y_1, y_{\ell+1}] \) is a subdivision of \( C(k, \ell) \), because \( T[y_1, y] \) has length at least \( k - 2 \) as \( \text{lvl}(y) \geq k + \ell - 3 \). This is a contradiction.

Henceforth \( y_{\ell+1} \geq_T z_{\ell+1} \). Observe that all the \( z_j \), \( 1 \leq j \leq \ell + 1 \) are in \( T[y_{\ell+1}, y_1] \). Thus, by the Pigeonhole principle, there exists \( i, j \geq 1 \) such that \( y_{i+1} \geq_T z_{j+1} \geq_T z_j \geq_T y_i \geq_T z_{j-1} \).

If \( \text{lvl}(z_{j-1}) \geq \text{lvl}(y_i) + \ell - 1 \), then \( T[y_i, z_{j-1}] \circ (z_{j-1}, z_j) \) has length at least \( \ell \). Hence its union with \( (y_i, y_{i+1}) \circ T[y_{i+1}, z_j] \), which has length greater than \( k \), is a subdivision of \( C(k, \ell) \), a contradiction.

Thus \( \text{lvl}(z_{j-1}) < \text{lvl}(y_i) + \ell - 1 \) (in particular, in this case, \( j > 1 \) and \( i > 2 \)). Therefore, by definition of \( A' \), \( \text{lvl}(y_i) \geq \text{lvl}(z_j) + k - 1 \) and \( \text{lvl}(y_{i+1}) \geq \text{lvl}(z_{j-1}) + k - 1 \). Hence both \( T[z_{j-1}, y_{i+1}] \) and \( T[z_j, y_i] \) have length at least \( k - 1 \). So the union of \( T[z_{j-1}, y_{i+1}] \circ (y_{i+1}, y_i) \) and \( (z_{j-1}, z_j) \circ T[z_j, y_i] \) is a subdivision of \( C(k, k) \) (and thus of \( C(k, \ell) \)), a contradiction. \( \diamond \)

**Claim 21.4.** \( \chi(D_3) \leq k + \ell + 1 \).

*Subproof.* In this claim, it is important to note that \( k + \ell - 3 \geq k - 1 \) because \( \ell \geq 2 \). We use the fact that \( \text{lvl}(x) - \text{lvl}(y) \geq k - 1 \) if \( xy \) is an edge in \( A_3 \).

Suppose for a contradiction that \( \chi(D_3) \geq k + \ell + 1 \). By Theorem 6, \( D_3 \) contains a copy \( Q \) of \( P^-(k, \ell) \) which is the union of two disjoint dipaths which are disjoint except in there initial vertex \( y \), say \( Q_1 = (y_0, y_1, y_2, \ldots, y_k) \) and \( Q_2 = (z_0, z_1, z_2, \ldots, z_\ell) \) with \( y_0 = z_0 = y \).

Assume that a vertex of \( Q_1 - y \) is an ancestor of \( y \). Let \( i \) be the smallest index such that \( y_i \) is an ancestor of \( y \). If it exists, by definition of \( A_3 \), \( i \geq 2 \). Let \( x \) be the common ancestor of \( y_i \) and \( y_{i-1} \) in \( T \). By definition of \( A_3 \), \( y_i \) is not an ancestor of \( y_{i-1} \), so \( x \) is different from \( y_i \) and \( y_{i-1} \). Moreover by definition of \( A' \), \( \text{lvl}(y) - k \geq \text{lvl}(y_{i-1}) - k \geq \text{lvl}(y_i) - 1 \geq \text{lvl}(x) \). Hence \( T[x, y_{i-1}] \) and \( T[x, y] \) have length at least \( k \). Moreover these two dipaths are disjoint except in \( x \).
Therefore, the union of \( T[x, y_{i-1}] \) and \( T[x, y] \odot Q_1[y, y_{i-1}] \) is a subdivision of \( C(k, k) \) (and thus of \( C(k, \ell) \)), a contradiction.

Similarly, we get a contradiction if a vertex of \( Q_2 - y \) is an ancestor of \( y \). Henceforth, no vertex of \( V(Q_1) \cup V(Q_2) \setminus \{ y \} \) is an ancestor of \( y \).

Let \( x_1 \) be the least common ancestor of \( y \) and \( y_1 \). Note that \( |T[x_1, y]| \geq k \) so \( |T[x_1, y_1]| < k \), for otherwise \( G \) would contain a subdivision of \( C(k, k) \). Therefore \( lvl(y_1) - lvl(x_1) < k \). We define inductively \( x_2, \ldots, x_k \) as follows: \( x_{i+1} \) is the least common ancestor of \( x_i \) and \( y_i \). As above \( |T[x_i, y_{i-1}]| \geq k \) so \( lvl(y_i) - lvl(x_i) < k \). Symmetrically, let \( t_1 \) be the least common ancestor of \( y \) and \( z_1 \) and for \( 1 \leq i \leq \ell - 1 \), let \( t_{i+1} \) be the least common ancestor of \( t_i \) and \( z_i \). For \( 1 \leq i \leq \ell \), we have \( lvl(z_i) - lvl(t_i) < k \). Moreover, by definition all \( x_i \) and \( t_j \) are ancestors of \( y \), so they all are on \( T[u, y] \).

Let \( P_y \) (resp. \( P_z \)) be a shortest dipath in \( D \) from \( y_k \) (resp. \( z_k \)) to \( T[u, y] \cup Q_1[y_1, y_{k-1}] \cup Q_2[z_1, z_{\ell-1}] \). Note that \( P_y \) and \( P_z \) exist since \( D \) is strongly connected. Let \( y' \) (resp. \( z' \)) be the terminal vertex of \( P_y \) (resp. \( P_z \)). Let \( w_y \) be the last vertex of \( T[x_k, y_k] \) in \( P_y \) (possibly, \( w_y = y_k \)). Similarly, let \( w_z \) be the last vertex of \( T[t_k, z_k] \) in \( P_z \) (possibly, \( w_z = z_k \)). Note that \( P_y[w_y, y'] \) is a shortest dipath from \( w_y \) to \( y' \) and \( P_z[w_z, z'] \) is a shortest dipath from \( w_z \) to \( z' \).

If \( y' = y_j \) for \( 0 \leq j \leq k - 1 \), consider \( R = T[x_k, w_j] \odot P_y[w_j, y_j] \) is an \( (y_j, y_j) \)-dipath. By Proposition 19, \( R \) has length at least \( k \) because \( lvl(y_j) - lvl(x_k) \geq lvl(y_j) - lvl(y_k) + 1 \geq k \). Therefore the union of \( R \) and \( T[x_k, y] \cup Q_1[y, y_j] \) is a subdivision of \( C(k, k) \), a contradiction.

Similarly, we get a contradiction if \( z' \) is in \( \{ z_1, \ldots, z_{\ell-1} \} \). Consequently, \( P_y \) is disjoint from \( Q_1[y, y_{k-1}] \) and \( P_z \) is disjoint from \( Q_2[y, z_{\ell-1}] \).

If \( P_y \) and \( P_z \) intersect in a vertex \( s \). By the above statement, \( s \notin V(Q) \setminus \{ y_k, z_k \} \). Therefore the union of \( Q_1 \odot P_y[y_k, s] \) and \( Q_2 \odot P_z[z_{\ell}, s] \) is a subdivision of \( C(k, \ell) \), a contradiction. Henceforth \( P_y \) and \( P_z \) are disjoint.

Assume both \( y' \) and \( z' \) are in \( T[u, y] \). By symmetry, we can assume \( y' \geq T z' \) and then the union of \( Q_1 \odot P_y \odot T[y', z'] \) and \( Q_2 \odot P_z \) form a subdivision of \( C(k, \ell) \). This is a contradiction.

Henceforth a vertex among \( y' \) and \( z' \) is not in \( T[u, y] \). Let us assume that \( y' \) is not in \( T[u, y] \) (the case \( z' \notin T[u, y] \) is similar), and so \( y' = z_i \) for some \( 1 \leq i \leq \ell - 1 \). If \( lvl(y') \geq lvl(x_k) + k \), then both \( T[x_k, w_j \odot P_y[w_j, y']] \) and \( T[x_k, y] \odot Q_2[y, z_i] \) have length at least \( k \) by Proposition 19, so their union is a subdivision of \( C(k, k) \), a contradiction. Hence \( lvl(x_k) \geq lvl(z_i) - k + 1 \geq lvl(z_{\ell-1}) \).

If \( z' = y_j \) for some \( j \), then necessarily \( lvl(z') \geq lvl(x_k) + k \geq lvl(t_{\ell}) + k \) and both \( T[t_{\ell}, w_j] \odot P_z[w_j, z'] \) and \( T[t_{\ell}, y] \odot Q_1[y, y_j] \) have length at least \( k \), so their union is a subdivision of \( C(k, k) \), a contradiction. Therefore \( z' \in T[u, y] \). The union of \( T[t_{\ell}, z'] \) and \( T[t_{\ell}, w_j] \odot P_z[w_j, z'] \) is not a subdivision of \( C(k, k) \) so by Proposition 19, \( lvl(z') \leq lvl(t_{\ell}) + k - 1 \leq lvl(z_{\ell-1}) \).

If \( lvl(z') \leq lvl(x_k) \), then the union of \( Q_1 \odot Q_2 \odot P_z \odot T[z', y_k] \) is a subdivision of \( C(k, \ell) \), a contradiction. Hence \( lvl(z') > lvl(x_k) \). Therefore \( lvl(y') = lvl(z_{i}) \leq lvl(x_k) + k - 1 \leq lvl(z') + k - 2 \leq lvl(z_{\ell}) + 2k - 3 \), which implies that \( i = \ell - 1 \) that is \( y' = z_i = z_{\ell-1} \). Now the union of \( T[x_1, y_1] \odot Q_1[y_1, y_k] \odot P_y \) and \( T[x_1, y] \odot Q_2[y, z_{\ell-1}] \) is a subdivision of \( C(k, \ell) \), a contradiction. 

Claims 21.1, 21.2, 21.3, and 21.4, together with Lemma 20 yield the result. \( \square \)
4.3 Better bound when \( \ell = 1 \)

We now improve on the bound of Theorem 21 when \( \ell = 1 \). To do so, we reduce the problem to digraphs having a Hamiltonian directed cycle. Let

\[
\phi(k, \ell) = \max \{ \chi(D) \mid D \in S-\text{Forb}(C(k, \ell)) \text{ and } D \text{ has a Hamiltonian directed cycle} \}.
\]

**Theorem 22.** Let \( k \) be an integer greater than 1. \( \chi(S-\text{Forb}(C(k, 1)) \cap S) \leq \max \{ 2k - 4, \phi(k, 1) \} \).

To prove this theorem, we shall use the following lemma.

**Lemma 23.** Let \( D \) be a digraph containing a directed cycle \( C \) of length at least \( 2k - 3 \). If there is a vertex \( y \) in \( V(D - C) \) and two distinct vertices \( x_1, x_2 \in V(C) \) such that for \( i = 1, 2 \), there is a \((x_i, y)\)-dipath \( P_i \) in \( D \) with no internal vertices in \( C \), then \( D \) contains a subdivision of \( C(k, 1) \).

**Proof.** Since \( C \) has length at least \( 2k - 3 \), then one of \( C[x_1, x_2] \) and \( C[x_2, x_1] \) has length at least \( k - 1 \). Without loss of generality, assume that \( C[x_1, x_2] \) has length at least \( k - 1 \). Let \( z \) be the first vertex along \( P_2 \) which is also in \( P_1 \). Then the union of \( C[x_1, x_2] \cap P_2[x_2, z] \) and \( P_1[x_1, z] \) is a subdivision of \( C(k, 1) \). \( \square \)

**Proof of Theorem 22.** Suppose for a contradiction that there is a strong digraph \( D \) with chromatic number greater than \( \max \{ 2k - 4, \phi(k, 1) \} \) that contains no subdivision of \( C(k, 1) \). Let us consider the smallest such counterexample in term of vertices.

All 2-connected components of \( D \) are strong, and one of them has chromatic number \( \chi(D) \). Hence, by minimality, \( D \) is 2-connected. Let \( C \) be a longest directed cycle in \( D \). By Bondy’s theorem (Theorem 10), \( C \) has length at least \( 2k - 3 \), and by definition of \( \phi(k, 1) \), \( C \) is not Hamiltonian.

Because \( D \) is strong, there is a vertex \( v \in C \) with an out-neighbour \( w \notin C \). Since \( D \) is 2-connected, \( D - v \) is connected, so there is a (not necessarily directed) oriented path in \( D - v \) between \( C - v \) and \( w \). Let \( Q = (a_1, \ldots, a_q) \) be such a path so that all its vertices except the initial one are in \( V(D) \setminus V(C) \). By definition \( a_q = w \) and \( a_1 \in V(C) \setminus \{ v \} \).

- Let us first assume that \( a_1a_2 \in A(D) \). Let \( t \) be the largest integer such that there is a dipath from \( C - v \) to \( a_t \) in \( D - v \). Note that \( t > 1 \) by the hypothesis. If \( t = q \), then by Lemma 23, \( C \) contains a subdivision of \( C(k, 1) \), a contradiction. Henceforth we may assume that \( t < q \). By definition of \( t \), \( a_{t+1}a_t \) is an arc. Let \( P \) be a shortest \((v, a_{t+1})\)-dipath in \( D \).

  Such a dipath exists because \( D \) is strong. By maximality of \( t \), \( P \) has no internal vertex in \((C - v) \cup Q[a_1, a_t]\). Hence, \( a_t \in D - C \) and there are an \((a_1, a_t)\)-dipath and a \((v, a_t)\)-dipath with no internal vertices in \( C \). Hence, by Lemma 23, \( D \) contains a subdivision of \( C(k, 1) \), a contradiction.

- Now, we may assume that any oriented path \( Q = (a_1, \ldots, a_q) \) from \( C - v \) to \( w \) starts with a backward arc, i.e., \( a_2a_1 \in A(D) \). Let \( W \) be the set of vertices \( x \) such that there exists a (not necessarily directed) oriented path from \( w \) to \( x \) in \( D - C \). In particular, \( w \in W \).

  By the assumption, all arcs between \( C - v \) and \( W \) are from \( W \) to \( C - v \). Since \( D \) is strong, this implies that, for any \( x \in W \), there exists a directed \((w, x)\)-dipath in \( W \). In other words,
be a subdivision of $k$ if $v$.

By Theorem 22, it suffices to prove $\chi$ for $k$. However, this leaves Corollary 26 unchanged.

Let $D_1$ be the digraph obtained from $D - W$ by adding the arc $vy$ (if it does not already exist). $D_1$ contains no subdivision of $C(k, 1)$, for otherwise $D$ would contain one (replacing the arc $vy$ by the dipath $(v, w) \odot T_w[w, a] \odot (a, y)$). Since $D_1$ is strong (because $D - W$ is strong), by minimality of $D$, $\chi(D_1) \leq \max\{2k - 4, \phi(k, 1)\}$.

Let $D_2$ be the digraph obtained from $D \cup \{vy\}$ by adding the arc $yy$. $D_2$ contains no subdivision of $C(k, 1)$, for otherwise $D$ would contain one (replacing the arc $yy$ by the dipath $C[y, y]$). Moreover, $D_2$ is strong, so by minimality of $D$, $\chi(D_2) \leq \max\{2k - 4, \phi(k, 1)\}$.

Consider now $D^*$ the digraph $D_1 \cup D_2$. It is obtained from $D$ by adding the two arcs $vy$ and $yy$ (if they did not already exist). Since $\{v, y\}$ is a clique-cutset in $D^*$, we get $\chi(D^*) \leq \max\{\chi(D_1), \chi(D_2)\} \leq \max\{2k - 4, \phi(k, 1)\}$. But $\chi(D) \leq \chi(D^*)$, a contradiction.

From Theorem 22, one easily derives an upper bound on $\chi(S\text{-Forb}(C(k, 1)) \cap S)$.

Corollary 24. $\chi(S\text{-Forb}(C(k, 1)) \cap S) \leq 2k - 1$.

Proof. By Theorem 22, it suffices to prove $\phi(k, 1) \leq 2k - 1$.

Let $D \in S\text{-Forb}(C(k, 1))$ with a Hamiltonian directed cycle $C = (v_1, \ldots, v_n, v_1)$. Observe that if $v_i v_j$ is an arc, then $j \in C[v_{i+1}, v_{i+k-1}]$ for otherwise the union of $C[v_i, v_j]$ and $(v_i, v_j)$ would be a subdivision of $C(k, 1)$. In particular, every vertex had both its in-degree and out-degree at most $k - 1$, and so degree at most $2k - 2$. As $\chi(D) \leq \Delta(D) + 1$, the result follows.

The bound $2k - 1$ is tight for $k = 2$, because of the directed odd cycles. However, for larger values of $k$, we can get a better bound on $\phi(k, 1)$, from which one derives a slightly better one for $\chi(S\text{-Forb}(C(k, 1)) \cap S)$.

3While this paper was under review, Kim et al. [14] showed $\phi(k, \ell) = k + \ell$, which improves on Theorem 25. However, this leaves Corollary 26 unchanged.
Theorem 25. $\phi(k, 1) \leq \max\{k + 1, \frac{3k-3}{2}\}$.

Proof. For $k = 2$, the result holds because $\phi(2, 1) \leq \phi(2, 2) \leq 3$ by Corollary 30.

Let us now assume $k \geq 3$. We prove by induction on $n$, that every digraph $D \in S\text{-Forb}(C(k, 1))$ with a Hamiltonian directed cycle $C = (v_1, \ldots, v_n, v_1)$ has chromatic number at most $\max\{k + 1, \frac{3k-3}{2}\}$, the result holding trivially when $n \leq \max\{k + 1, \frac{3k-3}{2}\}$.

Assume now that $n > \max\{k + 1, \frac{3k-3}{2}\} + 1$ All the indices are modulo $n$. Observe that if $v_i v_j$ is an arc, then $j \in C[v_{i+1}, v_{i+k-1}]$ for otherwise the union of $C[v_i, v_j]$ and $(v_i, v_j)$ would be a subdivision of $C(k, 1)$. In particular, every vertex had both its in-degree and out-degree at most $k - 1$.

Assume that $D$ contains a vertex $v_i$ with in-degree 1 or out-degree 1. Then $d(v_i) \leq k$. Consider $D_i$ the digraph obtained from $D - v_i$ by adding the arc $v_{i-1}v_{i+1}$. Clearly, $D_i$ has a Hamiltonian directed cycle. Moreover, is has no subdivision of $C(k, 1)$ for otherwise, replacing the arc $v_{i-1}v_{i+1}$ by $(v_{i-1}, v_i, v_{i+1})$ if necessary, yields a subdivision of $C(k, 1)$ in $D$. By the induction hypothesis, $D_i$ a max $\{k + 1, \frac{3k-3}{2}\}$-colouring which can be extended to $v_i$ because $d(v_i) \leq k$.

Henceforth, we may assume that $\delta^+(D), \delta^-(D) \geq 2$.

Claim 25.1. $d^+(v_i) + d^-(v_i) \leq 3k - n - 3$ for all $i$.

Subproof. Let $v_{i^+}$ be the first out-neighbour of $v_i$ along $C[v_{i+2}, v_{i-1}]$ and let $v_{i^-}$ be the last in-neighbour of $v_{i+1}$ along $C[v_{i-3}, v_i]$. There are $d^+(v_i) - 1$ out-neighbours of $v_i$ in $C[v_{i^+}, v_{i-1}]$ which must be in $C[v_{i^+}, v_{i+k-1}]$ by the above observation. Therefore $i^+ \leq i + k - d^+(v_i)$. Similarly, $i^- \geq i - k + d^-(v_{i+1})$.

- if $v_i \in C[v_{i^-}, v_{i^+}]$, $C[v_{i^-}, v_{i^+}]$ has length $i^+ - i^- \leq 2k - d^+(v_i) - d^-(v_{i+1})$. Hence $C[v_{i^+}, v_{i^-}]$ has length at least $n - 2k + d^+(v_i) + d^-(v_{i+1})$. But the union of $(v_i, v_{i^+}) \cup C[v_{i^+}, v_{i-1}] \cup (v_{i^-}, v_{i+1})$ and $(v_i, v_{i+1})$ is not a subdivision of $C(k, 1)$, so $C[v_{i^+}, v_{i^-}]$ has length at most $k - 3$. Hence, $k - 3 \geq n - 2k + d^+(v_i) + d^-(v_{i+1})$, so $d^+(v_i) + d^-(v_{i+1}) \leq 3k - n - 3$.

- otherwise, $v_{i^+} \in C[v_{i^-}, v_{i-1}]$ and $v_{i^-} \in C[v_i, v_{i+1}]$. Both $C[v_i, v_{i+1}]$ and $C[v_{i^-}, v_{i^+}]$ have length less than $n$ as $v_{i^-}v_{i+1}$ and $v_{i^+}v_i$ are arcs. Moreover, the union of these two dipaths is $C$ and their intersection contains the three distinct vertices $v_i, v_{i+1}, v_{i^-}$. Consequently, $n = |C| \leq |C[v_{i^-}, v_{i+1}]| + |C[v_i, v_{i^+}]| - 3 \leq 2k - 3$. Let $v_{i_0}$ be the last out-neighbour of $v_i$ along $C[v_{i-2}, v_{i-1}]$. All the out-neighbours of $v_i$ and all the in-neighbours of $v_{i+1}$ are in $C[v_i, v_{i_0}]$ which has length less than $n$ because $v_i v_{i_0}$ is an arc. Hence $d^+(v_i) + d^-(v_{i+1}) \leq k$, so $d^+(v_i) + d^-(v_{i+1}) \leq 3k - n - 3$ because $n \geq 2k - 3$.

But $n \geq \frac{3k-1}{2}$, so by the above claim, $d^+(v_i) + d^-(v_{i+1}) \leq \frac{3k-5}{2}$ for all $i$.

Summing these inequalities over all $i$, we get $\sum_{i=1}^{\frac{3k-5}{2}} d^+(v_i) + d^-(v_{i+1}) \leq \frac{3k-5}{2} \cdot n$. Thus $\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{n} (d^+(v_i) + d^-(v_i)) \leq \frac{3k-5}{2} \cdot n$. Therefore there exists an index $i$ such that $v_i$ has degree at most $\frac{3k-5}{2}$. Consider the digraph $D_i$ defined above. It is Hamiltonian and contains no subdivision of $C(k, 1)$. By the induction hypothesis, $D_i$ has a max $\{k + 1, \frac{3k-3}{2}\}$-colouring which can be extended to $v$ because $d(v_i) \leq \frac{3k-5}{2}$.

Corollary 26. Let $k$ be an integer greater than 1. Then $\chi(S\text{-Forb}(C(k, 1)) \cap S) \leq \max\{k + 1, 2k - 4\}$.  

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Proof. By Theorems 22 and 25, \( \chi(S-\text{Forb}(C(k,1)) \cap S) \leq \max\{2k - 4, k + 1, \frac{3k - 3}{2}\} = \max\{k + 1, 2k - 4\}. \)

5 Small cycles with two blocks in strong digraphs

5.1 Handle decomposition

Let \( D \) be a strongly connected digraph. A handle \( h \) of \( D \) is a directed path \( (s, v_1, \ldots, v_\ell, t) \) from \( s \) to \( t \) (where \( s \) and \( t \) may be identical) such that:

- \( d^-(v_i) = d^+(v_i) = 1 \), for every \( i \), and
- removing the internal vertices and arcs of \( h \) leaves \( D \) strongly connected.

The vertices \( s \) and \( t \) are the endvertices of \( h \) while the vertices \( v_i \) are its internal vertices. The vertex \( s \) is the initial vertex of \( h \) and \( t \) its terminal vertex. The length of a handle is the number of its arcs, here \( \ell + 1 \). A handle of length 1 is said to be trivial.

Given a strongly connected digraph \( D \), a handle decomposition of \( D \) starting at \( v \in V(D) \) is a triple \( (v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p}) \), where \( (D_i)_{0 \leq i \leq p} \) is a sequence of strongly connected digraphs and \( (h_i)_{1 \leq i \leq p} \) is a sequence of handles such that:

- \( V(D_0) = \{v\} \),
- for \( 1 \leq i \leq p \), \( h_i \) is a handle of \( D_i \) and \( D_i \) is the (arc-disjoint) union of \( D_{i-1} \) and \( h_i \), and
- \( D = D_p \).

A handle decomposition is uniquely determined by \( v \) and either \( (h_i)_{1 \leq i \leq p} \), or \( (D_i)_{0 \leq i \leq p} \).

The number of handles \( p \) in any handle decomposition of \( D \) is exactly \( |A(D)| - |V(D)| + 1 \). The value \( p \) is also called the cyclomatic number of \( D \). Observe that \( p = 0 \) when \( D \) is a singleton and \( p = 1 \) when \( D \) is a directed cycle.

A handle decomposition \( (v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p}) \) is nice if all handles except the first one \( h_1 \) have distinct endvertices (i.e., for any \( 1 < i \leq p \), the initial and terminal vertices of \( h_i \) are distinct).

A digraph is robust if it is 2-connected and strongly connected. The following proposition is well-known (see [4] Theorem 5.13).

**Proposition 27.** Every robust digraph admits a nice handle decomposition.

**Lemma 28.** Every strong digraph \( D \) with \( \chi(D) \geq 3 \) has a robust subdigraph \( D' \) with \( \chi(D') = \chi(D) \) and which is an oriented graph.

**Proof.** Let \( D \) be a strong digraph \( D \) with \( \chi(D) \geq 3 \). Let \( D' \) be a 2-connected components of \( D \) with the largest chromatic number. Each 2-connected component of a strong digraph is strong, so \( D' \) is strong. Moreover, \( \chi(D') = \chi(D) \) because the chromatic number of a graph is the maximum of the chromatic numbers of its 2-connected components. Now by Bondy’s Theorem
(Theorem 10), $D'$ contains a cycle $C$ of length at least $\chi(D') \geq 3$. This can be extended into a handle decomposition $(v,(h_i)_{1 \leq i \leq p},(D_i)_{0 \leq i \leq p})$ of $D$ such that $D_1 = C$. Let $D''$ be the digraph obtained from $D'$ by removing the arcs $(u,v)$ which are trivial handles $h_i$ and such that $(v,u)$ is in $A(D_{i-1})$, we obtain an oriented graph $D''$ which is robust and with $\chi(D'') = \chi(D') = \chi(D)$.  

Proposition 27 and Lemma 28 will be very useful to establish bounds on $\chi(S-Forb(C(k,\ell)) \cap S)$ for small values of $k$ and $\ell$. As a warming up, Proposition 27 implies easily that a robust digraph containing no subdivision of $C(1,2)$ is a directed cycle. Together with Lemma 28 and the fact that every directed cycles is 3-colourable, this implies $\chi(S-Forb(C(1,2)) \cap S) \leq 3$. But the directed cycles of odd length have chromatic number 3 and contain no subdivision of $C(1,2)$. Therefore, $\chi(S-Forb(C(1,2)) \cap S) = 3$. In the following subsections, we establish the exact values of $\chi(S-Forb(C(k,\ell)) \cap S) = 3$, when $(k,\ell)$ is $(2,2)$, $(1,3)$ and $(2,3)$.

5.2 $C(2,2)$

**Theorem 29.** Let $D$ be a strong digraph. If $\chi(D) \geq 4$, then $D$ contains a subdivision of $C(2,2)$.

**Proof.** By Lemma 28, we may assume that $D$ is robust.

By Proposition 27, $D$ has a nice handle decomposition. Consider a nice decomposition $(v,(h_i)_{1 \leq i \leq p},(D_i)_{0 \leq i \leq p})$ that maximizes the sequence $(\ell_1, \ldots, \ell_p)$ of the length of the handles with respect to the lexicographic order.

Let $q$ be the largest index such that $h_q$ is not trivial.

Assume first that $q \neq 1$. Let $s$ and $t$ be the initial and terminal vertex of $h_q$ respectively. There is an $(s,t)$-path $P$ in $D_{q-1}$. If $P = (s,t)$, let $r$ be the index of the handle containing the arc $(s,t)$. Obviously, $r < q$. Now replacing $h_r$ by the handle $h'_r$ obtained from it by replacing the arc $(s,t)$ by $h_q$ and replacing $h_q$ by $(s,t)$, we obtain a nice handle decomposition contradicting the maximality of $(v,(h_i)_{1 \leq i \leq p},(D_i)_{0 \leq i \leq p})$. Therefore $P$ has length at least 2. So $P \cup h_q$ is a subdivision of $C(2,2)$.

Assume that $q = 1$, that is $D$ has a hamiltonian directed cycle $C$. Let us call chords the arcs of $A(D) \setminus A(C)$. Suppose that two chords $(u_1,v_1)$ and $(u_2,v_2)$ cross, that is $u_2 \in C[u_1,v_1]$ and $v_2 \in C[v_1,u_1]$. Then the union of $C[u_1,u_2] \circ (u_2,v_2)$ and $(u_1,v_1) \circ C[v_1,v_2]$ forms a subdivision of $C(2,2)$.

If no two chords cross, then one can draw $C$ in the plane and all chords inside it without any crossing. Therefore the graph underlying $D$ is outerplanar and has chromatic number at most 3. 

Since the directed odd cycles are in $S-Forb(C(2,2))$ and have chromatic number 3, Theorem 29 directly implies the following.

**Corollary 30.** $\chi(S-Forb(C(2,2)) \cap S) = 3$. 


5.3 \( C(1, 3) \)

**Theorem 31.** Let \( D \) be a strong digraph. If \( \chi(D) \geq 4 \), then \( D \) contains a subdivision of \( C(1,3) \).

**Proof.** By Lemma 28, we may assume that \( D \) is robust. Thus, by Proposition 27, \( D \) has a nice handle decomposition. Consider a nice decomposition \((v,(h_i)_{1 \leq i \leq p},(D_i)_{0 \leq i \leq p})\) that maximizes the sequence \((\ell_1, \ldots, \ell_p)\) of the length of the handles with respect to the lexicographic order.

Let \( q \) be the largest index such that \( h_q \) is not trivial.

**Case 1:** Assume first that \( q \neq 1 \). Let \( s \) and \( t \) be the initial and terminal vertex of \( h_q \) respectively. Since \( D_{q-1} \) is strong, there is an \((s,t)\)-dipath \( P \) in \( D_{q-1} \). If \( P = (s,t) \), let \( r \) be the index of the handle containing the arc \((s,t)\). Obviously, \( r < q \). Now replacing \( h_r \) by the handle \( h_r' \) obtained from it by replacing the arc \((s,t)\) by \( V \) and replacing \( h_q \) by \((s,t)\), we obtain a nice handle decomposition contradicting the minimality of \((v,(h_i)_{1 \leq i \leq p},(D_i)_{0 \leq i \leq p})\). Therefore \( P \) has length at least 2. If either \( P \) or \( h_q \) has length at least 3, then \( P \cup h_q \) is a subdivision of \( C(1,3) \). Henceforth, we may assume that both \( P \) and \( h_q \) have length 2. Set \( P = (s,u,t) \) and \( h = (s,x,t) \). Observe that \( V(D) = V(D_{q-1}) \cup \{x\} \).

Assume that \( x \) has a neighbour \( t' \) distinct from \( s \) and \( t \). By directional duality (i.e., up to reversing all arcs), we may assume that \( x \rightarrow t' \). Considering the handle decomposition in which \( h_q \) is replaced by \((s,x,t')\) and \((s,t')\) by \((x,t)\), we obtain that there is a dipath \((s,u',t')\) in \( D_{q-1} \). Now, if \( u' = t \), then the union of \((s,x,t')\) and \((s,u,t,t')\) is a subdivision of \( C(1,3) \). Henceforth, we may assume that \( t \notin \{s,u,u',t'\} \). Since \( D_{q-1} \) is strong, there is a dipath \( Q \) from \( t \) to \( \{s,u,u',t'\} \), which has length at least one by the preceding assumption. Note that \( x \notin Q \) since \( Q \) is a dipath in \( D_{q-1} \). Whatever vertex of \( \{s,u,u',t'\} \) is the terminal vertex \( z \) of \( Q \), we find a subdivision of \( C(1,3) \):

- If \( z = s \), then the union of \((x,t')\) and \((x,t) \cup Q \cup (s,u',t')\) is a subdivision of \( C(1,3) \);
- If \( z = u \), then the union of \((s,u)\) and \( h_q \cup Q \) is a subdivision of \( C(1,3) \);
- If \( z = u' \), then the union of \((s,u')\) and \( h_q \cup Q \) is a subdivision of \( C(1,3) \);
- If \( z = t' \), then the union of \((s,x,t')\) and \((s,u,t) \cup Q \) is a subdivision of \( C(1,3) \).

**Case 2:** Assume that \( q = 1 \), that is \( D \) has a hamiltonian directed cycle \( C \). Assume that two chords \((u_1,v_1)\) and \((u_2,v_2)\) cross. Without loss of generality, we may assume that the vertices \( u_1, u_2, v_1 \) and \( v_2 \) appear in this order along \( C \). Then the union of \( C[u_2,v_1] \) and \((u_2,v_2) \cup C[v_2,u_1] \cup (u_1,v_1)\) forms a subdivision of \( C(1,3) \).

If no two chords cross, then one can draw \( C \) in the plane and all chords inside it without any crossing. Therefore the graph underlying \( D \) is outerplanar and has chromatic number at most 3. \( \square \)

Since the directed odd cycles are in \( S-\text{Forb}(C(1,3)) \) and have chromatic number 3, Theorem 31 directly implies the following.

**Corollary 32.** \( \chi(S-\text{Forb}(C(1,3)) \cap S) = 3. \)
5.4 $C(2, 3)$

**Theorem 33.** Let $D$ be a strong directed graph. If $\chi(D) \geq 5$, then $D$ contains a subdivision of $C(2, 3)$.

*Proof.* By Lemma 28, we may assume that $D$ is a robust oriented graph. Thus, by Proposition 27, $D$ has a nice handle decomposition. Let $HD = (\{h_i\}_{1 \leq i \leq p}, \{D_i\}_{1 \leq i \leq p})$ be a nice decomposition that maximizes the sequence $(\ell_1, \ldots, \ell_p)$ of the length of the handles with respect to the lexicographic order. Recall that $D_i$ is strongly connected for any $1 \leq i \leq p$. In particular, $h_1$ is a longest directed cycle in $D$. Let $q$ be the largest index such that $h_q$ is not trivial. Observe that for all $i > q$, $h_i$ is a trivial handle by definition of $q$ and, for $i \leq q$, all handles $h_i$ have length at least 2.

**Claim 33.1.** For any $1 < i \leq q$, $h_i$ has length exactly 2.

*Subproof.* For sake of contradiction, let us assume that there exists $2 \leq r \leq q$ such that $h_r = (x_1, \ldots, x_t)$ with $t \geq 4$. Since $D_{r-1}$ is strong, there is a $(x_1, x_t)$-dipath $P$ in $D_{r-1}$. Note that $P$ does not meet $\{x_2, \ldots, x_{r-1}\}$. If $P$ has length at least 2, then $P \cup h_r$ is a subdivision of $C(2, 3)$. If $P = (x_1, x_t)$, let $r'$ be the handle containing the arc $h_r$. Now the handle decomposition obtained from $HD$ by replacing $h_r$ by the handle derived from it by replacing the arc $(x_1, x_t)$ by $h_r$, and replacing $h_r$ by $(x_1, x_t)$, contradicts the maximality of HD. $\Box$

For $1 < i \leq q$, set $h_i = (a_i, b_i, c_i)$. Since $h_1$ is a longest directed cycle in $D$ and $\chi(D) \geq 5$, by Bondy’s Theorem, $h_1$ has length at least 5. Set $h_1 = (u_1, \ldots, u_5, u_1)$.

A clone of $u_i$ is a vertex whose unique out-neighbour in $D_q$ is $u_{i+1}$ and whose unique in-neighbour in $D_q$ is $u_{i-1}$ (indices are taken modulo $m$).

**Claim 33.2.** Let $v \in V(D) \setminus V(D_1)$. Let $1 < i \leq q$ such that $v = b_i$, the internal vertex of $h_i$. There is an index $j$ such that $b_j$ is a clone of $u_j$, that is $a_i = u_{j-1}$ and $c_i = u_{j+1}$.

*Subproof.* We prove the result by induction on $i$.

By the induction hypothesis (or trivially if $i = 2$), there exists $i^-$ and $i^+$ such that $a_i$ is a clone of $u_{i^-}$ and $c_i$ is a clone of $u_{i^+}$. If $i^+ \notin \{i^- + 1, i^- + 2\}$, then the union of $h_i$ and $(a_i, u_{i^-+1}, \ldots, u_{i^-1}, c_i)$ is a subdivision of $C(2, 3)$, a contradiction. If $i^+ = i^- + 1$, then $(a_i, b_i, c_i, h_1[u_{i^-+1}, \ldots, u_{i^-1}], a_i)$ is a cycle longer than $h_1$, a contradiction. Henceforth $i^+ = i^- + 2$. If $c_i$ is not $u_{i^-}$, then it is a clone of $u_{i^-}$. Thus the union of $(a_i, b_i, c_i, u_{i^-+1})$ and $(a_i, u_{i^-+1}, u_{i^-+2}, u_{i^-+1})$ is a subdivision of $C(2, 3)$, a contradiction. Similarly, we obtain a contradiction if $a_i \neq u_{i^-}$. Therefore, $a_i = u_{i^-+1}$ and $c_i = u_{i^-+2}$, that is $b_i$ is a clone of $u_{i^-+1}$. Moreover all $b_i$ for $i^- < i < i^+$ are not adjacent to $b_i$ and thus are still clones of some $u_j$. $\Box$

For $1 \leq i \leq m$, let $S_i$ be the set of clones of $u_i$.

**Claim 33.3.** All integers are taken modulo $m$.

(i) If $S_i \neq \emptyset$, then $S_{i-1} = S_{i+1} = \emptyset$.

(ii) If $x \in S_i$, then $N_D^+(x) = \{u_{i+1}\}$ and $N_D^-(x) = \{u_{i-1}\}$. 


Subproof. (i) Assume for a contradiction, that both $S_i$ and $S_{i+1}$ are non-empty, say $x_i \in S_i$ and $x_{i+1} \in S_{i+1}$. Then the union of $(u_{i-1}, x_i, x_{i+1}, u_{i+2})$ and $(u_{i-1}, x_i, u_{i+1}, u_{i+2})$ is a subdivision of $C(2,3)$, a contradiction.

(ii) Let $x \in S_i$. Assume for a contradiction that $x$ has an out-neighbour $y$ distinct from $u_{i+1}$. By (i), $y \notin S_{i-1}$, and $y \neq u_{i-1}$ because $D$ is an oriented graph. If $y \in S_j \cup \{u_j\}$, then $(x, y, h_1[u_{i+1}, u_{i-1}], x)$ is a directed cycle longer than $h$. If $y \in S_j \cup \{u_j\}$ for $j \notin \{i-2\}$, then the union of $(u_{i-1}, x, y, u_{j+1})$ and $h_1[u_{i-1}, u_{j+1}]$ is a subdivision of $C(2,3)$, a contradiction. If $y \in S_{i-2}$, then the union of $(x, y, u_{i-1})$ and $(x, h_1[u_{i+1}, u_{i-1}])$ is a subdivision of $C(2,3)$, a contradiction. If $y = u_j$ for $j \notin \{i-1, i, i+1\}$, then the union of $(u_{i-1}, x, y)$ and $h_1[u_{i-1}, y]$ is a subdivision of $C(2,3)$, a contradiction.

This implies that $q = 1$. Indeed, if $q \geq 2$, then there is $i \leq m$ such that $b_2 \in S_i$. But $D - b_q = D_{q-1}$ is strong, and $\chi(D - b_q) \geq 5$, because $\chi(D) \geq 5$ and $b_q$ has only two neighbours in $D$ by Claim 33.3-(ii). But then by minimality of $D$, $D - b_q$ contains a subdivision of $C(2,3)$, which is also in $D$, a contradiction.

Hence $m = |V(D)|$. Because $\chi(D) \geq 5$, $D$ is not outerplanar, so there must be $i < j < k < \ell < i + m$ such that $(u_i, u_k) \in A(D)$ and $(u_j, u_\ell) \in A(D)$. We must have $j = i + 1$ and $\ell = k + 1$ since otherwise $(u_i, \ldots, u_j, u_\ell)$ and $(u_i, u_k, \ldots, u_k)$ form a subdivision of $C(2,3)$. In addition, $k = j + 1$ since otherwise, $(u_j, u_\ell, \ldots, u_k)$ and $(u_j, \ldots, u_k)$ form a subdivision of $C(2,3)$. Therefore, any two “crossing” arcs must have their ends being consecutive in $D_1$. This implies that $N^+(u_j) = \{u_{j+1}, u_{j+2}\}$, $N^-(u_j) = \{u_{j-1}\}$, $N^+(u_k) = \{u_{k+1}\}$ and $N^-(u_k) = \{u_{k-1}, u_{k-2}\}$.

Now let $D'$ be the digraph obtained from $D - \{u_j, u_k\}$ by adding the arc $(u_i, u_j)$. Because $u_j$ and $u_k$ have only three neighbours in $D$, $\chi(D') \geq 5$. By minimality of $D$, $D'$ contains a subdivision of $C(2,3)$, which can be transformed into a subdivision of $C(2,3)$ in $D$ by replacing the arc $(u_i, u_j)$ by the directed path $(u_i, u_j, u_{k-1})$.

Since every tournament of order 4 does not contain $C(2,3)$ (which has order 5), we have the following.

**Corollary 34.** $\chi(S\text{-Forb}(C(2,3)) \cap \mathcal{S}) = 4$.

### 6 Cycles with four blocks in strong digraphs

Recall that $\hat{C}_4$ is the cycle on four blocks.

**Theorem 35.** Let $D$ be a digraph in $S\text{-Forb}(\hat{C}_4)$. If $D$ admits an out-generator, then $\chi(D) \leq 24$.

**Proof.** The general idea is the same as in the proof of Theorem 21.

Suppose that $D$ admits an out-generator $u$ and let $T$ be an BFS-tree with root $u$ (See Subsection 4.1.1.). We partition $A(D)$ into three sets according to the levels of $u$.

- $A_0 = \{(x, y) \in A(D) \mid lv(x) = lv(y)\}$;
- $A_1 = \{(x, y) \in A(D) \mid ||lv(x) - lv(y)|| = 1\}$;
- $A_2 = \{(x, y) \in A(D) \mid lv(y) \leq lv(x) - 2\}$.

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For $i = 0, 1, 2$, let $D_i = (V(D), A_i)$.

**Claim 35.1.** $\chi(D_0) \leq 3$.

**Subproof.** Suppose for a contradiction that $\chi(D) \geq 4$. By Theorem 6, it contains a $P^-(1,1)$ $(y_1,y_2,y_2)$, that is $y_1, y_1$ and $y_2, y_2$ are in $A(D_0)$. Let $x$ be the least common ancestor of $y_1$ and $y_2$ in $T$. The union of $T[y_1, y_1], (y_1, y_2), (y_2, y_2)$, and $T[y_1, y_2]$ is a subdivision of $\hat{C}_4$, a contradiction. ◊

**Claim 35.2.** $\chi(D_1) \leq 2$.

**Subproof.** Since the arc are between consecutive levels, then the colouring $\phi_1$ defined by $\phi_1(x) = \text{lvl}(x) \mod 2$ is a proper 2-colouring of $D_1$. ◊

Let $y \in V_i$ we denote by $N'(y)$ the out-degree of $y$ in $\bigcup_{0 \leq j \leq i-1} V_j$. Let $D' = (V,A')$ with $A' = \bigcup_{x \in V} \{(x,y), y \in N'(x)\}$ and $D_\chi = (V,A_\chi)$ where $A_\chi$ is the set of arc inside the level and from $V_i$ to $V_{i+1}$ for all $i$. Note that $A = A' \cup A_\chi$ and

**Claim 35.3.** $\chi(D_2) \leq 4$.

**Subproof.** Let $x$ be a vertex of $V(D)$. If $y$ and $z$ are distinct out-neighbours of $x$ in $D_2$, then their least common ancestor $w$ is either $y$ or $z$, for otherwise the union of $T[w,y], (x,y), (x,z)$, and $T[w,z]$ is a subdivision of $\hat{C}_4$. Consequently, there is an ordering $y_1, \ldots, y_p$ of $N^+(D_2)(x)$ such that the $y_i$ appear in this order on $T[u,x]$.

Let us prove that $N^+(y_i) = \emptyset$ for $2 \leq i \leq p-1$. Suppose for a contradiction that $y_i$ has an out-neighbour $z$ in $D_2$. Let $t$ be the least common ancestor of $y_1$ and $z$. If $t = z$, then the union of $(y_i, z) \circ T[z,y_1], (x,y_1), (x,y_p)$, and $T[y_i,y_p]$ is a subdivision of $\hat{C}_4$; if $t = y \neq z$, then the union of $(y_i, z), (x,y_1) \circ T[y_1,z], (x,y_p)$, and $T[y_i,y_p]$ is a subdivision of $\hat{C}_4$. Otherwise, if $t \notin \{y, z\}, T[t,y_1], T[t,z], (x,y_i) \circ (y_i, z) and (x,y_1)$ is a subdivision of $\hat{C}_4$.

Henceforth, in $D_2$, every vertex has at most two out-neighbours that are not sinks. Let $V_0$ be the set of sinks in $D_2$. It is a stable set in $D_2$. Furthermore $\Delta^+(D_2 - V_0) \leq 2$, so $D_2 - V_0$ is 3-colourable, because $D_2$ (and so $D_2 - V_0$) is acyclic. Therefore $\chi(D_2) \leq 4$. ◊

Claims 35.1, 35.2, 35.3, and Lemma 20 implies $\chi(D) \leq 24$. □

7 Further research

The upper bound of Theorem 21 can be lowered when considering 2-strong digraphs.

**Theorem 36.** Let $k$ and $\ell$ be two integers such that $k \geq \ell$, $k + \ell \geq 4$ and $(k, \ell) \neq (2,2)$. Let $D$ be a 2-strong digraph. If $\chi(D) \geq (k + \ell - 2)(k - 1) + 2$, then $D$ contains a subdivision of $C(k, \ell)$.

**Proof.** Let $D$ be a 2-strong digraph with chromatic number at least $(k + \ell - 2)(k - 1) + 2$. Let $u$ be a vertex of $D$. For every positive integer $i$, let $L_i = \{v \mid \text{dist}_D(u,v) = i\}$.

Assume first that $L_k \neq \emptyset$. Take $v \in L_k$. In $D$, there are two internally disjoint $(u,v)$-dipaths $P_1$ and $P_2$. Those two dipaths have length at least $k$ (and $\ell$ as well) since $\text{dist}_D(u,v) \geq k$. Hence $P_1 \cup P_2$ is a subdivision of $C(k, \ell)$. 18
choose a vertex $(n$ and out-degree at least $A$

Set Proof.

subdivision of $C$.

Therefore we may assume that $L_k$ is empty, and so $V(D) = \{u\} \cup L_1 \cup \cdots \cup L_{k-1}$. Consequently, there is $i$ such that $\chi(D[L_i]) \geq k + \ell - 1$. Since $k + l - 1 \geq 3$ and $(k - 1, \ell - 1) \neq (1, 1)$, by Theorem 6, $D[L_i]$ contains a copy $Q$ of $P^+(k - 1, \ell - 1)$. Let $v_1$ and $v_2$ be the initial and terminal vertices of $Q$. By definition, for $j \in \{1, 2\}$, there is a $(u, v_j)$-dipath $P_j$ in $D$ such that $V(P_j) \cap L_i = \{v_j\}$. Let $w$ be the last vertex along $P_1$ that is in $V(P_1) \cap V(P_2)$. Clearly, $P_1[w, v_1] \cup P_2[w, v_2] \cup Q$ is a subdivision of $C(k, \ell)$.

To go further, it is natural to ask what happens if we consider digraphs which are not only strongly connected but $k$-strongly connected ($k$-strong for short).

**Proposition 37.** Let $C$ be an oriented cycle of order $n$. Every $(n - 1)$-strong digraph contains a subdivision of $C$.

**Proof.** Set $C = (v_1, v_2, \ldots, v_n, v_1)$. Without loss of generality, we may assume that $(v_1, v_n) \in A(C)$. Let $D$ be an $(n - 1)$-strong digraph. Choose a vertex $x_1$ in $V(D)$. Then for $i = 2$ to $n$, choose a vertex $x_i$ in $V(D) \setminus \{x_1, x_2, \ldots, x_{i-1}\}$ such that $x_i - x_{i-1}$ is an arc in $D$ if $x_{i-1} x_i$ is an arc in $C$ and $x_i x_{i-1}$ is an arc in $D$ if $x_{i-1} x_i$ is an arc in $C$. This is possible since every vertex has in- and out-degree at least $n - 1$. Now, since $D$ is $(n - 1)$-strong, $D - \{x_2, \ldots, x_{n-1}\}$ is strong, so there exists a $(x_1, x_n)$-dipath $P$ in $D - \{x_2, \ldots, x_{n-1}\}$. The union of $P$ and $(x_1, x_2, \ldots, x_n)$ is a subdivision of $C$.

Let $S_p$ be the class of $p$-strong digraphs. Proposition 37 implies directly that $S-Forb(C) \cap S_p = \emptyset$ and so $\chi(S-Forb(C) \cap S_p) = 0$ for any oriented cycle $C$ of length $p + 1$. This yields the following problems.

**Problem 38.** Let $C$ be an oriented cycle and $p$ a positive integer. What is $\chi(S-Forb(C) \cap S_p)$ ?

Note that $\chi(S-Forb(C) \cap S_{p+1}) \leq \chi(S-Forb(C) \cap S_p)$ for all $p$, because $S_{p+1} \subseteq S_p$.

**Problem 39.** Let $C$ be an oriented cycle.

1) What is the minimum integer $p_C$ such that $\chi(S-Forb(C) \cap S_{p_C}) < +\infty$ ?

2) What is the minimum integer $p_C^0$ such that $\chi(S-Forb(C) \cap S_{p_C^0}) = 0$ ?

**References**


