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THE COST OF BOUNDARY CONTROLLABILITY FOR A PARABOLIC EQUATION WITH INVERSE SQUARE POTENTIAL

P. MARTINEZ AND J. VANCOSTENOBLE

ABSTRACT. The goal of this paper is to analyze the cost of boundary null controllability for the 1 – D linear heat equation with the so-called inverse square potential:

$$u_t - u_{xx} - \frac{\mu}{x^2}u = 0, \quad x \in (0, 1), \quad t \in (0, T),$$

where μ is a real parameter such that $\mu \leq 1/4$.

Since the works by Baras and Goldstein [2, 3], it is known that such problems are well-posed for any $\mu \leq 1/4$ (the constant appearing in the Hardy inequality) whereas instantaneous blow-up may occur when $\mu > 1/4$.

For any $\mu \leq 1/4$, it has been proved in [42] (via Carleman estimates) that the equation can be controlled (in any time $T > 0$) by a locally distributed control. Obviously, the same result holds true when one considers the case of a boundary control acting at $x = 1$. The goal of the present paper is to provide sharp estimates of the cost of the control in that case, analyzing its dependence with respect to the two parameters $T > 0$ and $\mu \in (-\infty, 1/4]$. Our proofs are based on the moment method and very recent results on biorthogonal sequences.

CONTENTS

I. Introduction and main results	2
I.1. Description of the problem	2
I.2. Main results and comments	3
II. Well-posedness of the problem	5
II.1. Preliminary transformation	5
II.2. Homogeneous boundary conditions and a source term	6
II.3. Non homogeneous boundary condition	7
III. Proof of the main results (Theorems I.1 and I.2)	8
III.1. Eigenvalues and eigenbasis of the Sturm-Liouville Problem	8
III.2. Useful estimates on eigenvalues and eigenfunctions	8
III.3. Transformation into a moment problem	9
III.4. Formal solution of the moment problem	10
III.5. Existence of a suitable biorthogonal family	10
III.6. Upper bounds of the cost: proof of (I. 4) and (I. 6)	11
III.7. Lower bound of the cost when $\mu \in [0, \frac{1}{4}]$: proof of (I. 5)	13
III.8. Lower bound of the cost when $\mu \leq 0$: proof of (I. 7)	14
IV. Technical part	16
IV.1. Proof of Proposition III.1	16
IV.2. Proof of Lemma III.1	20
IV.3. Proof of Lemma III.2	21
V. Appendix : elements of Bessel Theory	21

V.1.	Bessel's equation and Bessel's functions of order ν	21
V.2.	Fundamental solutions of Bessel's equation when $\nu \notin \mathbb{N}$	21
V.3.	Fundamental solutions of Bessel's equation when $\nu = n \in \mathbb{N}$	21
V.4.	Zeros of Bessel functions of order ν of the first kind	22
V.5.	Orthogonality property	23
	References	23

I. INTRODUCTION AND MAIN RESULTS

I.1. Description of the problem. In this paper, we are interested in the linear $1 - D$ heat equation with an inverse square potential (that arises for example in the context of combustion theory and quantum mechanics):

$$(I. 1) \quad \begin{cases} u_t - u_{xx} - \frac{\mu}{x^2}u = 0 & x \in (0, 1), t \in (0, T), \\ u(0, t) = 0 & t \in (0, T), \\ u(1, t) = H(t) & t \in (0, T), \\ u(x, 0) = u_0(x) & x \in (0, 1), \end{cases}$$

where $u_0 \in L^2(0, 1)$, $T > 0$ and μ is a real parameter. Here H represents some control term that aims to steer the solution to zero at time T . Our goal is not only to establish the existence of such control (which could be easily deduced from known results, see later) but also to provide sharp estimates of the cost of such control.

Since the works by Baras and Goldstein [2, 3], it is known that existence/non-existence of positive solutions is determined by the value of μ with respect to the constant $1/4$ appearing in the Hardy inequality [23, 34]:

$$(I. 2) \quad \forall z \in H_0^1(0, 1), \quad \frac{1}{4} \int_0^1 \frac{z^2}{|x|^2} dx \leq \int_0^1 |z_x|^2 dx.$$

When $\mu < 1/4$, the operator $z \mapsto -z_{xx} - \mu x^{-2}z$ generates a coercive quadratic form in $H_0^1(0, 1)$. This allows showing the well-posedness in the classical variational setting of the linear heat equation with smooth coefficients. For the critical value $\mu = 1/4$, the space $H_0^1(0, 1)$ has to be slightly enlarged as shown in [43] but a similar result of well-posedness occurs (see section II.2 for details). Finally, when $\mu > 1/4$, the problem is ill-posed (due to possible instantaneous blow-up) as proved in [2].

For these reasons, we concentrate on the two first cases and *we assume throughout this paper that μ satisfies $\mu \leq 1/4$.*

Recently, the null controllability properties of (I. 1) began to be studied. For any $\mu \leq 1/4$, it has been proved in [42] that such equations can be controlled (in any time $T > 0$) by a locally distributed control: $\forall \mu \leq 1/4, \forall u_0 \in L^2(0, 1), \forall T > 0, \forall 0 \leq a < b \leq 1$, there exists $h \in L^2((0, 1) \times (0, T))$ such that the solution of

$$(I. 3) \quad \begin{cases} u_t - u_{xx} - \frac{\mu}{x^2}u = h(x, t)\chi_{(a,b)}(x) & x \in (0, 1), t \in (0, T), \\ u(0, t) = 0 & t \in (0, T), \\ u(1, t) = 0 & t \in (0, T), \\ u(x, 0) = u_0(x) & x \in (0, 1), \end{cases}$$

satisfies $u(\cdot, T) \equiv 0$.

The proof in [42] is based on Carleman estimates. It also concerns the case of the N -dimensional equation with some restricting geometric condition on the region of the control, condition that has been later erased in [11]. After those first results,

several other works followed extending them in various situations. See for instance [40, 41, 9, 5, 16, 21].

Here we are interested in the study of null controllability using a boundary control acting at $x = 1$ (see problem (I. 1)). More precisely, our aim is to provide sharp estimates of the cost of controllability in that case, analyzing its dependence with respect to the two parameters $T > 0$ and $\mu \in (-\infty, 1/4]$.

Observe that the existence of a boundary control for (I. 1) could be deduced from [42], using for example the standard argument that consists in extending the domain $(0, 1)$ into $(0, 1 + \eta)$, applying the result of null controllability for (I. 3) in $(0, 1 + \eta)$ with a distributed control localized in $(1, 1 + \eta)$ and concluding by taking the trace at $x = 1$ of such a control function. However this method based on Carleman estimates would not provide optimal estimates of the cost of the control, in particular when $\mu \rightarrow -\infty$ (see Remark 3.5 in [42]).

For this reason, we turn here to methods based on decomposition in series and moment problems. These methods have been developed by Fattorini-Russell [13, 14], and have been successfully applied/adapted to obtain sharp results in quite simple geometric situations, we refer for example the reader to [1, 4, 6, 10, 18, 19, 22, 28, 29, 31, 38, 39].

Here we follow the classical strategy:

- first of all, we transform the controllability question into a moment problem;
- to solve this moment problem and get some upper estimate of the cost of the control, we construct some suitable biorthogonal family (with best possible upper bounds of the norm of its elements);
- to prove the sharpness of our estimate, we provide some lower bound of the cost of the control using lower bounds that are satisfied by any biorthogonal family.

Hence the tools that are required to solve our problem are :

- the construction of some suitable biorthogonal family (with best possible upper bounds of the norm of its elements);
- lower bounds of the norm of the elements of any biorthogonal family.

As seen in [6], the obtention of explicit and precise (upper and lower) estimates for such biorthogonal families is closely related to gap conditions on the eigenvalues, namely

$$\forall n, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}.$$

(Roughly speaking, the gap γ_{min} gives the upper estimates whereas the gap γ_{max} gives the lower estimates). Then here

- we solve the eigenvalue problem, and we express the eigenvalues and eigenfunctions using Bessel functions and their zeros;
- when $\mu \in [0, 1/4]$, the gap between the square root of successive eigenvalues satisfies some upper and lower bounds which are uniform with respect to the parameter μ ; this enables us to use the results of [6];
- but when $\mu \leq 0$, the eigenvalues do not satisfy a good uniform gap condition from above; we use here some new tools developed in [7] in order to treat cases where the eigenvalues do not satisfy a good uniform gap condition but satisfy some better asymptotic gap condition; these new results have already been applied in the context of the degenerate heat equation with a strong degeneracy in [8], and in the present case they help us to provide a suitable lower bound of the cost.

I.2. Main results and comments. Let us define the notion of cost of controllability. For any $T > 0$, $\mu \leq 1/4$ and $u_0 \in L^2(0, 1)$, we introduce the set of admissible

controls:

$$\mathcal{U}_{\text{ad}}(\mu, T, u_0) := \{H \in H^1(0, T) \mid u^{(H)}(T) = 0\},$$

where $u^{(H)}$ denotes the solution of (I. 1). Then we consider the controllability cost for any $u_0 \in L^2(0, 1)$

$$C^{H^1}(\mu, T, u_0) := \inf_{H \in \mathcal{U}_{\text{ad}}(\mu, T, u_0)} \|H\|_{H^1(0, T)}$$

which is the minimal value to drive u_0 to 0. Finally, we define the global notion of controllability cost:

$$C_{\text{bd-ctr}}^{H^1}(\mu, T) := \sup_{\|u_0\|_{L^2(0, 1)}} C^{H^1}(\mu, T, u_0).$$

Then we prove the following results

Theorem I.1. *Given $\mu \in [0, \frac{1}{4}]$, $T > 0$, and $u_0 \in L^2(0, 1)$, there exists $H \in H^1(0, T)$ such that the solution of (I. 1) satisfies $u(\cdot, T) \equiv 0$. Moreover, we have the following estimates of the controllability cost: there exists $0 < c < C$ both independent of $\mu \in [0, 1/4]$ and of $T > 0$ such that*

$$(I. 4) \quad C_{\text{ctr-bd}}(\mu, T) \leq C e^{C/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/C} \left(1 + \sqrt{\frac{1}{4}-\mu}\right),$$

and

$$(I. 5) \quad C_{\text{ctr-bd}}(\mu, T) \geq c e^{c/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/c}.$$

Theorem I.2. *Given $\mu \leq 0$, $T > 0$, and $u_0 \in L^2(0, 1)$, there exists $H \in H^1(0, T)$ such that the solution of (I. 1) satisfies $u(\cdot, T) \equiv 0$. Moreover, we have the following estimates of the controllability cost: there exists $0 < c < C$ both independent of $\mu \leq 0$ and of $T > 0$ such that*

$$(I. 6) \quad C_{\text{ctr-bd}}(\mu, T) \leq C e^{C/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/C} \left(1 + \sqrt{\frac{1}{4}-\mu}\right),$$

and

$$(I. 7) \quad C_{\text{ctr-bd}}(\mu, T) \geq c e^{c/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/c} e^{-\sqrt{\frac{1}{4}-\mu}^{4/3} (\ln \sqrt{\frac{1}{4}-\mu} + \ln \frac{1}{T})/c}.$$

Remark I.1. Several observations can be made from Theorems I.1 and I.2:

- we deduce from (I. 4) that the null controllability cost is uniformly bounded when $\mu \in [0, \frac{1}{4}]$, in particular it does not blow-up as $\mu \rightarrow 1/4$; this is natural and expected since it has been proved in [42] (in the case of a distributed control) that null controllability holds true for any $\mu \leq 1/4$; *however*, the proof of (I. 4) holds on the fact that $\frac{1}{4} - \mu \geq 0$, and the final estimate (I. 4) puts this in evidence; it also implies the following simpler estimate

$$C_{\text{ctr-bd}}(\mu, T) \leq 2C e^{C/T} e^{-T/C},$$

but where the condition $\frac{1}{4} - \mu \geq 0$ is hidden;

- (I. 5) allows to measure how good is the qualitative behavior given by the upper bound (I. 4), once again putting in evidence that the condition $\frac{1}{4} - \mu \geq 0$ is not to be forgotten; the proofs of (I. 4) and of (I. 5) give explicit values of all the coefficients, hence in particular of the coefficients of $1/T$ in the exponential factors, but in this paper we are mainly interested in the qualitative behavior with respect to μ and T ; once again, (I. 5) implies the following simpler estimate

$$C_{\text{ctr-bd}}(\mu, T) \geq c e^{c/T} e^{-2T/c},$$

but where the condition $\frac{1}{4} - \mu \geq 0$ is hidden;

- (I. 6) gives an upper bound interesting when $T \rightarrow 0$, $T \rightarrow \infty$, and $\mu \rightarrow -\infty$; it implies the more compact form

$$C_{ctr-bd}(\mu, T) \leq C e^{C/T} e^{-(1+|\mu|)T/C} \left(1 + \sqrt{|\mu|}\right),$$

but we stated (I. 6) to put it in parallel with (I. 4); note that (I. 6) implies that $C_{ctr-bd}(\mu, T) \rightarrow 0$ as $\mu \rightarrow -\infty$: this comes from the fact the parameter here has the good sign (it makes the energy decreasing); this could not have been deduced from the result obtained by Carleman estimates since those estimates does not allow us to take into account the "good" or "bad" sign of potential terms (the constant appearing in Carleman estimates would grow up as $\mu \rightarrow -\infty$, see Remark 3.5 in [42]);

- (I. 7) allows to measure how good is the qualitative behavior given by the upper bound (I. 6), the main difference coming from the last exponential factor; this allows us to precise the behavior of $C_{ctr-bd}(\mu, T) \rightarrow 0$ as $\mu \rightarrow -\infty$.

II. WELL-POSEDNESS OF THE PROBLEM

II.1. Preliminary transformation. Let $\mu \leq 1/4$ be given. To define the solution of the boundary value problem (I. 1), we transform it (as done for instance in [6] in the context of a degenerate parabolic equation) into a problem with homogeneous boundary conditions and a source term. Let us define

$$\forall x \in [0, 1], \quad p(x) := x^{q_\mu} \text{ where } q_\mu := \frac{1 + \sqrt{1 - 4\mu}}{2}.$$

Observe that $p(0) = 0$, $p(1) = 1$ and

$$p''(x) + \frac{\mu}{x^2} p(x) = 0.$$

Formally, if u is a solution of (I. 1), then the function defined by

$$(II. 1) \quad v(x, t) = u(x, t) - \frac{p(x)}{p(1)} H(t) = u(x, t) - x^{q_\mu} H(t)$$

is solution of

$$(II. 2) \quad \begin{cases} v_t - v_{xx} - \frac{\mu}{x^2} v = -\frac{p(x)}{p(1)} H'(t) & x \in (0, 1), t \in (0, T), \\ v(0, t) = 0 & t \in (0, T), \\ v(1, t) = 0 & t \in (0, T), \\ v(x, 0) = u_0(x) - \frac{p(x)}{p(1)} H(0) & x \in (0, 1). \end{cases}$$

Reciprocally, given $h \in L^2(0, T)$, consider the solution of

$$\begin{cases} v_t - v_{xx} - \frac{\mu}{x^2} v = -\frac{p(x)}{p(1)} h(t) & x \in (0, 1), t \in (0, T), \\ v(0, t) = 0 & t \in (0, T), \\ v(1, t) = 0 & t \in (0, T), \\ v(x, 0) = v_0(x) & x \in (0, 1). \end{cases}$$

Then the function u defined by

$$(II. 3) \quad u(x, t) = v(x, t) + \frac{p(x)}{p(1)} \int_0^t h(\tau) d\tau$$

satisfies

$$(II. 4) \quad \begin{cases} u_t - u_{xx} - \frac{\mu}{x^2}u = 0 & x \in (0, 1), t \in (0, T), \\ u(0, t) = 0 & t \in (0, T), \\ u(1, t) = \int_0^t h(\tau) d\tau & t \in (0, T), \\ u(x, 0) = v_0(x) & x \in (0, 1). \end{cases}$$

This motivates the fact that we will first establish results of well-posedness for an auxiliary problem with Dirichlet homogeneous boundary conditions and source term (see section II.2). Then we will define the solutions of the boundary value problem (I. 1) via this auxiliary problem (see section II.3).

II.2. Homogeneous boundary conditions and a source term. Let us first consider the system with homogeneous boundary conditions and a source term

$$(II. 5) \quad \begin{cases} w_t - w_{xx} - \frac{\mu}{x^2}w = f(x, t) & x \in (0, 1), t \in (0, T), \\ w(0, t) = 0 & t \in (0, T), \\ w(1, t) = 0 & t \in (0, T), \\ w(x, 0) = w_0(x) & x \in (0, 1). \end{cases}$$

We define $H_0^1(\mu)$ the Hilbert space obtained as the closure of $H_0^1(0, 1)$ with respect to the norm

$$\forall z \in H_0^1(0, 1), \quad \|z\|_\mu := \left(\int_0^1 \left(z_x^2 - \mu \frac{z^2}{x^2} \right) dx \right)^{1/2}.$$

(Thanks to the Hardy inequality (I. 2), this defines a norm for any value of the parameter $\mu \leq 1/4$).

In the sub-critical case $\mu < 1/4$, the norm $\|\cdot\|_\mu$ is equivalent to the standard norm of $H_0^1(0, 1)$ (see [43] p. 115). Therefore, $H_0^1(\mu) = H_0^1(0, 1)$ for any $\mu < 1/4$.

In the critical case $\mu = 1/4$, it has been proved (see [43] p. 127) that $H_0^1(\mu = 1/4)$ is strictly larger than $H_0^1(0, 1)$:

$$H_0^1(0, 1) \subsetneq H_0^1(\mu = 1/4).$$

Observe that, if we denote $H^1(\mu)$ the Hilbert space obtained as the completion of $H^1(0, 1)$ with respect to the norm $\|z\|_{L^2(0,1)} + \|z\|_\mu$, we have

$$H_0^1(\mu) = \{z \in H^1(\mu) \mid z(0) = 0 = z(1)\}.$$

Let us define the operator $L_\mu : D(L_\mu) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ by:

$$D(L_\mu) = \{z \in H_0^1(\mu) \mid -z_{xx} - \mu \frac{z}{x^2} \in L^2(0, 1)\}, \quad L_\mu z = -z_{xx} - \mu \frac{z}{x^2}.$$

We also define

$$H^2(\mu) = \{z \in H^1(\mu) \mid -z_{xx} - \mu \frac{z}{x^2} \in L^2(0, 1)\}$$

so that $D(L_\mu) = H^2(\mu) \cap H_0^1(\mu)$.

It can be proved that, for any $\mu \leq 1/4$, L_μ is self-adjoint with compact inverse. In particular, we have:

Theorem II.1. (see [43])

Assume $\mu \leq 1/4$. There exists an orthonormal basis $(\Phi_k)_{k \geq 1}$ of $L^2(0, 1)$ constituted of eigenvectors of L_μ with eigenvalues sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty,$$

so that

$$\begin{cases} L_\mu \Phi_k = \lambda_k \Phi_k & \text{in } (0, 1), \\ \Phi_k(0) = 0 = \Phi_k(1). \end{cases}$$

Besides L_μ generates an analytic semi-group of contractions in $L^2(0, 1)$ for the equation (II. 5) (see also [43]). As a consequence, for any $w_0 \in L^2(0, 1)$ and $f \in L^2((0, 1) \times (0, T))$, problem (II. 5) is well-posed :

Definition II.1. a) Given $w_0 \in L^2(0, 1)$ and $f \in L^2((0, 1) \times (0, T))$, one defines the mild solution of (II. 5)

$$w \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_0^1(\mu))$$

as the one given by the variation formula:

$$w(x, t) = e^{tL_\mu} w_0 + \int_0^t e^{(t-s)L_\mu} f(x, s) ds.$$

b) Moreover, we say that a function

$$w \in \mathcal{C}^0([0, T]; H_0^1(\mu)) \cap H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(L_\mu))$$

is a strict solution of (II. 5) if it satisfies the equation a.e. in $(0, 1) \times (0, T)$ and the boundary and initial conditions for all $t \in [0, T]$ and $x \in [0, 1]$.

Notice that, if $w_0 \in H_0^1(\mu)$, then the mild solution of (II. 5) is also the unique strict solution.

II.3. Non homogeneous boundary condition. Next we turn to the boundary value problem (I. 1). Let H be given in $H^1(0, T)$. The results of section II.2 apply in particular to problem (II. 5) when one chooses

$$(II. 6) \quad f(x, t) = -\frac{p(x)}{p(1)} H'(t) \quad \text{and} \quad v_0(x) = u_0(x) - \frac{p(x)}{p(1)} H(0).$$

This allows us to define in a suitable way the solution of (I. 1):

Definition II.2. a) We say that $u \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(\mu))$ is the mild solution of (I. 1) if v defined by (II. 1) and (II. 6) is the mild solution of (II. 2).

b) We say that $u \in \mathcal{C}^0([0, T]; H^1(\mu)) \cap H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^2(\mu))$ is the strict solution of (I. 1) if v defined by (II. 1) and (II. 6) is the strict solution of (II. 2).

We deduce

Proposition II.1. a) Given $u_0 \in L^2(0, 1)$ and $H \in H^1(0, T)$, problem (I. 1) admits a unique mild solution.

b) Given $u_0 \in H^1(\mu)$ such that $u_0(0) = 0$ and $H \in H^1(0, T)$ such that $H(0) = u_0(0)$, problem (I. 1) admits a unique strict solution. (In particular, this holds true when $u_0 \in H_0^1(\mu)$, $H \in H^1(0, T)$ such that $H(0) = 0$).

The proof of Proposition II.1 follows immediately noting that

$$\tilde{H}(x, t) := \frac{p(x)}{p(1)} H(t)$$

satisfies

$$\tilde{H} \in \mathcal{C}^0([0, T]; H^1(\mu)) \cap H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^2(\mu)).$$

Remark II.1. In the case $u_0 \in L^2(0, 1)$, we have : for any $\tau > 0$, $v(\cdot, \tau) \in H^2(\mu) \cap H_0^1(\mu)$. Therefore the solution of (II. 2) is strict on $[\tau, T]$. The same is true for the solution of (I. 1).

III. PROOF OF THE MAIN RESULTS (THEOREMS I.1 AND I.2)

In this section, we prove our main result. In order to straight the main lines of the proof, some computations are postponed in the technical part in section IV.

III.1. Eigenvalues and eigenbasis of the Sturm-Liouville Problem. In order to transform the question of null controllability into a moment problem, let us first determine the eigenfunctions and eigenvalues associated to the operator L_μ . This means that we aim at solving the following boundary value problem for any suitable $\lambda \in \mathbb{R}$:

$$(III. 1) \quad \begin{cases} -\phi''(x) - \frac{\mu}{x^2}\phi(x) = \lambda\phi(x) & x \in (0, 1), \\ \phi(0) = 0 = \phi(1). \end{cases}$$

As this problem is closely related to Bessel's equations, we will need to use throughout this paper some standard definitions, notations and properties coming from Bessel's theory. For reader convenience, we recall and summarize the elements that are useful in the appendix in section V.

Concerning problem (III. 1), we prove (see section IV.1 for the proof):

Proposition III.1. *Assume $\mu \leq 1/4$ and define*

$$\nu(\mu) := \sqrt{\frac{1}{4} - \mu}.$$

We denote by $J_{\nu(\mu)}$ the Bessel function of first kind of order $\nu(\mu)$ (see section V) and we denote $0 < j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,n} < \dots \rightarrow +\infty$ as $n \rightarrow +\infty$ the sequence of positive zeros of $J_{\nu(\mu)}$. Then the admissible eigenvalues λ for problem (III. 1) are

$$\forall n \geq 1, \quad \lambda_{\mu,n} = (j_{\nu(\mu),n})^2$$

and the corresponding (normalized) eigenfunctions are

$$\forall n \geq 1, \quad \Phi_{\mu,n}(x) = \frac{1}{|J'_{\nu(\mu)}(j_{\nu(\mu),n})|} \sqrt{x} J_{\nu(\mu)}(j_{\nu(\mu),n}x), \quad x \in (0, 1).$$

Moreover the family $(\Phi_{\mu,n})_{n \geq 1}$ forms an orthonormal basis of $L^2(0, 1)$.

III.2. Useful estimates on eigenvalues and eigenfunctions. Next we give some estimates on the eigenvalues (proved later in section IV.2) that will be useful in the analysis of the problem:

Lemma III.1. (i) *When $\mu \in [0, 1/4]$, then*

$$(III. 2) \quad \forall n \geq 1, \quad \frac{7\pi}{8} \leq \sqrt{\lambda_{\mu,n+1}} - \sqrt{\lambda_{\mu,n}} \leq \pi.$$

(ii) *When $\mu \leq 0$, then*

$$(III. 3) \quad \forall n \geq 1, \quad \pi \leq \sqrt{\lambda_{\mu,n+1}} - \sqrt{\lambda_{\mu,n}},$$

and

$$(III. 4) \quad \forall n > \nu(\mu), \quad \sqrt{\lambda_{\mu,n+1}} - \sqrt{\lambda_{\mu,n}} \leq 2\pi.$$

Observe that (III. 2) and (III. 3) ensure a lower estimate of the gap that is uniform for any $\mu \leq 1/4$:

$$\forall \mu \in [0, 1/4], \forall n \geq 1, \quad \frac{7\pi}{8} \leq \sqrt{\lambda_{\mu,n+1}} - \sqrt{\lambda_{\mu,n}}.$$

This will enable us to use the standard methods developed by Fattorini-Russell [13, 14] to give an upper estimate of the cost. More precisely, we use a result proved in [6] that makes explicit the one obtained in [13, 14] (in short time). For this part, we will be able to treat the two cases $\mu \in [0, 1/4]$ and $\mu \leq 0$ in the same way.

As for the obtention of a lower estimate of the controllability cost, we will have to treat separately the two cases $\mu \in [0, 1/4]$ and $\mu \leq 0$, using extensions of Guichal [20], that we proved in [6] and [7].

III.3. Transformation into a moment problem. Following the strategy initiated by Fattorini and Russell [13, 14], we reduce here the controllability question to some moment problem. In this part, we analyze the problem with formal computations.

First, we expand the initial condition $u_0 \in L^2(0, 1)$: there exists $(\beta_{\mu,n}^0)_{n \geq 1} \in \ell^2(\mathbb{N}^*)$ such that

$$u_0(x) = \sum_{n \geq 1} \beta_{\mu,n}^0 \Phi_{\mu,n}(x).$$

Next we expand the solution u of (I. 1):

$$u(x, t) = \sum_{n \geq 1} \beta_{\mu,n}(t) \Phi_{\mu,n}(x), \quad x \in (0, 1), \quad t \geq 0 \quad \text{with} \quad \sum_{n \geq 1} \beta_{\mu,n}(t)^2 < +\infty.$$

Therefore the controllability condition $u(\cdot, T) = 0$ becomes

$$\forall n \geq 1, \quad \beta_{\mu,n}(T) = 0.$$

On the other hand, we observe that $w_{\mu,n}(x, t) := \Phi_{\mu,n}(x)e^{\lambda_{\mu,n}(t-T)}$ is solution of the adjoint problem:

$$(III. 5) \quad \begin{cases} (w_{\mu,n})_t + (w_{\mu,n})_{xx} + \frac{\mu}{x^2} w_{\mu,n} = 0 & x \in (0, 1), \quad t > 0, \\ w_{\mu,n}(0, t) = 0 & t > 0 \\ w_{\mu,n}(1, t) = 0 & t > 0. \end{cases}$$

Multiplying (I. 1) by $w_{\mu,n}$ and (III. 5) by u , we obtain

$$\left[\int_0^1 u w dx \right]_0^T - \int_0^T [u_x w]_0^1 dt + \int_0^T [w_x u]_0^1 dt = 0.$$

Taking into account that $u(T) \equiv 0$ and the boundary conditions satisfied by u and w , we obtain

$$- \int_0^1 u_0 \Phi_{\mu,n} e^{-\lambda_{\mu,n} T} dx + \int_0^T \Phi'_{\mu,n}(1) e^{\lambda_{\mu,n}(t-T)} H(t) dt = 0.$$

Therefore the question reduces into the following moment problem : find H such that

$$(III. 6) \quad \forall n \geq 1, \quad r_{\mu,n} \int_0^T H(t) e^{\lambda_{\mu,n} t} dt = \beta_{\mu,n}^0,$$

where we set

$$r_{\mu,n} = \Phi'_{\mu,n}(x=1).$$

As we look for a control H belonging to $H^1(0, T)$ such that $H(0) = 0 = H(T)$, we rather write the problem satisfied by $H'(t)$ which is simply obtained by some integration by part in time:

$$(III. 7) \quad \forall n \geq 1, \quad -\frac{r_{\mu,n}}{\lambda_{\mu,n}} \int_0^T H'(t) e^{\lambda_{\mu,n} t} dt = \beta_{\mu,n}^0.$$

III.4. **Formal solution of the moment problem.** Set artificially

$$\lambda_{\mu,0} := 0,$$

and assume for a moment that we are able to construct a family $(\sigma_{\mu,m}^+)_{m \geq 0}$ of functions $\sigma_{\mu,m}^+ \in L^2(0,T)$, which is biorthogonal to the family $(e^{\lambda_{\mu,n}t})_{n \geq 0}$, which means that:

$$(III. 8) \quad \forall m, n \geq 0, \quad \int_0^T \sigma_{\mu,m}^+(t) e^{\lambda_{\mu,n}t} dt = \delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Then let us define

$$(III. 9) \quad K(t) = - \sum_{m=1}^{+\infty} \frac{\lambda_{\mu,n} \beta_{\mu,n}^0}{r_{\mu,n}} \sigma_{\mu,m}^+(t)$$

and

$$H(t) = \int_0^t K(\tau) d\tau.$$

Then it is easy to show that, at least formally, K solves the moment problem for the derivative (III. 7).

Moreover, if $K \in L^2(0,T)$, then clearly $H \in H^1(0,T)$, $H' = K$, and $H(0) = 0$. Moreover $H(T) = 0$ thanks to the additional property that the family $(\sigma_{\mu,m}^+)_{m \geq 1}$ is orthogonal to $e^{\lambda_{\mu,0}t} = 1$. So H will be in $H^1(0,T)$ such that $H(0) = 0 = H(T)$ and will satisfy the moment problem (III. 6).

It remains to check that all this makes sense. For this purpose, we will have to prove the existence of a biorthogonal family $(\sigma_{\mu,m}^+)_{m \geq 0}$ together with suitable L^2 bounds (so that we can prove that $K \in L^2(0,T)$).

III.5. **Existence of a suitable biorthogonal family.** We will use the following result:

Theorem III.1. (see Theorem 2.4 in [6]) Assume that

$$\forall n \geq 0, \quad \lambda_n \geq 0,$$

and that there is some $\gamma_{min} > 0$ such that

$$(III. 10) \quad \forall n \geq 0, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{min}.$$

Then there exists a family $(\sigma_m^+)_{m \geq 0}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 0}$ in $L^2(0,T)$:

$$(III. 11) \quad \forall m, n \geq 0, \quad \int_0^T \sigma_m^+(t) e^{\lambda_n t} dt = \delta_{mn}.$$

Moreover, it satisfies: there is some universal constant C_u independent of T , γ_{min} and m such that, for all $m \geq 0$, we have

$$(III. 12) \quad \|\sigma_m^+\|_{L^2(0,T)}^2 \leq C_u e^{-2\lambda_m T} e^{C_u \frac{\sqrt{\lambda_m}}{\gamma_{min}}} e^{\frac{C_u}{\gamma_{min}^2 T}} B^*(T, \gamma_{min}),$$

with

$$(III. 13) \quad B^*(T, \gamma_{min}) = \frac{C_u}{T} \max\left\{T \gamma_{min}^2, \frac{1}{T \gamma_{min}^2}\right\}.$$

Remark III.1. Theorem 2.4 in [6] is formulated in the following way:

$$(III. 14) \quad \|\sigma_m^+\|_{L^2(0,T)}^2 \leq C_u e^{-2\lambda_m T} e^{C_u \frac{\sqrt{\lambda_m}}{\gamma_{min}}} B(T, \gamma_{min}),$$

with

$$(III. 15) \quad B(T, \gamma_{min}) = \begin{cases} \left(\frac{1}{T} + \frac{1}{T^2 \gamma_{min}^2} \right) e^{\frac{C_u}{\gamma_{min}^2 T}} & \text{if } T \leq \frac{1}{\gamma_{min}^2}, \\ C_u \gamma_{min}^2 & \text{if } T \geq \frac{1}{\gamma_{min}^2}, \end{cases}$$

and this is clearly equivalent to (III. 12)-(III. 13). Its proof is based on complex analysis tools, combining the approach of Seidman-Avdonin-Ivanov [38] with the addition of some parameter, as in Tucsnak-Tenenbaum [39].

As we have already noted (see Lemma III.1), the eigenvalues of the problem satisfy for all $\mu \leq 1/4$:

$$\forall n \geq 1, \quad \sqrt{\lambda_{\mu, n+1}} - \sqrt{\lambda_{\mu, n}} \geq 7\pi/8.$$

Define artificially

$$\lambda_{\mu, 0} := 0.$$

Then, for all $\mu \leq 1/4$,

$$\sqrt{\lambda_{\mu, 1}} - \sqrt{\lambda_{\mu, 0}} = j_{\nu(\mu), 1} \geq 3\pi/4,$$

using the fact that, thanks to (V. 5) and (V. 6), one can easily prove that $j_{\nu, 1} \geq 3\pi/4$ for all $\nu \geq 0$.

Therefore we can apply Theorem III.1 to the family $(e^{\lambda_{\mu, n} t})_{n \geq 0}$ provided that we choose $\gamma_{min} = \min(7\pi/8, 3\pi/4) = 3\pi/4$. We obtain that there exists a family $(\sigma_{\mu, m}^+)_{m \geq 0}$ biorthogonal to $(e^{\lambda_{\mu, n} t})_{n \geq 0}$ in $L^2(0, T)$, and such that

$$\|\sigma_{\mu, m}^+\|_{L^2(0, T)}^2 \leq C e^{-2\lambda_{\mu, m} T} e^{C\sqrt{\lambda_{\mu, m}}} \tilde{B}(T)$$

with

$$\tilde{B}(T) = \max\left(1, \frac{1}{T^2}\right) e^{C/T} \text{ for all } T > 0.$$

III.6. Upper bounds of the cost: proof of (I. 4) and (I. 6). Let us first state the preliminary Lemma (see section IV.3 for its proof):

Lemma III.2. *For all $\mu \leq 1/4$ and all $n \geq 1$, $r_{\mu, n} = (-1)^n j_{\nu(\mu), n}$.*

Then define

$$(III. 16) \quad K(t) := - \sum_{m=1}^{\infty} \frac{\lambda_{\mu, m} \beta_{\mu, m}^0}{r_{\mu, m}} \sigma_{\mu, m}^+(t), \quad \text{and} \quad H(t) := \int_0^t K(\tau) d\tau,$$

and let us check that H is an admissible control that drives the solution of (I. 1) to 0 in time T :

- first we check that $K \in L^2(0, T)$: let us write

$$\sum_{m=1}^{\infty} \frac{|\lambda_{\mu, m} \beta_{\mu, m}^0|}{|r_{\mu, m}|} \|\sigma_{\mu, m}^+\|_{L^2(0, T)} \leq \left(\sum_{m=1}^{\infty} |\beta_{\mu, m}^0|^2 \right)^{1/2} \left(\sum_{m=1}^{\infty} \frac{|\lambda_{\mu, m}|^2}{|r_{\mu, m}|^2} \|\sigma_{\mu, m}^+\|_{L^2(0, T)}^2 \right)^{1/2}.$$

Since $|r_{\mu, m}|^2 = (j_{\nu(\mu), m})^2 = \lambda_{\mu, m}$, it follows that

$$\sum_{m=1}^{\infty} \frac{|\lambda_{\mu, m} \beta_{\mu, m}^0|}{|r_{\mu, m}|} \|\sigma_{\mu, m}^+\|_{L^2(0, T)} \leq C \|u_0\|_{L^2(0, 1)} \left(\sum_{m=1}^{\infty} \lambda_{\mu, m} e^{-2\lambda_{\mu, m} T} e^{C\sqrt{\lambda_{\mu, m}}} \tilde{B}(T) \right)^{1/2}$$

which is finite. This implies that $K \in L^2(0, T)$. Therefore we have $H \in H^1(0, T)$ with of course $H(0) = 0$. And the fact that $H(T) = 0$ follows from (III. 11) with $n = 0$;

- next, we check that $H' = K$ satisfies the moment problem (III. 7):

$$\begin{aligned} \forall n \geq 1, \quad & -\frac{r_{\mu,n}}{\lambda_{\mu,n}} \int_0^T H'(t) e^{\lambda_{\mu,n} t} dt \\ &= \frac{r_{\mu,n}}{\lambda_{\mu,n}} \int_0^T \left(\sum_{m=1}^{\infty} \frac{\lambda_{\mu,m} \beta_{\mu,m}^0}{r_{\mu,m}} \sigma_{\mu,m}^+(t) \right) e^{\lambda_{\mu,n} t} dt \\ &= \frac{r_{\mu,n}}{\lambda_{\mu,n}} \sum_{m=1}^{\infty} \frac{\lambda_{\mu,m} \beta_{\mu,m}^0}{r_{\mu,m}} \delta_{mn} = \beta_{\mu,n}^0; \end{aligned}$$

- finally we check that the solution of (I. 1) satisfies $u(T) = 0$: multiplying the first equation of (I. 1) by $w_{\mu,n}(x, t) := \Phi_{\mu,n}(x) e^{\lambda_{\mu,n}(t-T)}$ and integrating by parts, we obtain that

$$\forall n \geq 1, \quad \int_0^1 u(x, T) \Phi_{\mu,n}(x) dx = 0,$$

hence $u(T) \equiv 0$.

Therefore H is an admissible control, and it follows that

$$C_{ctr-bd} \leq \frac{\|H\|_{H^1(0,T)}}{\|u_0\|_{L^2(0,1)}} \leq C \frac{\|K\|_{L^2(0,T)}}{\|u_0\|_{L^2(0,1)}},$$

hence

$$\begin{aligned} C_{ctr-bd} &\leq C \left(\sum_{m=1}^{\infty} \frac{|\lambda_{\mu,m}|^2}{|r_{\mu,m}|^2} \|\sigma_{\mu,m}^+\|_{L^2(0,T)}^2 \right)^{1/2} \\ &\leq C \sqrt{\tilde{B}(T)} \left(\sum_{m=1}^{\infty} \lambda_{\mu,m} e^{-2\lambda_{\mu,m} T} e^{C\sqrt{\lambda_{\mu,m}}} \right)^{1/2}. \end{aligned}$$

But $\lambda_{\mu,m} = (j_{\nu(\mu),m})^2$ and

$$C\sqrt{\lambda_{\mu,m}} \leq \lambda_{\mu,m} T + \frac{C'}{T} = j_{\nu(\mu),m}^2 T + \frac{C'}{T}.$$

One deduces that

$$\begin{aligned} C_{ctr-bd} &\leq C \sqrt{\tilde{B}(T)} e^{C'/T} \left(\sum_{m=1}^{\infty} (j_{\nu(\mu),m})^2 e^{-j_{\nu(\mu),m}^2 T} \right)^{1/2} \\ &\leq C e^{C''/T} \left(\sum_{m=1}^{\infty} (j_{\nu(\mu),m})^2 e^{-j_{\nu(\mu),m}^2 T} \right)^{1/2}. \end{aligned}$$

Next we use the following Lemma:

Lemma III.3. *There is some constant (independent of ν and of Y) such that :*

$$(III. 17) \quad \forall \nu \geq 0, \forall T > 0, \quad \sum_{m=1}^{\infty} (j_{\nu,m})^2 e^{-j_{\nu,m}^2 T} \leq C \frac{1+\nu^2}{T^{3/2}} e^{-(1+\nu^2)T/C}.$$

Proof of Lemma III.3. It is based on classical analysis estimates, see [8]. \square

Applying Lemma III.3, it follows that:

$$C_{ctr-bd} \leq C e^{C''/T} \frac{1+\nu}{T^{3/4}} e^{-(1+\nu^2)T/(2C)},$$

which implies (I. 4) and (I. 6). \square

III.7. **Lower bound of the cost when $\mu \in [0, \frac{1}{4}]$: proof of (I. 5).** Given $m \geq 1$, consider $u_0 = \Phi_{\mu, m}$, and let H_m be any control that drives the solution of (I. 1) to 0 in time T . Then (III. 6) gives that

$$\forall n \geq 1, \quad \int_0^T \left(r_{\mu, m} H_m(t) \right) e^{\lambda_{\mu, n} t} dt = \delta_{mn}.$$

Hence the sequence $(r_{\mu, m} H_m)_{m \geq 1}$ is biorthogonal to the set $(e^{\lambda_{\mu, n} t})_{n \geq 1}$. There exist several lower bounds in the literature for such biorthogonal sequences, in particular Guichal [20] and Hansen [22]. In this case we are going to use the following generalization of Guichal [20], proved in [6]:

Theorem III.2. (Theorem 2.5 in [6]) Assume that

$$\forall n \geq 1, \quad \lambda_n \geq 0,$$

and that there is some $0 < \gamma_{\min} \leq \gamma_{\max}$ such that

$$(III. 18) \quad \forall n \geq 1, \quad \gamma_{\min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{\max}.$$

Then there exists $c_u > 0$ independent of T , and m such that: any family $(\sigma_m^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$ satisfies:

$$(III. 19) \quad \|\sigma_m^+\|_{L^2(0, T)}^2 \geq e^{-2\lambda_m T} e^{\frac{1}{2\gamma_{\max}^2 T}} b(T, \gamma_{\max}, m),$$

with

$$(III. 20) \quad b(T, \gamma_{\max}, m) = \frac{c_u^2}{C(m, \gamma_{\max}, \lambda_1)^2 T} \left(\frac{1}{2\gamma_{\max}^2 T} \right)^{2m} \frac{1}{(4\gamma_{\max}^2 T + 1)^2}.$$

and

$$(III. 21) \quad C(m, \gamma_{\max}, \lambda_1) = m! 2^{m + \lceil \frac{2\sqrt{\lambda_1}}{\gamma_{\max}} \rceil + 1} \left(m + \left\lfloor \frac{2\sqrt{\lambda_1}}{\gamma_{\max}} \right\rfloor + 1 \right).$$

(The proof of Theorem III.2 is a natural generalization of the Hilbertian techniques used in [20].)

When $\mu \in [0, \frac{1}{4}]$, we are in position to apply Theorem III.2. Indeed, using (III. 2), we see that the assumption (III. 18) is satisfied with

$$\gamma_{\min} := \frac{7\pi}{8}, \quad \text{and} \quad \gamma_{\max} := \pi.$$

We are then in position to apply Theorem III.2, and we obtain that any family $(\sigma_{\mu, m}^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_{\mu, n} t})_{n \geq 1}$ in $L^2(0, T)$ satisfies:

$$(III. 22) \quad \|\sigma_{\mu, m}^+\|_{L^2(0, T)}^2 \geq e^{-2\lambda_{\mu, m} T} e^{\frac{32}{81\pi^2 T}} b(T, \gamma_{\max}, m),$$

with the expression of $b(T, \gamma_{\max}, m)$ given in (III. 20).

In the following, we apply this inequality for $m = 1$. Observe that, for $\nu \in [0, 1/2]$ and $n = 1$, (V. 5) gives

$$\frac{3\pi}{4} \leq \pi \left(\frac{3}{4} + \frac{\nu}{2} \right) \leq j_{\nu, 1} \leq \pi \left(1 + \frac{1}{4} \left(\nu - \frac{1}{2} \right) \right) \leq \pi.$$

Hence $(3\pi/4)^2 \leq \lambda_{\mu, 1} \leq \pi^2$, and $\lambda_{\mu, 1} \leq C(1 + \nu(\mu))$.

In particular, choosing $m = 1$ in (III. 22) and using $\lambda_{\mu, 1} \geq (3\pi/4)^2$, we obtain that there exists c_u independent of $T > 0$ and $\mu \in [0, \frac{1}{4}]$ such that

$$b(T, \gamma_{\max}, 1) \geq \frac{c_u}{T^3(1+T)^2}.$$

Next we deduce

$$\|\sigma_{\mu, 1}^+\|_{L^2(0, T)}^2 \geq \frac{c_u}{T^3(1+T)^2} e^{-2\lambda_{\mu, 1} T} e^{\frac{32}{81\pi^2 T}}.$$

Hence

$$\|r_{\mu,1}H_1\|_{L^2(0,T)}^2 \geq \frac{c_u}{T^3(1+T)^2} e^{-2\lambda_{\mu,1}T} e^{\frac{32}{81\pi^2 T}}.$$

Since $|r_{mu,1}| = j_{\nu(\mu),1} \leq \pi$, we obtain that

$$C_{ctr-bd} \geq \frac{\sqrt{c_u}}{\pi T^{3/2}(1+T)} e^{-\lambda_{\mu,1}T} e^{\frac{16}{81\pi^2 T}},$$

which, using the fact that $\lambda_{\mu,1} \leq C(1+\nu(\mu))$, proves (I. 5). \square

III.8. Lower bound of the cost when $\mu \leq 0$: proof of (I. 7). In this case, one still could apply Theorem III.2 but this would not give a good result. Indeed, in this case we have

$$\sqrt{\lambda_{\mu,n+1}} - \sqrt{\lambda_{\mu,n}} = j_{\nu(\mu),n+1} - j_{\nu(\mu),n},$$

and since $\mu \leq 0$, we have $\nu(\mu) \geq \frac{1}{2}$. Hence we deduce from Komornik-Loreti [25] p. 135 that the sequence $(j_{\nu(\mu),n+1} - j_{\nu(\mu),n})_{n \geq 1}$ is nonincreasing and converges to π , hence

$$\pi \leq j_{\nu(\mu),n+1} - j_{\nu(\mu),n} \leq j_{\nu(\mu),2} - j_{\nu(\mu),1}.$$

This says that assumption (III. 18) is satisfied, and that the best values for γ_{min} and γ_{max} are:

$$\gamma_{min} = \pi, \quad \gamma_{max} = j_{\nu(\mu),2} - j_{\nu(\mu),1}.$$

But it follows from (V. 7) that there exists $a > 0$ such that

$$j_{\nu,2} - j_{\nu,1} \sim a\nu^{1/3} \quad \text{as } \nu \rightarrow +\infty.$$

Hence

$$j_{\nu(\mu),2} - j_{\nu(\mu),1} \sim a\nu(\mu)^{1/3} \rightarrow +\infty \quad \text{as } \mu \rightarrow -\infty.$$

Therefore applying Theorem III.2 does not give a good result (since now $\gamma_{max} \rightarrow +\infty$ as $\mu \rightarrow -\infty$).

On the other hand, from point (iii) of Lemma III.1, Hence

$$(III. 23) \quad \forall \mu \leq 0, \forall n \geq N_*, \quad \sqrt{\lambda_{\mu,n+1}} - \sqrt{\lambda_{\mu,n}} \leq \gamma_{max}^*$$

with

$$(III. 24) \quad N_* := [\nu(\mu)] + 1 \quad \text{and} \quad \gamma_{max}^* := 2\pi.$$

In that context, when there is a 'bad' upper global gap γ_{max} , and a 'good' (much smaller) asymptotic upper gap γ_{max}^* , it is interesting to use the following extension of Theorem III.2:

Theorem III.3. (*Theorem 2.2 in [7]*) Assume that

$$\forall n \geq 1, \quad \lambda_n \geq 0,$$

and that there are $0 < \gamma_{min} \leq \gamma_{max}^* \leq \gamma_{max}$ such that

$$(III. 25) \quad \forall n \geq 1, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max},$$

and

$$(III. 26) \quad \forall n \geq N_*, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}^*.$$

Then any family $(\sigma_m^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$ satisfies:

$$(III. 27) \quad \|\sigma_m^+\|_{L^2(0,T)}^2 \geq e^{-2\lambda_m T} e^{\frac{2}{T(\gamma_{max}^*)^2}} b^*(T, \gamma_{max}, \gamma_{max}^*, N_*, \lambda_1, m)^2,$$

where b^* is rational in T (and explicitly given in Lemma 4.4 of [7]).

(The proof of Theorem III.3 is a natural generalization of the Hilbertian techniques used in [20]; Theorem III.3 was motivated by several problems where it appears that the global gap is quite bad, while there is a much better asymptotic gap, see [8] for an application to degenerate parabolic equations.)

We are in position to apply Theorem III.3: indeed, (III. 25) is satisfied with

$$\gamma_{min} := \pi, \quad \text{and} \quad \gamma_{max} := j_{\nu(\mu),2} - j_{\nu(\mu),1},$$

and (III. 26) is satisfied choosing N_* and γ_{max}^* as in (III. 24). Then, applying Theorem III.3, we obtain that any family $(\sigma_{\mu,m}^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_{\mu,n}t})_{n \geq 1}$ in $L^2(0, T)$ satisfies

$$\|\sigma_{\mu,m}^+\|_{L^2(0,T)}^2 \geq e^{-2\lambda_{\mu,m}T} e^{\frac{2}{4\pi^2 T}} b^*(T, \gamma_{max}, \gamma_{max}^*, N_*, \lambda_{\mu,1}, m)^2$$

with an explicit value of b^* (see Lemma 4.4 in [7]): when $m \leq N_*$, we have (III. 28)

$$b^*(T, \gamma_{max}, \gamma_{max}^*, N_*, \lambda_1, m) = C^* \frac{\sqrt{1+T\lambda_1}}{\sqrt{T}} \frac{(T(\gamma_{max}^*)^2)^{K_*+K'_*+2}}{(1+(T(\gamma_{max}^*)^2))^{N_*+K_*+K'_*+3}},$$

where

$$K_* = \left[\frac{2\sqrt{\lambda_1} + (N_* + m)\gamma_{max}}{\gamma_{max}^*} \right] - N_* + 2,$$

$$K'_* = \left[\frac{\gamma_{max}}{\gamma_{max}^*} (N_* - m) \right] - N_* + 2,$$

$$C^* = \frac{1}{(N_* + K_* + K'_* + 3)!} \frac{c_u(\gamma_{max}^*)^{2(N_*-1)}}{C^{(+)}C^{(-)}},$$

where

$$C^{(+)} = \left(\frac{\gamma_{max}}{\gamma_{max}^*} \right)^{N_*-1} \frac{(N_* + m + \lfloor \frac{2\sqrt{\lambda_1}}{\gamma_{max}^*} \rfloor + 1)!}{(m + \lfloor \frac{2\sqrt{\lambda_1}}{\gamma_{max}^*} \rfloor + 1)! \left(\lfloor \frac{2\sqrt{\lambda_1} + (N_* + m)\gamma_{max}}{\gamma_{max}^*} \rfloor + 1 \right)! (2m + \lfloor \frac{2\sqrt{\lambda_1}}{\gamma_{max}^*} \rfloor + 1)!},$$

and

$$C^{(-)} = \left(\frac{\gamma_{max}}{\gamma_{max}^*} \right)^{N_*-1} \frac{(m-1)!(N_*-m)!}{(1 + \lfloor \frac{2\gamma_{max}}{\gamma_{max}^*} (N_* - m) \rfloor)!}.$$

These expressions seem to be a little frightening, but we are looking for the behavior as $\mu \rightarrow -\infty$ i.e. $\nu(\mu) \rightarrow +\infty$, and this is not difficult to study (see [8]), and to obtain that, when $m = 1$:

$$b^*(T, \gamma_{max}, \gamma_{max}^*, N_*, \lambda_{\mu,1}, 1) \geq e^{-C\nu(\mu)^{4/3}(\ln \nu(\mu) + \ln \frac{1}{T})} \frac{\sqrt{1+T}}{\sqrt{T}},$$

hence

$$\|\sigma_{\mu,1}^+\|_{L^2(0,T)}^2 \geq \underline{b}(T, \mu, 1)^2,$$

with

$$(III. 29) \quad \underline{b}(T, \mu, 1) := e^{-\lambda_{\mu,1}T} e^{\frac{1}{4\pi^2 T}} \frac{\sqrt{1+T}}{\sqrt{T}} e^{-C\nu(\mu)^{4/3}(\ln \nu(\mu) + \ln \frac{1}{T})}.$$

This gives the following lower bound of the cost when $\mu \leq 0$:

$$C_{ctr-bd} \geq \frac{1}{|r_{\mu,1}|} \underline{b}(T, \mu, 1) = \frac{1}{j_{\nu(\mu),1}} \underline{b}(T, \mu, 1)$$

So, using (V. 6), we get

$$\begin{aligned} C_{ctr-bd} &\geq \frac{8}{\pi(7+2\nu(\mu))} e^{-\lambda_{\mu,1}T} e^{\frac{1}{4\pi^2 T}} e^{-C\nu(\mu)^{4/3}(\ln \nu(\mu) + \ln \frac{1}{T})} \frac{\sqrt{1+T}}{\sqrt{T}} \\ &\geq e^{-\pi^2(\frac{3}{4} + \frac{\nu(\mu)}{2})^2 T} e^{\frac{1}{4\pi^2 T}} e^{-C'\nu(\mu)^{4/3}(\ln \nu(\mu) + \ln \frac{1}{T})} \frac{\sqrt{1+T}}{\sqrt{T}} \\ &\geq e^{-\frac{\pi^2}{2}\nu(\mu)^2 T} e^{-\frac{9\pi^2}{8}T} e^{\frac{1}{4\pi^2 T}} e^{-C'\nu(\mu)^{4/3}(\ln \nu(\mu) + \ln \frac{1}{T})}. \end{aligned}$$

This proves (I. 7).

IV. TECHNICAL PART

IV.1. Proof of Proposition III.1. Let us first observe that any admissible eigenvalue λ satisfies $\lambda > 0$ (use for instance Theorem II.1). Therefore in the following we assume $\lambda \neq 0$. Using the following changes of variables,

$$\phi(x) = \sqrt{x}\psi(\sqrt{\lambda}x) \quad \text{and} \quad y = \sqrt{\lambda}x,$$

one can easily see that ϕ satisfies (III. 1) if and only if ψ is solution of

$$(IV. 1) \quad \begin{cases} y^2\psi''(y) + y\psi'(y) + \left(y^2 - \left(\frac{1}{4} - \mu\right)\right)\psi(y) = 0 & y \in (0, \sqrt{\lambda}), \\ \psi(0) = 0 = \psi(\sqrt{\lambda}). \end{cases}$$

Hence ψ is a solution of the Bessel equation (V. 1) (see section V.1) of order

$$\nu(\mu) := \sqrt{\frac{1}{4} - \mu}.$$

Let us now solve (IV. 1).

IV.1.a. Case 1 : $\nu(\mu) \notin \mathbb{N}$. Let us first treat the case $\nu(\mu) \notin \mathbb{N}$. Observe that, in that case, $\mu \neq 1/4$ so $H_0^1(\mu) = H_0^1(0, 1)$.

The space of solutions of the differential equation

$$(IV. 2) \quad y^2\psi''(y) + y\psi'(y) + (y^2 - \nu(\mu))\psi(y) = 0$$

is a vector space of dimension 2. Since $\nu(\mu) \notin \mathbb{N}$, a fundamental system of solutions of (IV. 2) is given by the Bessel's functions of the first kind : $J_{\nu(\mu)}$ and $J_{-\nu(\mu)}$, (see section V.2).

Hence solutions of equation (IV. 2) are linear combinations of $J_{\nu(\mu)}$ and $J_{-\nu(\mu)}$:

$$\forall y \in (0, \sqrt{\lambda}), \quad \Psi(y) = C_+ J_{\nu(\mu)}(y) + C_- J_{-\nu(\mu)}(y),$$

with $C_+, C_- \in \mathbb{R}$. Thus

$$\forall x \in (0, 1), \quad \Phi(x) = C_+ \sqrt{x} J_{\nu(\mu)}(\sqrt{\lambda}x) + C_- \sqrt{x} J_{-\nu(\mu)}(\sqrt{\lambda}x).$$

We will denote

$$(IV. 3) \quad \Phi_+(x) := \sqrt{x} J_{\nu(\mu)}(\sqrt{\lambda}x), \quad \Phi_-(x) := \sqrt{x} J_{-\nu(\mu)}(\sqrt{\lambda}x).$$

Using the development in series of $J_{\nu(\mu)}$ and $J_{-\nu(\mu)}$ (see section V.2), we obtain:

$$(IV. 4) \quad \Phi_+(x) = \sum_{m=0}^{\infty} c_{\nu(\mu),m}^+ (\sqrt{\lambda})^{2m+\nu(\mu)} x^{2m+\nu(\mu)+1/2},$$

$$(IV. 5) \quad \Phi_-(x) = \sum_{m=0}^{\infty} c_{\nu(\mu),m}^- (\sqrt{\lambda})^{2m-\nu(\mu)} x^{2m-\nu(\mu)+1/2}.$$

We will denote

$$(IV. 6) \quad \tilde{c}_{\nu(\mu),m}^+ := c_{\nu(\mu),m}^+ (\sqrt{\lambda})^{2m+\nu(\mu)}, \quad \tilde{c}_{\nu(\mu),m}^- := c_{\nu(\mu),m}^- (\sqrt{\lambda})^{2m-\nu(\mu)},$$

in such a way that

$$(IV. 7) \quad \Phi_+(x) = \sum_{m=0}^{\infty} \tilde{c}_{\nu(\mu),m}^+ x^{2m+\nu(\mu)+1/2}, \quad \Phi_-(x) = \sum_{m=0}^{\infty} \tilde{c}_{\nu(\mu),m}^- x^{2m-\nu(\mu)+1/2}.$$

Then, let us verify that $\Phi_+ \in H^1(0, 1)$:

$$(IV. 8) \quad \Phi_+(x)^2 \underset{x \rightarrow 0}{\sim} \left(\tilde{c}_{\nu(\mu),0}^+ \right)^2 x^{2\nu(\mu)+1} \underset{x \rightarrow 0}{\rightarrow} 0.$$

So we have $\Phi_+ \in L^2(0, 1)$. Next

$$\Phi_+'(x)^2 \underset{x \rightarrow 0}{\sim} \frac{\left(\tilde{c}_{\nu(\mu),0}^+ \right)^2 (\nu(\mu) + 1/2)^2}{x^{1-2\nu(\mu)}}.$$

Since $\nu(\mu) > 0$, we have $1 - 2\nu(\mu) < 1$ and $\Phi_+' \in L^2(0, 1)$. Hence $\Phi_+ \in H^1(0, 1)$.

Next we look at Φ_- :

$$\Phi_-(x)^2 \underset{x \rightarrow 0}{\sim} \frac{\left(\tilde{c}_{\nu(\mu),0}^- \right)^2}{x^{2\nu(\mu)-1}}.$$

We deduce that $\Phi_- \in L^2(0, 1)$ if only if $\nu(\mu) < 1$ i.e. $\mu > -3/4$. Moreover, even in the case $\mu > -3/4$, we have $\Phi_-' \notin L^2(0, 1)$ since

$$\Phi_-'(x)^2 \underset{x \rightarrow 0}{\sim} \frac{\left(\tilde{c}_{\nu(\mu),0}^- \right)^2 (-\nu(\mu) + 1/2)^2}{x^{2\nu(\mu)+1}}$$

with $\nu(\mu) > 0$. So Φ_- never belongs to $H^1(0, 1)$.

In conclusion, we have $\Phi_+ \in H^1(0, 1)$ but $\Phi_- \notin H^1(0, 1)$. It implies that $C_- = 0$ and so

$$\Phi(x) = C_+ \Phi_+(x) = C_+ \sqrt{x} J_{\nu(\mu)}(\sqrt{\lambda}x).$$

Now let us take into account the boundary conditions. From (IV. 8), we have $\Phi_+(0) = 0$ so the condition $\Phi(0) = 0$ is automatically satisfied. Next the condition $\Phi(1) = 0$ turns into $C_+ J_{\nu(\mu)}(\sqrt{\lambda}) = 0$ which means that, in order to get a non trivial solution, $\sqrt{\lambda}$ has to be a zero of $J_{\nu(\mu)}$. So there exists some n such that

$$\lambda = (j_{\nu(\mu),n})^2.$$

Finally,

$$\forall x \in (0, 1), \quad \Phi(x) = C_+ \sqrt{x} J_{\nu(\mu)}(j_{\nu(\mu),n}x).$$

Reciprocally, for all $n \geq 1$ and all $C_+ \in \mathbb{R}$, we have a solution of (IV. 1). Hence we have solved (IV. 1).

Now consider

$$\forall n \geq 1, \quad \tilde{\Phi}_{\mu,n}(x) := \sqrt{x} J_{\nu(\mu)}(j_{\nu(\mu),n}x).$$

The orthogonality properties of the Bessel's functions imply that $(\tilde{\Phi}_{\mu,n})_n$ forms an orthogonal family of $L^2(0, 1)$: indeed, using (V. 8) in section V.5, we have

$$\begin{aligned} \int_0^1 \tilde{\Phi}_{\mu,n}(x) \tilde{\Phi}_{\mu,m}(x) dx &= \int_0^1 x J_{\nu(\mu)}(j_{\nu(\mu),n}x) J_{\nu(\mu)}(j_{\nu(\mu),m}x) dx \\ &= \delta_{nm} [J_{\nu(\mu)+1}(j_{\nu(\mu),n})]^2 = \delta_{nm} [J'_{\nu(\mu)}(j_{\nu(\mu),n})]^2, \end{aligned}$$

where we also used the fact that Bessel function of the first kind satisfy the following recurrence formulae (see [44, p. 45, relation (4)]):

$$x J'_\nu(x) - \nu J_\nu(x) = -x J_{\nu+1}(x),$$

hence $J_{\nu+1}(j_{\nu,n}) = -J'_\nu(j_{\nu,n})$.

Therefore one can normalize the eigenfunctions $\tilde{\Phi}_{\mu,n}$ into

$$\Phi_{\mu,n}(x) = \frac{1}{|J'_{\nu(\mu)}(j_{\nu(\mu),n})|} \sqrt{x} J_{\nu(\mu)}(j_{\nu(\mu),n}x), \quad x \in (0,1), \quad n \geq 1.$$

Finally the family $(\Phi_{\mu,n})_{n \geq 1}$ forms an orthonormal basis of $L^2(0,1)$ since they are the eigenfunctions of the operator L_μ .

IV.1.b. *Case 2* : $\nu(\mu) = n(\mu) \in \mathbb{N}^*$. Let us assume that $\nu(\mu) = n(\mu) \in \mathbb{N}^*$. Observe that, once again in that case, $\mu \neq 1/4$ so $H_0^1(\mu) = H_0^1(0,1)$.

Since $\nu(\mu) \in \mathbb{N}^*$, a fundamental system of solutions of (IV. 2) is given by the Bessel's functions of the first kind and the Bessel's functions of the second kind : $J_{\nu(\mu)}$ and $Y_{\nu(\mu)}$, (see section V.3). So, if we denote

$$(IV. 9) \quad \Phi_{+,-}(x) := \sqrt{x} Y_{n(\mu)}(\sqrt{\lambda}x),$$

then Φ takes the form $\Phi = C_+ \Phi_+ + C_{+,-} \Phi_{+,-}$. We proved in case 1 that $\Phi_+ \in H^1(0,1)$. It remains to study if $\Phi_{+,-} \in H^1(0,1)$. Using the decomposition in series of $Y_{n(\mu)}$ (see (V. 4) in section V.3), we get

(IV. 10)

$$\Phi_{+,-}(x) = \frac{2}{\pi} \Phi_+(x) \log \left(\frac{\sqrt{\lambda}x}{2} \right) + \sum_{m=0}^{n(\mu)-1} \hat{a}_m x^{2m-n(\mu)+1/2} + \sum_{m=0}^{+\infty} \hat{b}_m x^{n(\mu)+2m+1/2},$$

where

$$\hat{a}_m := -\frac{1}{\pi} \frac{(n(\mu) - m - 1)!}{m!} \left(\frac{\sqrt{\lambda}}{2} \right)^{2m-n(\mu)}$$

and

$$\hat{b}_m := -\frac{1}{\pi} \frac{(-1)^m}{m!(n(\mu) + m)!} \left(\frac{\sqrt{\lambda}}{2} \right)^{2m+n(\mu)} \left[\frac{\Gamma'(m+1)}{\Gamma(m+1)} + \frac{\Gamma'(m+n(\mu)+1)}{\Gamma(m+n(\mu)+1)} \right].$$

In the following, we study these three functions that appear in the formula of $\Phi_{+,-}$. First, let us study

$$(IV. 11) \quad \Phi_{+,-,1}(x) := \frac{2}{\pi} \Phi_+(x) \ln \left(\frac{\sqrt{\lambda}x}{2} \right).$$

It satisfies

$$\Phi_{+,-,1}(x) \underset{x \rightarrow 0}{\sim} \frac{2}{\pi} \tilde{c}_{n(\mu),0}^+ x^{n(\mu)+1/2} \ln \left(\frac{\sqrt{\lambda}x}{2} \right).$$

Hence we deduce that $\Phi_{+,-,1} \in L^2(0,1)$. Moreover

$$\Phi'_{+,-,1}(x) = \frac{2}{\pi} \Phi'_+(x) \ln \left(\frac{\sqrt{\lambda}x}{2} \right) + \frac{2}{\pi} \Phi_+(x) \frac{1}{x} \underset{x \rightarrow 0}{\sim} \frac{2}{\pi} \frac{\tilde{c}_{n(\mu),0}^+ (n(\mu) + 1/2)}{x^{1/2-n(\mu)}} \ln \left(\frac{\sqrt{\lambda}x}{2} \right).$$

Since $n(\mu) \geq 1$, we deduce that $\Phi'_{+,-,1} \in L^2(0,1)$ and finally $\Phi_{+,-,1} \in H^1(0,1)$.

Next we study

$$\Phi_{+,-,2}(x) := \sum_{m=0}^{n(\mu)-1} \hat{a}_m x^{2m-n(\mu)+1/2}.$$

It satisfies

$$\Phi_{+,-,2}(x) \underset{x \rightarrow 0}{\sim} \hat{a}_0 x^{-n(\mu)+1/2}, \quad \Phi'_{+,-,2}(x) \underset{x \rightarrow 0}{\sim} \hat{a}_0 (-n(\mu) + 1/2) x^{-n(\mu)-1/2},$$

hence $\Phi_{+,-,2} \notin H^1(0,1)$, since $\hat{a}_0 \neq 0$ and $n(\mu) > 0$.

Finally, let us observe that

$$(IV. 12) \quad \Phi_{+,-,3}(x) := \sum_{m=0}^{+\infty} \hat{b}_m x^{n(\mu)+2m+1/2}$$

satisfies

$$\Phi_{+,-,3}(x) \sim_0 \hat{b}_0 x^{n(\mu)+1/2}, \quad \Phi'_{+,-,3}(x) \sim_0 \hat{b}_0 (n(\mu) + 1/2) x^{n(\mu)-1/2}.$$

Since $n(\mu) \geq 1$, we deduce that $\Phi_{+,-,3} \in H^1(0, 1)$.

Finally $\Phi_{+,-} = \Phi_{+,-,1} + \Phi_{+,-,2} + \Phi_{+,-,3} \notin H^1(0, 1)$. Consequently $\Phi = C_+ \Phi_+ + C_{+,-} \Phi_{+,-} \in H^1(0, 1)$ necessarily implies that $C_{+,-} = 0$. Then $\Phi = C_+ \Phi_+$ and we are in the same position as in case 1 and the conclusion is the same.

IV.1.c. *Case 3* : $\nu(\mu) = 0$ (i.e. $\mu = 1/4$). Let us assume that $\nu(\mu) = 0$. Observe that, in that case, $\mu = 1/4$ so $H_0^1(1/4) \neq H_0^1(0, 1)$. We recall that

$$H_0^1(1/4) = \{z \in H^1(1/4) \mid z(0) = 0 = z(1)\}$$

where $H^1(1/4)$ is the Hilbert space obtained as the completion of $H^1(0, 1)$ with respect to the norm $\|z\|_{L^2(0,1)} + \|z\|_{1/4}$ with

$$\|z\|_{1/4} := \left(\int_0^1 \left(z_x^2 - \frac{1}{4} \frac{z^2}{x^2} \right) dx \right)^{1/2}.$$

As in case 2, since $\nu(1/4) = 0 \in \mathbb{N}$, Φ takes the form $\Phi = C_+ \Phi_+ + C_{+,-} \Phi_{+,-}$.

In the following, we will first prove that $\Phi_+ \in H^1(1/4)$ whereas $\Phi_{+,-} \notin H^1(1/4)$. This implies that $\Phi = C_+ \Phi_+$ otherwise $\Phi \notin H^1(1/4)$. Then we take into account the boundary conditions coming from the fact that $\Phi \in H_0^1(1/4)$ and the conclusion is the same as in case 1.

So let us first prove that $\Phi_+ \in H^1(1/4)$. Taking $\nu(\mu) = 0$ in (IV. 7), we get

$$(IV. 13) \quad \Phi_+(x) = \tilde{c}_{0,0}^+ x^{1/2} + \tilde{c}_{0,1}^+ x^{5/2} + o(x^{5/2}) = \tilde{c}_{0,0}^+ x^{1/2} \left(1 + \frac{\tilde{c}_{0,1}^+}{\tilde{c}_{0,0}^+} x^2 + o(x^2) \right) \text{ as } x \rightarrow 0,$$

and also

$$(IV. 14) \quad \begin{aligned} \Phi_+'(x) &= \frac{\tilde{c}_{0,0}^+}{2} x^{-1/2} + \frac{5\tilde{c}_{0,1}^+}{2} x^{3/2} + o(x^{3/2}) \\ &= \frac{\tilde{c}_{0,0}^+}{2} x^{-1/2} \left(1 + \frac{5\tilde{c}_{0,1}^+}{\tilde{c}_{0,0}^+} x^2 + o(x^2) \right) \text{ as } x \rightarrow 0. \end{aligned}$$

It follows that

$$\Phi_+'(x)^2 - \frac{1}{4} \frac{\Phi_+(x)^2}{x^2} \underset{x \rightarrow 0}{\sim} 2\tilde{c}_{0,0}^+ \tilde{c}_{0,1}^+ x.$$

Hence $\Phi_+ \in H^1(1/4)$.

Next, let us prove that $\Phi_{+,-} \notin H^1(1/4)$. In this case, the first sum in the decomposition of Y_0 has to be taken equal to zero (see section V.3 in appendix), hence we have $\Phi_{+,-} = \Phi_{+,-,1} + \Phi_{+,-,3}$. As in case 2, we study separately the two functions $\Phi_{+,-,1}$ and $\Phi_{+,-,3}$.

From (IV. 11) and (IV. 13), we get

$$(IV. 15) \quad \Phi_{+,-,1}(x) = \frac{2}{\pi} \tilde{c}_{0,0}^+ x^{1/2} \ln \left(\frac{\sqrt{\lambda} x}{2} \right) \left(1 + \frac{\tilde{c}_{0,1}^+}{\tilde{c}_{0,0}^+} x^2 + o(x^2) \right) \text{ as } x \rightarrow 0.$$

And from (IV. 13) and (IV. 14), we compute

$$(IV. 16) \quad \Phi'_{+,-,1}(x) = \frac{\tilde{c}_{0,0}^+}{\pi} x^{-1/2} \ln \left(\frac{\sqrt{\lambda x}}{2} \right) \left[1 + \frac{2}{\ln \left(\frac{\sqrt{\lambda x}}{2} \right)} + o(x^{3/2}) \right] \text{ as } x \rightarrow 0.$$

It follows that

$$\begin{aligned} \Phi'_{+,-,1}(x)^2 - \frac{1}{4} \frac{\Phi_{+,-,1}(x)^2}{x^2} &= \frac{(\tilde{c}_{0,0}^+)^2}{\pi^2 x} \ln \left(\frac{\sqrt{\lambda x}}{2} \right)^2 \left[\frac{4}{\ln \left(\frac{\sqrt{\lambda x}}{2} \right)} + o(x^{3/2}) \right] \\ &\sim_0 \frac{4(\tilde{c}_{0,0}^+)^2}{\pi^2} \frac{1}{x} \ln \left(\frac{\sqrt{\lambda x}}{2} \right) \notin L^1(0, 1). \end{aligned}$$

Hence $\Phi_{+,-,1} \notin H^1(1/4)$.

Next we prove that $\Phi_{+,-,3} \in H^1(1/4)$. Indeed we can compute from (IV. 12):

$$\begin{aligned} \Phi_{+,-,3}(x) &= \hat{b}_0 x^{1/2} \left[1 + \frac{\hat{b}_1}{\hat{b}_0} x^2 + o(x^2) \right], \\ \Phi'_{+,-,3}(x) &= \frac{1}{2} \hat{b}_0 x^{-1/2} \left[1 + \frac{5\hat{b}_1}{\hat{b}_0} x^2 + o(x^2) \right]. \end{aligned}$$

Hence

$$\Phi'_{+,-,3}(x)^2 - \frac{1}{4} \frac{\Phi_{+,-,3}(x)^2}{x^2} \sim_0 2\hat{b}_0 \hat{b}_1 x \in L^1(0, 1).$$

Finally this implies that $\Phi_{+,-} \notin H^1(1/4)$ which ends the proof. \square

IV.2. Proof of Lemma III.1. The proof of Lemma III.1 directly follows from the following one:

Lemma IV.1.

- (i) For any $\nu \in [0, 1/2]$ and any $n \geq 1$, $7\pi/8 \leq j_{\nu,n+1} - j_{\nu,n} \leq \pi$.
- (ii) For any $\nu \geq 1/2$ and any $n \geq 1$, $\pi \leq j_{\nu,n+1} - j_{\nu,n}$.
- (iii) For any $\nu \geq 1/2$ and any $n > \nu$, $j_{\nu,n+1} - j_{\nu,n} \leq 2\pi$.

Indeed, using that $\lambda_{\mu,n} = (j_{\nu(\mu),n})^2$, we get Lemma III.1. So it remains to prove Lemma IV.1.

(i) First consider $\nu \in [0, \frac{1}{2}]$. Using (V. 5), we get

$$j_{\nu,n+1} - j_{\nu,n} \geq \pi \left(\frac{7}{8} + \frac{\nu}{4} \right).$$

Since $\nu \geq 0$, it follows that

$$j_{\nu,n+1} - j_{\nu,n} \geq \frac{7\pi}{8}.$$

Moreover it is classical (see [25] p. 135) that, when $\nu \in [0, \frac{1}{2}]$, the sequence $(j_{\nu,n+1} - j_{\nu,n})_{n \geq 1}$ is nondecreasing and converges to π when $n \rightarrow \infty$. This completes the proof of (i) of Lemma III.1.

(ii) Next consider $\nu \geq 1/2$. Here we use the fact (see section V.4) that, for $\nu \geq \frac{1}{2}$, the sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ is nonincreasing and converges to π . This ensures that $j_{\nu,n+1} - j_{\nu,n} \geq \pi$.

(iii) Finally, (iii) is proved in Lemma 5.1 of [8] (in the spirit of Komornik-Loreti [25] p. 135, using classical Sturm theory for second order differential equations).

IV.3. **Proof of Lemma III.2.** We compute

$$\Phi'_{\mu,n}(x) = \frac{J_{\nu(\mu)}(j_{\nu(\mu),n}x)}{2|J'_{\nu(\mu)}(j_{\nu(\mu),n})|\sqrt{x}} + \frac{\sqrt{x}J'_{\nu(\mu)}(j_{\nu(\mu),n}x)j_{\nu(\mu),n}}{|J'_{\nu(\mu)}(j_{\nu(\mu),n})|}.$$

Hence

$$r_{\mu,n} = \Phi'_{\mu,n}(1) = \frac{J'_{\nu(\mu)}(j_{\nu(\mu),n})j_{\nu(\mu),n}}{|J'_{\nu(\mu)}(j_{\nu(\mu),n})|} = (-1)^n j_{\nu(\mu),n}.$$

V. APPENDIX : ELEMENTS OF BESSEL THEORY

For reader convenience, we recall here the definitions concerning Bessel's equations and functions together with some useful properties of these functions and of their zeros. *Throughout this section, we assume that $\nu \in \mathbb{R}_+$.*

V.1. **Bessel's equation and Bessel's functions of order ν .** The Bessel's functions of order ν are the solutions of the following differential equation (see [44, section 3.1, eq. (1), p. 38] or [27, eq (5.1.1), p. 98]):

$$(V. 1) \quad x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, \quad x \in (0, +\infty).$$

The above equation is called *Bessel's equation for functions of order ν* . Of course the fundamental theory of ordinary differential equations says that the solutions of (V. 1) generate a vector space S_ν of dimension 2. In the following we recall what can be chosen as a basis of S_ν .

V.2. **Fundamental solutions of Bessel's equation when $\nu \notin \mathbb{N}$.** Assume that $\nu \notin \mathbb{N}$. When looking for solutions of (V. 1) of the form of series of ascending powers of x , one can construct two series that are solutions:

$$\sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(\nu + m + 1)} \left(\frac{x}{2}\right)^{\nu+2m} \quad \text{and} \quad \sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(-\nu + m + 1)} \left(\frac{x}{2}\right)^{-\nu+2m},$$

where Γ is the Gamma function (see [44, section 3.1, p. 40]). The first of the two series converges for all values of x and defines the so-called Bessel function of order ν and *of the first kind* which is denoted by J_ν :

$$(V. 2) \quad J_\nu(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^+ x^{2m+\nu}, \quad x \geq 0,$$

(see [44, section 3.1, (8), p. 40] or [27, eq. (5.3.2), p. 102]). The second series converges for all positive values of x and is evidently $J_{-\nu}$:

$$(V. 3) \quad J_{-\nu}(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m - \nu + 1)} \left(\frac{x}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^- x^{2m-\nu}, \quad x > 0.$$

When $\nu \notin \mathbb{N}$, the two functions J_ν and $J_{-\nu}$ are linearly independent and therefore the pair $(J_\nu, J_{-\nu})$ forms a fundamental system of solutions of (V. 1), (see [44, section 3.12, eq. (2), p. 43]).

V.3. **Fundamental solutions of Bessel's equation when $\nu = n \in \mathbb{N}$.** Assume that $\nu = n \in \mathbb{N}$. When looking for solutions of (V. 1) of the form of series of ascending powers of x , one sees that J_n and J_{-n} are still solutions of (V. 1), where J_n is still defined by (V. 2) and

$$J_{-n}(x) = \sum_{m \geq n} \frac{(-1)^m}{m! \Gamma(-n + m + 1)} \left(\frac{x}{2}\right)^{-n+2m}.$$

However now $J_{-n}(x) = (-1)^n J_n(x)$, hence J_n and J_{-n} are linearly dependent, (see [44, section 3.12, p. 43] or [27, eq. (5.4.10), p. 105]). The determination of a fundamental system of solutions in this case requires further investigation. In this

purpose, one introduces the Bessel's of order ν and *of the second kind*: among the several definitions of Bessel's functions of second order, we recall here the definition by Weber. The Bessel's functions of order ν and *of second kind* are denoted by J_ν and defined by (see [44, section 3.54, eq. (1)-(2), p. 64] or [27, eq. (5.4.5)-(5.4.6), p. 104]):

$$\begin{cases} \forall \nu \notin \mathbb{N}, & Y_\nu(x) := \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \\ \forall n \in \mathbb{N}, & Y_n(x) := \lim_{\nu \rightarrow n} Y_\nu(x). \end{cases}$$

For any $\nu \in \mathbb{R}_+$, the two functions J_ν and Y_ν always are linearly independent, see [44, section 3.63, eq. (1), p. 76]. In particular, in the case $\nu = n \in \mathbb{N}$, the pair (J_n, Y_n) forms a fundamental system of solutions of the Bessel's equation for functions of order n .

In the case $\nu = n \in \mathbb{N}$, it will be useful to expand Y_n under the form of a series of ascending powers. This can be done using Hankel's formula, see [44, section 3.52, eq. (3), p. 62] or [27, eq. (5.5.3), p. 107]:

$$(V. 4) \quad \forall n \in \mathbb{N}^*, \quad Y_n(x) = \frac{2}{\pi} J_n(x) \log\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n} \\ - \frac{1}{\pi} \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{x}{2}\right)^{n+2m} \left[\frac{\Gamma'(m+1)}{\Gamma(m+1)} + \frac{\Gamma'(m+n+1)}{\Gamma(m+n+1)} \right],$$

where $\frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the Gamma function, and satisfies $\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$ (here γ denotes Euler's constant) and

$$\frac{\Gamma'(m+1)}{\Gamma(m+1)} = 1 + \frac{1}{2} + \dots + \frac{1}{m} - \gamma \quad \text{for all } m \in \mathbb{N}.$$

In the case $n = 0$, the first sum in (V. 4) should be set equal to zero.

V.4. Zeros of Bessel functions of order ν of the first kind. The function J_ν has an infinite number of real zeros which are simple with the possible exception of $x = 0$ ([44, section 15.21, p. 478-479 applied to $C_\nu = J_\nu$] or [27, section 5.13, Theorem 2, p. 127]). We denote by $(j_{\nu,n})_{n \geq 1}$ the strictly increasing sequence of the positive zeros of J_ν :

$$0 < j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,n} < \dots$$

and we recall that

$$j_{\nu,n} \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

We will also use the following bounds on the zeros, proved in Lorch and Muldoon [30]:

$$(V. 5) \quad \forall \nu \in [0, \frac{1}{2}], \forall n \geq 1, \quad \pi(n + \frac{\nu}{2} - \frac{1}{4}) \leq j_{\nu,n} \leq \pi(n + \frac{\nu}{4} - \frac{1}{8}),$$

$$(V. 6) \quad \forall \nu \geq \frac{1}{2}, \forall n \geq 1, \quad \pi(n + \frac{\nu}{4} - \frac{1}{8}) \leq j_{\nu,n} \leq \pi(n + \frac{\nu}{2} - \frac{1}{4}).$$

We also mention the following inequality from [36]

$$(V. 7) \quad \forall \nu > 0, \forall n \geq 1, \quad \nu - \frac{a_n}{2^{1/3}} \nu^{1/3} < j_{\nu,n} < \nu - \frac{a_n}{2^{1/3}} \nu^{1/3} + \frac{3}{20} a_n^2 \frac{2^{1/3}}{\nu^{1/3}}$$

where a_n is the n -th negative zero of the Airy function.

Moreover, we recall that it is classical ([25] p. 135) that

- if $\nu \in [0, \frac{1}{2}]$, the sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ is nondecreasing and converges to π ,
- if $\nu \geq \frac{1}{2}$, the sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ is nonincreasing and converges to π .

V.5. Orthogonality property. For all $\nu \geq 0$, Bessel functions satisfy the following orthogonality property (see [27, eq. (5.14.4) and (5.14.6), p. 129]):

$$(V. 8) \quad \int_0^1 x J_\nu(j_{\nu,n}x) J_\nu(j_{\nu,m}x) dx = \begin{cases} \frac{1}{2} [J_{\nu+1}(j_{\nu,n})]^2 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

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